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Specific Luminosity Limit of e+e- Colliding Rings

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ABSTRACT

The maximum luminosity achievable in a flat beam, e+e- circular collider depends on nonlinear effects that limit particle lifetimes and are amenable to study only by computer simulation. But, for intemediate currents the luminosity dependence is governed by beam profile distortion that is unambiguously described by a linear equation that is exactly solvable with pencil and paper. This equation describes the "parametric pumping" of the vertical betatron amplitude of each particle by its own (inexorable) horizontal amplitude. For given tunes Q_x and Q_y, and given beam height σ_{y0} (due to extraneous but well-understood sources) this parametric oscillation is either stable or unstable; the beam current at the transition point can be expressed as a threshold tune shift value $\xi_{y,\mathrm{thr.}}(Q_x, Q_y; \sigma_{y0})$. The value of σ_{y0} (in a well-tuned-up ring) is typically small enough that this pumping phenomenon governs the "specific luminosity" (luminosity/current). Once the threshold is passed the luminosity may increase, but the specific luminosity "saturates".

An initially-only-conjectured dependence of *maximum* luminosity on "damping decrement" δ_y is, by now, fairly well established empirically. The present parametric pumping model yields the related, but not equivalent, dependence of *specific* luminosity on δ_y. For typical, but favorably chosen, tune combinations the model predicts $\xi_{y,\mathrm{thr.}}$ to be proportional to $\delta_y^{1/2}$. For unfavorable tunes the exponent in this relation is 1, and there may be "excellent" tune combinations for which the exponent is 1/3. How much the luminosity can be increased by increasing beam currents beyond the saturation value is not addressed.

1. The Beam-Beam Deflection

The dependence of vertical beam-beam deflection $\Delta y'$ on vertical displacement y is shown in Fig. 1.1. The beam-beam tune shift parameter ξ_y is defined to be the tune shift caused by this force acting on a small amplitude particle. The angular deflection $\Delta y'$ and the tune shift ΔQ_y caused by a quadupole of strength q at a place where the beta function is β_y are given by

$$\Delta y' = qy, \quad \text{and} \quad \Delta Q_y = \frac{\beta_y q}{4\pi} \ . \tag{1.1}$$

Eliminating q from these relations, and setting $\Delta Q_y = \xi_y$, yields the formula

$$\Delta y' = \frac{4\pi\xi_y}{\beta_y}\, y \ . \tag{1.2}$$

This dependence is labeled “equivalent quadrupole” in the figure.

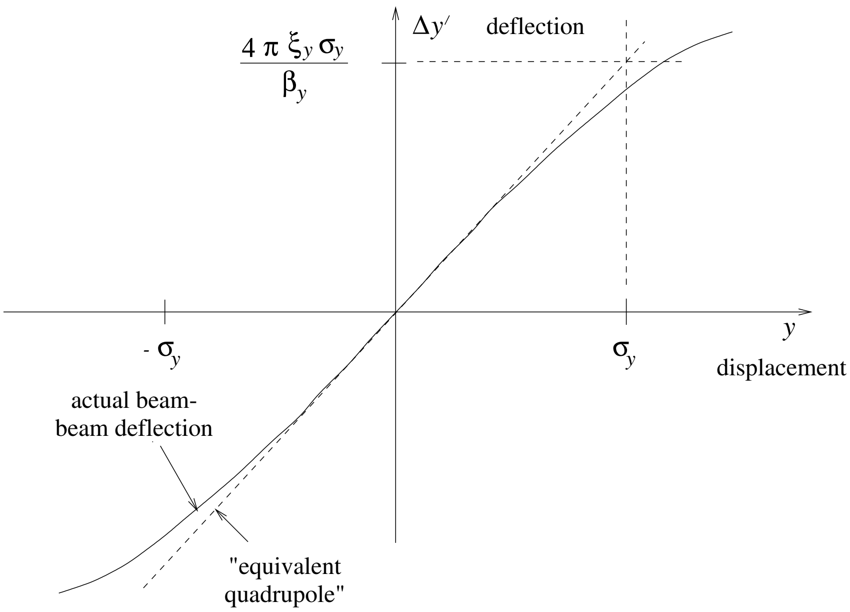

Figure 1.1: Dependence of vertical deflection $\Delta y'$ on vertical displacement y. The deflection of an “equivalent” quadrupole of strength $q = 4\pi\xi_y/\beta_y$ is also shown.

Consider a “typical particle” for which the vertical phase space components, just before colliding with the opposing bunch, are $y_- = \sigma_y$, $y'_- = 0$, so its Courant-Snyder invariant is $\epsilon_{y,CS} = \sigma_y^2/\beta_y$. The graph shows that in passing through the other beam at the intersection point, the particle’s deflection is almost $4\pi\xi_y\sigma_y/\beta_y$; (the defect is 14%.) For this particle the effect of the beam-beam impulse on the Courant-Snyder invariant is

$$\epsilon_{y,CS} \to \frac{\sigma_y^2}{\beta_y} + \beta_y \left(\frac{4\pi\xi_y}{\beta_y}\right)^2 \sigma_y^2 = \epsilon_{y,CS} \left(1 + (4\pi\xi_y)^2\right) \ . \tag{1.3}$$

A tune shift parameter $\xi_y \approx 1/(4\pi)$ therefore causes a rough doubling of the Courant-Snyder invariant of the particle. This formulation makes it all the more impressive when tune shifts approaching 0.1 are achieved, for example with flat beams at LEP and round beams at CESR. It seems that the beam-beam tune shift parameter might better have been defined with an extra factor of 4π since that would yield the mnemonically more satisfactory value of 1 as the tune shift parameter that causes a rough doubling of the Courant-Snyder invariant. As ξ is *in fact* defined, it is therefore important to keep in mind that $\xi = 0.1$ is a *big* value.

2. Beam-Beam Observations from Existing Storage Rings

Fig. 2.1 shows beam-beam tune shift data, available in 1983, from PETRA and CESR, extrapolated in both directions, to encompass both electrons and protons. This analysis was initially performed during the LEP design phase to predict the luminosities to be expected, and the projection has proved to be quite accurate for LEP. The maximum tune shift parameter for VLLC (a post-LEP circular collider) can conservatively be predicted to be at least $\xi_y^{\max} \approx 0.12$. On the other hand, the extrapolation to the small damping decrements relevant for proton colliders has already been contradicted by Tevatron performance. This does not, however, contradict the theory presented in this paper, which applies only to flat, not round, beams.

Other data from existing colliding rings is shown in Figs. 2.2 and Fig. 2.3. Comments concerning the relevance to the present paper are given in the captions. In an ideal (perfectly decoupled) ring the beam width is much greater than the beam height. Since the horizontal motion is "hot" and the vertical "cold" any mechanism that couples these motions tends to affect the vertical motion a lot, and the horizontal motion hardly at all.

The observed beam-beam phenomenology is that, when colliding with the other beam the horizontal beam distributions is largely independent of beam current, but, above some threshold current, the beam height increases proportional to beam current. This causes the beam-beam tune shift parameter to "saturate" and no longer increase with increasing

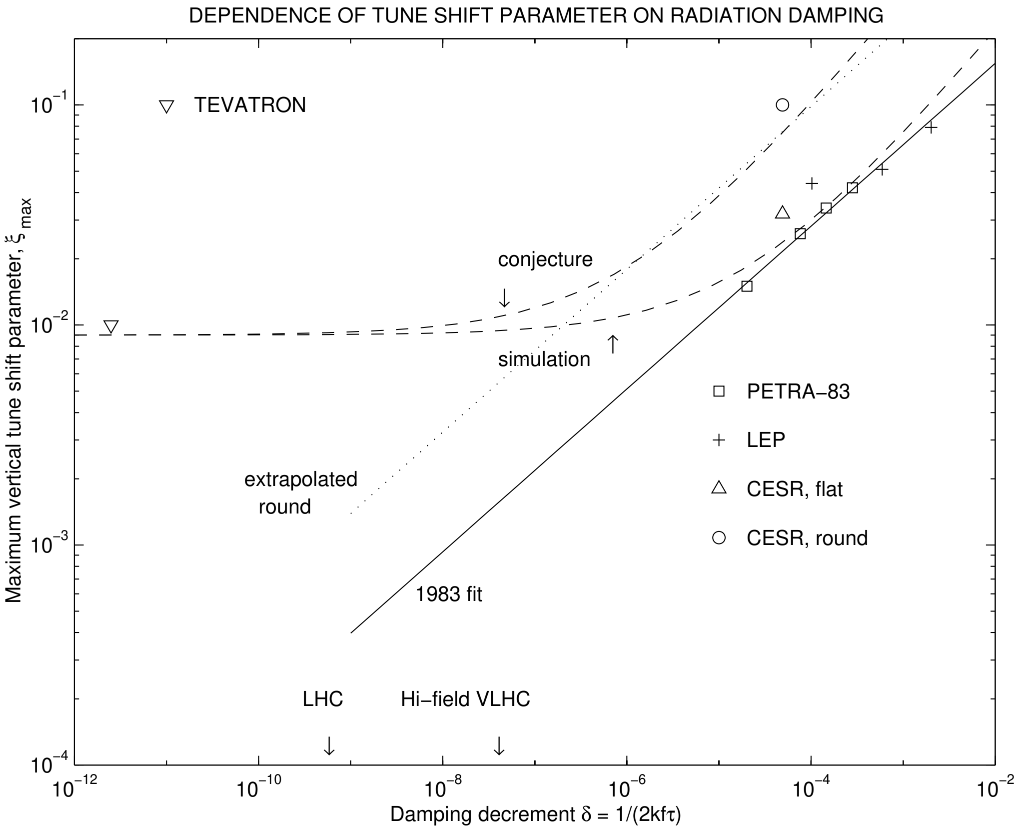

Figure 2.1: Dependence of maximum vertical tuneshift parameter $\xi_{\max}$ on damping decrement $1/(2kf\tau)$, where k is number of bunches, f is revolution frequency, and τ is damping time. The line labeled "1983 fit" was conjectured in 1983 by Keil and Talman (Part. Accel. 14, 109 (1983)) based on data available at the time; it describes well LEP data that was acquired subsequently. The curve labeled "simulation" linking the ultralow (proton) and ultrahigh (electron) regions is due to Peggs (private communication). The curve labeled "conjecture" is my fit (adjusting a parameter in the Peggs formula) to the Tevatron point and a (slightly downward adjusted) round beam CESR point.

beam current. This behavior at LEP is exhibited in Fig. 2.3, copied from D. Brandt et al. According to the theory in the present paper, this behavior would set in already at arbitrarily small beam current in a perfect ring but this behavior is masked by any beam height σ_{y0} present due to single beam effects, especially coupling. This picture is supported by observed behavior in which improving the decoupling reduces the threshold current at which saturation sets in. When running LEP at highest energy, 100 Gev, no saturation was observed up to the highest possible beam current. This might contradict the model being presented but the authors note that the coupling coefficent could not be reduced below $\kappa = 0.8\%$, The present paper contains nothing that could account for the saturation of ξ_x suggested by Fig. 2.3.

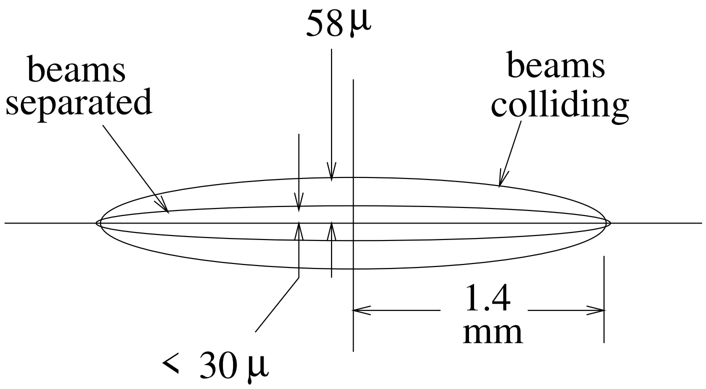

Figure 2.2: Beam profiles (r.m.s. sizes represented by ellipses) measured using synchrotron light impinging on video camera during operation of CESR. The r.m.s. beam heights with beams not in collision were not greater than 30 μ which was the optical resolution of the viewing apparatus. That the horizontal profiles are unaffected corresponds to the assumption in the paper that this motion is "inexorable". The beam height enlargement is shown in this paper to be due to "parametric pumping" of vertical oscillations by the horizontal oscillations.

3. "Subharmonic" Parametric Excitation of Vertical Oscillations

We now turn to the analysis of beam-beam distortion. The leading parametric resonance in mechanical oscillators occurs for drive frequency equal to twice the natural frequency (so the response is a "subharmonic" of the drive.) This phenomenon is clearly explained by, for example, Landau and Lifshitz, *Mechanics*; other than employing difference equations rather than differential equations, the present treatment mirrors their treatment. The need for difference equations arises because of the impulsive nature of beam-beam interactions. For the same reason the phenomenon of "aliasing", without changing the essence, increases the number of possible resonances and alters the vocabulary.

From a pedagogical point of view the reader unfamiliar with difference equations might profit from first reading Appendix A, which uses difference equations to solve for betatron

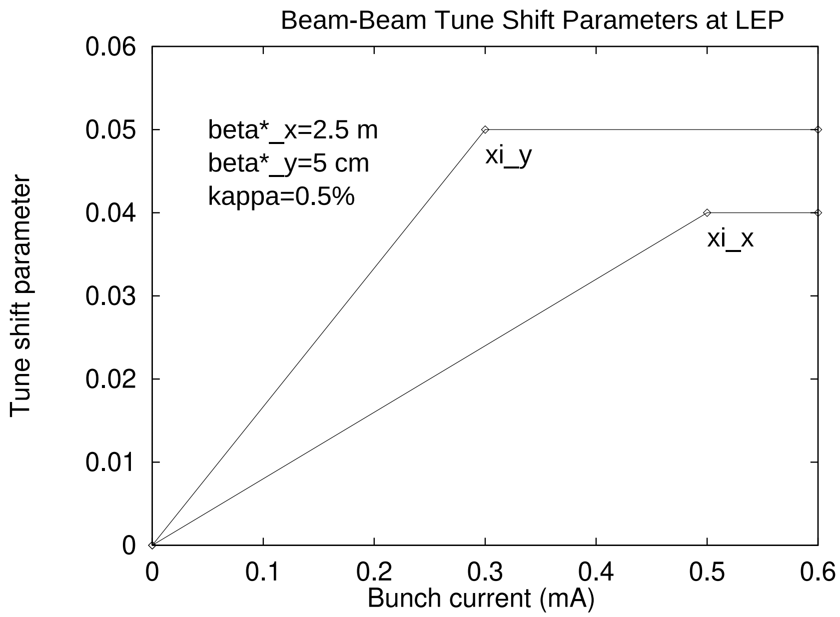

Figure 2.3: Dependence of ξ_x and ξ_y on beam current. Data from LEP running at 65 GeV, D. Brandt et. al., Rep. Prog. Phys. **63** (2000) 939-100. Similar behavior is observed at CESR, though saturation of ξ_x (with increasing current) is not observed at CESR. Also, saturation of ξ_y was not observed for operationally practical beam currents during highest energy running at LEP.

response to an external shaker. Because that drive is "direct" the analysis is simpler than this section requires. Higher order parametric resonances are analysed in Appendix C.

The vertical beam-beam deflection, given previously by Eq. (1.2), actually depends also on the horizontal displacement. Because the beams are ribbon-shaped, and the horizontal profile is Gaussian the deflection, on turn t, is given by†‡

$$\Delta y'_t = \frac{4\pi\xi_y}{\beta_y}\,\exp\left(-\frac{a_x^2\cos^2\mu_x t}{2}\right)\,y_t\ , \tag{3.1}$$

where a_x is the horizontal particle amplitude in units of the r.m.s. beam width and ξ_y is now to be interpreted as the value of the tune shift parameter at $x = 0$. It is appropriate to Fourier expand the nonlinear exponential function;

$$\Delta y'_t = \frac{4\pi\xi_y}{\beta_y}\,\left(\sum_{n=0}^{\infty}\left(\frac{B_0}{2} + B_n\cos\left(2n\mu_x t\right)\right)\right)\,y_t \tag{3.2}$$

† Though t could stand for "time" it more appropriately stands for "turn number" and can only take integer values.

‡ "Tunes" Q and phase advances per turn $\mu = 2\pi Q$ will be interchanged as convenient, and without warning, in this paper, and either may be referred to as a "frequency".

n	$B_n(0)$	$B_n(1)$	$B_n(2)$	$B_n(3)$	$B_n(4)$	$B_n(5)$
0	2.	1.58	.932	.575	.414	.326
1	0.	.196	.416	.422	.358	.299
2	0.	.0122	.0999	.199	.235	.231
3	0.	.000509	.0163	.0680	.122	.151
4	0.	.0000159	.00201	.0180	.0519	.0854
5	0.	.397e-6	.000200	.00390	.0185	.0420
6	0.	.827e-8	.0000165	.000710	.00566	.0182
7	0.	.148e-9	.118e-5	.000112	.00151	.00703
8	0.	.231e-11	.733e-7	.0000154	.000359	.00245

Table 3.1: Fourier coefficients $B_n(a_x)$ as given by Eq. (3.3).

The coefficients B_n can be evaluated in terms of (modified) Bessel functions I_n using an integral from Watson, *Bessel Functions,* 6.22(4); the result is

$$B_n = 2\exp\left(-\frac{a_x^2}{4}\right) I_n\left(-\frac{a_x^2}{4}\right), \quad n = 0, 1, 2, \dots\ . \tag{3.3}$$

Values of B_n are given in Table 3.1. The first row and first column are shown only for completeness. B_0 can (and will) be set to zero as far as the mechanism of this paper is concerned. Because the "effective tune" of the vertical gradient acting on the particle under study is $2\mu_x$, the leading "subharmonic" resonance occurs for $\mu_x \approx \mu_y$ which, in accelerator jargon, is a "difference resonance".

What with frequency aliasing it is possible for any of the terms in the sum (3.2) to "resonate" with a pre-existing vertical betatron oscillation;

$$y_t = a_t \cos\left((\mu_y + \varepsilon_n)\, t\right) + b_t \sin\left((\mu_y + \varepsilon_n)\, t\right) \equiv a_t \cos\tilde{\mu}_y + b_t \sin\tilde{\mu}_y \tag{3.4}$$

where the "pulled" frequency offset ε_n will be defined shortly. The quantity $\mu_y + \varepsilon_n$ will be systematically replaced by $\tilde{\mu}_y$ throughout this paper, even though this suppresses the (essential) index n. The coefficients a_t and b_t are "variation of constants" coefficients whose variation will be arranged later to satisfy the equation of motion. They are assumed to vary slowly with t; that is, their fractional changes per revolution are small compared to 1. If they are treated as depending on a continuous variable t, then

$$a_{t\pm1} \approx a_t \pm \dot{a}_t, \text{ and } b_{t\pm1} \approx b_t \pm \dot{b}_t\ . \tag{3.5}$$

Combining Eqs. (3.2) and (3.4) yields

$$\begin{aligned}
\frac{\Delta y'_t}{4\pi\xi_y/\beta_y} &= \sum_{n=1}^{\infty} B_n \cos(2n\mu_x t)\ \left(a_t\ \cos(\tilde{\mu}_y t) + b_t\ \sin(\tilde{\mu}_y t)\right) \\
&= \sum_{n=1}^{\infty} \frac{B_n}{2}\ \left(a_t \cos\left(\left(2n\mu_x - \mu_y - \varepsilon_n^{(-)}\right) t\right) - b_t \sin\left(\left(2n\mu_x - \mu_y - \varepsilon_n^{(-)}\right) t\right)\right) \\
&+ \sum_{n=1}^{\infty} \frac{B_n}{2}\ \left(a_t \cos\left(\left(2n\mu_x + \mu_y + \varepsilon_n^{(+)}\right) t\right) + b_t \sin\left(\left(2n\mu_x + \mu_y + \varepsilon_n^{(+)}\right) t\right)\right) .
\end{aligned} \tag{3.6}$$

Any of these terms can potentially cause resonance, with the frequency offset $\varepsilon_n^{(\pm)}$ quantifying the "distance from resonance". These phase offsets are defined by the following relations (for which the overall signs are not significant.)

$$\begin{aligned}
2n\mu_x + \mu_y + \varepsilon_n^{(+)} &= -\left(\mu_y + \varepsilon_n^{(+)}\right),\ \text{or}\ \varepsilon_n^{(+)} = n\mu_x + \mu_y\ , \\
2n\mu_x - \mu_y - \varepsilon_n^{(-)} &= +\left(\mu_y + \varepsilon_n^{(-)}\right),\ \text{or}\ \varepsilon_n^{(-)} = n\mu_x - \mu_y\ .
\end{aligned} \tag{3.7}$$

Presumably one of these possibilities, say $n,-$, will dominate over all others. From here on the index n will be taken to indicate this particular dominant case, and Eq. (3.6) becomes

$$\Delta y'_t = \frac{4\pi\xi_y}{\beta_y}\,\frac{B_n}{2}\ \left(a_t \cos(\tilde{\mu}_y t) - b_t \sin(\tilde{\mu}_y t)\right)\ . \tag{3.8}$$

The difference equation describing weakly damped betatron motion is derived in Appendix A. Setting damping decrement δ_y temporarily to zero, Eq. $(A.4)$ becomes

$$y_{t+1} - 2C_y y_t + y_{t-1} = S_y\, 2\pi\xi_y\, B_n\ \left(a_t \cos(\tilde{\mu}_y t) - b_t \sin(\tilde{\mu}_y t)\right); \tag{3.9}$$

(abbreviations $C_y = \cos\mu_y$ and $S_y = \sin\mu_y$ are employed here.) In preparation for substituting Eq. (3.4) into this equation, using Eqs. (3.5), we obtain

$$\begin{aligned}
y_{t+1} = &\ (a_t + \dot{a}_t)\left(\cos\tilde{\mu}_y \cos(\tilde{\mu}_y t) - \sin\tilde{\mu}_y \sin(\tilde{\mu}_y t)\right) \\
&+\ \left(b_t + \dot{b}_t\right)\left(\sin\tilde{\mu}_y \cos(\tilde{\mu}_y t) + \cos\tilde{\mu}_y \sin(\tilde{\mu}_y t)\right) \\
y_{t-1} = &\ (a_t - \dot{a}_t)\left(\cos\tilde{\mu}_y \cos(\tilde{\mu}_y t) + \sin\tilde{\mu}_y \sin(\tilde{\mu}_y t)\right) \\
&+\ \left(b_t - \dot{b}_t\right)\left(-\sin\tilde{\mu}_y \cos(\tilde{\mu}_y t) + \cos\tilde{\mu}_y \sin(\tilde{\mu}_y t)\right)
\end{aligned} \tag{3.10}$$

Performing these substitutions, and requiring that the sine and cosine terms vanish separately, yields the equations

$$\begin{aligned}
-\dot{a}_t \sin\tilde{\mu}_y + b_t \cos\tilde{\mu}_y - C_y b_t - S_y\,\pi\xi_y\, B_n\ (-b_t) = 0 \\
\dot{b}_t \sin\tilde{\mu}_y + a_t \cos\tilde{\mu}_y - C_y a_t - S_y\,\pi\xi_y\, B_n\ (a_t) = 0
\end{aligned} \tag{3.11}$$

We seek a solution for which a_t and b_t exhibit time dependence of the form $\exp(st)$;

$$\begin{aligned} s\,a_t - \frac{\cos\tilde{\mu}_y - C_y + S_y\,\pi\xi_y\,B_n}{\sin\tilde{\mu}_y}\,b_t = 0 \\ \frac{\cos\tilde{\mu}_y - C_y - S_y\,\pi\xi_y\,B_n}{\sin\tilde{\mu}_y}\,a_t + s\,b_t = 0 \end{aligned} \tag{3.12}$$

The requirement for such a solution to exist is that the determinant formed from the coefficients must vanish; this yields

$$s^2 = \frac{-\left(\cos\tilde{\mu}_y - C_y\right)^2 + \left(S_y\,\pi\xi_y\,B_n\right)^2}{\sin^2\tilde{\mu}_y} \approx -\varepsilon_n^2 + \pi^2\xi_y^2\,B_n^2\ . \tag{3.13}$$

In the last step it has been assumed that $\varepsilon_n << 1$. In this form the condition for *unstable* motion is that s^2 be positive, which requires†

$$-\pi\xi_y\,B_n < \varepsilon_n < \pi\xi_y\,B_n\ . \tag{3.14}$$

It is customary to call such excluded regions "stop bands".

By setting δ_y to zero we have been neglecting damping so far and have found that, even with no damping, if ε_n lies outside this range, the motion will be stable—the horizontal oscillation will not "pump up" vertical oscillations. But, in an ideal electron storage ring, if there were no cross-plane coupling or other extraneous source of vertical excitation, ξ_y would be infinite because the vertical beam height would vanish. (This uses the result that synchrotron-radiated photons are emitted precisely in the forward direction; since their typical angle is $1/\gamma$ this is an excellent, but not perfect assumption.) In this ideal limit the stability condition would be violated for any finite beam current. In this limit the parametric pumping that is being described blows up the beam until condition (3.14) is satisfied.

In fact there *is* damping, as represented by $\delta_y \neq 0$. The threshold of instability is therefore determined by the condition that the (positive) growth rate given by Eq. (3.13) is equal to δ_y;

$$\sqrt{-\varepsilon_n^2 + \pi^2\xi_y^2\,B_n^2} = \delta_y, \quad \text{or} \quad \varepsilon_n = \sqrt{\pi^2\xi_y^2\,B_n^2 - \delta_y^2}\ . \tag{3.15}$$

† The stability limits (3.14), though not the growth rate in the interior, could have been determined by setting $\dot{a}_t = \dot{b}_t = 0$ in Eq. (3.11). The justification is that amplitude neither grows nor shrinks at the ends of the range.

The band of instability is therefore given by

$$-\sqrt{\pi^2\xi_y^2\,B_n^2-\delta_y^2}<\varepsilon_n<\sqrt{\pi^2\xi_y^2\,B_n^2-\delta_y^2}\ . \tag{3.16}$$

For $\delta_y > \pi|\xi_y B_n|$ there is no unstable band at all. It is in this role that δ_y has its greatest influence on the beam-beam interaction for flat beams. (This can be contrasted with a case discussed in Appendix B, for which δ_y has little influence.)

4. Tune Scan of Stop Bands

To indicate one possible use of these formulas, consider the largest coefficient in Table 3.1, $B_1(a_x = 1) = 0.42$ and, in particular, the task of choosing tunes to avoid vertical beam growth from this resonance for a particle with this (highly probable) horizontal amplitude. Assuming Q_y is given, and temporarily taking δ_y to be negligibly small, according to Eq. (3.14), the lowest order sum resonance establishes an excluded "stop band"

$$\Delta Q_x = \xi_y B_n \ = 0.42\xi_y. \tag{4.1}$$

Since the available fractional tune range is only $\Delta Q_x = 0.5$ this sets an upper limit on ξ_y of order 1 (at least with negligible δ_y.) This (and the corresponding sum resonance) are probably close to the worst possible cases. Tune ranges excluded for other n values or for other reasons (nonlinear, synchrobetatron, etc.) have been ignored. Also there are higher order parametric resonance (analysed in Appendix C) that can cause vertical beam growth.

By scanning all possible values of Q_x and Q_y, with δ_y still set to zero, one can identify favorable and unfavorable tune regions based on their stop band widths. For first order (meaning soon-to-be-introduced index r has value 1) parametric resonance one can evaluate all possible values of ξ_y satisfying Eq. (3.14) and from them define

$$\tilde{\xi}_y\,(Q_x,Q_y)=\min_{n\pm,\,a_x}\left[\frac{2}{B_n\,(a_x)}\,|nQ_x\pm Q_y-\ \text{nearest integer}|\right], \tag{4.2}$$

where a_x is allowed to range over values having appreciable probability, for example from 1 to 4, and n ranges over positive integers. The result of doing this is shown in Table 4.1.

Table 4.1 indicates strong tune dependence, as one would expect, with many tune combinations contra-indicated. Many are exactly zero, though this is a binning artifact.

Q_x / Q_y	.025	.075	.125	.175	.225	.275	.325	.375	.425	.475
.025	0.	.14	.38	.62	.85	.96	0.	.98	0.	.04
.075	0.	0.	.14	.38	.19	0.	.16	.16	.38	.04
.125	0.	.043	0.	0.	0.	.16	.85	0.	0.	.19
.175	0.	.043	.14	0.	.14	0.	.62	.16	.043	.81
.225	.85	0.	.043	0.	0.	.14	.38	.043	.16	1.1
.275	1.1	.16	.043	.19	.14	0.	0.	.043	0.	.85
.325	1.3	.19	.16	.043	0.	.14	0.	.14	.16	0.
.375	1.6	0.	0.	.043	.16	0.	0.	0.	.14	0.
.425	1.8	1.1	.16	.16	0.	.043	.38	.14	0.	0.
.475	2.0	0.	.19	0.	.043	.043	.62	.19	.14	0.

Table 4.1: $\tilde{\xi}_y(Q_x, Q_y)$ given by Eq. (4.2) for the lower left quadrant of fractional tunes. Other quadrants can be obtained by mirroring in integers or half integers. For the values listed the "stop band width" just overlaps the assumed ± 0.01 beam tune spread. Values exactly 0.0 occur where the table grid matches resonance lines.

Other tunes appear to be very favorable, but a table like this, based as it is on "stop band widths" is misleading for several reasons. The least essential of these is that far finer binning would be necessary for the table to be useful. Another is that higher order resonances ($r \neq 1$, in the notation of Appendix C) have not been included. Probably most important of all is that aliasing is not properly accounted for by Eq. (4.2). The point is that resonance is caused by equality of *cosines* of tunes rather than equality of the tunes themselves. Tune combinations that require large values of n to bring the factor $|nQ_x \pm Q_y -$ nearest integer$|$ in Eq. (4.2) close to zero may give a small difference of cosines for a much smaller value of n. This is illustrated by resonance diagrams Fig. 5.1 to be discussed in the next section. Also, any effect of damping decrement has not yet been included. For these reasons it is not useful to refine this particular table. Rather, just a few combinations will be considered in the rest of the paper. Still, in spite of all these reasons, (except neglect of δ_y) the values given in Table 4.1 constitute upper limits for ξ_y.

These estimates make no allowance as yet for the damping decrement δ_y. From Eq. (3.16) one sees non-vanishing δ_y reduces the resonance band to $2\sqrt{(\pi\xi_y B_n)^2 - \delta_y^2}$. Since typical values of δ_y are in the range 10^{-4} to 10^{-3} or, at most, 10^{-2} (for highest

Ring	Q_x/IP	Q_y/IP	n	r	$10^4\delta_y$	$B_{n,\mathrm{max.}}$	$(\pi B_n)^{-1}\,\delta_y^{1/(1+\lvert r-1\rvert)}$
LEP-46	.58	.04	1	2	1.02	0.42	0.0077
LEP-65	.57	.04	1	2	5.9	0.42	0.018
LEP-46	.59	.04	1	2	20.4	0.42	0.034
CESR(1)	.52	.58	3	1	0.49	0.24	0.00006
CESR(2)			4	2		0.052	0.042
PEP-LER	.570	.642			1.21		
PEP-HER	.618	.638	4	2	1.96	0.052	0.086

Table 5.1: Parameters of some circular, flat beam, e+e- colliding rings.

energy operation at LEP), the δ_y^2 correction term is negligible for strong resonances, such as those listed in the first few rows of Table 3.1. Such resonances have to be avoided by judicious choice of tunes. Let us assume that tunes have been selected which satisfy this requirement.

5. Single Resonance Dominance

Tune plane/resonance diagrams for important *linear* resonances are shown in Fig. 5.1. Tune combinations for a few existing colliding beam facilities are shown in Table 5.1 as well as in Fig. 5.1. The KEK B-factory is not shown because its large crossing angle complicates the simple picture of this paper.

The master formula governing exact resonance is Eq. (C.3). Expressed in terms of tunes it is

$$Q_y = \pm\frac{2n}{1+r}\,Q_x \tag{5.1}$$

where n is a positive integer and r is any integer. Here Q_x and Q_y are fractional tunes. The restriction to fractional tunes in Fig. 5.1 is enforced by "periodic boundary conditions". When a line terminates on an integer boundary another line with the same slope starts from the same location on the opposite boundary. To help in identifying lines, only cases with the plus sign are exhibited in the upper figure and the key gives Eq. (5.1) for each line. The lower figure contains these lines plus the mirrored lines obtained with the minus sign in Eq. (5.1).

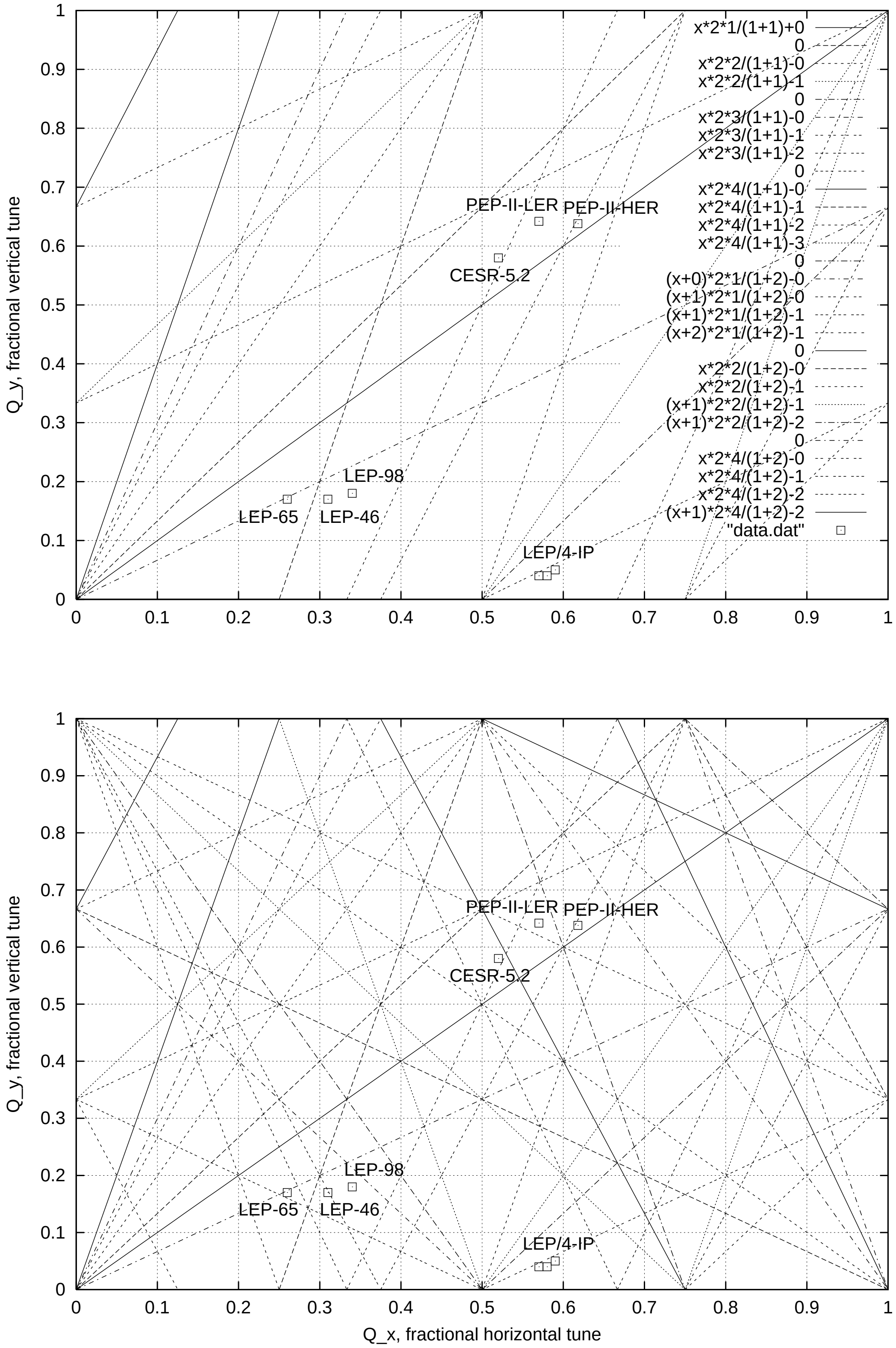

Figure 5.1: Linear parametric beam-beam resonances.

The parametric growth mechanism that has been analysed is very powerful since it causes the vertical amplitude of single particles to grow exponentially. It is not necessary for all horizontal amplitudes to be resonant. Again referring to Table 3.1, it is therefore not possible to exclude whole rows at a time by choice of tunes. Rather, because of the horizontal tune spread, particular horizontal amplitudes can resonate. This leads to the realization that the tune ranges to be avoided have to be expanded by the range of horizontal tunes. There is also a spread of vertical tunes, but its effect is more complicated. This spread increases the probability that some particle will be resonant, but the accompanying "detuning" tends to moderate, and may even reverse, the amplitude growth of that particular particle.

It is almost a cliché that ξ_x and ξ_y are "tune spreads" rather than "tune shifts". Even though this is not very precise, let us accept that the horizontal tune spread is ξ_x. Also part of the lore, and fairly well supported empirically, is that, for optimal performance, $\xi_x \approx \xi_y$. Accepting these rules of thumb, and recognizing that tune shifts exceeding, say, 0.02 are routine in existing rings, one will accept that the operating point has a spread of roughly this size. As a result, one is *always* running under the influence of at least one of the parametric resonances in Table 3.1 or possibly one of the higher order resonances analysed in Appendix C.

The model to be adopted therefore assumes that it is impossible to avoid all resonances and for any given operating conditions it is necessary only to identify and analyse the one that is dominant. Entries to this effect have been made in Table 5.1. In the case of CESR there are two possible assignments labeled (1) and (2). It is likely, however, that (2) is the appropriate assignment, since Q_y increases more rapidly than Q_x with increasing beam current which moves the operating point away from (1). However the presence of such a damaging resonance so close to the normal operating point suggests an experiment to test the formulas of this paper—strong growth of the beam height and corresponding reduction of the specific luminosity is predicted when Q_x is increased at fixed Q_y.

Since it is assumed, no matter what the tunes are, that a particular resonance is dominant, the onset of beam growth is controlled by δ_y. For the lowest order ($r = 1$)

resonance, according to Eq. (3.16), the instability sets in at

$$\xi_{n,1} = \frac{\delta_y}{\pi B_n} \tag{5.2}$$

where the second subscript stands for $r = 1$. This is a special case of formulas for $\xi_{y\mathrm{sat.}}$ derived in the appendix which take the form

$$\xi_{n,r} = \frac{1}{\pi B_n}\, \delta_y^{1/(1+|r-1|)}\, T_{n,r}^{1/(1+|r-1|)} \tag{5.3}$$

where $T_{n,r}$ is a trigonometric function of the tunes, whose value is approximately 1. For example, from Eq. (C.13)

$$\xi_{n,0} = \frac{\delta_y^{1/2}}{\pi B_n} \sqrt{\frac{C_y - \cos 2\tilde{\mu}_y}{S_y}}. \tag{5.4}$$

Values obtained from Eq. (5.3) with $T_{n,r} = 1$ are included in Table 5.1.

6. Conclusions and Conjectures

Tune combinations for which $r = 1$ yield such small values of $\xi_{y,\mathrm{thr.}}$ it seems reasonable to suppose they have always been, and will always be, avoided operationally. This is the basis of the statement in the abstract that "for unfavorable tunes $\xi_{y,\mathrm{thr.}} \sim \delta_y$." For "once-removed" resonances, $r = 0$ or $r = 2$, $\xi_{y,\mathrm{thr.}} \sim \sqrt{\delta_y}$. This is the most prevalent case in Table 5.1 and it may be "generic" for "good" tune combinations. No assignment has been made for PEP-LER in Table 5.1 since none of the resonance lines of Fig. 5.1 come close to that operating point. This is one (feeble) basis for the comment in the abstract that there may be "excellent" points for which $\xi_y \sim \sqrt[3]{\delta_y}$.

For the straight line fit of Fig. 2.1, the exponent is 3/8. But this particular exponent was picked on the basis of data available two decades ago and was not determined with great accuracy even then. The parametric pumping model seems to be in at least semi-quantitative agreement and gives what is, to me, a persuasive explanation of the prominent influence of the damping decrement on luminosity.

The absolute $\xi_{y,\mathrm{sat.}}$ entries in the last column of Table 5.1 are intended to be only semi-quantitative. To be regarded as predictions the entries would have to be made more carefully and correct values of $T_{n,r}$ included. One reason this has not been done is that the higher order calculation of Appendix C seems to be not entirely self-consistent and still higher order calculations have not been attempted. (Since the equations are linear it should be possible to do this using Maple or Mathematica.) There is no ambiguity about the exponent in Eq. (5.3), however, once the dominant resonance is identified.

There is one way in which the growth described so far is "too powerful". It is that the exponential growth of y amplitude diverges to infinity, which is clearly unphysical. Apart from this violating the precondition that $y << \sigma_x$, an effect that has been left out, which moderates this behavior, is the nonlinearity as a function of y. (See Fig. 1.1.) As individual particles come into resonance their amplitudes build, but this growth is accompanied by detuning, that eventually defeats the resonance. In the process the Courant-Snyder invariant of the particular particle will have been "heated"† and the particle will be left in a state that contributes appreciably to the beam height (at least for a time comparable with the equilibration time.) A detailed dynamical description of this mechanism would have amplitudes tending to "pile up" a bit at amplitudes near the stability boundary. The distribution would therefore be non-Gaussian. Another way the resonance can be moderated is that the "vertical heating" is accompanied by "horizontal cooling". Since these mechanisms contradict the assumptions of the model and rely on the detailed dynamical evolution of the beam distributions, they are hard to calculate. This is why the present model cannot predict *maximum* luminosities.

In contrast with the inherently nonlinear behavior mentioned in the previous paragraph the model described in this paper is *linear*, in spite of the (obviously nonlinear) Gaussian beam profile that figures so prominently. The point here is that the horizontal motion is robust in spite of the nonlinearity in x and the equation describing y *is* linear.

Certain other effects, which have been neglected in this paper, may be subject to similar analysis. Certainly the presence of horizontal dispersion at the interaction point would cause horizontal motion. All formulas in this paper would still apply (after introducing the synchrotron tune Q_s by the replacement $Q_x \rightarrow Q_s$.) This effect would be significant if the transverse motion accomanying this "synchrotron oscillation" is comparable with σ_x.

† When the resonance curve of an oscillator becomes multiple-valued because of nonlineariy, it is possible for a large-amplitude (and hence unstable) particle to jump discontinuously to a stable point of different amplitude. Since this process is emittance nonconserving, it contributes to the growth of vertical beam size. A dynamical theory that calculates the absolute beam size caused by these two effects (parametric pumping plus discontinuous jumps in Courant-Snyder invariant) is not available, but computer simulations have born out the essential features of this model with semi-quantitative accuracy.

Even with no dispersion there are mechanisms that couple the longitudinal and vertical motion[‡] and may be subject to similar equations. One of these is the "hourglass effect"—it would become significant for $\beta_y \lesssim \sigma_s$. Another is the crossing angle (call it Θ) effect—the usual criterion ($\Theta\sigma_s > \sigma_x$) probably identifies the region of importance of this effect.

Finally a gratuitous comment on a seemingly unrelated topic. There is a school of thought suggesting that a next-generation, very large hadron collider should use flat beams. This paper shows this will not be possible because of the extraordinarily small values of δ_y under even the most optimistic assumptions.

[‡] Joe Rogers has reminded me.

Appendices

A. Excitation of Vertical Betatron Motion by an External Shaker

The method of difference equations will be employed. To illustrate this method, before applying it to the actual problem, it will be used in this section to calculate the vertical motion induced by the "direct drive" due to an external "shaker". As well as introducing the method of analysis, the equations of motion and an example of aliasing, this introduces the important damping decrement δ_y and shows how it influences the motion. It will, however, turn out that the influence of δ_y on parametric drive (the main topic of the paper) is very different from its influence on direct drive (the topic of this section.)

The deflection caused by the external drive on the t'th turn is

$$\Delta y'_t = F_E \cos \mu_E t. \tag{A.1}$$

We postulate a small "damping decrement" δ_y, so that the once-around transfer map in "Twiss form" is

$$\begin{pmatrix} y \\ y' - \Delta y'/2 \end{pmatrix}_{t+1} = \exp(-\delta_y) \begin{pmatrix} C_y + \alpha_y S_y & \beta_y S_y \\ -\gamma_y S_y & C_y - \alpha_y S_y \end{pmatrix} \begin{pmatrix} y \\ y' + \Delta y'/2 \end{pmatrix}_t \tag{A.2}$$

and a similar equation can be written for backwards propagation from t to $t-1$. Note that y' is evaluated at the middle of the shaker. We are using the notation $C_y \equiv \cos \mu_y$ and $S_y \equiv \sin \mu_y$ and are intentionally using the subscript t as a turn index to be suggestive of the time measured in units of the revolution period. It will however always be an integer. For these two maps the top equations are

$$\begin{aligned} y_{t+1} &= \exp(-\delta_y) \left[(C_y + \alpha_y S_y)\, y_t + \beta_y S_y \left(y' + \Delta y'/2 \right)_t \right] \\ y_{t-1} &= \exp(+\delta_y) \left[(C_y - \alpha_y S_y)\, y_t - \beta_y S_y \left(y' - \Delta y'/2 \right)_t \right] \end{aligned} \tag{A.3}$$

By treating δ_y as small and by addition of the equations Eq. $(A.3)$ one eliminates y' and obtains

$$y_{t+1} - 2C_y y_t + y_{t-1} = \beta_y S_y \Delta y'_t - \delta_y \left(y_{t+1} - y_{t-1} \right) \tag{A.4}$$

After solving this for y_t it will be possible to obtain y'_t from the equation

$$y'_t = \frac{y_{t+1} - y_{t-1} - 2\alpha_y S_y y_t + \delta_y \left(y_{t+1} + y_{t-1} \right)}{2\beta_y S_y} \tag{A.5}$$

which is obtained by subtracting Eqs. $(A.3)$.

As usual with driven oscillations we expect a response at the drive frequency. i.e.

$$y_t = A\cos\mu_E t + B\sin\mu_E t \tag{A.6}$$

where any "transient" (i.e. any solution of the homogeneous equation which is obtained by setting the drive term of Eq. $(A.4)$ to zero.) has been neglected. In electron accelerators this neglect is justified by the existence of true damping. Even in proton accelerators where true damping is negligible, it can be justified by decoherence, or, as it is called, Landau damping. Substituting into Eq. $(A.4)$ and equating the "in-phase" and the "out-of-phase" coefficients separately to zero, one obtains

$$\begin{aligned} A &= \frac{\beta_y S_y\left(C_E - C_y\right)/2}{\left(C_E - C_y\right)^2 + \delta_y^2 S_E^2} F_E \\ B &= \frac{\beta_y S_y S_E \delta_y/2}{\left(C_E - C_y\right)^2 + \delta_y^2 S_E^2} F_E \end{aligned} \tag{A.7}$$

For near-resonance analysis we define[†]

$$\varepsilon = \mu_E - \mu_y \tag{A.8}$$

(Be sure not to misinterpret frequency difference ε as an emittance, for which the symbol is ϵ.) Substituting into Eq. $(A.6)$ and neglecting terms containing $\varepsilon\delta_y$ we obtain

$$\begin{aligned} y_t &= \frac{F_E\beta_y/2}{\varepsilon^2 + \delta_y^2}\left[-\varepsilon\cos\mu_E t + \delta_y\sin\mu_E t\right] \\ &= -\frac{F_E\beta_y}{2\sqrt{\varepsilon^2+\delta_y^2}}\cos\left(\mu_E t + \phi\right)\ , \end{aligned} \tag{A.9}$$

where $\phi = \tan^{-1}(\delta_y/\varepsilon)$, $\sin\phi = \delta_y/\sqrt{\varepsilon^2+\delta_y^2}$, and $\cos\phi = \varepsilon/\sqrt{\varepsilon^2+\delta_y^2}$. Taking $\alpha_y = 0$, the slope is given by

$$\begin{aligned} y_t' &= \frac{F_E/2}{\varepsilon^2+\delta_y^2}\left(\delta_y\cos\mu_E t + \varepsilon\sin\mu_E t\right) \\ &= \frac{F_E}{2\sqrt{\varepsilon^2+\delta_y^2}}\sin\left(\mu_E t + \phi\right)\ . \end{aligned} \tag{A.10}$$

[†] It is the equality of cosines, rather than the equality of tunes, that causes resonance. To handle this all tunes can be aliased into fractional tunes in a range from 0 to 0.5. This effectively reduces the resonance-free fractional tune landscape by a factor of 2.

These equations should be reminiscent of driven simple harmonic motion though they are the solution of the difference equations Eq. $(A.2)$. Except nearly on resonance, the "in-phase" $\cos \mu_E t$ term of Eq. $(A.9)$ is dominant, but for small ε, the "out-of-phase" $\sin \mu_E t$ dominates. The response always "lags", with phase angle ϕ varying from zero to $-\pi$ as the drive frequency varies from zero to infinity. With $\phi = -\pi/2$ at resonance, the response changes sign in passing from below to above the resonance. The CS invariant of the motion is

$$\epsilon_{y,CS} = \frac{\beta_y F_E^2/4}{\varepsilon^2 + \delta_y^2} \ . \qquad (A.11)$$

For small deflections the averaged change in $\epsilon_{y,CS}$ due to the shaker is

$$\begin{aligned} \langle \Delta\epsilon_{y,CS}^{(S)} \rangle \approx \langle 2y'_t \Delta y'_t \rangle &= \langle \frac{\beta_y F_E}{\varepsilon^2 + \delta_y^2} \ (\delta_y \cos \mu_E t + \varepsilon \sin \mu_E t) \ F_E \cos \mu_E t \rangle \\ &= \frac{\beta_y F_E^2 \delta_y /2}{\varepsilon^2 + \delta_y^2} \ . \end{aligned} \qquad (A.12)$$

The averaged *fractional* change is therefore

$$\frac{\langle \Delta\epsilon_{y,CS}^{(S)} \rangle}{\epsilon_{y,CS}} = 2\delta_y \ . \qquad (A.13)$$

This can be compared to the fractional change due to damping

$$\frac{\Delta\epsilon_{y,CS}^{(D)}}{\epsilon_{y,CS}} = -2\delta_y \ . \qquad (A.14)$$

The fact that these changes are equal but opposite is consistent with the equilibrium.

B. Centroid Response of a Bunch of Particles Having Broad Tune Spread

As given by Eq. $(A.9)$, the response of a single particle depends prominently on δ_y; very small δ_y is associated with very strong response over a very narrow tune band. It is, however, possible for this dependence to be masked in the coherent response of a bunch of particles having a broad distribution of tunes.

Suppose a beam bunch consists of N particles whose tunes, rather than being equal, are spread according to a given probability distribution. When expressed in terms of ε this probability distribution is $P_\varepsilon(\varepsilon)$. The response of the entire bunch is

$$Y_t = \sum_{i=1}^{N} y_t \left(\varepsilon^{(i)} \right) = N \int_{-\infty}^{\infty} P_\varepsilon \left(\varepsilon \right) \ y_t \left(\varepsilon \right) \ d\varepsilon \ . \qquad (B.1)$$

If the tunes are distributed uniformly over range $\Delta\varepsilon$ this becomes

$$\begin{aligned} Y_t &= \frac{N}{\Delta\varepsilon}\frac{F_E\beta_y}{2}\,\delta_y\,\sin\mu_E t\int_{-\Delta\varepsilon/2}^{\Delta\varepsilon/2}\frac{d\varepsilon}{\varepsilon^2+\delta_y^2} \\ &= \frac{NF_E\beta_y}{\Delta\varepsilon}\tan^{-1}\frac{\Delta\varepsilon}{2\delta_y}\,\sin\mu_E t\ . \end{aligned} \tag{B.2}$$

In the circumstance that $\delta_y << \Delta\varepsilon$, this becomes

$$Y_t \approx \frac{NF_E\beta_y\pi}{\Delta\varepsilon}\,\sin\mu_E t\ ; \tag{B.3}$$

in this case the visible response is independent of δ_y. This example shows that the dependence of oscillatory phenomena on damping decrement is not "universal" and may be hidden from external view.

C. Appendix: Higher Order Parametric Resonances

Eq. (3.4) was not the most general possibility for parametric resonance. For example, suppressing the t subscripts to free up a position for Fourier indices, let us seek a solution of the form†

$$y = a_0 + \sum_{m=1}^{3} a_m\,\cos\left(m\tilde{\mu}_y t\right) + \sum_{m=1}^{3} b_m\,\sin\left(m\tilde{\mu}_y t\right), \tag{C.1}$$

truncated, at least for the time being, at $m = 3$. Extra terms appear in Eq. (3.6). Suppressing the summation over n,

$$\begin{aligned} \frac{\Delta y_t'}{4\pi\xi_y/\beta_y}\frac{2}{B_n} &= 2a_0\cos\left(2n\mu_x t\right) + \\ &+\sum_{m=1}^{3} a_m\,\cos\left(2n\mu_x - m\tilde{\mu}_y t\right) + \sum_{m=1}^{3} a_m\,\cos\left(2n\mu_x + m\tilde{\mu}_y t\right) \\ &-\sum_{m=1}^{3} b_m\,\sin\left(2n\mu_x - m\tilde{\mu}_y t\right) + \sum_{m=1}^{3} b_m\,\sin\left(2n\mu_x + m\tilde{\mu}_y t\right) \end{aligned} \tag{C.2}$$

† Ordinarily an *ansatz* like $(C.1)$ would be made in preparation for finding nonlinear harmonics, intending to truncate higher Fourier terms. Here, because the drive is parametric, the equations will remain linear. There will be a certain amount of "leakage" into high order terms that will be neglected in "hand calculation", but this is mainly a question of convenience, and there is no possibility of the chaotic motion that characterizes nonlinear equation. This may be somewhat academic as the exponential growth the equations can exhibit will inevitably lead to amplitudes for which nonlinearity becomes important and the assumptions of the model lose their validity.

These lead to definitions, like (3.7), that pick out tune combinations for which the perturbed frequency matches the fundamental frequency. Recalling that $\mu_y + \varepsilon_n \equiv \tilde{\mu}_y$,

$$2n\mu_x - r\tilde{\mu}_y = \tilde{\mu}_y, \quad \text{or} \quad 2n\mu_x = (1+r)\,\tilde{\mu}_y, \tag{C.3}$$

where r is another integer. Since this not the only possibility; the notation no longer identifies the particular offset ε_n that is being defined. Then Eq. $(C.2)$ becomes

$$\begin{aligned} \frac{\Delta y_t'}{4\pi\xi_y/\beta_y}\frac{2}{B_n} &= 2a_0\cos\left((1+r)\,\tilde{\mu}_y t\right) + \\ &+ \sum_{m=1}^{3} a_m \cos\left((1+r-m)\,\tilde{\mu}_y t\right) + \sum_{m=1}^{3} a_m \cos\left((1+r+m)\,\tilde{\mu}_y t\right) \\ &- \sum_{m=1}^{3} b_m \sin\left((1+r-m)\,\tilde{\mu}_y t\right) + \sum_{m=1}^{3} b_m \sin\left((1+r+m)\,\tilde{\mu}_y t\right). \end{aligned} \tag{C.4}$$

Let us expand Eq. $(C.4)$ for general r;

$$\begin{aligned} \frac{\Delta y_t'}{4\pi\xi_y/\beta_y}\frac{2}{B_n} &= 2a_0\cos\left((1+r)\,\tilde{\mu}_y t\right) + \\ &+ a_1\cos\left((r-0)\,\tilde{\mu}_y t\right) + a_2\cos\left((r-1)\,\tilde{\mu}_y t\right) + a_3\cos\left((r-2)\,\tilde{\mu}_y t\right) \\ &+ a_1\cos\left((r+2)\,\tilde{\mu}_y t\right) + a_2\cos\left((r+3)\,\tilde{\mu}_y t\right) + a_3\cos\left((r+4)\,\tilde{\mu}_y t\right) \\ &- b_1\sin\left((r-0)\,\tilde{\mu}_y t\right) - b_2\sin\left((r-1)\,\tilde{\mu}_y t\right) - b_3\sin\left((r-2)\,\tilde{\mu}_y t\right) \\ &+ b_1\sin\left((r+2)\,\tilde{\mu}_y t\right) + b_2\sin\left((r+3)\,\tilde{\mu}_y t\right) + b_3\sin\left((r+4)\,\tilde{\mu}_y t\right) \end{aligned} \tag{C.5}$$

The case $r = 1$ was previously called "lowest order". Let us try $r = 0$, so $2n\mu_x = \tilde{\mu}_y$, or $\epsilon_n = 2n\mu_x - \mu_y$;

$$\begin{aligned} \frac{\Delta y_t'}{4\pi\xi_y/\beta_y}\frac{2}{B_n} &= a_1 + (2a_0 + a_2)\,\cos\left(\tilde{\mu}_y t\right) + (a_1 + a_3)\,\cos\left(2\tilde{\mu}_y t\right) \\ &+ b_2\,\sin\left(\tilde{\mu}_y t\right) + (b_1 + b_3)\,\sin\left(2\tilde{\mu}_y t\right) \end{aligned} \tag{C.6}$$

where terms with argument $3\tilde{\mu}_y t$ and $4\tilde{\mu}_y t$ have been dropped. Eqs. (3.10) now acquire extra terms and, dropping b_3, Eqs. (3.11) generalize to multiple equations;

$$\begin{aligned} & a_0 - C_y a_0 - S_y\,\pi\xi_y\,B_n\,(a_1) = 0, \\ & -\dot{a}_1\sin\tilde{\mu}_y + b_1\cos\tilde{\mu}_y - C_y b_1 - S_y\,\pi\xi_y\,B_n\,(b_2) = 0, \\ & \dot{b}_1\sin\tilde{\mu}_y + a_1\cos\tilde{\mu}_y - C_y a_1 - S_y\,\pi\xi_y\,B_n\,(2a_0 + a_2) = 0, \\ & -\dot{a}_2\sin 2\tilde{\mu}_y + b_2\cos 2\tilde{\mu}_y - C_y b_2 - S_y\,\pi\xi_y\,B_n\,(b_1) = 0, \\ & \dot{b}_2\sin 2\tilde{\mu}_y + a_2\cos 2\tilde{\mu}_y - C_y a_2 - S_y\,\pi\xi_y\,B_n\,(a_1) = 0, \end{aligned} \tag{C.7}$$

From the first equation we obtain

$$a_0 = \frac{S_y\,\pi\xi_y\,B_n}{1-C_y}\,a_1. \tag{C.8}$$

As mentioned in an earlier footnote, at the stability limits the derivative terms vanish. Using this and $\cos\tilde{\mu}_y \approx C_y - S_y\varepsilon_n$ and $\cos(2\tilde{\mu}_y) \approx \cos 2\mu_y$ these equations become

$$\begin{aligned} b_1\varepsilon_n &= \, -\pi\xi_y\,B_n\,b_2, \\ a_1\varepsilon_n &= \, -\pi\xi_y\,B_n\,(2a_0 + a_2)\,, \\ b_2\,(C_y - \cos 2\tilde{\mu}_y) &= \, -S_y\,\pi\xi_y\,B_n\,b_1, \\ -a_2\,(C_y - \cos 2\tilde{\mu}_y) &= S_y\,\pi\xi_y\,B_n\,a_1, \end{aligned} \tag{C.9}$$

which yield

$$a_2 = \frac{-S_y\pi\xi_y B_n}{C_y - \cos 2\tilde{\mu}_y}\,a_1, \quad b_2 = \frac{-S_y\pi\xi_y B_n}{C_y - \cos 2\tilde{\mu}_y}\,b_1. \tag{C.10}$$

The stability limits are unbalanced;

$$\epsilon_{n1} = \frac{S_y\,(\pi\xi_y B_n)^2}{C_y - \cos 2\tilde{\mu}_y}, \quad \epsilon_{n2} = -S_y\,(\pi\xi_y B_n)^2 \left(\frac{2}{1-C_y} + \frac{1}{C_y - \cos 2\tilde{\mu}_y}\right). \tag{C.11}$$

This "higher order" resonance, with $n = 1, r = 0$, requires the same relation between Q_x and Q_y as the "lowest order" resonance with $n = 2, r = 1$, but the numerical factor and resonant denominators are different. Compared to the limit given in Eq. (3.14) these acquire factors of order $\pi\xi_y B_n$. Referring to values of B_n given in Table 3.1, and expecting the factor $\pi\xi_y$ to not exceed, say, 0.3, the only values of n likely to be significant will probably not exceed a few, and only if one of the denominators is small. For this particular resonance the resonant denominators are the same as would correspond to vertical third integer (and integer) nonlinear resonances (as well as $2n\mu \approx \mu_y$).† Taking account of the other resonances of the same order, several of these higher order parametric resonances are candidates to dominate the growth of the vertical beam size.

To incorporate damping decrement δ_y one should first solve for the growth rate, as in Eq. (3.13), from the condition that the determinant of the matrix of coefficients vanishes.

† It seems to me to be potentially significant that the effect of the pumping can "pull" the vertical tune toward a nonlinear resonance, in this case third integer (and integer). This can be seen from the final denominator in Eq. (C.11). However, this requires two resonance conditions to be approximately satisfied. Otherwise it is probably always justified to replace $\tilde{\mu}_y$ by μ_y in all formulas like Eqs. (C.10), (C.11), (C.17), and (C.18).

This should then be set equal to the δ_y to find the stability limits in the presence of damping, as in Eq. (3.15). Not wishing to work out the determinant, I conjecture that it is valid to mimic Eq. (3.16), to obtain

$$-\sqrt{\epsilon_{n1}^2 - \delta_y^2} < \varepsilon_n < \sqrt{\epsilon_{n2}^2 - \delta_y^2}, \tag{C.12}$$

or with ϵ_{n1} and ϵ_{n2} reversed, as appropriate. Expressed inversely, these formulas predict the dependence of ξ_y on δ_y with everything else held constant; for example the first limit yields

$$\xi_y = \frac{\delta_y^{1/2}}{\pi B_n}\sqrt{\frac{C_y - \cos 2\tilde{\mu}_y}{\sin\tilde{\mu}_y}}. \tag{C.13}$$

For $r = 2$ we have a different resonance; $2n\mu_x = 3(\mu_y + \epsilon_n)$ or $\epsilon_n = (2/3)n\mu_x - \mu_y$.

$$\begin{aligned}
\frac{\Delta y_t'}{4\pi\xi_y/\beta_y}\,\frac{2}{B_n} &= 2a_0\cos(3\tilde{\mu}_y t) + \\
&+ a_1\,\cos(2\tilde{\mu}_y t) + a_2\,\cos(\tilde{\mu}_y t) + a_3 + a_4\,\cos(\tilde{\mu}_y t) + a_5\,\cos(2\tilde{\mu}_y t) + a_6\,\cos(3\tilde{\mu}_y t) \\
&+ a_1\,\cos(4\tilde{\mu}_y t) + a_2\,\cos(5\tilde{\mu}_y t) + a_3\,\cos(6\tilde{\mu}_y t) \\
&- b_1\,\sin(2\tilde{\mu}_y t) - b_2\,\sin(\tilde{\mu}_y t) + b_4\,\sin(\tilde{\mu}_y t) + b_5\,\sin(2\tilde{\mu}_y t) + b_6\,\sin(3\tilde{\mu}_y t) \\
&+ b_1\,\sin(4\tilde{\mu}_y t) + b_2\,\sin(5\tilde{\mu}_y t) + b_3\,\sin(6\tilde{\mu}_y t)
\end{aligned} \tag{C.14}$$

Eqs. $(C.7)$ change to 13 equations in 13 unknowns;

$$\begin{aligned}
&a_0 - C_y a_0 - S_y\,\pi\xi_y\,B_n\;(a_3) = 0, \\
&-\dot{a}_1\sin\tilde{\mu}_y + b_1\cos\tilde{\mu}_y - C_y b_1 - S_y\,\pi\xi_y\,B_n\;(-b_2 + b_4) = 0, \\
&\dot{b}_1\sin\tilde{\mu}_y + a_1\cos\tilde{\mu}_y - C_y a_1 - S_y\,\pi\xi_y\,B_n\;(a_2 + a_4) = 0, \\
&-\dot{a}_2\sin 2\tilde{\mu}_y + b_2\cos 2\tilde{\mu}_y - C_y b_2 - S_y\,\pi\xi_y\,B_n\;(-b_1 + b_5) = 0, \\
&\dot{b}_2\sin 2\tilde{\mu}_y + a_2\cos 2\tilde{\mu}_y - C_y a_2 - S_y\,\pi\xi_y\,B_n\;(a_1 + a_5) = 0, \\
&-\dot{a}_3\sin 3\tilde{\mu}_y + b_3\cos 3\tilde{\mu}_y - C_y b_3 - S_y\,\pi\xi_y\,B_n\;(b_6) = 0, \\
&\dot{b}_3\sin 3\tilde{\mu}_y + a_3\cos 3\tilde{\mu}_y - C_y a_3 - S_y\,\pi\xi_y\,B_n\;(2a_0 + a_6) = 0, \\
&-\dot{a}_4\sin 4\tilde{\mu}_y + b_4\cos 4\tilde{\mu}_y - C_y b_4 - S_y\,\pi\xi_y\,B_n\;(b_1) = 0, \\
&\dot{b}_4\sin 4\tilde{\mu}_y + a_4\cos 4\tilde{\mu}_y - C_y a_4 - S_y\,\pi\xi_y\,B_n\;(a_1) = 0, \\
&-\dot{a}_5\sin 5\tilde{\mu}_y + b_5\cos 5\tilde{\mu}_y - C_y b_5 - S_y\,\pi\xi_y\,B_n\;(b_2) = 0, \\
&\dot{b}_5\sin 5\tilde{\mu}_y + a_5\cos 5\tilde{\mu}_y - C_y a_5 - S_y\,\pi\xi_y\,B_n\;(a_2) = 0, \\
&-\dot{a}_6\sin 6\tilde{\mu}_y + b_6\cos 6\tilde{\mu}_y - C_y b_6 - S_y\,\pi\xi_y\,B_n\;(b_3) = 0, \\
&\dot{b}_6\sin 6\tilde{\mu}_y + a_6\cos 6\tilde{\mu}_y - C_y a_6 - S_y\,\pi\xi_y\,B_n\;(a_3) = 0,
\end{aligned} \tag{C.15}$$

From these we get immediately

$$a_0 = \frac{S_y \pi \xi_y B_n}{1 - C_y}\, a_3,\;\; a_4 = \frac{S_y \pi \xi_y B_n}{\cos 4\tilde{\mu}_y - C_y}\, a_1,\;\; a_5 = \frac{S_y \pi \xi_y B_n}{\cos 5\tilde{\mu}_y - C_y}\, a_2,\;\; a_6 = \frac{S_y \pi \xi_y B_n}{\cos 6\tilde{\mu}_y - C_y}\, a_3,$$

$$b_3 = 0, \quad b_4 = \frac{S_y \pi \xi_y B_n}{\cos 4\tilde{\mu}_y - C_y}\, b_1, \quad b_5 = \frac{S_y \pi \xi_y B_n}{\cos 5\tilde{\mu}_y - C_y}\, b_2, \quad b_6 = \frac{S_y \pi \xi_y B_n}{\cos 6\tilde{\mu}_y - C_y}\, b_3. \tag{C.16}$$

leaving 5 equations for a_1, a_2, a_3, b_2, and b_3.

$$\begin{aligned}
\epsilon_n &= -\frac{S_y \left(\pi \xi_y B_n\right)^2}{\cos 4\tilde{\mu}_y - C_y} - \pi \xi_y \, B_n \, \frac{a_2}{a_1}, \\
a_2 &\left(\cos 2\tilde{\mu}_y - C_y - \frac{\left(S_y \pi \xi_y B_n\right)^2}{\cos 5\tilde{\mu}_y - C_y} \right) = S_y \, \pi \xi_y \, B_n \, a_1, \\
a_3 &\left(\cos 3\tilde{\mu}_y - C_y - \frac{\left(S_y \pi \xi_y B_n\right)^2}{\cos 6\tilde{\mu}_y - C_y} \right) = S_y \, \pi \xi_y \, B_n \, 2a_0, \\
\epsilon_n &= -\frac{S_y \left(\pi \xi_y B_n\right)^2}{\cos 4\tilde{\mu}_y - C_y} + \pi \xi_y \, B_n \, \frac{b_2}{b_1}, \\
b_2 &\left(\cos 2\tilde{\mu}_y - C_y - \frac{\left(S_y \pi \xi_y B_n\right)^2}{\cos 5\tilde{\mu}_y - C_y} \right) = -S_y \, \pi \xi_y \, B_n \, b_1.
\end{aligned} \tag{C.17}$$

Both limits are given by†

$$\epsilon_n = -S_y \left(\pi \xi_y B_n\right)^2 \left(\frac{1}{\cos 4\tilde{\mu}_y - C_y} + \frac{1}{\cos 2\tilde{\mu}_y - C_y - \left(S_y \pi \xi_y B_n\right)^2 / \left(\cos 5\tilde{\mu}_y - C_y\right)} \right). \tag{C.18}$$

This has the same order of magnitude as the $r = 0$ case but, of course, the tune ranges for which it is significant are entirely different. For this particular resonance the resonant denominators are the same as would correspond to vertical fifth integer (and integer) nonlinear resonances (as well as $(2/3)n\mu \approx \mu_y$).

† In the body of the paper it was stated that this calculation seems to not be self-consistent. If the limits of the stop band are identical there would seem to be no stop band. Clearly the analysis of this, and higher order, resonances has to be refined.