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Integrable Boundary Conditions and Reflection Matrices for the $O(N)$ Nonlinear Sigma Model

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Abstract

We find new integrable boundary conditions, depending on a free parameter g, for the $O(N)$ nonlinear σ model, which are of nondiagonal type, that is, particles can change their "flavor" through scattering off the boundary. These boundary conditions are derived from a microscopic boundary lagrangian, which is used to establish their integrability, and exhibit integrable flows between diagonal boundary conditions investigated previously. We solve the boundary Yang-Baxter equation, connect these solutions to the boundary conditions, and examine the corresponding integrable flows.

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1 Introduction

One of the main problems in boundary integrable field theories is to find boundary conditions that preserve the integrability of a given bulk integrable field theory, that is, integrable boundary conditions. This has been carried out for several models by direct inspection of higher-spin conserved charges [1, 2, 3, 4]. In [3] (see also [5, 6]) it has been shown that models with $O(N)$ or $SO(N)$ symmetry admit a family of $N+1$ integrable boundary conditions, which are obtained by requiring n field components to satisfy Neumann and the remaining $N-n$ to satisfy Dirichlet, $n = 0, 1, \ldots, N$. These conditions are of diagonal type, which means that scattering off the boundary will not change the flavor of the incoming particle. We will refer to this type of boundary scattering problem as the diagonal case. A natural question one would ask then is if there are more integrable boundary conditions for these models with $O(N)$ symmetry, or if the diagonal case exhaust all possibilities.

The purpose of this paper is to study the possible integrable boundary conditions for models with $O(N)$ global symmetry. We find nondiagonal integrable boundary conditions for the $O(N)$ nonlinear σ (nlσ) model. These are genuinely nondiagonal boundary conditions in the sense that they can not be put into diagonal form by a redefinition (global rotation) of the fields. These conclusions should carry out to other models, like the $SO(N)$ Gross-Neveu and principal chiral models.

The structure of this paper is as follows. In the next section we review some of the generalities of boundary integrable field theories which are relevant for our discussion. We also review the Goldschmidt-Witten argument, and use it to establish the integrability of the $O(N)$ nonlinear σ model. In section 3 we look at a very simple example, to illustrate some of the issues that are relevant in our analysis. In section 4 we introduce the new integrable boundary conditions for the nonlinear σ model, which are the main concern of this paper. In section 5 we compute the nondiagonal reflection matrices associated to these boundary conditions by solving the boundary Yang-Baxter equation, in particular we look at the $O(2)$ case, and present a solution for the $O(2N)$ case to which we do not know what are the corresponding boundary conditions. Finally in section 6 we present our conclusions and some possible directions for further work.

2 Integrable Boundary Field Theory

Let us consider a bulk integrable field theory defined by an action

$$S = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_0 \; \mathcal{L}_B(\phi, \partial\phi), \tag{2.1}$$

where $\mathcal{L}_B(\phi, \partial\phi)$ is the bulk Lagrangian, which depends locally on the field ϕ and its partial derivatives [1]. We say that it may be defined by a local action because there are several bulk integrable models that are not, such as perturbed conformal field theories.

As is well known, in two-dimensions integrability has several important consequences: in a multi-particle scattering process there is no particle production and the set on in and out momenta is the same, the S-matrix factorizes into a product of two-body S-matrices, and the two-body S-matrix satisfies the Yang-Baxter equation, besides the usual requirements of unitarity and crossing-symmetry. These conditions, together with the knowledge of the symmetries of the S-matrix, often allow one to determine the two-body S-matrix exactly, up to CDD factors.

When we consider this theory on the half-line, the introduction of a boundary may easily break the conservation of bulk charges, as it is seen in the simple case of linear momentum. Therefore we are not assured that the resulting model will be integrable, and we must specify boundary conditions. These boundary conditions correspond to boundary dynamics, which may be described by some local lagrangian. The boundary version of a given bulk model is defined by the action

$$S = \int_{-\infty}^{0} dx_1 \int_{-\infty}^{+\infty} dx_0 \, \mathcal{L}_B(\phi, \partial\phi) + \int_{-\infty}^{+\infty} dx_0 \, \mathcal{L}_b(\phi|_{x=0} \, , \dot{\phi}\Big|_{x=0}) \; , \tag{2.2}$$

where $\mathcal{L}_b(\phi|_{x=0} \, , \dot{\phi}\Big|_{x=0})$ is the boundary action, depending only on the values of the field ϕ and its time derivatives, at $x = 0$ [2]. The boundary Lagrangian defines the boundary dynamics, and it's easy to see that it induces boundary conditions through equations of motion. For example, for a scalar boson we have

$$\partial_1 \phi|_{x=0} - \frac{\partial \mathcal{L}_b}{\partial \phi}\bigg|_{x=0} + \partial_0 \, \frac{\partial \mathcal{L}_b}{\partial \dot{\phi}}\bigg|_{x=0} = 0 \; . \tag{2.3}$$

In order to have a boundary integrable model, we have to pick a suitable boundary action such that a combination of bulk conserved charges is still conserved. In order to discuss boundary conserved charges, we briefly review the bulk case.

A local (bulk) conservation law of spin s is written as

$$\partial_- J_+^{(s+1)} = \partial_+ R_-^{(s-1)} \qquad \text{and} \qquad \partial_+ J_-^{(s+1)} = \partial_- R_+^{(s-1)} \; , \tag{2.4}$$

and from these equations, one easily establishes the conservation of the following charges

$$Q_+ = \int_{-\infty}^{+\infty} dx_1 \, (J_+^{(s+1)} - R_-^{(s-1)}) \qquad \text{and} \qquad Q_- = \int_{-\infty}^{+\infty} dx_1 \, (J_-^{(s+1)} - R_+^{(s-1)}) \tag{2.5}$$

[1] Here the field ϕ is symbolic and may denote a collection of fields, bosonic or fermionic fields and so on.

[2] We use an upper dot for time derivative at the boundary.

In proving that these charges are conserved we have to use the fact that we can discard surface terms. When we restrict our model to the half-line we can not do that with the surface term at $x=0$. On the other hand, if the following condition [1] is satisfied

$$J_-^{(s+1)} - J_+^{(s+1)} + R_-^{(s-1)} - R_+^{(s-1)}|_{x=0} = \frac{d}{dt}\Sigma(t) \tag{2.6}$$

where $\Sigma(t)$ is a *local* field, then

$$\widetilde{Q} = \int_{-\infty}^{0} dx_1 \, (J_-^{(s+1)} + J_+^{(s+1)} - R_-^{(s-1)} - R_+^{(s-1)}) - \Sigma(t) \tag{2.7}$$

is a conserved charge. For example, in light-cone coordinates the energy-momentum charges are $Q_\pm = E \pm P$, but after the introduction of a boundary only $Q = Q_+ + Q_- = 2E$ is conserved, with $\Sigma(t) = 0$.

We should look for the possible boundary conditions that we can impose to a given model such that 2.6 holds. Once this is done one should look for solutions of the boundary Yang-Baxter equation (bYBe) which can be related to the proposed boundary conditions. We describe the bYBe in the next subsection

2.1 The Scattering Matrix: Generalities

Boundary integrability is encoded in the boundary Yang-Baxter equation (bYBe) in a similar way as bulk integrability is encoded in the Yang-Baxter equation (YBe). We should stress that, unlike the bulk case, we do not know what is the symmetry group at the boundary a priori. In the bulk case the knowledge of the bulk global symmetries greatly simplify the form of the S-matrix. Unfortunately we do not have a similar set up for the boundary case, and therefore the analysis of the possible structure of the reflection matrix will have to rely on general physical arguments. Recently this issue has been addressed in [10] for the case of the nonlinear Schroedinger model. We review now a few well-known concepts. For a more complete discussion, see [13]

The Hilbert space of in (out) asymptotic states is spanned by the multi-particle states

$$|A_{i_1}(\theta_1), A_{i_2}(\theta_2) \ldots A_{i_n}(\theta_n)\rangle_{in(out)} = A_{i_1}(\theta_1) A_{i_2}(\theta_2) \ldots A_{i_n}(\theta_n) |0\rangle_{in(out)} \ , \tag{2.8}$$

where the $\{A_i(\theta)\}$ are the Faddeev-Zamolodchikov (FZ) operators that create the one-particle asymptotic states, the $\{\theta_i\}$ are the rapidities and $\theta_{i_1} > \theta_{i_2} > \ldots > \theta_{i_n}$ for in-states and the other way around for out-states, and $|0\rangle_{in(out)}$ is the in (out) vacuum. We assume henceforth that the in and out vacuum are the same.

Multiparticle scattering processes factorize in a product of two-body S-matrix, which is defined in terms of the FZ operators by

$$A_i(\theta_1) A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2) A_l(\theta_2) A_k(\theta_1) \ . \tag{2.9}$$

The YBe is obtained by requiring the associativity of the algebra defined by 2.9

In defining the reflection matrix we have to change the Hilbert space. Boundary scattering processes are defined in terms of the boundary state $|0\rangle_B = B|0\rangle$, where B is the so-called boundary-state operator [1].

The Hilbert space is spanned by $|A_{i_1}(\theta_1)\ldots A_{i_n}(\theta_n)\rangle_B = A_{i_1}(\theta_1)\ldots A_{i_n}(\theta_n)|0\rangle_B$, where the in (out) states are obtained by setting $\theta_1 > \ldots > \theta_n > 0$ ($\theta_1 < \ldots < \theta_n < 0$). Similarly to the bulk case, the reflection matrix is defined by

$$A_i(\theta)B = R_i^j(\theta)A_j(-\theta)B \ . \tag{2.10}$$

We represent the S-matrix and the reflection matrix R_i^j graphically as in figure 1.

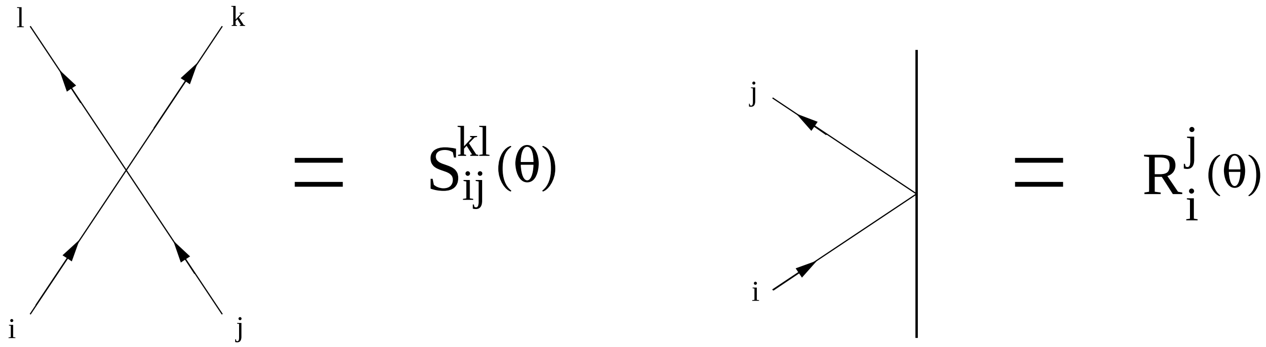

Fig. 1 The S-matrix and the reflection matrix

The bYBe is obtained by requiring the compatibility of 2.10 with 2.9 by looking at the process $|A_i(\theta_1)A_j(\theta_2)\rangle_B \to |A_k(-\theta_2)A_l(-\theta_1)\rangle_B$.

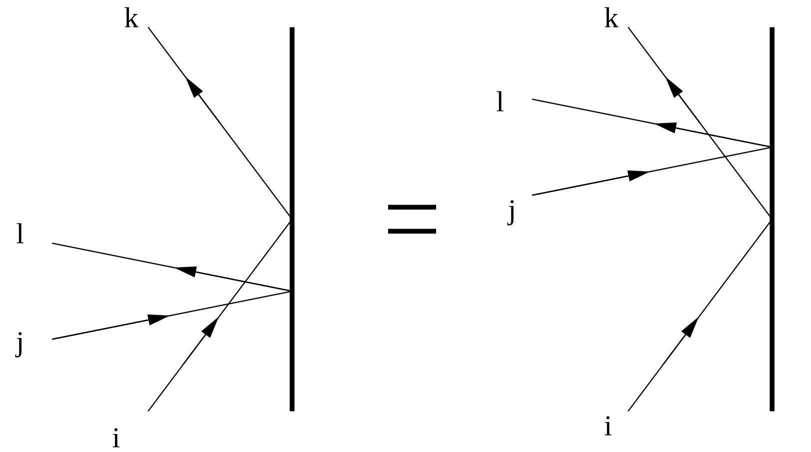

Fig. 2 The boundary Yang-Baxter equation.

In terms of the two-body S-matrix and reflection matrix, it reads

$$S_{ji}^{nm}(\theta)R_n^p(\theta_1)S_{mp}^{ql}(\theta_+)R_q^k(\theta_2) = R_i^p(\theta_2)S_{jp}^{nm}(\theta_+)R_n^q(\theta_1)S_{mq}^{kl}(\theta) \ , \tag{2.11}$$

where we introduced the variables $\theta_+ = \theta_1 + \theta_2$ and $\theta = \theta_1 - \theta_2$, and sum over m, n, p and q.

Besides the bYBe, one must impose unitary and crossing-symmetry conditions on the reflection matrix. We quote them here, and refer the reader to [1] for more details. The unitarity condition is

$$R_i^k(\theta) R_k^j(-\theta) = \delta_i^j \ , \tag{2.12}$$

and the boundary crossing symmetry is

$$R_i^j(\frac{i\pi}{2} - \theta) = S_{kl}^{ij}(2\theta) R_k^l(\frac{i\pi}{2} + \theta) \ , \tag{2.13}$$

where we are assuming implicitly that the particles are invariant under charge conjugation. For the general case we refer to [1] and the appendix of [12].

2.2 The $O(N)$ Nonlinear Sigma Model

In this subsection we briefly review the main features of the $O(N)$ nonlinear sigma model.

The nlσ model is defined by the following Lagrangian

$$\mathcal{L}_{nl\sigma} = \frac{1}{2g^2} \partial \vec{\mathbf{n}} \cdot \partial \vec{\mathbf{n}} \tag{2.14}$$

where $\vec{\mathbf{n}}$ is an N-dimensional vector subject to the constraint $\vec{\mathbf{n}} \cdot \vec{\mathbf{n}} = 1$. This constraint can be implemented by means of a Lagrange multiplier λ in the action. It is straightforward to see that the equation of motion for $\vec{\mathbf{n}}$ in light-cone variables ($x_\pm = x_0 \pm x_1$) is

$$\partial_+ \partial_- \vec{\mathbf{n}} = -\vec{\mathbf{n}} (\partial_+ \vec{\mathbf{n}} \cdot \partial_- \vec{\mathbf{n}}) \ . \tag{2.15}$$

This will be used in establishing the integrability of the boundary conditions of section 4.

The main features of the nlσ model is that it is classically conformal, asymptotically free, and displays dynamical mass-generation.

2.3 The Goldschmidt-Witten Argument

We can use the fact that the nlσ model is classically conformally invariant in order to show that it is indeed an integrable model at the quantum level. Starting with the classical conformal symmetry and exploring the structure of the possible anomalies that appear after quantization, Goldschmidt and Witten [9], and earlier Polyakov [7], have shown that some conserved currents are not spoiled by quantization, that is, there *are* anomalies, but their structure is such that one can rewrite the quantum corrections to the conservation law as a total derivative, allowing us to write down non-trivial conserved charges. This argument can be used to establish the integrability of the Gross-Neveu model [8] and of the principal chiral model [9]. In this subsection we review the Goldschimdt-Witten argument and write down the conserved currents that will be of use to establish the integrability of the new boundary conditions for the nlσ model.

The Goldschmidt-Witten (GW) argument starts by observing that the trace of the energy-momentum of a classically conformal-invariant theory vanishes, that is $T_{+-} = T_{-+} = 0$. This implies that the conservation of energy-momentum reads

$$\partial_- T_{++} = 0 \qquad \text{and} \qquad \partial_+ T_{--} = 0 \tag{2.16}$$

which, on its turn has as a consequence that for any integer n

$$\partial_-(T_{++})^n = 0 \qquad \text{and} \qquad \partial_+(T_{--})^n = 0 \ . \tag{2.17}$$

Therefore, we generate towers of classically conserved currents.

After we quantize the theory the equations 2.17 do not make sense any more, since we are considering products of operators at the same point, and these operators should be redefined. In general these equations will be spoiled by anomalies. The right hand side of 2.17 will be nonzero, but there are still several constraints we should impose on the possible terms that are generated: whatever comes due to anomalies has to have the correct dimension, Lorentz weight [3] , and group theoretic properties as the left-hand side. By analyzing all the possible terms that may appear on the right-hand side of 2.17, with $n = 2$, GW showed that they can all be written as total derivatives, which implies that we have a nontrivial conserved current.

For the nlσ model they showed that the possible anomalies can be written as

$$\partial_+(T_{--})^2 = c_1\ \partial_+(\partial_-^2\vec{\mathbf{n}}\cdot\partial_-^2\vec{\mathbf{n}}) + c_2\ \partial_-(\partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\ \partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}) + c_3\ \partial_-(\partial_-^3\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}})\ , \tag{2.18}$$

with some constants $\{c_i\}$ and an analogous equation for $\partial_+(T_{--})^2$.

In [3] we used these charges to establish the integrability of the aforementioned diagonal boundary conditions. We will see that we can use them to establish the integrability of nondiagonal boundary conditions too.

2.4 The Exact S-matrix

Once the nlσ model has been established to be integrable we can proceed and compute its two-body S-matrix. This has been done in [11], and we quote it here for further reference. Since the bulk is $O(N)$ invariant, the S-matrix has to be of the form

$$S_{ij}^{kl}(\theta) = \sigma_1(\theta)\delta_{ij}\delta^{kl} + \sigma_2(\theta)\delta_i^k\delta_j^l + \sigma_3(\theta)\delta_i^l\delta_j^k \tag{2.19}$$

[3]If under a Lorentz transformation parametrized by rapidity α, a quantity $\phi \to e^{p\alpha}\phi$, we say that ϕ has Lorentz weight p.

where the $\sigma_i(\theta)$ are determined by the solving the YBe [11], unitarity, and crossing-symmetry. In [11] a large-N expansion check has been performed to show that this is indeed the exact S-matrix of the $O(N)$ nlσ model. The $\sigma_i(\theta)$ are given by

$$\sigma_1(\theta) = -\frac{i\lambda}{i\pi-\theta}\sigma_2(\theta) \quad , \quad \sigma_3(\theta) = -\frac{i\lambda}{\theta}\sigma_2(\theta) \quad \text{and}$$
$$\sigma_2(\theta) = \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{2\pi}+\frac{i\theta}{2\pi})\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})\Gamma(\frac{\lambda}{2\pi}-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+\frac{\lambda}{2\pi}-\frac{i\theta}{2\pi})\Gamma(-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})\Gamma(1+\frac{\lambda}{2\pi}+\frac{i\theta}{2\pi})} . \tag{2.20}$$

where $\lambda = 2\pi/(N-2)$. We will also need the isoscalar S-matrix element $\sigma_I(\theta) = N\sigma_1(\theta)+\sigma_2(\theta)+\sigma_3(\theta)$, which can be written as

$$\sigma_I(\theta) = -\frac{(i\pi+\theta)(i\lambda+\theta)}{(i\pi-\theta)\theta}\sigma_2(\theta) . \tag{2.21}$$

This will be useful when we solve the boundary crossing unitarity condition in section 5.

For $N=2$ one has to be careful, since $\lambda\to\infty$ but $\sigma_2(\theta)\to 0$. In this limit we get

$$\sigma_1(\theta) = -\frac{2i\pi}{i\pi-\theta}f(\theta) \quad , \quad \sigma_2(\theta) = 0 \quad , \quad \text{and} \quad \sigma_3(\theta) = -\frac{2i\pi}{\theta}f(\theta) \tag{2.22}$$

where the function $f(\theta)$ is given by

$$f(\theta) = \frac{\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})}{\Gamma(-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})} . \tag{2.23}$$

The $O(2)$ nlσ model is not a simple massless free boson, as the map $n_1 = \cos(\sqrt{g}\phi)$ and $n_2 = \sin(\sqrt{g}\phi)$ would suggest. It is actually the sine-Gordon model at $\beta^2 = 8\pi$, the sine-Gordon potential being a marginally-relevant perturbation at this point, and it describes the Kosterlitz-Thouless point of the classical XY model.

3 A Very Simple Example

Before we analyze the nondiagonal boundary conditions for the nlσ model, let us look at a very simple example of nondiagonal scattering, consisting of 2 free bosons ϕ_1 and ϕ_2, with equal mass m, on the half-line, which captures some of the main physical aspects for the case of the nlσ model. The action we consider is

$$S = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{0} dx_1 \;\; \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{2}m_2^2\phi_2^2 + \int_{-\infty}^{\infty} \mathcal{L}_b(\phi_1,\dot{\phi}_1,\phi_2,\dot{\phi}_2) , \tag{3.1}$$

where $\mathcal{L}_b$ is the boundary action, to be chosen shortly.

Clearly any quadratic form at the boundary can be solved exactly, being obviously integrable. We have consider two possibilitites [4]

$$\mathcal{L}_b^1 = g_1 \phi_1 \phi_2 \tag{3.2}$$

$$\mathcal{L}_b^2 = g_2 \phi_1 \dot{\phi}_2 \tag{3.3}$$

Recall that a boundary Lagrangian $\mathcal{L}_b(\phi_i)$ induces the following boundary condition

$$\left.\frac{\partial \phi_i}{\partial x}\right|_{x=0} = \left.\frac{\partial \mathcal{L}_b}{\partial \phi_i}\right|_{x=0} . \tag{3.4}$$

Since the masses of the two particles are the same, which is required to have nondiagonal scattering, we can define new fields $\eta_\pm$ by

$$\eta_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2) , \tag{3.5}$$

which leave the bulk action with the same form but takes the boundary action to

$$\mathcal{L}_b^1 = g_1(\eta_+^2 - \eta_-^2) \tag{3.6}$$

and therefore we conclude that the boundary condition induced by 3.2 is not really nondiagonal.

On the other hand, for the second boundary Lagrangian it is not possible to find an orthogonal transformation which diagonalizes the boundary Lagrangian. In this sense we can say that the boundary Lagrangian II corresponds to nondiagonal boundary scattering.

It is fairly easy to find the reflection amplitudes for 3.3. The boundary conditions induced by 3.3 are (using g for g_2)

$$\left.\frac{\partial \phi_1}{\partial x}\right|_{x=0} = g \left.\frac{\partial \phi_2}{\partial t}\right|_{x=0} \tag{3.7}$$

$$\left.\frac{\partial \phi_2}{\partial x}\right|_{x=0} = -g \left.\frac{\partial \phi_1}{\partial t}\right|_{x=0} \tag{3.8}$$

By using the mode expansion for the fields,

$$\phi_i(x,t) = \int d\theta (A_i(\theta) e^{(im\cosh(\theta)t - im\sinh(\theta)x)} + A_i^\dagger(\theta) e^{(-im\cosh(\theta)t + im\sinh(\theta)x)}) \tag{3.9}$$

where the $A_i(\theta)$ are the FZ operators, and the definition of the reflection matrix $R_i^j(\theta)$ given in 2.10, we find the scattering amplitudes to be

$$R_i^j(\theta) = \begin{pmatrix} \frac{\sinh^2(\theta) - g^2\cosh^2(\theta)}{\sinh^2(\theta) + g^2\cosh^2(\theta)} & \frac{g\sinh(2\theta)}{g^2\cosh^2(\theta) + \sinh^2(\theta)} \\ \frac{-g\sinh(2\theta)}{g^2\cosh^2(\theta) + \sinh^2(\theta)} & \frac{\sinh^2(\theta) - g^2\cosh^2(\theta)}{\sinh^2(\theta) + g^2\cosh^2(\theta)} \end{pmatrix} . \tag{3.10}$$

[4] The choice $\mathcal{L}_b = g_1\phi_1^2 + g_2\phi_2^2$ is trivial, since there is no coupling between the two fields. We are also disregarding possible boundary mass terms.

Notice that, as expected, the diagonal amplitudes are the same, and the off-diagonal ones have opposite signs. Moreover if we expand the elements of the reflection matrix in powers of the coupling constant g, the diagonal elements are even functions of g, whereas the off-diagonal ones are odd functions of g. This can be understood by noticing that each flavor change at the boundary is accompanied by a factor of g, and so in order to have diagonal (nondiagonal) boundary scattering we need an even (odd) number of g's. We will use some of the intuition from this very simple model in the study of the nondiagonal boundary conditions for the nlσ model.

4 Boundary Conditions for the nlσ Model

As we have seen in section 3 there are essentially two possibilities we could try for the boundary conditions in the nlσ model. One of them, 3.2, is not even integrable in this case. We are left then with the boundary conditions induced by a boundary lagrangian of the form 3.3,

$$S_{nl\sigma} = S_B + \int_{-\infty}^{+\infty} dx_0 \frac{1}{2} M_{ij} n_i n_j \ , \tag{4.11}$$

where we have to have $M_{ij} = -M_{ji}$, and the indices i and j run through a subset of $\{1, 2, \ldots, N\}$. For the remaining indices, which correspond to particles scattering diagonally, we choose Dirichlet boundary conditions

$$\partial_0 n_j|_{x=0} = 0 \ . \tag{4.12}$$

The reson for this choice will become clear in the next section. From 4.11 we get the following boundary conditions for the non-diagonally scattering fields

$$\partial_1\, n_i|_{x=0} = M_{ij} \partial_0\, n_j|_{x=0} \ . \tag{4.13}$$

We have to check that the condition 2.6 is satisfied for these boundary conditions, that is

$$\begin{aligned}
&(\partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}})^2 - (\partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}})^2 + c_1(\partial_+^2\vec{\mathbf{n}}\cdot\partial_+^2\vec{\mathbf{n}} - \partial_-^2\vec{\mathbf{n}}\cdot\partial_-^2\vec{\mathbf{n}}) + \\
&+c_2(\partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\ \partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}} - \partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\ \partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}) + \\
&+c_3(\partial_+^3\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}} - \partial_-^3\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}})\Big|_{x=0} = \frac{d}{dt}\Sigma(t)
\end{aligned} \tag{4.14}$$

for some local field $\Sigma(t)$. We discuss this condition term by term. The first term can be simplified to

$$(\partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}})^2 - (\partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}})^2 = 8\ \vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_1\ (\vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_0 + \vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_1)\ , \tag{4.15}$$

where the subscripts 0 and 1 denote time and space derivatives, respectively. The "c_1" term can be simplified to

$$\partial_+^2\vec{\mathbf{n}}\cdot\partial_+^2\vec{\mathbf{n}} - \partial_-^2\vec{\mathbf{n}}\cdot\partial_-^2\vec{\mathbf{n}} = 16\ \vec{\mathbf{n}}_{00}\cdot\vec{\mathbf{n}}_{01} - 8\ \vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_1\ (\vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_0 - \vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_1)\ , \tag{4.16}$$

the "c_2" term to

$$\partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\ \partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}} - \partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\ \partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}} = 4\ \vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_1\ (\vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_0 - \vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_1) \tag{4.17}$$

and the "c_3" term to

$$\partial_+^3\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}} - \partial_-^3\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}} = 8\ \partial_0(\vec{\mathbf{n}}_{01}\cdot\vec{\mathbf{n}}_0 - \vec{\mathbf{n}}_{00}\cdot\vec{\mathbf{n}}_1) - 2\ \vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_1\ (\vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_0 - \vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_1) \tag{4.18}$$

We see from these that we have to have $\vec{\mathbf{n}}_0\cdot\vec{\mathbf{n}}_1 = 0$ in order that 2.6 is satisfied. Moreover, if we consider 4.12 and 4.13, we immediately see that the one "bad" remaining term $16\ \vec{\mathbf{n}}_{00}\cdot\vec{\mathbf{n}}_{01}$ vanishes. This establishes the integrability of 4.12 and 4.13.

Since M is an antisymmetric matrix, there is a real-orthogonal matrix O, such that O^tMO is block-diagonal, each block being an antisymmetric 2×2 matrix. If M has k zero eigenvalues we readily see that its structure will have $O(2)\times O(2)\times\ldots\times O(2)\times O(k)$ symmetry, where there are $(N-k)/2$ $O(2)$'s (and therefore $N-k$ must be even). The $O(2)$-symmetric blocks correspond to interactions between pairs of field components at the boundary, that is, a particle of a given type, say i, can scatter diagonally or onto some other fixed particle of type j, where the pairing (i,j) is fixed.

The question we should try to answer now is, in which ways can we break the boundary symmetry? In the diagonal case, the bulk $O(N)$ symmetry is broken to $O(k)\times O(N-k)$ at the boundary, with $k=0,1,\ldots,N$. We will be looking at solutions that respect this symmetry, and so we should consider matrices M symmetric under $O(k)\times O(N-k)$. But this implies that the only case we should consider is $O(2)\times O(N-2)$, where the $O(2)$-symmetric part corresponds to non-diagonal scattering, and the remaining $O(N-2)$ to diagonal scattering, that is, the components n_i satisfy

$$\begin{aligned}
&\partial_1 n_1|_{x=0} = g\,\partial_0 n_2|_{x=0} \qquad , \qquad \partial_1 n_2|_{x=0} = -g\,\partial_0 n_1|_{x=0}\\
&\partial_0 n_j|_{x=0} = 0 \qquad \text{for} \qquad j=3,4,\ldots,N\ .
\end{aligned} \tag{4.19}$$

We have chosen the diagonally-scattering field components to satisfy Dirichlet boundary conditions, but at this point it seems that we could have chosen them to satisfy Neumann. As we will see later, the solutions of the boundary Yang-Baxter equations are such that in the "diagonal" limits the correct choice for the diagonally-scattering components is indeed Dirichlet boundary conditions.

In [14] Corrigan and Sheng have established the classical integrability of the Neumann boundary condition for the $O(N)$ nlσ model, for all N, and found one extra boundary condition for $N=3$. In [3] we showed that this special boundary condition was integrable at the quantum level to. We will show now, that it is actually related to the ones we propose in this paper.

The explicit form of the Corrigan-Sheng boundary condition for the $O(3)$ nlσ model is

$$\partial_1 \vec{\mathbf{n}}|_{x=0} = -k \times \partial_0 \vec{\mathbf{n}}|_{x=0} + (\vec{\mathbf{n}} \cdot k \times \partial_0 \vec{\mathbf{n}})\, \vec{\mathbf{n}}|_{x=0} \qquad \text{and} \qquad k \cdot \partial_0 \vec{\mathbf{n}}|_{x=0} = 0 \tag{4.20}$$

where k is an arbitrary vector. Let us choose $k = g\ (0,0,1)$, where g is a coupling constant. This means that $\partial_0 n_3 = 0$ and so n_3 is a constant at the boundary. This, together with the constraint $n_1^2 + n_2^2 + n_3^2 = 1$, imply $n_1^2 + n_2^2 =$ constant at $x = 0$. The Corrigan-Sheng boundary condition reduces to

$$\begin{aligned} &\partial_1 n_1|_{x=0} = \quad \tilde{g}\ \partial_0 n_2|_{x=0} \qquad , \qquad \partial_1 n_2|_{x=0} = -\tilde{g}\ \partial_0 n_1|_{x=0} \\ &\partial_0 n_3|_{x=0} = 0\ . \end{aligned} \tag{4.21}$$

This is the same boundary condition we are considering in the $O(3)$ nlσ model case.

5 Reflection Matrices

In this section we will compute the reflection matrices associated to the boundary condition identified previously. We will concentrate on the nlσ model, since the ones for the GN and PCM models, can be found by attaching suitable CDD factors. Since we are interested in the integrable flows for these boundary conditions, we will revisit the diagonal case before.

5.1 The Diagonal Case

In [15] the reflection matrices associated to the diagonal boundary conditions found in [3] were computed (see also [16]). One question one may ask is if there are integrable *diagonal* flows, that is, if it is possible to go from a configuration with, say, M components satisfying Neumann and the remaining $N-M$ Dirichlet boundary conditions, to some other other configuration with M' Neumann and $N-M'$ Dirichlet boundary conditions. We will see that this is not possible.

Let us suppose that there are two flavors i and j that scatter diagonally. Consider the scattering process $|A_i(\theta_1)A_j(\theta_2)\rangle \to |A_i(\theta_2)A_j(\theta_1)\rangle$. Let's call the reflection amplitude for the i particle $R_1(\theta)$, and for the j particle $R_2(\theta)$ The bYBe for this process is, therefore

$$\begin{aligned} &R_1(\theta_1)R_2(\theta_2)\sigma_2(\theta_+)\sigma_3(\theta_-) + R_2(\theta_2)R_2(\theta_1)\sigma_3(\theta_+)\sigma_2(\theta_-) = \\ &= R_1(\theta_1)R_1(\theta_2)\sigma_2(\theta_-)\sigma_3(\theta_+) + R_1(\theta_2)R_2(\theta_1)\sigma_3(\theta_-)\sigma_2(\theta_+)\ . \end{aligned} \tag{5.22}$$

Dividing by $R_2(\theta_1)R_2(\theta_2)\sigma_2(\theta_+)\sigma_2(\theta_-)$ and introducing $X(\theta) = R_1(\theta)/R_2(\theta)$, and $\delta_i(\theta) = \sigma_i(\theta)/\sigma_2(\theta)$, we get

$$(X(\theta_1) - X(\theta_2))\delta_3(\theta_-) = (X(\theta_1)X(\theta_2) - 1)\delta_3(\theta_+) \ . \tag{5.23}$$

By taking the limit $\theta_2 \to \theta_1$, we arrive at the following differential equation

$$\frac{d}{d\theta}X(\theta) = \frac{X^2(\theta) - 1}{2\theta} \ . \tag{5.24}$$

The solutions for 5.24 are $X(\theta) = 1$ which means that the amplitudes $R_1 = R_2$, or

$$X(\theta) = \frac{c - \theta}{c + \theta} \ . \tag{5.25}$$

Notice that this result is true for any pair of diagonally scattering flavors. Using 5.25 we will show now that there are no integrable diagonal flows.

Suppose there are 3 diagonally scattering flavors, and call them $1, 2$ and 3. From 5.25 we know that

$$\frac{R_1(\theta)}{R_2(\theta)} = \frac{c - \theta}{c + \theta} \quad , \quad \frac{R_1(\theta)}{R_3(\theta)} = \frac{c' - \theta}{c' + \theta} \quad \text{and} \quad \frac{R_2(\theta)}{R_3(\theta)} = \frac{c'' - \theta}{c'' + \theta} \tag{5.26}$$

with some constants c, c' and c''. But these equations are incompatible, unless one of the ratios is 1. This proves that we can not have more than 2 different types of reflection amplitudes in the diagonal case. Moreover, as it was shown in [3], if we assume that there are only 2 types of reflection amplitudes, the constant c in 5.24 can be fixed using the bYBe to be $c = -i\frac{\pi}{2}\frac{N-2M}{N-2}$.

Now we can easily show that there are no diagonal integrable flows. Since the ratios are fixed by the bYBe, we simply can not go diagonally from a reflection matrix where there are k particles satisfying Neumann boundary condition, to one where there are $k' \neq k$.

5.2 The Nondiagonal Case

In this subsection we use the bYBe 2.11 in order to find the reflection matrices corresponding to the boundary conditions identified earlier. Using the intuition gained in section 3 we have the following ansatz for the reflection matrix

$$R = \begin{pmatrix} A(\theta) & B(\theta) & 0 & 0 & \cdots \\ -B(\theta) & A(\theta) & 0 & 0 & \cdots \\ 0 & 0 & R_0(\theta) & 0 & \cdots \\ 0 & 0 & 0 & R_0(\theta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ , \tag{5.27}$$

where A, B and R_0 are to be determined.

Let us consider the scattering process $|A_1(\theta_1)A_i(\theta_2)\rangle \to |A_i(-\theta_1)A_1(-\theta_2)\rangle$, where i is any of the diagonally scattering particles, that is $i > 2$ The bYBe for this process is

$$\begin{aligned} R_0(\theta_1)R_0(\theta_2)\sigma_3(\theta_+)\sigma_2(\theta) + R_0(\theta_2)A(\theta_1)\sigma_2(\theta_+)\sigma_3(\theta) = R_0(\theta_1)R_0(\theta_2)\sigma_3(\theta)\sigma_2(\theta_+) + \\ +A(\theta_1)A(\theta_2)\sigma_2(\theta)\sigma_3(\theta_+) - B(\theta_1)B(\theta_2)\sigma_2(\theta)\sigma_3(\theta_+) \ . \end{aligned} \tag{5.28}$$

Dividing this equation by $R_0(\theta_1)R_0(\theta_2)\sigma_2(\theta_+)\sigma_2(\theta)$, and defining $X(\theta) = A(\theta)/R_0(\theta)$ and $Y(\theta) = B(\theta)/R_0(\theta)$, we get

$$(X(\theta_1) - X(\theta_2))\delta_3(\theta) = (X(\theta_1)X(\theta_2) - 1)\delta_3(\theta_+) - Y(\theta_1)Y(\theta_2)\delta_3(\theta_+) \ , \tag{5.29}$$

and by taking the limit $\theta_1 \to \theta_2$, we obtain the following differential equation

$$\frac{d}{d\theta}X(\theta) = \frac{X^2(\theta) - Y^2(\theta) - 1}{2\theta} \ . \tag{5.30}$$

Notice that if $Y(\theta) = 0$ this equation reduces to the one for the diagonal case 5.24. Since we have two unknown functions now, we need one more equation. This is accomplished by the bYBe for $|A_1(\theta_1)A_i(\theta_2)\rangle \to |A_i(-\theta_1)A_2(-\theta_2)\rangle$, which reads

$$\begin{aligned} R_0(\theta_2)B(\theta_1)\sigma_2(\theta_+)\sigma_3(\theta) = R_0(\theta_1)B(\theta_2)\sigma_2(\theta_+)\sigma_3(\theta) + \\ +A(\theta_1)B(\theta_2)\sigma_2(\theta)\sigma_3(\theta_+) + B(\theta_1)A(\theta_2)\sigma_2(\theta)\sigma_3(\theta_+) \ . \end{aligned} \tag{5.31}$$

After dividing by $R_0(\theta_1)R_0(\theta_2)\sigma_2(\theta)\sigma_2(\theta_+)$, and taking the limit $\theta_1 \to \theta_2$, we get

$$\frac{d}{d\theta}Y(\theta) = \frac{X(\theta)Y(\theta)}{\theta} \ . \tag{5.32}$$

We can easily solve equations 5.30 and 5.32 by introducing $Z_\pm(\theta) = X(\theta) \pm iY(\theta)$, which satisfy

$$\frac{d}{d\theta}Z_\pm(\theta) = \frac{Z_\pm^2(\theta) - 1}{2\theta} \ . \tag{5.33}$$

This is the same equation we obtained earlier for the diagonal case 5.24. The solution for $X(\theta)$ and $Y(\theta)$ are, therefore

$$X(\theta) = \frac{1}{2}\left(\frac{c-\theta}{c+\theta} + \frac{c'-\theta}{c'+\theta}\right) \qquad \text{and} \qquad Y(\theta) = \frac{1}{2i}\left(\frac{c-\theta}{c+\theta} - \frac{c'-\theta}{c'+\theta}\right) \tag{5.34}$$

Notice that the purely diagonal case corresponds to $Y(\theta) = 0$, that is $c = c'$, and we recover 5.25. One aspect that may look puzzling is the fact that, at this point, we have two constants to be fixed, c and c', but only one coupling constant g.

In order to find a further constraint involving c and c', consider the scattering process $|A_1(\theta_1)A_1(\theta_2)\rangle \to |A_1(-\theta_1)A_2(-\theta_2)\rangle$. The bYBe reads

$$\begin{aligned}&A(\theta_1)B(\theta_2)(\sigma_2(\theta_+)\sigma_3(\theta)+\sigma_2(\theta)\sigma_2(\theta_+))+A(\theta_2)B(\theta_1)(\Sigma(\theta_+)\sigma_3(\theta)-\sigma_1(\theta_+)\sigma_2(\theta))=\\&A(\theta_1)B(\theta_2)(\Sigma(\theta)\Sigma(\theta_+)+\sigma_1(\theta)\sigma_1(\theta_+))+A(\theta_2)B(\theta_1)(\Sigma(\theta)\sigma_3(\theta_+)-\sigma_1(\theta)\sigma_2(\theta_+)+\\&(N-2)R(\theta_1)B(\theta_2)\sigma_1(\theta)\sigma_1(\theta_+))\end{aligned}\tag{5.35}$$

After dividing by $R(\theta_1)R(\theta_2)\sigma_2(\theta)\sigma_2(\theta_+)$, we plug the solutions 5.34 into 5.35 to obtain, after an elementary but tedious calculation, that

$$c+c'=-i\pi\frac{N-4}{N-2}\tag{5.36}$$

We have checked that the expressions 5.34 together with 5.36 solve all the remaining bYBe's.

There are two ways to get diagonal reflection matrices from 5.34 and the constraint 5.36. The first one is by setting $c=c'$, and we obtain the result for the diagonal case with 2 fields satisfying Neumann boudary condition and the remaining $N-2$ Dirichlet [3]. This is the deciding factor that made us choose Dirichlet boundary conditions for the diagonally-scattering fields in section 4. The other way to have diagonal reflection matrices is by having both c and c' go to ∞. In this case we have $X(\theta)=1$ and we conclude that all field components satisfy the same boundary condition, that is, Dirichlet.

Therefore if we write $c=-i\pi(N-4)/(2N-4)+\xi(g)$ and $c'=-i\pi(N-4)/(2N-4)-\xi(g)$, where $\xi(g)$ is a (unknown) function of the boundary coupling constant, we have that, for $g=0$ (2 Neumann, $N-2$ Dirichlet) we should have $\xi(0)=0$ (and so $X(\theta)=(-i\pi(N-4)/(2N-4)-\theta)/(-i\pi(N-4)/(2N-4)+\theta))$, and for $g\to\infty$ (all Dirichlet), $X(\theta)=1$ and so $\xi(\infty)=\infty$. We see, then, that there are integrable flows between diagonal reflection matrices, and that the reflection matrices we found have the correct behaviour, as compared to the one dictated by the microscopic boundary Lagrangian.

All that is left to do now is to solve the boundary unitarity and boundary crossing-symmetry conditions, which will be dealt with shortly.

Based on the "very simple example" studied before and the solution we found for the case of 2 nondiagonal fields, we are tempted at constructing other solutions of the bYBe in a similar fashion as the one just done. As we will see now, there are no such solutions.

Consider the following ansatz for the $O(5)$ model

$$R(\theta)=\begin{pmatrix} A(\theta) & B(\theta) & 0 & 0 & 0 \\ -B(\theta) & A(\theta) & 0 & 0 & 0 \\ 0 & 0 & A(\theta) & B(\theta) & 0 \\ 0 & 0 & -B(\theta) & A(\theta) & 0 \\ 0 & 0 & 0 & 0 & R_0(\theta) \end{pmatrix}, \tag{5.37}$$

This reflection matrix should correspond to the following boundary action

$$S_b=\int_{-\infty}^{+\infty} dx_0 \frac{g}{2}(n_1\dot{n}_2-n_2\dot{n}_1+n_3\dot{n}_4-n_4\dot{n}_3) \tag{5.38}$$

In principle we could have two different coupling constants, one for the fields n_1 and n_2 and another for n_3 and n_4, but we are choosing them to be equal in order to simplify the argument. Since this case is a special point of the general case, the absence of solutions in this case will establish our claim that there are no solutions with more than 2 nondiagonal fields. We will look at the $O(5)$ case, but the argument is the same for $O(N)$ with more than on nondiagonal block.

The bYBe for the processes $|A_1(\theta_1)A_5(\theta_2)\rangle \to |A_5(-\theta_1)A_2(-\theta_2)\rangle$ and $|A_1(\theta_1)A_5(\theta_2)\rangle \to |A_5(-\theta_1)A_2(-\theta_2)\rangle$ are essentially the same as in the case for 2 nondiagonal fields only, and so the ratios A/R_0 and B/R_0 have the same form as the ones given in 5.34. If we consider now the bYBe for the process $|A_5(\theta_1)A_5(\theta_2)\rangle \to |A_1(-\theta_1)A_1(-\theta_2)\rangle$ we find that there is no solution besides the diagonal one ($c=c'$) for the constants c and c', and therefore there is no solution of the bYBe consistent with 5.37.

Before we proceed, we should mention another solution of the bYBe for even N, but to which we do not know how to assign microscopic boundary conditions.

Let us consider the following block-diagonal ansatz for the reflection matrix

$$R(\theta)=\begin{pmatrix} A(\theta) & B(\theta) & 0 & 0 & \cdots \\ -B(\theta) & A(\theta) & 0 & 0 & \cdots \\ 0 & 0 & A(\theta) & B(\theta) & \cdots \\ 0 & 0 & -B(\theta) & A(\theta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.39}$$

The bYBe for the process $|A_1(\theta_1)A_3(\theta_2)\rangle \to |A_3(-\theta_1)A_2(-\theta_2)\rangle$ is

$$\begin{aligned} A(\theta_2)B(\theta_1)\sigma_2(\theta_+)\sigma_3(\theta) &= A(\theta_1)B(\theta_2)\sigma_2(\theta_+)\sigma_3(\theta)+ \\ &+ \left(A(\theta_1)B(\theta_2)+A(\theta_2)B(\theta_1)\right)\sigma_2(\theta)\sigma_3(\theta_+)\ . \end{aligned} \tag{5.40}$$

Dividing this equation by $\sigma_2(\theta_+)\sigma_2(\theta)A(\theta_1)A(\theta_2)$, and taking the limit $\theta_1\to\theta_2$, we get

$$\frac{d}{d\theta}\left(\frac{B(\theta)}{A(\theta)}\right)=\frac{1}{\theta}\left(\frac{B(\theta)}{A(\theta)}\right)\ , \tag{5.41}$$

whose solution is

$$\frac{B(\theta)}{A(\theta)} = \alpha\ \theta\ , \tag{5.42}$$

where α is a constant. We have checked that this alone solves all the remaining bYBe's, and therefore α is left as a free-parameter. All we need to do is to fix $A(\theta)$ using unitarity and boundary crossing symmetry. Since their solution is quite straightforward, we will not quote it here. The only diagonal reflection matrix that can be obtained from this solution is by taking $\alpha = 0$, which corresponds to all components satisfying Neumann boundary conditions. If $\alpha \to \infty$ we have a block diagonal solution, each block being antisymmetric. Note that one must be careful in taking $\alpha \to \infty$, since unitarity and crossing symmetry will give a prefactor with an overall dependence on α, which precisely cancels the α dependence of $B(\theta)$ in this limit.

5.3 The $O(2)$ Case

In the limit $N \to 2$ we can solve the bYBe completely. In this subsection we will not write down the bYBe's explicitly, since they can be derived quite easily, and the $O(2)$ case is not our main interest. We will describe which equations one needs to solve to fix the structure of the reflection matrix completely, in any case.

Initially consider the general form for the reflection matrix

$$R(\theta) = \begin{pmatrix} f(\theta) & w(\theta) \\ h(\theta) & g(\theta) \end{pmatrix} \ . \tag{5.43}$$

The bYBe for $|A_1(\theta_1)A_1(\theta_2)\rangle \to |A_1(-\theta_1)A_1(-\theta_2)\rangle$ gives

$$h(\theta_1)w(\theta_2) = h(\theta_2)w(\theta_1)\ . \tag{5.44}$$

If these amplitudes are non-zero, we have to have

$$w(\theta) = \alpha h(\theta)\ , \tag{5.45}$$

where α is a constant. Next, the bYBe for $|A_1(\theta_1)A_1(\theta_2)\rangle \to |A_2(-\theta_1)A_2(-\theta_2)\rangle$ requires

$$\frac{d}{d\theta}\left(\frac{f(\theta)}{g(\theta)}\right) = \frac{1}{2\theta}\left(\left(\frac{f(\theta)}{g(\theta)}\right)^2 - 1\right) \tag{5.46}$$

whose solution is

$$\frac{f(\theta)}{g(\theta)} = \frac{\beta-\theta}{\beta+\theta} \tag{5.47}$$

for some constant β. Finally, the bYBe for $|A_1(\theta_1)A_1(\theta_2)\rangle \to |A_2(-\theta_1)A_1(-\theta_2)\rangle$ demands

$$g(\theta) = \gamma h(\theta)\frac{\beta+\theta}{\theta}\ . \tag{5.48}$$

The reflection matrix is, then,

$$R(\theta) = h(\theta) \begin{pmatrix} \gamma^{\frac{\beta-\theta}{\theta}} & \alpha \\ 1 & \gamma^{\frac{\beta+\theta}{\theta}} \end{pmatrix} . \tag{5.49}$$

To obtain the solution where we would have set $h(\theta) = 0$ (or equivalently $w(\theta) = 0$) in 5.44, all we should do is to set $\alpha = 0$ in 5.49. We have checked that 5.49 solves all the remaining bYBe's. The function $h(\theta)$ may be fixed by unitarity and boundary crossing-unitarity, but again, we are not going to carry out this computation here.

One aspect that may look puzzling is the fact that we know that the $O(2)$ nlσ model is related to the sine-Gordon model, and 5.49 has 3 independent parameters, unlike the boundary sine-Gordon model, where one has only 2 independent parameters. The reason for that is that we are not looking at the sG model at arbitrary coupling constant, but at $\beta^2 = 8\pi$, and as pointed out in [1], at special points for the coupling constant (like this one) there are more solutions than the ones presented there for arbitrary β.

We now go back to the solution of unitarity and boundary-crossing symmetry conditions for the reflection matrix 5.27. The next two subsections refer to this reflection matrix.

5.4 Boundary Unitarity

The boundary-unitarity condition implies that

$$R_0(\theta)R_0(-\theta) = 1 \ , \tag{5.50}$$

and

$$A(\theta)A(-\theta) - B(\theta)B(-\theta) = 1 \qquad \text{and} \qquad A(-\theta)B(\theta) + A(\theta)B(-\theta) = 0 \ . \tag{5.51}$$

By dividing 5.51 by $1 = R(\theta)R(-\theta)$, and using the explicit forms 5.34, we see that 5.51 is satisfied trivially. Therefore, all we have to solve from unitarity is 5.50.

5.5 Boundary Crossing Unitarity

The boundary-crossing unitarity condition for the diagonal part of the reflection matrix is

$$R_0(\frac{i\pi}{2} - \theta) = ((N-2)\sigma_1(2\theta) + \sigma_2(2\theta) + \sigma_3(2\theta))R_0(\frac{i\pi}{2} + \theta) + 2\sigma_1(2\theta)A(\frac{i\pi}{2} + \theta) \ . \tag{5.52}$$

Using 5.34 and 5.36, we can rewrite 5.52 as

$$R_0(\frac{i\pi}{2} - \theta) = R_0(\frac{i\pi}{2} + \theta)(\sigma_I(2\theta) + 2\sigma_1(2\theta)(X(\frac{i\pi}{2} + \theta) - 1)) \ . \tag{5.53}$$

All the other boundary-crossing unitarity equations are equivalent to this one.Using the parametreization for c and c' introduced after equation 5.36, and after some elementary, but tedious, computation, we can cast 5.53 into a quite compact form

$$R_0(\frac{i\pi}{2}-\theta) = R_0(\frac{i\pi}{2}+\theta)\sigma_I(2\theta)\left[\frac{\theta+\xi-\frac{i\lambda}{2}}{\theta-\xi+\frac{i\lambda}{2}}\right]\left[\frac{\theta-\xi-\frac{i\lambda}{2}}{\theta+\xi+\frac{i\lambda}{2}}\right] \tag{5.54}$$

Finally we can fix $R_0(\theta)$ by solving 5.54 and 5.50

$$\begin{aligned} R_0(\theta) = &-\frac{\Gamma(\frac{1}{4}+\frac{\lambda+2i\xi}{4\pi}+\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{\lambda+2i\xi}{4\pi}-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{4}+\frac{\lambda+2i\xi}{4\pi}-\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{\lambda+2i\xi}{4\pi}+\frac{i\theta}{2\pi})}\times \\ &\frac{\Gamma(\frac{1}{4}+\frac{\lambda-2i\xi}{4\pi}+\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{\lambda-2i\xi}{4\pi}-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{4}+\frac{\lambda-2i\xi}{4\pi}-\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{\lambda-2i\xi}{4\pi}+\frac{i\theta}{2\pi})}K(\theta) \end{aligned} \tag{5.55}$$

where $K(\theta)$ is the solution of 5.54 and 5.50 without the ξ-dependent factors [15],

$$K(\theta) = -\frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(1+\frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4}+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4}-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(1-\frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4}+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4}+\frac{i\theta}{2\pi})} \tag{5.56}$$

This fixes the reflection matrix completely.

This solution for the reflection matrix reduces to the appropriate diagonal solutions in the limits $\xi \to 0$ (2 Neumann, $N-2$ Dirichlet) and $\xi \to \infty$ (all Dirichlet).

It would be interesting to study the analytical structure of this reflection amplitude, but this requires the knowledge of the explicit form of the function $\xi(g)$.

6 Conclusions

We have found new integrable boundary conditions for the $O(N)$ nlσ model which are nondiagonal and depend on one free parameter. The integrable boundary conditions presented here break the $O(N)$ symmetry to $O(2)\times O(N-2)$. A similar conclusion has been reached for the $SU(N)$ spin chain in [17]. These boundary conditions, together with the diagonal ones proposed in [15], are argued to exhaust the possible integrable boundary conditions for the $O(N)$ nlσ model. We also found solutions of the bYBe which depend on one parameter and that can be associated to these boundary conditions, and saw that they do satisfy the appropriate limits, which correspond to the coupling constant $g \to 0$ or ∞. We found the most general solution of the bYBe for the $O(2)$ nlσ model, which is described by a three-parameter family, unlike the well-known two-parameter solution of the boundary sine-Gordon. This is understood by noting that the $O(2)$ nlσ model corresponds to the $\beta^2 = 8\pi$ sine-Gordon model, and as pointed out in[1], at this point, and in general for points where $\lambda = 8\pi/\beta^2 - 1$, there are additional solutions.

In section 4 we established the integrability of the nondiagonal boundary conditions, by inspecting the spin-4 charges proposed in [9]. There were no constraints on how many nondiagonal blocks one can have, whereas we found that there are solutions of the bYBe only in the case of one single nondiagonal block. One possible explanation for that is that by inspecting higher spin charges there would be further constraints on the possible boundary conditions, but this is still to be checked.

Recently, MacKay and Short [6] have studied the principal chiral model with a boundary and found an interesting relationship between their boundary conditions and the theory of symmetric spaces. Their solutions, though, are quite different from ours, and some work should be done in trying to clarify their relationship.

There are several directions to be pursued now. The S-matrix for the elementary excitations in the $SO(N)$ principal chiral and $O(N)$ Gross-Neveu models, are, up to CDD factors, the same as the one for the $O(N)$ nlσ model, and therefore the bYBe's for all the three models is the same. This means that it should be possible to find microscopic lagrangians which provide nondiagonal boundary conditions for these other models, similar to the ones presented here, also depending on one free parameter.

Another problem that arises naturally is to find the exact form of the function $\xi(g)$, which is, in general quite hard (as for example, in the case of the boundary sine-Gordon model). A related problem is to perform a large-N expansion for the reflection matrices (diagonal and nondiagonal) as a check of their validity. Unfortunately a framework of how to perform boundary perturbation theory is yet to be developed. Finally, the study of the thermodynamics of the boundary nlσ model would be an interesting, if somewhat challenging, problem.

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