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The large M limit of Non-Commutative Open Strings at strong coupling

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Abstract

Two dimensional Non-Commutative Open String (NCOS) theory, well-defined perturbatively, may also be studied at strong coupling and for large D-string charge by making use of the Holographic duality. We analyze the zero mode dynamics of a closed string in the appropriate background geometry and map the results onto a sector of strongly coupled NCOS dynamics. We find an elaborate classical picture that shares qualitative similarities with the SL(2,R) WZW model. In the quantum problem, we compute propagators and part of the energy spectrum of the theory; the latter involves interesting variations in the density of states as a function of the level number, and energies scaling inversely with the coupling. Finally, the geometry exhibits a near horizon throat, associated with NCOS dynamics, yet it is found that the whole space is available for Holography. This provides a setting to extend the Maldacena duality beyond the near horizon limit.

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1 Introduction and Summary

Within the realm of perturbation theory it was originally formulated in, string theory has been known to entail unusual and sometimes remarkable dynamics, such as non-local interactions and interesting spectra. Certain puzzles that had plagued theoretical physics for many years were successfully addressed in this perturbative framework due to such unique attributes of the theory. In more recent years, sometimes through the use of supersymmetry, certain non-perturbative aspects of the theory have been explored, revealing yet richer and more remarkable dynamics.

An important new tool in this program is the so-called Holographic duality [1]-[4] that allows the controlled study of the theory beyond a perturbative expansion. It is believed that this principle is a general one; that it applies to the theory expanded about many of its multitude of vacua. In particular, our goal in this work is to focus on Non-Commutative Open Strings (NCOS) [5]-[8] within Wound String theory [9, 10, 11]. The latter corresponds to a certain sector of string theories that is characterized by non-relativistic dispersion relations. In this setting, a particular class of non-perturbative solitons, longitudinal D-strings bounded with fundamental string charge, involve particularly interesting dynamics; they can be described by a two dimensional, Lorentz invariant theory of open strings, with the time and space coordinates non-commuting. This is a useful setup as it focuses onto an important attribute of string theories, non-locality, in a framework that is relatively simple. And of most interest to us, through the use of the Holographic duality, this non-local dynamics is computationally accessible at strong coupling. It is worthwhile emphasizing that this is a quite unusual situation; whereby *stringy dynamics* can be explored non-perturbatively.

The Holographic duality in this context establishes a strong-weak coupling map between two string theories[2]. This map is not well understood, but may hold important treasures for one's intuition. There have been suggestions that this correspondence may be formulated directly on the world sheet level [11]; and indications of rich dynamics in the theory at strong coupling [14, 15, 16, 17]. It even appears that the system is a prime setting to understand Holography beyond the usual near-horizon scaling limit [11].

In conventional approaches to computations in holographic duals, one has a field theory on one side, and perturbative string theory (in practice a supergravity) on the other. The useful probes are often fields in the supergravity propagating in some curved background geometry [2, 18, 19] (see however [20]). In the NCOS analogue, a central role must be played by probes of the dual geometry that are strings. The novelty in the problem then is that it involves studying the dynamics of closed strings in curved backgrounds, and mapping results

[2] A system that is most similar to this in this regard is that of Little String theory [12, 13]; however, in that context, there has not been a good understanding of what constitutes a perturbative expansion of the theory.

from this non-linear sigma model onto physical quantities in NCOS theory.

In this work, we present an attempt in understanding strongly coupled dynamics in NCOS theory using this Holographic duality. We do this by studying, to a leading order in a semi-classical approximation scheme, the dynamics of an unexcited IIB closed string in the background geometry cast about a bound state of D strings and fundamental strings. This involves the NCOS (or Wound String theory) scaling limit, and the background space of interest becomes conformal to $AdS_3 \times S^7$ (see also discussion in [21]), in the presence of an NSNS B-field. Effects from the RR fields, and certain corrections arising from the running dilaton, are found subleading to the dynamics we consider. To be more concrete, our space is mapped by: seven coordinates on a seven sphere; one radial coordinate we call v (which is chosen dimensionless); and a time and a space coordinate associated with the two dimensional NCOS theory, that we denote by t and y. We then have in the perturbative NCOS setting the non-commutation relation $[t, y] = i\theta$, with non-locality in time measured by the scale θ. The latter also sets the length scale in the NCOS Virasoro spectrum. Furthermore, we compactify the y coordinate on a circle of circumference Σ.

The AdS_3, of curvature scale set by θ, is then parameterized by the coordinates v, t and y. An important role is however played by the conformal factor, which introduces additionally a throat in the geometry at the radial coordinate $v^3 \sim 1/G$; the dimensionless variable G being interpreted in the dual NCOS theory as the open string coupling. We refer to this throat as the non-commutativity throat. Figure 1 shows a depiction of the string frame scalar curvature, as a function of v and y. Note that this curvature is bounded everywhere, being in particular zero at $v = 0$ and $v \to \infty$. However, this does not mean that we can trust this background for all v; as there are other conditions that need to be met to paint a reliable picture of the dynamics.

Hence, the problem at hand is as follows. We consider the center of mass motion of a closed string wrapping the cycle y in the geometry shown in Figure 1, with zero angular momentum on the seven sphere. We analyze the motion first classically, then quantum mechanically. This leads to several interesting predictions about the dual NCOS theory; about a theory of strings at strong coupling $G \gg 1$. In the subsequent paragraphs, we summarize our main results.

1.1 Classical dynamics

The classical dynamics has similarities to a related system, the SL(2,R) WZW model [22, 23, 24]. The main differences arise from the presence of the non-commutativity throat, which also makes the full problem in this case considerably more involved to solve for. The center of mass dynamics of the closed string is however simple to unravel, as it reduces to the dynamics of a point particle in a certain two dimensional curved space (the $v - t$ plane), in

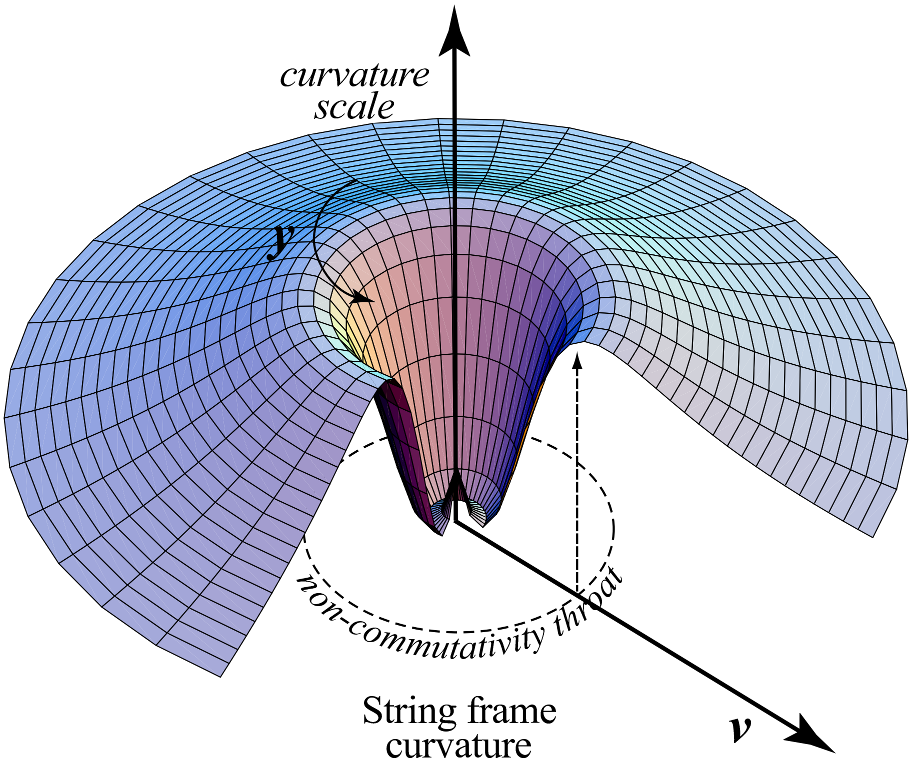

Figure 1: *The string frame curvature (in IIB string units) is finite everywhere, becoming zero at the center* $v = 0$ *and at asymptotic infinity; and has a throat at* $v^3 \sim 1/G$. *It is found that the center* $v = 0$ *repulses incoming strings.*

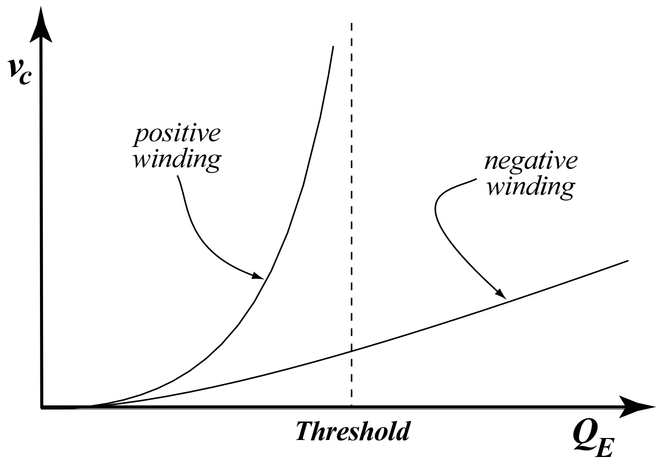

Figure 2: *The maximum extent v_c that the closed string reaches in the radial direction v as a function of its energy Q_E. The 'threshold' refers to the mass of the corresponding wound string in the Wound string theory, $\omega\Sigma/(4\pi\alpha_e)$; with ω being the winding number, and α_e the NCOS string scale related to θ as in (2).*

the presence of a v dependent potential and magnetic field. We find an elaborate picture.

There are two sets of solutions that describe bounded motion, differing by the orientation of the closed string along y (the analogues of 'short strings' in the WZW model, pointed out already in [7, 11]). The string falls toward the horizon at $v = 0$, initially accelerating, then decelerating as it gets *repulsed* from the origin. It comes to a halt at the origin and reverses direction, but cannot escape to infinity. Instead, it reverses course again at some finite $v = v_c$, and falls back in. The period of this oscillatory motion is finite in proper time; but infinite as measured by the time variable t, because of the infinite redshift associated with the horizon. More interestingly, v_c is not necessarily within the non-commutativity throat! Depending on the classical energy of the closed string, v_c can readily extend to asymptotic infinity. Figure 2 shows a plot of this maximum extent explored by the closed string as a function of its energy Q_E. Positive (negative) winding refers to the fact that the string is oriented parallel (anti-parallel) to the background B-field. The throat region was believed to be the analogue of the near horizon region of the Maldacena duality [1, 25]; perhaps being the 'boundary' of the holographic projection. It was then suggestive that such bound state solutions would be confined to within the throat area[3], being duals to states in the NCOS theory. We see however that this is not the case. Figure 3 shows a plot of the dynamics; the two cases of different winding orientation are qualitatively similar in this respect.

[3]This suggestion was made in [11] by analyzing the dynamics at small velocities.

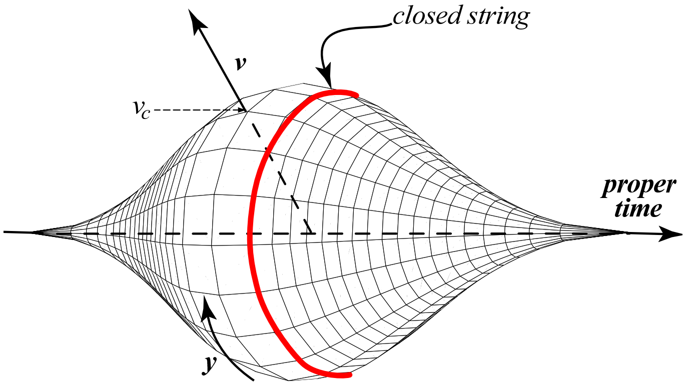

Figure 3: *Depiction of the closed string world sheet as a function of proper time for the bounded motions; both possible winding orientations for the string generate qualitatively similar plots. Note the deceleration effect near $v \sim 0$; the closed string is repulsed from the center.*

If the energy of the string is high enough, only the positively wound closed string can escape to infinity. The threshold energy is given by the mass of the wound string in Wound String theory, which is consistent with one's intuition. We call the corresponding scenario the scattering solution (the analogue of 'long strings' in the SL(2,R) WZW model). As shown in Figure 4, in finite proper time, the closed string can be made to scatter off the horizon, if its initial energy is high enough. That the world-sheet theory at asymptotic infinity coincides with that of the wound string was already shown in [11]. Hence, this presents a setting whereby one can define asymptotic on shell states corresponding to certain vertex operator insertions in the dual NCOS theory. This however also presents a puzzle, as the time of scattering appears to be infinite in the variable t, due to the horizon redshift effect already encountered above.

This summarizes the classical dynamics of the problem. Next, we present some of the results concerning the quantum mechanical treatment.

1.2 A spectrum

Whenever one encounters bounded classical dynamics, one should expect the possibility of a quantized energy spectrum in the quantum mechanical problem. And indeed, due to subtle cancelations that appear in the computation of the path integral of this problem, we argue that the bounded dynamics leads to a prediction for part of the spectrum of the strongly coupled NCOS theory. Figure 5 summarizes these results in a plot. For high energies, the energy levels scale as $N^{3/2}/\sqrt{G}$ for the negatively wound case, with N being the level

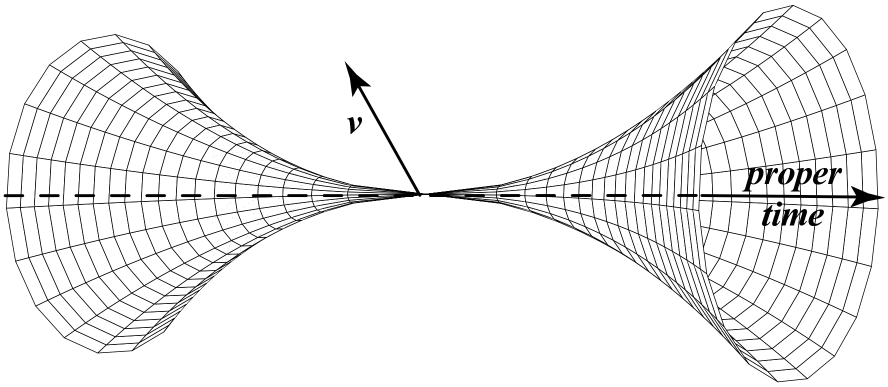

Figure 4: *The closed string world sheet as a function of proper time for the scattering scenario with positive winding.*

number, and G the NCOS coupling; while they scale as N^3/G for the positively wound case for low energies (the lower region in the figure, where the two spectra become degenerate). The threshold energy indicated in the figure is the mass of the corresponding wound closed string, which comes out correctly from our analysis [7]. And the shaded region is to indicate that the computation can be trusted for a window of energies; this is widened by increasing the number of D-strings M, and/or increasing the cycle size Σ in NCOS string units. The analytical form for this spectrum is presented in the text.

An interesting observation is that all physical quantities we compute, classically or quantum mechanically, have all instances of the variable counting number of longitudinal D-strings appearing only in a certain combination with the original coupling of the theory. Denoting the open string coupling by G_o, and by M the number of D-strings, the effective coupling is argued to be $G \equiv G_o\sqrt{M}$ for large M [16]. More importantly, the strict limit $M \to \infty$ with fixed G is *regular*, with all observables remaining finite and with the background geometry becoming reliable for all v. In particular, the center, generally associated with a singularity, is almost flat; as if the non-commutativity throat plays the role of a regulator of the singularity for $M \to \infty$. Consequently, the spectrum is well-defined for all energies. This constitutes one of the main results of this work; a computation of part of the non-perturbative spectrum of a string theory at strong coupling and in the large M limit.

At finite M, the reliability of our result hinges on certain assumptions about regularity of the dynamics near $v \sim 0$, where string interactions become important. We discuss this in some detail in the text; for large but finite M, we have energy levels that may receive corrections through an expansion in inverse powers of M. We also argue for the possibility for transitions between these various levels with different winding number sectors; the computed

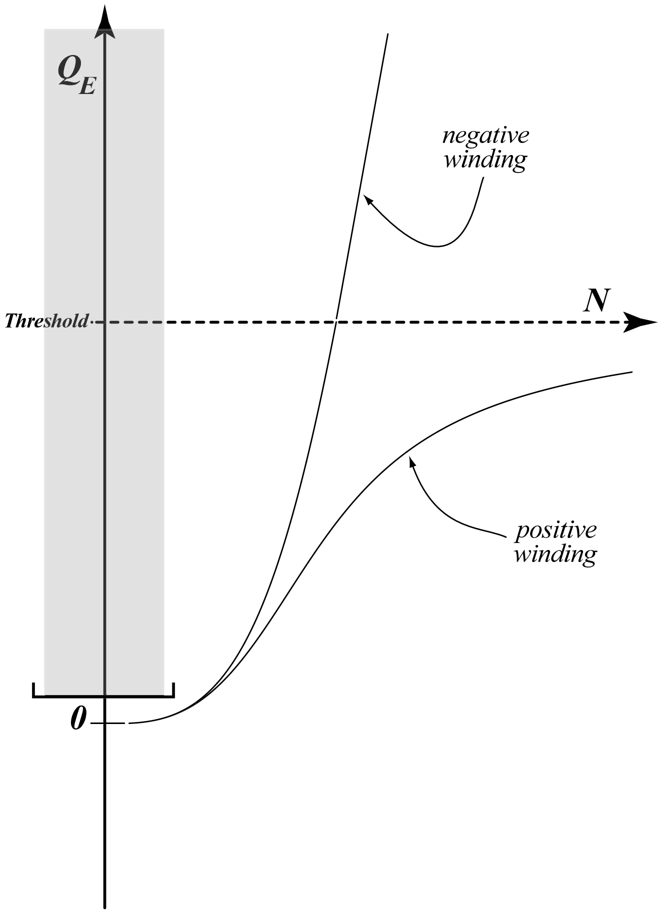

Figure 5: *The quantized spectrum associated with the bounded dynamics of the closed string. Q_E is mass of the state, and N is an integer denoting the the level number. This is a numerical plot of equations (56), (71) and (75). The threshold corresponds to the rest mass of the corresponding wound string, $Q_E = \omega\Sigma/(4\pi\alpha_e)$. The shaded area indicates the window of energies for which the spectrum may be reliable, as determined by the values of M for the upper bound, and either Σ or M for the lower bound (depending on whether $\sqrt{M}$ is smaller or bigger than Σ/l_e). As a general rule, the shaded area widens for large M and large Σ/l_e. For this plot, we have chosen $G \sim 110$, $\Sigma \sim 23 l_e$, $\omega \sim 100$, and the full scale on the N axis is around 10^4. The shaded area is drawn for $M = 1.5 \times 10^5$; the upper bound is beyond the figure. For $M \to \infty$, with all other parameters finite, this shaded region encompasses the whole spectrum.*

spectrum being however still a good first order layout of the dynamics. Finally, note also that we should superimpose, towering on each mass level depicted in the figure, another spectrum corresponding to excitations of the closed string about the center of mass.

1.3 Propagators

In the NCOS theory at strong coupling, it then seems that one has an interesting spectroscopy of states whose masses scale inversely with the NCOS coupling. These objects perhaps may be accorded a size, and corresponding breathing modes, that correlate with location in the radial direction v in the dual picture. In the sector we confine our analysis, all these states carry zero total momentum in the y direction, as well as zero angular momentum on the seven sphere. They can be attributed a 'winding number' correlating with the number of times the correponding closed string winds the y direction in the dual description. We mentioned also scattering processes whereby a closed string in Wound String theory, represented perhaps as a local vertex operator insertion in the NCOS theory, scatters off the longitudinal D-strings. All of this dynamics may be probed by computing propagators, which we also do. For propagations of duration T in NCOS string units, up to a 'size' (or location in the v coordinate) v_2, we find that the propagators diverge at $T \sim 1/v_2$; and we find amplitudes that scale as $\sqrt{G}$ and G.

1.4 Beyond the near horizon regime

Another matter we focus on is an attempt to understand in what sense Holography is being implemented in this setting; and, in particular, to what extent is one studying this phenomenon beyond the near-horizon limit that Holography is usually attributed with. We observe that, while the throat is not a boundary of dynamics between what we associate with NCOS theory, and what we qualify as Wound closed string theory, it does however correspond to an energy scale at which all running physical parameter of the theory scale as those of the NCOS theory. The picture that seems to emerge is that the whole of the space is holographically encoded in the NCOS theory, but there is a screen at some definite location in the bulk, where the non-commutative throat sits, that we may still associate with a 'projection' plane. While the Wound string theory may be viewed as living at asymptotic infinity only. This strongly suggests that there is a controlled approach in which the Maldacena duality can be extended beyond the near horizon.

1.5 Outline

As this work was possibly more pleasant to type than it will be to read, we have tried to organize the presentation such that many computational matters are collected in appendices. Section 2 presents the notation and the setup of the problem. Section 3 describes the classical dynamics. Section 4 concentrates on the details of the quantum problem; while Sections 5 and 6 collect the results and present some analysis. Section 7 contains the observations with regards to the role of the throat in the geometry. And Section 8 discusses various loose ends, extensions and suggestions for future work. The casual reader may focus on Sections 2, 3, 5 and 8 (and Appendix E if the need arises). Appendices A-E summarize certain computational details.

2 Preliminaries

In this section, we review the two dimensional NCOS theory and establish notational matters. The geometry dual to the strongly coupled NCOS theory is described first, with more background material on the subject collected in Appendix A. We then outline the regime of validity of the setup.

2.1 Background geometry

Two dimensional NCOS theory describes the dynamics of a bound state of D strings and fundamental strings in IIB string theory. It can also be attained as a subsector of Wound String theory in the presence of longitudinal D-strings.

NCOS theory is parameterized by an integer M that counts the number of D-strings; a string coupling G restricted to a finite range $0 \leq G \leq M/(32\pi^2)$; a string length l_e that sets the scale for (a) the spacing of the levels in the Virasoro tower of free open string excitations; and (b) for a parameter θ that measures non-commutativity of the coordinates

$$[t, y] = i\ \theta\ . \tag{1}$$

Here, t and y denote the coordinates of this two dimensional theory. In our conventions, we have

$$\theta = 2\pi\alpha_e \equiv 2\pi {l_e}^2\ . \tag{2}$$

We will also choose to compactify the y coordinate on a circle of size Σ.

This theory has a well defined perturbative expansion for $G \ll 1$ inherited from the parent string theory. Novel features of the dynamics can be attributed to the non-commutation relation (1).

The strong coupling dynamics of the two dimensional NCOS theory can be described via a dual setup, by studying IIB closed string theory in the background geometry cast about a bound state of D-strings and fundamental strings. A particular scaling limit is needed to establish this correspondence and it is briefly outlined in Appendix A. The resulting background is given by the metric (in the string frame)

$$ds_{str}^2 = \Omega^2 \left\{ \frac{v^2}{8\pi^2\alpha_e} \left(-dt^2 + \Sigma^2 dy^2\right) + \frac{dv^2}{v^2} + 4 d\Omega_7^2 \right\} , \tag{3}$$

where

$$\Omega^2 \equiv 8\pi^2 \alpha' \frac{\sqrt{G}}{\sqrt{v}} \sqrt{1 + G v^3} , \tag{4}$$

and α' is the string scale in the associated IIB theory. The space is conformal to $AdS_3 \times S^7$ [4]. As such, dynamics in this background will have qualitative similarities to that of the SL(2,R) WZW system. Note also that we have rescaled the y coordinate such that it is compact of size 1. Furthermore, our choice of coordinates is such that v is dual to energy in the NCOS theory in NCOS string units (*i.e.* v is dimensionless); this is often termed the UV-IR map in the Holographic duality [26]. The large v region, corresponding to the UV in the NCOS theory, is where, if the conformal factor Ω was missing, the timelike AdS_3 boundary would be sitting.

The dilaton in this geometry runs as

$$e^{\phi} = \left(32\pi^2\right)^2 \frac{G^{3/2}}{M} \frac{1 + G v^3}{v^{3/2}} . \tag{5}$$

We also have an axion field

$$\chi = \frac{M}{\left(32\pi^2 G\right)^2} \frac{1}{1 + G v^3} . \tag{6}$$

There is the RR two-form gauge field

$$A_{ty} = -\frac{\alpha'}{\alpha_e} \Sigma \frac{M}{\left(32\pi^2 G\right)^2} ; \tag{7}$$

and, most importantly, the NSNS B-field

$$B_{ty} = \frac{\alpha'}{\alpha_e} \Sigma \; G v^3 . \tag{8}$$

[4] We work in the analogue of the Poincare patch of AdS_3 [18]; the motivation for this is that, in the S-dual picture, the corresponding geometry describes strongly coupled two dimensional SU(N) gauge theory with an electric flux; with a Hamiltonian canonical to our time variable t.

IIB string theory in this background is expected to be holographically encoded into two dimensional NCOS theory. We will focus on probing the dynamics in this geometry using a closed string that wraps the cycle y. Figure 1 shows a plot of the geometry at hand. The Penrose diagram is identical to that of the Poincare patch of AdS_3 (see for example [18]).

2.2 Regime of validity

Given that the geometry given by (3) arises in a low energy limit of perturbative IIB string theory, we need to determine the regime where this setup is reliable. First, the curvature scale must not be stringy, which leads to the condition

$$\sqrt{\frac{v}{G}}\frac{-2+7Gv^3}{(1+Gv^3)^{3/2}} \ll \frac{1}{\alpha'} \Rightarrow \begin{cases} v \gg 1/G & \text{for } Gv^3 \gg 1 \text{ and } G \ll 1 \\ v \ll G & \text{for } Gv^3 \ll 1 \text{ and } G \ll 1 \end{cases} , \tag{9}$$

with no restriction for $G \gg 1$ (see the curve labeled (a) in Figure 6). The coordinate v is dual to energy in the NCOS theory, in units of l_e; hence, this is a statement restricting energy scale. Requiring that the closed string coupling be small yields the additional condition

$$v \gg \frac{G}{M^{2/3}}\left(1+Gv^3\right)^{2/3} \Rightarrow \begin{cases} v \ll M^{2/3}G^{-5/3} & \text{for } Gv^3 \gg 1 \\ v \gg M^{-2/3}G & \text{for } Gv^3 \ll 1 \end{cases} , \tag{10}$$

or the curve labeled (b) in the figure below. On the low energy (small v) side, the compact cycle y becomes of stringy size unless

$$v^3 \gg \frac{1}{G}\left(\frac{l_e}{\Sigma}\right)^4 , \tag{11}$$

i.e. curve (c) in Figure 6. The region between these three curves is the arena where we will confine our calculations.

A key observation that we will rely on is that each of these three curves is controlled my one of three independent parameters. We first choose $G \gg 1$ so as to be safely onto the right of curve (a). We then choose $M \gg 1$ to create a hierarchically wide regime of energies between the two flanks of curve (b). And, finally, we make $\Sigma/l_e \gg 1$ so as to push curve (c) towards parametrically smaller values of v. The patch of spacetime then available for us extends from the largest values of v to the smallest, a window controlled by M and Σ.

Outside this domain, our calculations may be extended by applying a sequence of dualities, as in [16]. However, for low or large enough energies, and for finite M, we would eventually have to reach regions of space of stringy curvature scales (or Planckian scales in an M theory), for both large (or small v). We will argue later that, within the bounds of the

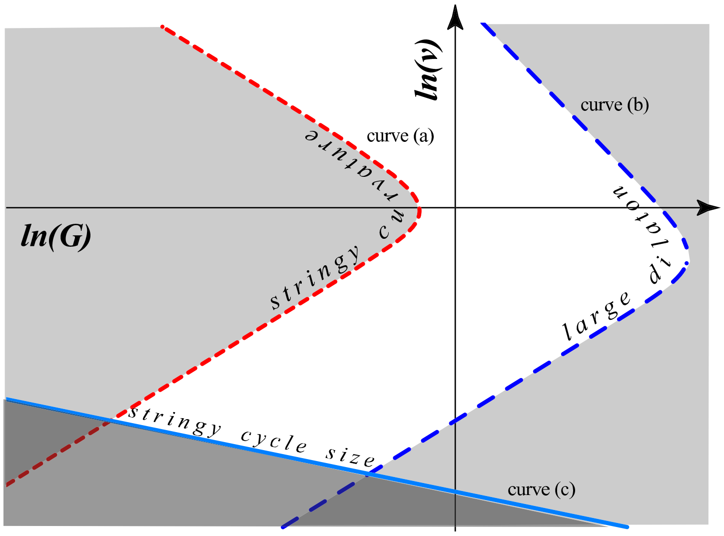

Figure 6: *The regime of validity of the background geometry used in the text, as a function of the coupling G and the energy scale (or coordinate) v. The white area is where we confine our calculations. To the left, perturbative NCOS dynamics can be used to study the system. The corner in curve (a) is at $G \sim 1$; that of curve (b) is at $G \sim \sqrt{M}$. They both occur along the line $Gv^3 \sim 1.$, i.e. the non-commutativity throat of Figure 1. The right intersection point of curves (b) and (c) is around $G \sim l_e\sqrt{M}/\Sigma$, which arises in our discussion on several occasions. For a more complete analysis of this setup in a thermodynamic setting, see [16].*

space of parameters we will be working in, our conclusions to leading order in an expansion in $1/M$ may be insensitive to the details behind these stringy walls at large and small v. We will have to assume however well-behaved boundary data in these asymptotic regions.

More interestingly, we note that the only explicit dependence on M appears in the condition dictated by curve (b); and it is such that, for $M \to \infty$, with $G > 1$ and all other parameters held finite [5], curve (b) imposes no restrictions on v. We are left with curve (c) to worry about, which sets a lower bound on v. We then need to apply two duality transformations; first, a T-duality on y, and then, for yet lower values of v, we need to lift to M-theory. Following [16], we find that the M-lift in the T-dual theory is needed at

$$v \sim \frac{G^{5/9}}{M^{4/9}} \left(\frac{l_e}{\Sigma}\right)^{4/9} . \tag{12}$$

[5] Note that this also corresponds to the large N limit in the S-dual two dimensional SU(N) gauge theory (see equations (106) and (107)).

But this goes to zero as $M \to \infty$, with all other parameters held fixed. The is due to the fact that our metric and B-field do not depend on M explicitly, yet the dilaton scales inversely with it. Hence, in the T-dual IIA theory, as in the original IIB theory, the $M \to \infty$ limit renders the dilaton and string interactions negligible. And the geometry can be trusted *for all* v, as there are no other restrictions that arise (see [16] for the details). The physical conclusions will be unchanged under the application of the duality transformation, as this corresponds to changing framework within the same parent theory; describing the same dynamics with other degrees of freedom. And all physical observables we compute will be found independent of M. This strict limit is then indeed regular, and our computations are reliable when restricted to this regime. This also gives us partial confidence that the finite M case, for fixed but large values of M, gives a faithful picture of the dynamics to leading order in an expansion involving inverse powers of M.

We also note that a number of papers have studied dynamics of the theory onto the left of Figure 6, where a perturbative expansion in G can be trusted [27, 28].

3 Classical dynamics

In this section, we probe the geometry given by (3) by studying the dynamics of a closed string wrapping the cycle y. In the first subsection, we set up the action, ansatz and classical equations of motion. In the second subsection, we focus on the specific case of interest, solve the corresponding dynamics, and analyze the resulting classical motion of the wrapped string.

3.1 Equations and ansatz

The classical motion of a closed string in a curved background is described by the action (see for example [29])

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-h} \left\{ \left(-h^{ab} G_{\mu\nu}(X) + \varepsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \mathcal{L}_{dil} + \mathcal{L}_{ferm} \right\} \,, \qquad (13)$$

where h_{ab} is the worldsheet metric. The coupling to the dilaton and RR fields is given by

$$\mathcal{L}_{dil} = \alpha' R^{(2)} \phi(X) \,, \quad \mathcal{L}_{ferm} = \mathcal{L}_{free} + \mathcal{L}_{RR} \,, \qquad (14)$$

where $\mathcal{L}_{free}$ contains the kinetic term for the fermions (and the kappa symmetry term in the spacetime formalism), while $\mathcal{L}_{RR}$ contains the coupling of the worldsheet theory to the background RR fields through fermions[6]. The details of the fermionic contributions will be

[6] The latter would involve terms of the form $\partial_\mu \chi \bar{\mathcal{V}} C \Gamma^\mu \mathcal{V}$ and $\partial_{[\alpha} A_{\beta\gamma]} \bar{\mathcal{V}} C \Gamma^{\alpha\beta\gamma} \mathcal{V}$ that couple the spin fields to the axion and D-string gauge field. In this respect, it is wiser to formulate such a setup from the outset

irrelevant, as we will focus in this work on classical bosonic dynamics on the worldsheet, with the fermion fields set to zero[7] . Furthermore, the coupling to the dilaton is subleading, being weighed by a power of the string length α'. The dynamics we will find can be plugged back into our energy-momentum tensor, to analyze the justification of dropping the term given by $\mathcal{L}_{dil}$. One then finds the condition (9). Hence, for the setup we will be concerned with, valid within the region depicted in Figure 6, both $\mathcal{L}_{dil}$ and $\mathcal{L}_{ferm}$ may be ignored.

The worldsheet theory is conformal by definition. The metric h_{ab} can be fixed, while supplementing the equations of motion with the first class constraint

$$G_{\mu\nu}\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} h_{cd} G_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + O(\alpha') = 0 \; ; \tag{15}$$

otherwise known as the string on-shell condition. Note that we have dropped from these equations contributions from the worldsheet quantum field theory of order α' and beyond[8]. Also, there is no contribution from the background B field.

The equations of motion that follow from (13) are

$$\nabla_a \nabla^a X^\gamma = -\Gamma^\gamma_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + \frac{1}{2} G^{\gamma\alpha} \varepsilon^{ab} H_{\alpha\mu\nu} \partial_a X^\mu \partial_b X^\nu \; , \tag{16}$$

with $\Gamma^\gamma_{\mu\nu}$ the Christoffel variables associated with the background metric $G_{\mu\nu}$[9]. And the field strength H is defined as

$$H_{\alpha\mu\nu} \equiv 3\partial_{[\alpha} B_{\mu\nu]} \; . \tag{17}$$

With the choice of coordinates given by the background (3), we write the worldsheet scalars X^μ as

$$X^\mu \in \{t(\tau,\sigma), v(\tau,\sigma), y(\tau,\sigma), \Theta(\tau,\sigma)\} \; , \tag{18}$$

where Θ denotes all angle variables on the seven sphere of (3). We also fix the worldsheet metric to $h_{ab} = \eta_{ab}$.

The classical system described by equation (16) can be solved by [32, 33]

$$X^\mu(\sigma+\tau) \text{ or } X^\mu(\sigma-\tau) \; , \tag{19}$$

with the spacetime supersymmetry formalism, since the vertex operators coupling to the RR fields are then more transparent (see also [30]).

[7]One effect at this level however may be additional degeneracies in the spectrum from fermionic zero modes.

[8]Such terms typically appear as quantum corrections to all components of the worldsheet energy-momentum tensor; in addition to correcting its trace, where they add to the equations of motion of IIB supergravity, as the condition for scale invariance on the worldsheet.

[9]Throughout, we conform to the conventions in [31].

for otherwise arbitrary functions, irrespective of the background fields. Generally, we loose however the naive principle of superposition. In special backgrounds one may find solutions with simultaneously both left and right moving modes.

We focus instead on dynamics subject to the ansatz

$$t(\tau,\sigma) \to t(\tau) \ , \quad v(\tau,\sigma) \to v(\tau) \ , \quad y(\tau,\sigma) \to \pm\omega\frac{\sigma}{\Sigma} \ , \quad \Theta(\tau,\sigma) \to 0 \ . \tag{20}$$

ω is the number of windings of the closed string on y (σ has size Σ). We have chosen

$$\omega \geq 1 \ , \tag{21}$$

and will account for both orientations explicitly by the $\pm$ sign in subsequent equations. The upper choice will be referred to as positive winding, the lower as negative. Positive winding corresponds to the closed string oriented parallel to the background B field.

Our ansatz corresponds to the center of mass motion of the closed string. Furthermore, if we were to consider the slight generalization

$$y(\tau,\sigma) \to \pm\omega\frac{\sigma}{\Sigma} + y(\tau) \ , \tag{22}$$

it is easy to check that the constraint from the off-diagonal elements of the worldsheet energy-momentum tensor requires $y(\tau)$ to be constant. Roughly speaking, we cannot put momentum on the probe string without wiggling it.

Subject to this ansatz, and in the background geometry given by (3)-(8), the Lagrangian becomes

$$\mathcal{L} = \pm\omega\frac{\Sigma}{2\pi\alpha_e}Gv^3\dot{t} + \frac{\Sigma}{4\pi}\frac{\Omega^2}{\alpha'}\frac{\dot{v}^2}{v^2} - \frac{\Sigma}{32\pi^3\alpha_e}\frac{\Omega^2}{\alpha'}v^2\dot{t}^2 - \frac{\Sigma}{32\pi^3\alpha_e}\frac{\Omega^2}{\alpha'}v^2\omega^2 \ ; \tag{23}$$

i.e. a classical system describing a point particle, in the presence of a background gauge field and a potential, evolving in proper time τ (see [32] for a more general treatment); we denote

$$\dot{v} \leftrightarrow \frac{dv}{d\tau} \ , \quad \dot{t} \leftrightarrow \frac{dt}{d\tau} \ . \tag{24}$$

And we leave the Ω factor arbitrary in this subsection.

The Hamiltonian becomes

$$H = \frac{\pi}{\Sigma}\frac{\alpha'}{\Omega^2}v^2\Pi_v^2 - \frac{8\pi^3\alpha_e}{\Sigma v^2}\frac{\alpha'}{\Omega^2}\left(\Pi_t \mp \omega\frac{\Sigma}{2\pi\alpha_e}Gv^3\right)^2 + \omega^2\frac{\Sigma}{32\pi^3\alpha_e}\frac{\Omega^2}{\alpha'}v^2 \ , \tag{25}$$

with the canonical momenta given by

$$\Pi_v = \frac{\Sigma}{2\pi}\frac{\Omega^2}{\alpha'}\frac{\dot{v}}{v^2} \ ; \tag{26}$$

$$\Pi_t = \pm \frac{\Sigma}{2\pi\alpha_e} v^2 \left(G\omega v \mp \frac{\Omega^2}{8\pi^2\alpha'} \dot{t} \right) \; . \tag{27}$$

The equation of motion for the radial coordinate v is then

$$2\frac{\Omega^2}{v^2}\ddot{v} - 2\frac{\Omega^2}{v^2}\dot{v}^2 \left(\frac{1}{v} - \frac{\Omega'}{\Omega} \right) + \frac{v^2\Omega^2}{4\pi^2\alpha_e} \left(\frac{1}{v} + \frac{\Omega'}{\Omega} \right) \dot{t}^2 \mp 6\omega G \frac{\alpha'}{\alpha_e} v^2 \dot{t} + \omega^2 \frac{v^2\Omega^2}{4\pi^2\alpha_e} \left(\frac{1}{v} + \frac{\Omega'}{\Omega} \right) = 0 \; ; \tag{28}$$

while for the time coordinate t, it is

$$\dot{\Pi}_t = 0 \; ; \tag{29}$$

i.e. Π_t is the Nöether charge associated with the Killing vector field ∂_t.

The constraint equation becomes

$$\frac{1}{2}\frac{\Omega^2}{v^2}\dot{v}^2 - \frac{v^2\Omega^2}{16\pi^2\alpha_e}\dot{t}^2 + \omega^2 \frac{v^2\Omega^2}{16\pi^2\alpha_e} = 0 \; , \tag{30}$$

i.e. the vanishing of the worldsheet Hamiltonian $H = 0$, which can also be interpreted as the on-shell condition for a particle in a certain curved background. As the system is constrained, one of the two solutions of (28) must be dropped in view of (30). Of course, consistent dynamics requires that the constraint evolves properly by the equations of motion, as it does.

We solve the t equation (29) trivially by introducing a constant of motion E

$$\Pi_t = \frac{\Sigma}{4\pi\alpha_e} E \; . \tag{31}$$

E is a dimensionless parameter related to the energy of the probe Q_E as seen by the dual NCOS theory (canonical to δt) as in

$$Q_E \equiv -\Pi_t \; . \tag{32}$$

The factor Σ introduced in (31) helps to make our intermediate equations somewhat less cluttered. Note the negative sign in (32), which we justify below. Using (31) and (27), we then find

$$\dot{t} = \frac{4\pi^2\alpha'}{\Omega^2} \frac{-E \pm 2G\omega v^3}{v^2} \; . \tag{33}$$

This allows us to solve the constraint equation (30) for $\dot{v}$ [10]

$$\dot{v}^2 = \frac{2\pi^2\alpha'^2}{\alpha_e\Omega^4} \left(\left(E \mp 2G\omega v^3 \right)^2 - \omega^2 \frac{v^4\Omega^4}{16\pi^4\alpha'^2} \right) \; . \tag{34}$$

[10]In taking the square root of (34), we choose conventionally $\dot{v} < 0$ throughout, *i.e.* in-bound motion. The reader may assume that in all subsequent equations where this ambiguity arises this choice has been made.

The two possible signs for $\dot{v}$ correspond to motion inward and outward from the center of the geometry located at $v = 0$. It is easy to check that this solution satisfies (28).

One way to think of the situation is in analogy to Light-Cone gauge fixing in flat backgrounds. We would like to gauge fix say some $X^+ \sim \tau$ using the residual conformal symmetry on the worldsheet; residual after fixing $h_{ab} = \eta_{ab}$. We could do this in flat space since this choice would solve the equations of motion. In our curved background, the analogous fixing of the residual conformal symmetry, which we still have, is solving for the time coordinate t using the time translational invariance of the background. This is given implicitly by (33). Once equation (34) is integrated, we obtain $t(\tau)$ from (33). And (30) is nothing but the constraint that in the flat space case allows us to solve for X^-. These basic ideas are illustrated also when setting up the Fateev-Popov determinant in the path integral in Appendix B. Finally, note that, because the system is constrained, we loose one constant of motion. The dynamics is parameterized by E, and the initial and final positions of the variable v; we denote them by v_1 and v_2, arising as integration limits in (33) and (34).

3.2 Analysis of the solutions

A more instructive way to view the dynamics is to push the analogy with a point particle further. We are solving for

$$\frac{\dot{v}^2}{2} + V(v) = \mathcal{E} = 0 \; , \tag{35}$$

with

$$V(v) \equiv -\frac{\pi^2 \alpha'^2}{\alpha_e \Omega^4} \left(\left(E \mp 2G\omega v^3 \right)^2 - \omega^2 \frac{v^4 \Omega^4}{16\pi^4 \alpha'^2} \right) \; . \tag{36}$$

By studying the shape of this potential $V(v)$, we see the bulk properties of the motion. At this point, we will need to use the explicit form of Ω given by (4).

From equations (36) and (4), we see that

$$V(v) \to \frac{E^2}{64\pi^2 \alpha_e G^2} \frac{v}{v_c^3} \text{ as } v \to \infty \; , \tag{37}$$

due to a subtle cancelation of terms, irrespective of the winding orientation. We have introduced the parameter

$$v_c^3 \equiv \frac{E^2}{4G\left(\omega^2 \pm \omega E\right)} \; , \tag{38}$$

whose significance will become apparent below. In the opposite limit, we have

$$V(v) \to -\frac{E^2}{64\pi^2 \alpha_e G} v \text{ as } v \to 0 \; . \tag{39}$$

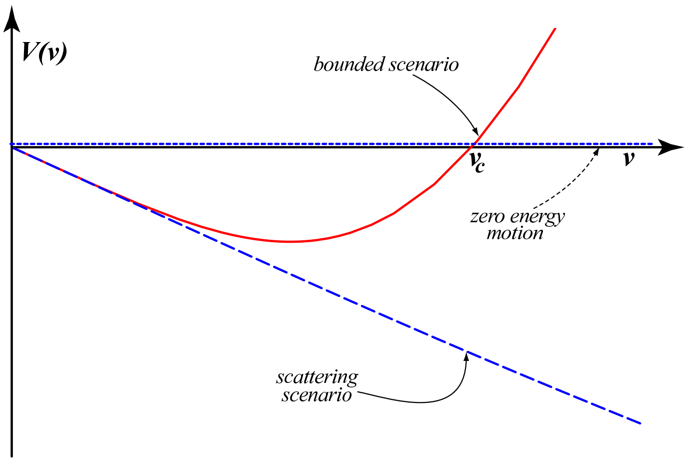

Figure 7: *The potential given by equation (36) as a function of the radial coordinate v. The 'particle' has zero total (worldsheet) energy, corresponding to the on-shell constraint condition on the dynamics. Two solutions are associated with the bounded scenario, with both orientations for closed string winding; and one solution with positive winding corresponds to a scattering process.*

for both orientations. And finally

$$V(v) = 0 \Rightarrow v = v_c \text{ or } v = 0 \ . \tag{40}$$

It is now easy to plot the potential $V(v)$. There are two qualitatively different cases. For $v_c^3 > 0$, we have bounded motion with $0 \leq v \leq v_c$. While for $v_c < 0$, we have unbounded motion $0 \leq v$ (see Figure 7).

Using (4), one easily gets the explicit form of the evolution of the variables

$$\dot{t} = \frac{-E \pm 2G\omega v^3}{2\sqrt{G}v^{3/2}\sqrt{1+Gv^3}} \ . \tag{41}$$

$$\dot{v}^2 = \frac{E^2}{32\pi^2\alpha_e G}\frac{1-(v/v_c)^3}{1+Gv^3}v \ . \tag{42}$$

One then integrates (42), and substitutes in (41). Note that $\dot{v}$ vanishes for all v for $E = 0$.

We next determine the physical relevance of the various possible solutions. We need to require that

$$\frac{dt}{d\tau} \geq 0 \Rightarrow -E \geq \mp 2G\omega v^3 \ . \tag{43}$$

First, consider the case $v_c^3 < 0$, *i.e.* unbounded motion. It is then easy to see that the only possibility consistent with (43) is for the winding of the string to be parallel to the B-field; negative winding is not possible, as it corresponds to opposite propagation in times t and τ. Furthermore, the combination of the two conditions $v_c^3 < 0$ and (43) in the case of positive winding leads to $E < -\omega$.

Next consider the case $v_c^3 > 0$, *i.e.* bounded motion. We then find that the positive winding solution must satisfy $-\omega < E < 0$, complementing the parameter space established by the unbounded case. However, looking at the negative winding case, we now find that it is also a possible solution to the dynamics. For this scenario, assuming $E > 0$ leads to

$$E^2 \geq 2E\omega \text{ and } E \leq \omega \; ; \tag{44}$$

a contradiction. While $E < 0$ leads to the consistent scenario.

To validate the proposed dynamics, we also look at the 'acceleration' at two critical points in the motion. We find

$$\ddot{v}|_{v=0} = \frac{E^2}{64\pi^2\alpha_e G} \geq 0 \; ; \tag{45}$$

and

$$\ddot{v}|_{v=v_c} = -\frac{3E^2}{64\pi^2\alpha_e G\left(1+Gv_c^3\right)} \leq 0 \; . \tag{46}$$

From (45), we see that the string is repulsed from the origin $v \sim 0$, where a horizon sits.

We can now put the story together, depicting it as in Figures 3 and 4, in analogy to classical dynamics that has risen in the context of the SL(2,R) WZW model. We have two oscillating solutions which we call bound states, and one unbounded motion that reaches asymptotically larger values of v; we call the latter a scattering solution. We summarize the scenarios as follows

$$\left\{ \begin{array}{lll} v_c^3 < 0 \;\Rightarrow & \text{unbounded dynamics with positive winding} & \Rightarrow \quad E < -\omega \\ v_c^3 > 0 \;\Rightarrow & \text{bounded dynamics with positive winding} & \Rightarrow \; -\omega < E < 0 \\ v_c^3 > 0 \;\Rightarrow & \text{bounded dynamics with negative winding} & \Rightarrow \quad E < 0 \end{array} \right. \tag{47}$$

We note that for all cases, we have

$$E < 0 \; . \tag{48}$$

However, in our chosen convention for E through (31), there is a sign difference between real energy Q_E and E as seen in (32). Hence, the Nöether energy Q_E is positive definite for all cases. The origin of the sign flip will be seen below. It is also significant that the threshold energy $E = -\omega$ is then identified in the dual picture as $Q_E = \omega\Sigma/(4\pi\alpha_e)$, the mass of the closed string in Wound String theory, with the additional $1/2$ factor for the wound string tension that has already been noted in [7]. This indicates that we have identified in this dual picture what corresponds to energy in NCOS theory with the correct numerical factor.

The two solutions with positive winding have an attractive interpretation in the dual NCOS theory. We will argue below that the scattering solution is seen as a closed string in Wound string theory scattering off the bound state of D strings and fundamental strings. In the NCOS theory, it is seen as an insertion of vertex operators, and the propagation of a non-perturbative open string resonance in between[11]. On the other hand, if the corresponding open string state does not have enough energy to create the appropriate closed string, as in the case of the bounded solution with positive winding and $-\omega < E < 0$, it is a non-perturbative state within NCOS theory.

The negatively wound solution however is at first sight a pathological case. It appears to correspond to a non-perturbative state in NCOS theory that cannot leave the system no matter how high an energy in attains. For now, we want to check the relevance of these solutions quantum mechanically, with the hope that perhaps the negatively wound state becomes metastable.

4 Quantum effects

The task in this section is to understand the quantum mechanics of the classical dynamics we discussed above . It is a problem of quantum mechanics as we have reduced the system to that of a point particle moving in a curved space subject to an electric field and a constraint. Such a treatment however assumes that quantum fluctuations that can excite modes on the probe closed string outside our ansatz are negligible. The quantum mechanics we study is that of the center of mass in a two dimensional cross-section of the whole spacetime. We will address the relevance of any additional corrections in the Discussion section.

The propagator for the system is given by the path integral[12]

$$\langle v_2, \Delta t | v_1, 0\rangle \equiv \mathcal{G}\left(v_1, v_2, \Delta t\right) \sim \int \mathcal{D}v\mathcal{D}\Pi_v e^{i\int_1^2 dt\,\left(\Pi_v \frac{dv}{dt} + \Phi_t\right)}\ , \tag{50}$$

[11]We note that in the embedding Wound string theory, we are not working in a decoupling limit; our discussion naturally maps to the analogue of extending the Maldacena duality beyond the near horizon region (see discussion along this line of thought in [11].)

[12] We normalized the asymptotic states as in

$$\langle v_2, t | v_1, t\rangle = \delta(v_2 - v_1)\ , \tag{49}$$

without the conventional $1/\sqrt{|g(v_1,t)|}$ factor on the right (g being the determinant of the metric). This is so that we may directly look at $|\mathcal{G}|^2$ as probability of propagation, without the need to multiply with the measure in the curved space.

where

$$\Phi_t = \frac{v^2}{\sqrt{8\pi^2\alpha_e}}\sqrt{\Pi_v^2 + \omega^2 \frac{\Sigma^2}{32\pi^4\alpha_e}\frac{\Omega^4}{\alpha'^2}} \pm \omega \frac{\Sigma}{2\pi\alpha_e} G v^3 \ . \tag{51}$$

The latter is just the constraint solved for $\Phi_t \to \Pi_t$. This expression is such that the propagator is to be viewed as a function of the initial and final positions in v, that we call v_1 and v_2, and the time separation Δt. It would tell us the probability to propagate from v_1 to v_2 in time Δt. Note that we were careful in writing this propagator in the Hamiltonian formalism, as we are dealing with a path integral in curved space. The derivation of (50) is given in detail in Appendix B. As seen from equation (50), the system effectively evolves in time t according to $-\Phi_t$. This is the origin of the minus sign appearing in (32).

The result of this path integration is, to order $\hbar^2$

$$\mathcal{G}\left(v_1, v_2, \Delta t\right) \sim \sqrt{|\Delta|} e^{iS_{cl} + iS_{dW}} \ . \tag{52}$$

S_{cl} is the action evaluated at the classical solutions given by (41) and (42), with the appropriate boundary conditions; $|\Delta|$ is determinant of the propagation operator in this curved space; and S_{dW} is a term proposed by DeWitt [34] and is generally needed for path integrals in curved spaces. Corrections to (52) start at order $\hbar^2$. In the subsequent subsections, we elaborate on these three quantities and compute them.

4.1 Leading contribution

The leading contribution is obtained by substituting in the exponent of (50) the classical action evaluated at the solutions given by (41) and (42). This is because the extremum of the exponential is at these solutions for given boundary conditions, as shown in Appendix B. Changing integration variable to v, we then get from (23), (41) and (42)

$$S_{cl} = \pm\omega G \Sigma \sqrt{\frac{2}{\alpha_e}\frac{E \pm 2\omega}{-E}} \int_{v_1}^{v_2} dv \ \frac{v}{\sqrt{1-(v/v_c)^3}} \ . \tag{53}$$

The dependence of this expression on Δt is implicitly through the variable E, obtained by integrating and inverting

$$\Delta t = \sqrt{8\pi^2}\frac{l_e}{E}\int_{v_1}^{v_2} dv \ \frac{-E \pm 2\omega G v^3}{v^2\sqrt{1-(v/v_c)^3}} \ ; \tag{54}$$

the latter is simply (41) divided by the square root of (42). From (42), we also have an expression for the amount of proper time it takes for the string to propagate between v_1 and

v_2

$$\Delta\tau = \sqrt{32\pi^2}\frac{\sqrt{G\alpha_e}}{E}\int_{v_1}^{v_2} dv \; \frac{\sqrt{1+Gv^3}}{\sqrt{v}\sqrt{1-(v/v_c)^3}} \ . \tag{55}$$

All these expressions can be solved in terms of various hypergeometric functions. The more interesting physics is however in their asymptotic forms; which we now write down.

We will distinguish two processes: a positive winding scattering scenario; and two bound state scenarios with both possible winding orientations. For the scattering scenario, we take v_1 very large, and v_2 very small. For the bound state cases, we take $v_1 = v_c$ and v_2 small. Making use of the appropriate boundary conditions in v in each case, as well as on E as given by (47), we find:

- For the bound states, we have

$$S_{cl}^{bnd} = -\frac{\sqrt{2\pi}}{3}\frac{\Gamma(2/3)}{\Gamma(7/6)}\frac{G}{|E|}\frac{\Sigma}{l_e}v_c^2\left(\mp\omega|E| + 2\omega^2\right) \ , \tag{56}$$

  which is finite for both orientations of winding.

- The action for the scattering process becomes

$$S_{cl}^{scatt} \simeq \frac{\Sigma}{l_e}\sqrt{2G\omega}\frac{|E|-2\omega}{\sqrt{|E|-\omega}}\sqrt{v_1} - \frac{\Sigma}{\sqrt{2\alpha_e}}G\omega\frac{|E|-2\omega}{|E|}v_2^2 \ , \tag{57}$$

  and is large for v_1 big.

- The time Δt_{bnd} for half the trip in the bound state scenario is given by

$$2\Delta t_{bnd} \simeq \frac{\sqrt{32\pi^2\alpha_e}}{v_2} \ , \tag{58}$$

  for both winding orientations. The divergence, as $v_2 \to 0$, is essentially the infinite redshift that an external observer generically witnesses as a probe approaches a horizon. It will play a crucial role in regulating the bound state spectrum later.

- The scattering time in the same coordinate is given by

$$2\Delta t_{scatt} \simeq \frac{\sqrt{128\pi^2 G\omega\alpha_e}}{\sqrt{|E|}}\sqrt{v_1} + \frac{\sqrt{32\pi^2\alpha_e}}{v_2} \ , \tag{59}$$

  which is large for both big v_1 and small v_2.

- The half period in proper time $\Delta\tau_{bnd}$ that the string takes to oscillate in the bound state scenario is

$$2\Delta\tau_{bnd} = 8\pi^2 \frac{\sqrt{2G\alpha_e}}{3|E|} \sqrt{v_c}\sqrt{1+Gv_c^3} \ , \tag{60}$$

  which is finite unlike (58).

- The proper time elapsed during the scattering process is

$$2\Delta\tau_{scatt} \simeq 16\pi \frac{\sqrt{2\alpha_e}}{E} G\sqrt{|v_c|^3}\sqrt{v_1} \ , \tag{61}$$

  and increases unboundedly with large v_1.

We will use all these ingredients in upcoming sections in attempting to understand strongly coupled dynamics in the dual NCOS theory. Let us first however compute the subleading terms to the propagator.

4.2 Sub-leading contributions

4.2.1 The determinant

The determinant Δ appearing in (52) is nothing more that

$$|\Delta| = \left| \frac{\partial^2 S_{cl}}{\partial v_1 \partial v_2} \right| \ . \tag{62}$$

This is a well known statement from quantum mechanics, yet not properly advertised or emphasized in most textbooks. The origin of this important relation is unitarity [35, 36, 37]. It can be derived assuming *only* unitary evolution by the corresponding Hamiltonian. Alternatively, one can see it by deriving a first order current conservation differential equation that is satisfied by Δ [38]. For given boundary conditions, in the setting of quantum mechanics, equation (62) is essentially established by making use of the existence and uniqueness theorem for the corresponding differential equation[13].

While equation (62) makes it easy to compute the determinant of the propagation operator, we need to be careful about one aspect of this computation. In differentiating the

[13] A particularly elegant approach is also described in [35] (and see references therein), where the authors use the uncertainty relation, and a simple classical phase space probability estimate to derive (62).

classical action, we need to keep in mind that one holds t fixed, *not* E, which in turn implicitly depends on t, as well as v_1 and v_2. This requires some mild juggling of partial derivatives, that we show in Appendix D. The result is, for the canonical momentum,

$$\mathcal{P} \equiv \left(\frac{\partial S_{cl}}{\partial v_1}\right)_{t,v_2} = \frac{\Sigma}{\sqrt{2\alpha_e}}|E|\frac{\sqrt{1-(v_1/v_c)^3}}{v_1^2} \, , \tag{63}$$

as the reader may check with respect to Π_v (see (26) and (42)). And, more importantly,

$$\Delta = \left(\frac{\partial \mathcal{P}}{\partial v_2}\right)_{t,v_1} = \frac{\Sigma}{l_e}\left(\frac{|E| \pm 2G\omega v_1^3}{v_1^2\sqrt{1-(v_1/v_c)^3}}\right)\left(\frac{|E| \pm 2G\omega v_2^3}{v_2^2\sqrt{1-(v_2/v_c)^3}}\right)\frac{|E|}{\sqrt{32}G\omega^2}\mathcal{I}^{-1} \, , \tag{64}$$

with

$$\mathcal{I} \equiv \int_{v_1}^{v_2} dv \, \frac{v\,(1+Gv^3)}{(1-(v/v_c)^3)^{3/2}} \, . \tag{65}$$

This last integral can in turn be written down in terms of hypergeometric functions. We note that it is finite as $v_2 \to 0$. This expression, being the real part of the propagator, tells us about a physical amplitude as a function of the corresponding boundary data.

Let us look at (64) for the two interesting cases:

- For the bound state scenario, we have

$$\Delta_{bnd} \simeq 2^{1/6} 3 \frac{\Sigma}{l_e}\omega^{1/3}\frac{G^{1/3}}{|E|^{2/3}}\frac{(\omega \mp |E|)^{4/3}(\pm E^2 - 2\omega|E|)}{E^2 \mp 4\omega|E| + 4\omega^2}\frac{1}{v_2^2} \, ; \tag{66}$$

  which is divergent as $v_2 \to 0$.

- For scattering, we get

$$|\Delta_{scatt}| \sim \frac{\Sigma}{l_e}\frac{E^2}{4\sqrt{2}G\omega}\frac{1}{v_1 v_2^2 |v_c|^3} \, . \tag{67}$$

4.2.2 The DeWitt term

It was shown by DeWitt that, in evaluating the path integral for a particle in curved space, coupled to a gauge field, and an arbitrary potential, the resulting propagator does not satisfy the Schrödinger equation for the corresponding Hamiltonian. To assure that $\mathcal{G}(v_1, v_2, t)$ is the correct propagator for the Hamiltonian (25), DeWitt showed that we need to add a term S_{dW} as in (52), given by[14]

$$S_{dW} = \int d\tau\ \frac{\mathcal{R}}{12} = \frac{1}{16}\frac{\sqrt{2\alpha_e}}{\Sigma}\frac{1}{|E|}\int_{v_1}^{v_2} dv\ \frac{1+2Gv^3\,(3+Gv^3)}{(1+Gv^3)^2\sqrt{1-(v/v_c)^3}}\ , \tag{68}$$

where $\mathcal{R}$ is the scalar curvature associated with the metric

$$g_{vv} = \Sigma\frac{\Omega^2}{\pi\alpha' v^2}\ ,\quad g_{tt} = -\Sigma\frac{v^2\Omega^2}{8\pi^3\alpha_e\alpha'}\ , \tag{69}$$

that appears in our Hamiltonian. The physical meaning of this curvature term has long been a mystery[15]. The puzzle arises when one notices that this term carries more powers in $\hbar$ (which is not easy to see when one sets $\hbar = 1$). Restoring powers of $\hbar$ in the exponent of (52), we should have written

$$\frac{1}{\hbar}\left(S_{cl} + \frac{\hbar^2}{12}\int\ d\tau\ \mathcal{R}\right)\ . \tag{70}$$

In our problem, we will instead count powers of $\hbar$ through powers of the coupling G. Now, let us look at the two cases of interest:

- For the bound state problem, we get

$$S_{dW}^{bnd} = -\frac{\sqrt{\pi}}{8\sqrt{2}}\frac{\Gamma(4/3)}{\Gamma(5/6)}\frac{l_e}{\Sigma}\frac{1}{|E|}G^2 v_c^7\ \mathcal{S}\ ; \tag{71}$$

  where we define

$$\mathcal{S} \equiv \left(\frac{1}{x^2}\right){}_2F_1\left(\frac{1}{3},2,\frac{5}{6},-x\right) + \left(\frac{12}{5}\frac{1}{x}\right){}_2F_1\left(\frac{4}{3},2,\frac{11}{6},-x\right) + \left(\frac{32}{55}\right){}_2F_1\left(2,\frac{7}{3},\frac{17}{6},-x\right)\ , \tag{72}$$

  with

$$x \equiv G|v_c|^3\ . \tag{73}$$

- For the scattering scenario, we have

$$S_{dW}^{scatt} \simeq \frac{1}{128G}\frac{l_e}{\Sigma}\frac{1}{|E|}\left(\frac{2|E|}{\sqrt{\pm\omega|E|-\omega^2}}\frac{1}{\sqrt{v_1}} + \sqrt{G}v_2\right)\ , \tag{74}$$

  which interestingly goes to zero for large v_1 and small v_2.

[14] In this approach, the constraint can be imposed on the Hilbert space, instead of being solved for from the outset. See the discussion about this in Appendix E.

[15] In his classic book on path integration, Shulman writes "If you like excitement, conflict, and controversy, especially when nothing very serious is at stake, then you will love the history of quantization on curved spaces." Our problem can certainly be attributed by some of these qualifications.

5 The bound state problem

In this section, we focus on the bound state scenario, with two possible orientations for the closed string winding. We set $v_1 = v_c$ and choose v_2 small throughout.

To determine the spectrum within the Bohr-Sommerfeld approximation, we use the condition

$$S_{tot}^{bnd} = \pi N \text{ with } N \gg 1 \ , \tag{75}$$

with

$$S_{tot}^{bnd} = S_{cl}^{bnd} + S_{dW}^{bnd} \ , \tag{76}$$

evaluated for the bounded motion $v \in \{0, v_c\}$. N is taken as a large integer. The action in (76) is, remarkably, finite and corresponds to a non-zero phase picked up by the closed string during a round trip. The left hand side of (75) is a function of E, ω, G, and Σ/l_e, and this equation then gives us the quantization of E, and hence of Q_E. There are several issues with regards to this statement that are unusual. First, the infinite redshift at the horizon $v_2 \sim 0$ implies that, while the string takes finite proper time for a round trip between v_c and $v_2 \sim 0$, it would take infinite time in the time variable t. The dynamics is however novel in that that the closed string gets *repulsed* from the horizon. Furthermore, the limit $v_2 \to 0$ is problematic for finite M in view of (11). These issues are addressed in detail in Appendix E and the Discussion section. The reader concerned about them is urged to consult the appendix at this point.

While we have an analytical expressions for the level spectrum given by (75), along with (56) and (71), it is instructive to look at physically interesting limits, so as to write more transparent expressions. Two cases stand out

$$\left\{ \begin{array}{llll} |E| \ll \omega & \Rightarrow Q_E \ll \omega \frac{\Sigma}{4\pi\alpha_e} & \Rightarrow x \sim \frac{|E|^2}{4\omega^2} \ll 1 \\ |E| \gg \omega & \Rightarrow Q_E \gg \omega \frac{\Sigma}{4\pi\alpha_e} & \Rightarrow x \sim \frac{|E|}{4\omega} \gg 1 \end{array} \right. , \tag{77}$$

where x was defined in (73). Physically, these limits correspond respectively to energies much below and above the threshold of creating the corresponding wound closed string in Wound String theory.

We then find

$$S_{cl}^{bnd} = -\frac{\Sigma}{l_e} 2^{7/6} \pi^{1/2} \frac{\Gamma(2/3)}{\Gamma(1/6)} \left\{ \begin{array}{ll} G^{1/3}\omega^{2/3}|E|^{1/3} & x \ll 1 \\ \frac{1}{2} G^{1/3}\omega^{1/3}|E|^{2/3} & x \gg 1 \end{array} \right. ; \tag{78}$$

$$S_{dW}^{bnd} = -\frac{l_e}{\Sigma} \frac{\pi^{1/2}}{2^{19/6}} \frac{\Gamma(4/3)}{\Gamma(5/6)} \left\{ \begin{array}{ll} \frac{1}{2} \frac{1}{G^{1/3}\omega^{2/3}|E|^{1/3}} & x \ll 1 \\ \frac{1}{G^{1/3}\omega^{1/3}|E|^{2/3}} & x \gg 1 \end{array} \right. . \tag{79}$$

The expansion parameter in each limit is then the same for both expressions. For $x \ll 1$, the expansion in $\hbar$ becomes an expansion in $G|E|\omega^2$; while for $x \gg 1$, it becomes $G|E|^2\omega$. Putting (78) and (79) together in (76), we find

$$Q_E l_e \sim \left(\frac{l_e}{\omega\Sigma}\right)^2 \frac{N^3}{G} \quad \text{for } x \ll 1 \text{ with the two winding possibilities degenerate} \tag{80}$$

$$Q_E l_e \sim \sqrt{\frac{l_e}{\omega\Sigma}} \frac{N^{3/2}}{\sqrt{G}} \quad \text{for } x \gg 1 \text{ for negative winding only} \tag{81}$$

The DeWitt term corrects only the numerical coefficients in front of these expressions; the scaling with respect to all physical parameters of the theory is fixed by the leading term S_{cl}^{bnd}. From (77), (80) can be trusted for $N \ll N_{max}$ with $N_{max} \equiv \omega(\Sigma/l_e)G^{1/3}$; while (81) can be trusted for $N \gg N_{max}$. Beyond these two asymptotic regimes, one needs to use the full analytic forms given by (56), (71) and (75), which we plot in Figure 5. Note also that the spectrum does not have any explicit dependence on M.

From Figure 5, we see how the density of levels increases as we approach the threshold energy from below, in the case of a positively wound closed string. Much below the threshold, where the two windings correspond to degenerate levels, the energy scales as N^3. For the negatively wound case, we go past the threshold energy with a peculiar $N^{3/2}$ scaling with the level number. It also appears that the negatively wound states have always higher energies than the positively wound ones for same N. If we could trust the spectrum for low enough energies, we would also be identifying a ground state with energy slightly above zero, scaling with respect to the parameters of the theory as shown in equation (80). Finally, we note that the spacing of energy levels increases with N as $\delta Q_E/\delta N \sim N^2$ for positive winding and for energies well below the threshold, and as $\delta Q_E/\delta N \sim \sqrt{N}$ for negative winding and high energies.

Even with the full form of the quantization condition depicted in the Figure, we need to be careful in trusting the spectrum, given the restrictions on the spacetime imposed by (9), (10) and (11). In particular, we need to make sure that the region of space explored by the closed string lies within the area delineated by the curves of Figure 6. For example, for (80), we need $\alpha_e/(4\pi\Sigma^3) \ll Q_E \ll \omega G^2\Sigma/(4\pi\alpha_e)$ for $G \ll 1$ (corresponding to the lower small triangular region in Figure 6); $\omega/(4\pi\Sigma) \ll Q_E \ll \omega\Sigma/(4\pi\alpha_e)$ for $1 \ll G \ll \sqrt{M}l_e/\Sigma$; and $\omega\Sigma G^2/(4\pi\alpha_e M) \ll Q_E \ll \omega\Sigma/(4\pi\alpha_e)$ for $\sqrt{M}l_e/\Sigma \ll G$; while for the second expression, one needs $\omega\Sigma/(4\pi\alpha_e) \ll Q_E \ll \omega\Sigma M^2/(4\pi\alpha_e G^4)$. One should not make too much of these additional conditions; the only relevant point is that there are certain restrictions and we can arrange a large hierarchy of energies where our expression for energy levels can be trusted by changing G, M, and Σ/l_e. This is shown in the figure above with a shaded region. The important points are that (a) the energy levels scale inversely with the coupling G, and hence are non-perturbative in character; (b) the scaling with the integer N is large enough that the higher levels will be spaced more and more; (c) that the spectrum is regular in the limit $M \to \infty$; a regime which circumvents complications that affect the dynamics near $v \sim 0$; (d) that there exists states with the 'wrong' winding orientation that cannot escape to infinity and yet explore energies above the threshold; and (e) that, for positive winding, the transition point between freed closed strings and NCOS states is the threshold energy. As pointed out earlier, this threshold energy appears with the correct additional $1/2$ factor that has already been attributed to the closed string tension [7].

Most interestingly, the $M \to \infty$ limit, with all other parameters of the theory held fixed, is regular; this renders the whole of space, from $v = 0$ to $v \to \infty$, reliable for our computations. In this strict limit, the spectrum is in principle good for all energies as to the issues pointed out in Section 2.2, but except for issues having to do with reliability of a WKB-like approximation. These latter matters are controllable and are discussed in Appendix E as well as the Discussion section. For finite but large M, we argue in the Discussion section that this spectrum may constitute a first order estimate of energy levels in an expansion in $1/M$.

Let us next look at the propagators in the bound state problem. We emphasize that the region explored by the bound state motion is not confined to the throat $Gv^3 < 1$, but can involve the whole of the space; as seen from equation (38), for both windings, v_c ranges from 0 to ∞, for the allowed values for E as given in (47). Expression (52) for the propagator tells us about the probability of propagation in the v direction (see footnote at the beginning of Section 4 with regards to the measure); in the dual NCOS theory, this presumably translates to probability for a corresponding NCOS state to live and 'breath' for some time; location in the v direction being mapped onto size of a soliton in the NCOS theory. Note also that these states carry zero total momentum in the y direction. In order to interpret (52) as a probability amplitude, we need however to eliminate all instances of E in the propagator in favor of Δt. This is identical to the situation that arises say in AdS_3 where one tries to find the correlators of two operators in the conformal field theory by looking at a geodesic motion in the bulk, *i.e.* at a propagator in the bulk. We would then need to invert (54) to write E as a function of Δt; a difficult task that we will not be able to do. The problem here is that the motion explores the space all the way near the center; whereas in the AdS_3 case for example, the corresponding geodesic motion lives near the boundary, and leads to a trivial relation $t \sim 1/E$. We cannot do this here as the motion is not confined near a boundary, about which we would expand. For the case of the scattering solution that we discuss in the next section, the problem will be the same, as the closed string falls from infinity all the way to the horizon. In an effort to unravel the dynamics, we will then look at the two limits $|E| \gg \omega$ and $|E| \ll \omega$, and expand the relations between t and E in these regimes only.

The expression given by (58) is not enough and one needs the subleading term to extract the energy dependence. This is a subtle limit to take, which we do carefully by taking the limits on energy first, then integrating (54). The procedure is outlined briefly in Appendix

C. The result are simple relations (for $v_1 = v_c$)

$$|E|^{1/3} = 2\sqrt{G}\omega\sqrt{\frac{D_1 v_2}{1 - T v_2}} \text{ for } |E| \ll \omega \; ; \tag{82}$$

and

$$|E|^{1/3} = 4G\omega\frac{D_2 v_2}{1 - T v_2} \text{ for } |E| \gg \omega \; . \tag{83}$$

D_1 and D_2 are numerical constants given by $D_1 = \sqrt{\pi}\Gamma(2/3)/\Gamma(1/6)$ and $D_2 = 2D_1$. And T is defined as

$$T \equiv \frac{\Delta t_{bnd}}{\sqrt{8\pi^2\alpha_e}} \; , \tag{84}$$

i.e. the time in NCOS string units. In these expressions, we have also expanded in powers of $v_2/v_c \ll 1$. These are not the typical relation $|E| \sim 1/T$; as $v_2 \to 0$, and $T \to \infty$, with the product $T v_2$ getting close to 1, we have E remaining finite. Note also that v_2 is the IR cutoff in the dual NCOS theory, so that the $T v_2$ is a natural combination. In particular, we have the requirement $T v_2 \leq 1$; *i.e.* we should not probe dynamics for times longer than the one set by the IR cutoff. We have also verified these asymptotic expansions numerically, and found they are excellent approximations to the exact form within the regimes of interest[16].

The interesting quantity to look at is the real part of the propagator (52), $|\mathcal{G}|^2 \sim |\Delta|$. Let us focus on the two interesting regimes in expression (66):

$$|\Delta_{bnd}| \sim \left\{ \begin{array}{ll} \omega^{2/3}\frac{\Sigma}{l_e}G^{1/3}\frac{|E|^{1/3}}{v_2^2} & \text{for } |E| \ll \omega \\ \omega^{1/3}\frac{\Sigma}{l_e}G^{1/3}\frac{|E|^{2/3}}{v_2^2} & \text{for } |E| \gg \omega \end{array} \right. . \tag{85}$$

Eliminating E between (82), (83) and (85), we get

$$|\mathcal{G}|^2 \sim \omega\frac{\Sigma}{l_e}\frac{\sqrt{G}}{v_2^{3/2}}\frac{1}{\sqrt{1-T v_2}} \quad \text{for } v_1^3 \ll \frac{1}{G} \tag{86}$$

$$|\mathcal{G}|^2 \sim \omega\frac{\Sigma}{l_e}G\frac{1}{1-T v_2} \quad \text{for } v_1^3 \gg \frac{1}{G} \tag{87}$$

Note that, in these expressions, v_c is the initial position $v_1 = v_c$; and we need to be careful to take $v_2 \ll v_c$, in particular for (86). Also, the dependence on T includes contribution from v_c; so that our expression is a function of v_2 and T independently, with fixed initial position of propagation at the classical turning point. We interpret these expressions as the probability for a corresponding configuration in NCOS theory to spread in size from its most compact form to a size $1/v_2$ in a time T.

[16] Note that the exponent in (52) could in principle be used to extract the spectrum of states in an alternative way. One would Fourier transform in the time variable t dual to energy in the NCOS theory; then, look for poles in energy. This is technically a much more involved approach in this case for obvious reasons.

6 The scattering problem

The scattering solution exists only for positive winding. The closed string starts at v large, where perhaps it can be used to define an asymptotic on-shell state within Wound string theory, and falls to $v \to 0$, while decelerating as it approaches the horizon. It would then bounce back to infinity. For fixed initial position v_1, this process takes finite proper time, as seen from (61). In the time variable t, which we associate with the NCOS theory, this process seems to take infinite time. This presents a problem in trying to interpret the process as scattering within the dual NCOS dynamics.

The natural suggestion of this setup is that this solution describes a closed string in Wound String theory scattering off the bound state of D-strings and fundamental strings. This correlates well with the fact that strings of only positive winding can undergo this process. Given that v_c for the positively wound bound state approaches infinity as $Q_E \to \omega\frac{\Sigma}{4\pi\alpha_e}$, which is the threshold of creating a wound closed string (see equation (38)), we may expect that in the dual NCOS theory the scattering process is encoded holographically by the insertion of the appropriate local vertex operators at large values of v, in the UV. Hence in this case, Δ given by (67) would tell us about the amplitude for the incoming closed string to break up on the D-strings, breath for a while through a resonance (with zero net momentum in the y direction), and leave the NCOS theory. We will carry on in computing this amplitude, keeping in mind that we do not have a satisfactory resolution of the issue that this process yet appears to take infinite times in the variable t . We consider in this section v_1 large and v_2 small throughout.

Using equation (59), we solve for E and find

$$|E| = 4G\omega\frac{v_1 v_2^2}{(Tv_2 - 1)^2} \ . \tag{88}$$

And taking the limit $|E| \gg \omega$ in (67), we get

$$|\Delta_{scatt}| \sim \frac{\Sigma}{\sqrt{2\alpha_e}}|E|\frac{1}{v_1 v_2^2} \sim \omega\frac{\Sigma}{l_e}\frac{G}{(Tv_2 - 1)^2} \ , \tag{89}$$

where in the last step, we used (88) to eliminate E. In particular, the divergences from v_1 and v_2, the IR and UV cutoff cancel. Another interesting aspect of this result is the linearity in the NCOS coupling. We can interpret this propagator as an amplitude in NCOS theory associated with a local insertion of a single vertex operator corresponding to the closed string in the parent Wound string theory; with v_2 being interpreted as the size of the final state, presumably a non-perturbative coherent state, in the NCOS theory. Yet, at strongly coupling, this amplitude scales linearly with the NCOS coupling. For the same technical reasons encountered above in inverting t as a function of E, the bound state spectrum is difficult to unravel from the scattering propagator.

For the reader's convenience, we write in this regime the leading phase of the propagator. We find, after eliminating E

$$S_{cl}^{scatt} \sim 2\omega \frac{\Sigma}{l_e} G \frac{v_1 v_2}{T v_2 - 1} \ , \tag{90}$$

Hence, there is a suggestive correlation between the UV and IR cutoffs, as they appear in the product $v_1 v_2$ that perhaps could be held finite in a controlled limit.

7 The holographic duality for NCOS

In this section, we collect several observations suggesting that we may associate a screen, located at finite v in the bulk space, with the dual NCOS theory, that holographically encodes dynamics in the whole of space. Consider an observer siting at the non-commutativity throat

$$v_o^3 = \frac{\kappa}{G} \ , \tag{91}$$

where κ is an arbitrary numerical constant. If this observer was to measure, locally, energy of our projectile, it would be

$$\varepsilon = |G_{00}|^{1/2} \left. \frac{dt}{d\tau} \right|_{v_0} \sim |E| + \text{constant} \tag{92}$$

i.e. all instances of G disappear from (92). Furthermore, evaluating the metric (3) at v_0, we get

$$G_{\mu\nu}|_{v_o} = \frac{G_s^2}{g_{str}} \sqrt{\kappa}\sqrt{1+\kappa} \ . \tag{93}$$

While the dilaton is

$$\left. e^{\phi} \right|_{v_o} = G_s \frac{1+\kappa}{\sqrt{\kappa}} \ ; \tag{94}$$

(see also (106)). And from the worldsheet action of the closed string in this region, the effective string length at this point is

$$\alpha'|_{v_0} = \frac{\alpha'}{|G_{00}|_{v_0}} = \frac{\alpha_e}{\sqrt{\kappa}\sqrt{1+\kappa}} \ . \tag{95}$$

These four expressions are, up to a numerical constant logged by κ, the NCOS energy scale, metric, (closed string) coupling, and string scale respectively, in *flat space*[17]. Hence, it is

[17]In particular, we remind the reader of the NCOS map $G_{\mu\nu} = G_s^2/g_{str}$ [5, 6, 7, 8]. We may also refer to this as the Wound String theory map adopted to the NCOS setting.

natural to think that the dual NCOS theory to our spacetime 'sits' at an energy scale $v \sim v_0$, which is the throat region of the spacetime geometry depicted in Figure 1. It is the region analogous to the 'boundary' of space in the Maldacena scaling limit in the cases with zero B-field. Note however that, for the bound state solutions, the closed string reaches up to an extent v_c, which can get larger than v_0 for high enough energies. And it then appears that the whole of space is available for holographic encoding.

Finally, it is important that, at $v \to \infty$, within the same framework, we also recover the conventionally different scaling limit of Wound String theory $g_{\mu\nu} = (-1, 1, \delta, \delta, \ldots)$, $\alpha' \sim \delta$, and $g_{str} \sim \frac{1}{\sqrt{\delta}}$, as shown explicitly in [11], with $\delta \sim 1/v^3$.

8 Discussion

Two aspects of our setup need more elaboration. The first has to do with the fact that our formalism neglects quantum fluctuations beyond the ansatz we used. We have been discussing the dynamics of the center of mass of the closed string, with zero angular momentum on the seven sphere. We showed that this ansatz can be classically solved for self-consistently. We studied the quantum mechanics with sum over paths restricted to this ansatz; yet the string has available to itself a much larger phase space for quantum fluctuations than that of a point particle in one dimension.

If we were to imagine how the spectrum we calculated would look like had we solved the full problem exactly, we may expect a tower of levels associated with the bounded motion of the center of mass, and, superimposed on this, another level spectrum associated with fluctuations on the string about the center mass. We may expect correlations between the spacings of these two classes of energy levels. The spectrum we have computed must then correspond to approximating part of this full spectrum. We should expect two additional effects: first, additional levels towering on each energy level we predict, corresponding to vibrations of the string itself; and second, possibly additions and corrections to the levels we computed from fluctuations of the zero modes that take us away from the ansatz; corrections, in particular, that come from the zero point energy of other oscillators. We suggest confining the results to large values of N, where the energy level spacings become large parametrically with N. Other fluctuations being independent of this, they become of less importance for the higher levels. Also, our WKB-like approximation is naturally improved for large N.

The other issue of concern has to do with the part of the motion of the closed string near the center $v \sim 0$. The problem here is that, even as we push Σ and, more importantly, M large enough so as to trust the computation to small values of the radial coordinate, we will eventually venture into a regime where the calculation breaks down near the repulsive horizon at $v = 0$ if M is finite; and the question is with regards to the effect of this forbidden region on the overall picture of the dynamics we painted.

As mentioned earlier, the strict limit $M \to \infty$, with G, Σ, and l_e fixed, is a well-defined regime with all physical measurables remaining finite. This is strong evidence that we have identified the coupling of the theory G correctly as in (107). More interestingly, in this regime, the issues that plague dynamics in the $v \sim 0$ region do not exist, as the geometry is reliable from $v = 0$ to $v \to \infty$. The spectrum computation is then controlled, up to the standard issues associated with the WKB-like approach discussed in Appendix E, and the restriction of the phase space to the $v - t$ plane discussed above; both matters are effects we can understand and control by taking the level number large. Note also that, for $M \to \infty$, there is a possibility that, the WKB-like calculation being coincidentally exact as it sometimes can be, and with the help of supersymmetry in controlling zero point energies of other oscillators in the system, we may have part of the exact spectrum of a string theory at strong coupling.

For finite M, the picture is more problematic. First, let us note that, in principle we can extend the calculation to smaller values in v by applying duality transformations. But this is bound to postpone the question instead of answering it; either into an S-dual picture, or strongly coupled IIA theory (after applying a T-duality). We instead will attempt to speculate within IIB theory as to the effect of the center of the geometry on the dynamics in the bulk for finite M.

Near the origin of the radial coordinate, the local string coupling is becoming big with smaller values of v, as can be seen from Figure 6. Furthermore, as the string falls onto the center, it is decelerating and its multiply wound strands are getting squeezed into a smaller area of space. As it spends longer and longer proper times in this region, we expect string interactions to become more and more important for this part of the motion. In the first quantized formalism we adopted, we should then consider processes where the closed string splits and joins, while conserving total winding number. One could describe these processes by introducing this surgery by hand in the center of the geometry, and representing the entire process by gluing the corresponding trajectories near the center, as classical solutions that interpolate between the various solutions we have written. The effect of these interactions is subleading to the overall dynamics, weighed by powers of the local string coupling, and correspondingly, as can be seen from equation (5), by powers of $1/M$, with M being finite but large. This suggests that we may describe the picture by the spectrum we have computed, by allowing possibly copious transitions between the various energy levels across different winding sectors. In the dual NCOS picture, this would correspond to non-perturbative states that can decay amongst each other, with their masses being estimated well to first order in $1/M$ by the quantum mechanical spectrum computed. This is a first look into what appears to be a complex and rich non-perturbative dynamics in NCOS theory at finite M. Somewhat similar phenomena may be at work in the SL(2,R) WZW model (see discussion in [22]).

Hence, the picture we have is that there exists states in the NCOS at strong coupling, with masses that scale inversely with the coupling. These objects, which carry zero momentum in the y direction, may have time dependent sizes; perhaps similar to breathing modes. There can be created in scattering processes involving a closed string of Wound String theory, perhaps represented as a vertex operator insertion in the NCOS theory. In certain asymptotic regimes, we wrote propagators of the NCOS theory, as a function of time and size. A general observation was that these have poles at times of order the final size $T \sim 1/v_2$. A linear scaling in the coupling, along with certain non-trivial cancelations of potential divergences in the UV, are some of the interesting features we encountered. To address these issues, it seems important that we understand the infinite redshift effect at the horizon tied to our choice of what we call energy or time.

Another interesting matter has to do with our observation that, in the $M \to \infty$ limit, we have a well-defined background geometry for all v. This may be viewed as a mechanism whereby non-commutativity, through the introduction of the throat, regulates the singularity that would otherwise arise at $v = 0$. The potential trouble gets replaced in the center with a patch of spacetime which is almost flat, much like a similar mechanism proposed in [39]. In this case, the regulator is non-commutativity in the infinite M limit. It would be interesting to pursue this line of thought further in other systems.

Beyond resolving some of these open questions, let us also comment briefly on potentially interesting future directions. First, understanding the dynamics of putting momentum along the y direction, as well as vibrations on the closed string, is important. The problem is not intractable, as one can adopt an approach of a perturbative expansion about the center of mass dynamics (if needed) to gain at least a hint of how the spectrum evolves. It is useful however, at this stage, to understand and include the effects of the RR fields. This may necessitate a more controlled approach by using the symmetries in the theory, in particular supersymmetry. There are interesting qualitative similarities between our problem and the SL(2,R) WZW model analyzed in [22]. In understanding that system, a crucial role is played by spectral flow, in a system that is an exactly solvable CFT. It would be interesting to see whether one can find an analogous operation that may be used to study the NCOS problem in a more mature manner. In this respect, an important simplification may be achieved by working in the strict limit $M \to \infty$.

Another line of thought is to attempt to understand dynamics of D-branes in this background. In particular, D-string motion may entail interesting information, the setting being S-dual to the one considered. And it would be desirable to better understand how one should think of holography in this context. It appears this system may be used to test and study the Holographic duality beyond the near horizon scaling limit; within Wound String theory, the energy regime we consider is a superset of the conventional regime associated with Holography [11]. In this respect, our analysis demonstrate that previous attempts to identify a criterion for Holography are too restrictive [40, 41, 42], as they confine themselves to short wavelength, point-like probes; geodesics, which are only part of the story in a picture involving strings. We hope to report on this in an upcoming work.

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10 Appendix A: The (N,M) string solution

In this appendix, we present, for the convenience of the reader, the background fields generated by the (N,M) string, and the associated NCOS limit. The parameter space in the parent IIB theory is given by

$$\{N, M, g_{str}, \alpha', \Sigma\} \ . \tag{96}$$

The metric can be found in [43, 16]

$$ds^2_{str} = g_{str}\sqrt{\frac{K}{L}}\left\{A^{-1/2}\left(-dt^2+\Sigma^2 dy^2\right)+A^{1/2}\left(dr^2+r^2 d\Omega_7^2\right)\right\} \ . \tag{97}$$

With the NSNS fields

$$B_{ty} = g_{str}^2 \Sigma \frac{N}{M} A^{-1} L^{-1/2} \ ; \tag{98}$$

$$e^{\phi} = g_{str} A^{1/2} \frac{K}{L} \ . \tag{99}$$

And the RR fields

$$A_{ty} = \Sigma\left(A^{-1}-1\right)L^{-1/2} \ ; \tag{100}$$

$$\chi = \frac{N}{M} A^{-1} \frac{A-1}{K} \ . \tag{101}$$

The various variables appearing in these expressions are defined as

$$A \equiv 1+\frac{q^6}{r^6} \ , \quad K \equiv 1 + A^{-1}\left(\frac{N g_{str}}{M}\right)^2 \ , \quad L \equiv 1+\left(\frac{N g_{str}}{M}\right)^2 \ , \tag{102}$$

with

$$q^6 \equiv \frac{32\pi^2}{g_{str}^2} M \alpha'^3 L^{1/2} \ . \tag{103}$$

This solution has manifest SL(2,Z) symmetry.

The NCOS limit is obtained by

$$\alpha' \to 0 \ , \quad g_{str}\alpha' = \alpha_e G_s^2 \text{ fixed} \quad , \quad \text{and } U \equiv \frac{r}{\alpha'} \text{ fixed} \ , \tag{104}$$

with α_e defined by this expression, and G_s defined below. In the main text, we have also performed a coordinate change

$$U^2 \equiv \frac{V}{l_e^{\,3}} \equiv \frac{M^2}{(32\pi^2)^3 G^3 \alpha_e} v \ , \tag{105}$$

to put the metric in a form conformal to $AdS_3 \times S^7$. Furthermore, v becomes energy scale in the UV-IR relation $v \sim \mu_{NCOS}$.

In this limit, the dual theory is 1+1 dimensional NCOS theory, with (closed) string coupling

$$G_s = \frac{M}{N} \ , \tag{106}$$

which is more conveniently written as

$$G \equiv \frac{G_o \sqrt{M}}{32\pi^2} \ , \quad G_o \equiv \sqrt{G_s} \ . \tag{107}$$

11 Appendix B: The path integral

We need to setup a path integral for the propagator of our quantum mechanical system, and evaluate it, subject to the first class constraint

$$C_1 = \frac{2\pi^2}{\Sigma^2}\frac{\alpha'^2}{\Omega^2} v^2 \Pi_v^2 - \frac{16\pi^4 \alpha_e}{\Sigma^2 v^2}\frac{\alpha'^2}{\Omega^2}\left(\Pi_t \mp \frac{\omega G \Sigma}{2\pi \alpha_e} v^3\right)^2 + \omega^2 \frac{v^2 \Omega^2}{16\pi^2 \alpha_e} = 0 \ . \tag{108}$$

The term 'first class' refers to the fact that C_1 commutes with the Hamiltonian (in this case, it *is* our Hamiltonian). The proper approach is to supplement the constraint with a secondary constraint that is consistent with the equations of motion [44]. In our case, we choose

$$C_2 = t - T(\tau) = 0 \ , \tag{109}$$

where

$$\frac{dT}{d\tau} = \left.\frac{dt}{d\tau}\right|_{v \to v_{cl}(\tau)} \ . \tag{110}$$

The Poisson bracket of the two constraints is then

$$\{C_1, C_2\} = -2\left(\Pi_t \mp \omega\frac{\Sigma}{2\pi\alpha_e}Gv^3\right) \ . \tag{111}$$

We now can write down the appropriate path integral, being careful to start in the Hamiltonian formalism[18]

$$\mathcal{G} \sim \int \mathcal{D}t\ \mathcal{D}\Pi_t\ \mathcal{D}v\ \mathcal{D}\Pi_v\ \delta(C_1)\delta(C_2)\sqrt{|\mathrm{Det}\,\{C_1, C_2\}|}e^{i\int_0^{\tau_0} d\tau\ \left(\dot{v}\Pi_v + \dot{t}\Pi_t - H(t,\Pi_t,v,\Pi_v)\right)} \ . \tag{112}$$

Using the well-known identity

$$\delta(C_1) \to \frac{1}{2\left|\Pi_t \mp \omega\frac{\Sigma}{2\pi\alpha_e}Gv^3\right|}\delta\left(\Pi_t - \Phi_t\right) \ , \tag{113}$$

with Φ_t given by solving for Π_t in $C_1 = 0$, the determinant in the measure cancels. We then have

$$\mathcal{G} \sim \int \prod_\tau dv(\tau)\ d\Pi_v(\tau)\ e^{i\sum_\tau\ \delta\tau\left(\Pi_v\dot{v} + \Phi_t\dot{T}\right)} \ . \tag{114}$$

We now perform a change of coordinates from τ to t, using $T(\tau)$

$$\int\ d\tau\left(\Pi_v\dot{v} + \Phi_t\dot{T}\right) = \int\ dt\left(\Pi_v\frac{dv}{dt} + \Phi_t\right) \ . \tag{115}$$

We then have the expressions given in (50) and (51).

The phase space has been reduced from four to two dimensions. On this subspace, the evolution operator is $-\Phi_t$, evolving life in the time variable t. The extremum of the integrand in (50) is at

$$\frac{dv}{dt} = -\frac{\delta\Phi_t}{\delta\Pi_v}\ ,\quad \frac{d\Pi_v}{dt} = \frac{\delta\Phi_t}{\delta v} \ . \tag{116}$$

The reader may check that these correspond to the equations of motion (28) and (29), subject to (30).

[18] If arranged in the Lagrangian picture in the naive way, a path integral in curved space gives the incorrect measure. This is because of the coordinate dependent factors multiplying the kinetic term (see for example [45]).

12 Appendix C: Some computational details

In this appendix, we collect a few details the reader may find useful in checking some of the formula appearing in the main text.

To compute many of the integrals we encountered, or to write down their asymptotic behaviors, the coordinate change

$$\begin{cases} \sinh(y) = v^{3/2}/|v_c|^{3/2} & \text{for } v_c^3 < 0 \\ \sin(y) = v^{3/2}/|v_c|^{3/2} & \text{for } v_c^3 > 0 \end{cases} \tag{117}$$

is useful. For example, the classical action becomes

$$S_{cl} = \pm \frac{2\sqrt{2}}{3}\frac{\Sigma}{l_e}\omega G \frac{E \pm 2\omega}{E}\sqrt{1-(v/v_c)^3}v_c^2 \; {}_2F_1\left(\frac{1}{2},\frac{1}{3},\frac{3}{2},1-\frac{v^3}{v_c^3}\right)\Bigg|_{v_1}^{v_2} . \tag{118}$$

While the integral given in (65) becomes

$$\mathcal{I} = -\frac{v_c^3}{6}\left\{4\left(-\frac{1}{v_c^3}-G\right)\frac{v^2}{\sqrt{1-(v/v_c)^3}} + \left(\frac{1}{v_c^3}+4G\right)v^2 \; {}_2F_1\left(\frac{2}{3},\frac{1}{2},\frac{5}{3},\frac{v^3}{v_c^3}\right)\right\}_{v_1}^{v_2} . \tag{119}$$

In finding the asymptotic expansion of (54), we rescale $v \to v/v_c \equiv y$, then note that the numerator of the integrand becomes $|E| \pm 2\omega G v^3 \to |E|$ for $|E| \ll \omega$; while it becomes $|E|(1-y^3/2)$ for $|E| \gg \omega$. We then evaluate the integral in each case in term of hypergeometric functions, which have well defined expansions for $v_2/v_c \ll 1$.

13 Appendix D: Calculation of the determinant

In this appendix, we show the computation of the determinant Δ appearing in (52). For this, we need to compute derivatives of S_{cl} while holding t, v_1, and v_2 fixed as needed. This means we have to eliminate E that appears in S_{cl}

$$\{S_{cl}(E, v_1, v_2), t(E, v_1, v_2)\} \to S_{cl}(t, v_1, v_2) . \tag{120}$$

For example, the canonical momentum to v is given by

$$\mathcal{P} \equiv \left(\frac{\partial S_{cl}}{\partial v_1}\right)_{t,v_2} = \left(\frac{\partial S_{cl}}{\partial v_1}\right)_{E,v_2} + \left(\frac{\partial S_{cl}}{\partial E}\right)_{v_1,v_2}\left(\frac{\partial E}{\partial v_1}\right)_{t,v_2} , \tag{121}$$

with

$$\left(\frac{\partial E}{\partial v_1}\right)_{t,v_2} = -\frac{(\partial t/\partial v_1)_{E,v_2}}{(\partial t/\partial E)_{v_1,v_2}} . \tag{122}$$

The expressions on the right hand sides can be derived in a straightforward manner using techniques found typically in thermodynamics textbooks. The determinant is also given by

$$\Delta \equiv \frac{\partial^2 S_{cl}}{\partial v_2 \partial v_1} = \left(\frac{\partial \mathcal{P}}{\partial v_2}\right)_{t,v_1} = \left(\frac{\partial \mathcal{P}}{\partial v_2}\right)_{E,v_1} - \left(\frac{\partial \mathcal{P}}{\partial E}\right)_{v_1,v_2} \frac{(\partial t/\partial v_2)_{E,v_1}}{(\partial t/\partial E)_{v_1,v_2}} . \tag{123}$$

We now use equations (53) and (54) to evaluate these derivatives. We get

$$\left(\frac{\partial t}{\partial v_1}\right)_{E,v_2} = -2\pi \frac{\sqrt{2\alpha_e}}{E} \frac{-E \pm 2G\omega v_1^3}{v_1^2 \sqrt{1-(v_1/v_c)^3}} . \tag{124}$$

And

$$\left(\frac{\partial t}{\partial E}\right)_{v_1,v_2} = 8\pi G \frac{\sqrt{2\alpha_e}}{E^3} \omega^2 \int_{v_1}^{v_2} dv \, \frac{v\,(1+Gv^3)}{\left(1-(v/v_c)^3\right)^{3/2}} . \tag{125}$$

From the classical action, we have

$$\left(\frac{\partial S_{cl}}{\partial v_1}\right)_{E,v_2} = \sqrt{2} G \frac{\Sigma}{l_e} \frac{2 \pm E}{E} \frac{v_1}{\sqrt{1-(v_1/v_c)^3}} . \tag{126}$$

And

$$\left(\frac{\partial S_{cl}}{\partial E}\right)_{v_1,v_2} = 2\sqrt{2} \frac{G}{E^2} \frac{\Sigma}{l_e} \omega^2 \int_{v_1}^{v_2} dv \, \frac{v\,(1+Gv^3)}{\left(1-(v/v_c)^3\right)^{3/2}} . \tag{127}$$

Note that the integrals in (125) and (127) are identical. This results in a cancelation in computing (121) that simplifies the problem considerably. Putting everything together, we get the expressions given in (63) and (64).

14 Appendix E: Comments on the Bohr-Sommerfeld approximation

In this appendix, we discuss the merits of the method we used in deriving the energy spectrum associated with the bounded motion of the closed string. We also use this opportunity to briefly discuss the problem from a slightly different angle.

We can think of our quantum mechanical problem in the Schrödinger picture as one corresponding to a Hilbert space spanned by energy eigenstates

$$H|\Psi\rangle = \mathcal{E}|\Psi\rangle \ , \tag{128}$$

with the additional prescription to project onto the space of states with zero 'energy' $\mathcal{E}$; *i.e.* we apply our constraint directly on the Hilbert space. We also have

$$\Pi_t|\Psi\rangle = \frac{\Sigma}{4\pi\alpha_e} E|\Psi\rangle \ , \tag{129}$$

since Π_t commutes with the Hamiltonian H.

Our Hamiltonian has the form

$$H = f(v)^2 \frac{\Pi_v^2}{2} + \dots \ . \tag{130}$$

The peculiar factor multiplying the momentum is not a concern, since one can always apply a canonical transformation *without* changing the energy spectrum. For example, using the generator

$$\mathcal{H}(v, P_v) : (v, \Pi_v) \to (W, P_v) \ , \tag{131}$$

with

$$\mathcal{H}(v, P_v) = P_v \int_{v_0}^{v} \frac{dv}{f(v)} \ , \tag{132}$$

puts the problem in the standard form

$$H \to \frac{P_v^2}{2} + \mathcal{V}(W) \ . \tag{133}$$

The corresponding Bohr-Sommerfeld quantization rule may be written as[19]

$$\mathcal{B}_1 + \mathcal{B}_2 = \pi N \ , \tag{134}$$

with N a large integer, and

$$\mathcal{B}_1 = \int P_v \ dW = \int \Pi_v dv = \frac{\Sigma}{\sqrt{2\alpha_e}} E \int_{v_2 \to 0}^{v_c} dv \ \frac{\sqrt{1 - (v/v_c)^3}}{v^2} \ ; \tag{135}$$

$$\mathcal{B}_2 = \int_{v_c}^{v2 \to 0} \Pi_t \ dt = \frac{\Sigma}{4\pi\alpha_e} E \Delta t_{bnd} \ . \tag{136}$$

[19] Note that we are not quantizing $\mathcal{E}$, which is fixed to zero by the constraint; this is a statement of quantization for E.

The reader may be concerned with the inclusion of $\mathcal{B}_2$. The necessity of this term can be seen from (50); the exponent in the propagator is the sum of $\mathcal{B}_1$ *and* $\mathcal{B}_2$. The physical origin of the quantization condition is periodicity of this expression, hence the requirement that the exponent of (50) be quantized. Since our Hamiltonian vanishes, the 'counter' of wavefunction nodes $\oint p\, dx$ is the same as the classical action. E appears in the quantization condition as a parameter tuning the shape of the potential of the one dimensional quantum mechanics problem; and the Bohr-Sommerfeld quantization determines for what potential is there a 'zero energy' state with an integer number of nodes.

The second more important concern the reader must have has to do with the fact that, for the bounded motion, we have limits of integration in (135) and (136) extending to $v_2 \to 0$. But the latter involves venturing into forbidden domains of the spacetime if M is taken finite, as dictated by (11). To appreciate the problem at hand further, let us look at the behavior of (135) and (136) near $v_2 \sim 0$.

For the first term in (134), we have

$$\mathcal{B}_1 \sim -\frac{\Sigma}{\sqrt{2\alpha_e}} E \frac{1}{v_2} + \text{finite} \ . \tag{137}$$

Hence it is divergent as $v_2 \to 0$, since the momentum canonical to v blows up at the origin (while $\dot{v}$ vanishes). This divergence arises in the IR of the NCOS theory.

Looking at the second contribution, we get, using (58) in (136)

$$\mathcal{B}_2 = \frac{\Sigma}{\sqrt{2\alpha_e}} E \frac{1}{v_2} + \text{finite} \ . \tag{138}$$

The origin of this divergence is the infinite redshift at $v_2 \sim 0$ with respect to the time variable t; a phenomenon well known from dynamics near a black hole horizon. And indeed this term cancels precisely the divergence arising in (137), leading to the finite action in (56) and (134). The important conclusion is that the region near $v_2 \sim 0$ contributes negligibly to (134) for $v_2 \ll 1$ in $\int_0^{v_2} dv$.

For finite M, we then consider taking M large (and $\Sigma/l_e \gg 1$ if needed), so that the spacetime is reliable up to very small values in v and the asymptotic expansions used above are therefore valid. Then, the contribution of the forbidden region, as we approach it from larger values of v, is parametrically small in the quantization equation (134).

All this still assumes that, to leading order in the dynamics of the closed string, we have bounded motion, with a turning point at v_c; in particular, that our ignorance of the details of the dynamics at the origin of the radial coordinate v does not change the fact that, to this level of semiclassical approximation, the closed string slows down to a virtual stop as it approaches the horizon, bounces back, and hence is effectively in a box. Note that there is no need however of any concern in this regard for the regime where $M \to \infty$, as discussed in Section 2.2. In that strict limiting case, the dynamics is reliable all the way to $v = 0$; and our picture of bounded motion is correct. Note that no observable, such as the action or v_c, associated with this dynamics depends on M explicitly; and, hence, the limit is regular.

For large but finite M, the regularity of the classical action near $v \sim 0$ gives us some confidence that, at least as we approach this unknown region, the dynamics appears well controlled. As we argue in the Discussion section, we may expect however that there will be processes involving the splitting and joining the closed string; these effects being subleading in a perturbative expansion in the local value of the dilaton, and hence involving an expansion in $1/M$. We then propose a scenario whereby these effects may be incorporated in our formalism as transitions between energy levels of different winding numbers. Hence, the spectrum we derive may still be the leading effect in determining the dynamics. More discussion about this may be found in the Discussion section.

Finally, it is important to emphasize that, despite the fact there is an infinite redshift at $v \sim 0$, one is able to extract a finite spectrum for the energy, due to the cancelations detailed earlier. Along this line of thought, let us also note that we have checked that the phase factor computed above is additive as the string oscillates between $v = 0$ and $v = v_c$ in finite proper time; as opposed to perhaps canceling because of some sign flip.

The optimistic reader may wonder whether using the Bohr-Sommerfeld condition was somewhat too reserved; why not apply the full WKB-like machinery, in an attempt to estimate the ground state energy of the spectrum near $N \sim 0$. In fact, expanding the spectrum for small N, we do indeed find some non-zero ground state energy. At issue however is a numerical shift on the right hand side of equation (134) $N + \text{constant}$, this constant often called the Maslov index which we have not computed. It can be determined by analyzing caustics of the classical trajectory, a subject that connects to Catastrophe theory. While as an Armenian, I readily relate to theories of catastrophes, venturing into this analysis is uncalled for in this case, for two reasons. First, the approximation scheme we have adopted (sometimes wrongfully referred to as WKB) is improved for $N \gg 1$, as is well known. Hence, the estimate for a ground state energy, with numerical accuracy, is unlikely to be reliable. The overly optimistic reader may suggest that, this spectrum, being related to center of mass dynamics, is perhaps 'protected' in some unknown sense and for some unknown reason (perhaps supersymmetry). While this is an interesting suggestion, it is difficult to understand controlled fluctuations without analyzing the symmetry principle at work, which we have not explored in our formalism; and such quantum fluctuations are otherwise generically likely to wash out our center of mass spectrum for small values of N. Hence, the $N \gg 1$ is needed for this purpose as well, and the issue of estimating the ground state energy is deferred to a more complete analysis that studies, amongst other effects, the role of supersymmetry.

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