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Probe Brane Dynamics and the Cosmological Constant

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Abstract

Recently a brane world perspective on the cosmological constant and the hierarchy problems was presented. Here, we elaborate on some aspects of that particular scenario and discuss the stability of the stationary brane solution and the dynamics of a probe brane. Even though the brane is unstable under a small perturbation from its stationary position, such instability is harmless when the 4-D cosmological constant is very small, as is the case of our universe. One may also introduce radion stabilizing potentials in a more realistic scenario.

I. INTRODUCTION

The smallness of the cosmological constant is an outstanding fine-tuning problem. Recent observations [1] indicate strongly that it has a positive value, implying that the small cosmological constant probably has a dynamical origin (versus a zero value due to a symmetry reason). Many attempts addressing this fine-tuning issue can be found in the literature. Here, we would like to elaborate on the recent brane world proposal of Ref [2,3], which originates from a modification of the Randall-Sundrum model [4].

In such a scenario [3], where the branes are stationary, an exponentially small cosmological constant appears rather naturally. The scenario also easily incorporates the solution

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to the hierarchy problem via warped geometry. However, the general moving brane case is complicated by the coupling of the brane motions and the metric. This problem is tremendously simplified in the limit of zero brane tension and zero brane charge (but with a finite non-zero ratio). Since the visible brane tension must be smaller than the Planck brane tension, we are led to consider the visible brane as a probe brane. In this approximation, the probe brane motion does not change the background metric, i.e., there is no back-reaction. It turns out that the resulting brane equation of motion is still rather non-trivial, but exact analytic solutions can be found. We shall discuss this case. In summary, the stationary branes are unstable: under small perturbations they will start moving. This conclusion is consistent with the result in other similar situations [5]. A stationary probe brane is stable if it is sitting at a minimum of the warp factor. This corresponds to a particle horizon in our scenario. Otherwise, an unstable mode is present and a perturbation away from the stationary position will grows like e^{Ht}, where H is the Hubble constant. So a stationary probe brane will start moving, eventually moving at the "speed of light" (an exact solution of the probe brane equation of motion). However, for a small effective 4-D cosmological constant (so a small H), the case we are interested in, the growth of the perturbation is very slow. Consider today's H, where $1/H$ is the cosmic size. A small perturabtion will only grow by one e-fold (say doubling) in the lifetime of the universe. That is, such an instability is harmless. We also discuss the implication of this result in the more general and realistic situations. In particular, the branes should be stabilized with a radion potential [6].

To check that there is no back reaction, and as an illustration, we consider explicitly a three 3-brane model, where the visible brane is sitting between the Planck brane and a third brane, which is beyond the particle horizon. As the third brane and the Planck brane slowly move apart, the effective 4-D cosmological constant decreases. As the visible (probe) brane moves away from the Planck brane, the effective electroweak scale decreases, so there is a constraint on such motion as well.

We also clarify the equivalence of two apparently different ways to obtain the effective 4-D cosmological constant given in Ref [3]: (i) via the determination of the Hubble constant H and (ii) via the 4-D effective action. In the latter approach, besides the contributions from the brane tensions and the bulk cosmological constants, the extra dimension curvature also contributes. In the large brane separations case, the various terms almost cancel, reproducing the exponentially small 4-D effective cosmological constant found by the other approach.

II. THE MODEL

Consider the action S consisting of a bulk action containing 5-D gravity, with metric g_{ab} ($a = 0,1,2,3,5$), and a set of worldvolume actions of 3-branes with dynamical coordinate fields X^a_n. This means the location of the n^{th} brane in the bulk is given by the embedding functions $x^a = X^a_n(\xi^\mu_n)$, and the n^{th} brane possesses a brane metric $\gamma^{\mu\nu}_n$ which is a function of the brane coordinates ξ^μ_n ($\mu = 0,1,2,3$). The n^{th} brane couples (with charge e_{ni}) to the 4-form potentials $A_i \equiv A_{ia_1\cdots a_4}(X_n)$ (with 5-form field strength $F_{(i)} = dA_i$). The action for such a system is [7,8,3]

$$S = \int d^5x\sqrt{|g|}\left[\frac{1}{2\kappa^2}R - \Lambda - \frac{1}{2\cdot 5!}F^2_{(i)}\right]$$

$$+\sum_n \left\{ -\frac{1}{2}\sigma_n \int d^4\xi_n \sqrt{|\gamma_n|} \left[\gamma_n^{\mu\nu} \partial_\mu X^a \partial_\nu X^b g_{ab}(X_n) - 2 \right] - \frac{e_{ni}}{4!} \int d^4\xi_n A_{ia_1\cdots a_4}(X_n) \partial_{\mu_1} X^{a_1} \cdots \partial_{\mu_4} X^{a_4} \epsilon^{\mu_1\cdots\mu_4} \right\} \tag{1}$$

where σ_n and e_{ni} are the tension and the charges (coupling to the 4-form potential A_i of the $n^{\rm th}$ brane), respectively. In the action S, the sum over i is implicit. The tensor density is totally antisymmetric with $\epsilon^{0123} = 1$. Varying the action S with respect to the metric g_{ab}, the 4-form potential A_i, the brane metric $\gamma_{\mu\nu}$ and the brane coordinates X^a, we have, respectively, the 5-D Einstein equation, the 4-form field equation, the induced metric relation and the brane equation of motion. In particular, the brane equation of motion for the $n^{\rm th}$ brane is given by

$$\sigma_n \sqrt{|\gamma_n|} \left[\nabla_\mu \nabla^\mu X_n^a + \Gamma^a_{bc} \partial_\mu X_n^b \partial_\nu X_n^c \gamma_n^{\mu\nu} \right] + \frac{e_{ni}}{(4)!} F^{ia}_{b_1\cdots b_4} \partial_{\mu_1} X_n^{b_1} \cdots \partial_{\mu_4} X_n^{b_4} \epsilon^{\mu_1\cdots\mu_4} = 0. \tag{2}$$

Solving the 4-form field equations, we see that the background field strengths are constants to be determined by the Einstein equation. For parallel stationary branes with an appropriate number of 4-form potentials, the complete set of equations have been solved in Ref [3]. Here we shall review some of the key features of the more general solution by considering the specific case of three parallel stationary 3-branes with the $5^{\rm th}$ dimension uncompactified. The two brane compactified case and the two brane orbifold case discussed in Ref [2,3] can be easily obtained from this case (by identifying two of the branes, and by orbifolding, respectively). The above brane equation of motion also has moving brane solutions. They are rather complicated and will be discussed elsewhere. In this note, we shall only consider the brane motion for a brane with zero brane tension and zero brane charge (but with a non-zero charge to tension ratio).

III. MOTION OF THE VISIBLE BRANE AS A PROBE BRANE

The Einstein equations were solved in Ref [2,3,9]. Starting with the metric ansatz

$$ds^2 = dy^2 + A(y)[-dt^2 + e^{2Ht}\delta_{ij} dx^i dx^j] \tag{3}$$

the Einstein equations for the bulk are

$$\begin{aligned} A'' &= 2H^2 + 4k^2 A \\ \frac{(A')^2}{A} &= 4H^2 + 4k^2 A \end{aligned} \tag{4}$$

This yields the general solution for the bulk

$$A(y) = \frac{H^2 \sinh^2[k_i(y - y_i)]}{k_i^2}, \tag{5}$$

in which $k_i \equiv \sqrt{\kappa^2 \Lambda_i / 6}$ and y_i are in general integration constants. Here, $-\Lambda_i$ are the bulk cosmological constants that depend on the constant 5-form field strengths,

$$-\Lambda_i = -\Lambda + \frac{1}{2}\sum_j \left(e_{Bj} + e_{0j} + e_{1j} + ... + e_{(i-1)j}\right)^2 . \qquad (6)$$

Here we require $\Lambda_i > 0$ for Anti-deSitter spaces between branes. This requires $-\Lambda$ to be negative enough. Depending on the charges of the branes, this will allow some number of bulk cosmological constants to be determined by the Einstein equation, or equivalently, by the boundary conditions at the branes (that is, the jump conditions [10]). There, the warp factor $A(y)$ is continuous across the branes while $A'(y)$ has a jump at the brane as a function of the brane tension.

Let us now consider the brane equation of motion for branes that may not be stationary. In general, brane motions and the metric are coupled in a way that do not allow a simple explicit analytic solution. To see what is going on, one may consider slow brane motion, or alternatively, the situation where the particular brane tension is negligible. Here we shall consider the latter case, i.e., a probe (or test) brane. In particular, we may consider the case where the visible brane is the probe brane. As we shall see more explicitly, as the probe brane tension $\sigma_1 \to 0$, both H and $A(y)$ are independent of the position of the visible brane. In this limit, the brane equation of motion is still non-trivial, but it turns out that explicit analytic solutions can be found. This is clearly easier than the full dynamics. Understanding the probe brane motion can give insight into the full dynamics. We could also study the problem perturbatively, treating the probe brane tension as a small parameter.

In the above scenario, we shall consider a set of branes with non-zero brane tensions and charges, so that $A(y)$ is solved. Next we introduce the visible brane, namely the σ_1 brane, which is coupled to only one 4-form potential, say A_1. Now we shall take both this charge, namely e_{11}, and tension σ_0 to zero, namely $e_{11} \to 0$ and $\sigma_1 \to 0$, with a finite nonzero charge-to-tension ratio, $e_{11}/\sigma_1 = Q$. The brane equation of motion (2) for such a probe brane (at y) reduces to

$$0 = \ddot{y} + 2A' - \frac{QE_1(A-\dot{y}^2)^{3/2}}{\sqrt{A}} + 3H\dot{y} - \frac{5A'}{2A}\dot{y}^2 - \frac{3H}{A}\dot{y}^3, \qquad (7)$$

in the presence of the background metric (3) and a constant background 5-form field strength E_1. Since all the probe brane tension and charges are zero, there is no back-reaction. As the probe brane moves, both H and the metric $A(y)$ remain unchanged, as long as the other branes (ones that have non-zero tensions and charges) are stationary.

There are two exact solutions to this equation :

• One corresponds to the equilibrium point

$$QE_1 = \frac{2A'(y)}{A(y)} \qquad \dot{y} = 0, \qquad (8)$$

presuming it exists. In our set up, the background field strength E_1 is an integration constant, so, for any position y for the brane, we may fix the value E_1 to satisfy the above equation. This simply reproduces the stationary solution we have been discussing. However, once E_1 is fixed, its value (and so the bulk cosmological constant) will stay constant, at least classically.

• The other solution is

$$\dot{y} = \pm\sqrt{A(y)}. \qquad (9)$$

This solution corresponds to 'motion at the speed of light'.

These are the two exact analytic solutions: one is stationary at the equilibrium point, and the other has the brane moving at 'light' speed. The equation of motion is an initial value problem, and has an infinite number of solutions. We have found that a stationary brane is a solution. Suppose this is the initial situation. Let us check the stability of this stationary solution by asking what happens if the probe brane is perturbed away from equilibrium. To find out, consider small perturbations around the equilibrium point, $y = y_0 + \delta y(t)$. Linearizing Eq. (7) implies

$$\begin{aligned} 0 &= \delta\ddot{y} + 3H\delta\dot{y} + [2A_0'' - QE_0A_0']\delta y \\ &= \delta\ddot{y} + 3H\delta\dot{y} + 2\Big[A_0'' - \frac{(A_0')^2}{A_0}\Big]\delta y, \end{aligned} \tag{10}$$

where the subscripts 0 mean evaluate the derivatives at $y = y_0$, and the second form of the equation uses the equilibrium relation to eliminate QE_1. Using the Einstein equation (4), the equation for linear perturbations can be reduced to

$$0 = \delta\ddot{y} + 3H\delta\dot{y} - 4H^2\delta y. \tag{11}$$

This is the equation of motion for the radion mode. This equations has solutions of the form $\delta y \propto e^{\sigma t}$ with

$$0 = \sigma^2 + 3H\sigma - 4H^2 = (\sigma - H)(\sigma + 4H), \tag{12}$$

implying two roots, $\sigma = -4H$, which is stable, and $\sigma = H$, which is unstable. Thus, the equilibrium point is unstable, and the probe brane will tend to move away from it if perturbed at all. This instability property is consistent with the conclusion of similar analyses in similar scenarios [5]. What happens is that a probe brane perturbed away from equilibrium accelerates until it approaches the motion at speed of 'light', and, according to that solution, eventually wants to settle at a zero of $A(y)$, either by passing through the Planck (the σ_0) brane or not.

The timescale for the brane to move is $\sim H^{-1}$, which is long compared to $\sim k^{-1}$ if $H \ll k$ but the instability may still grow on a timescale relevant for cosmology. For example, the value of A on the brane changes by a fractional amount $(A_0'/A_0)\delta y_+(0)e^{Ht}$, where the amplitude of the growing mode at $t = 0$ is $\delta y_+(0)$. The change is substantial after a time of order $H^{-1}\ln[A_0/A_0'|\delta y(0)|]$, which would imply, for example, considerable cosmological evolution of the Higgs mass scale.

The time evolution of the displacement should proceed exponentially at first, according the linear instability, and then asymptote toward motion at the speed of light. In this limit, the rate of change of A at the probe brane is governed by

$$\frac{\dot{A}}{HA} = \pm 2\sqrt{1 + \frac{k^2 A}{H^2}}, \tag{13}$$

and is always large on a cosmological timescale (and is actually large on the timescale k^{-1} generally).

Let us consider what happens when $Q \equiv 0$, i.e. when the charge-to-tension ratio vanishes exactly. In this case, eq. (7) becomes

$$0 = \ddot{y} + 2A' + 3H\dot{y} - \frac{5A'}{2A}\dot{y}^2 - \frac{3H}{A}\dot{y}^3, \tag{14}$$

and the equilibrium point(s) are the extrema of A, i.e., $A' = 0$ at $y = y_0$. Linearizing around this point implies the perturbation equation (which can also be read off from the first line of Eq. (10))

$$0 = \delta\ddot{y} + 3H\delta\dot{y} + 2A_0''\delta y, \tag{15}$$

which implies decaying oscillations when $A_0'' > 0$. Thus, the probe brane can sit stably at the minimum of A when it has $Q \equiv 0$. For the AdS metric, $A(y) = 0$ at the minimum, but in non-AdS, the minimum value of $A(y) \neq 0$ in general.

Eq(11) shows that this model is unstable (with tachyonic $m^2 = -4H^2$), but also suggests that this instability could be generic. We shall discuss stability calculations that support this supposition elsewhere. This instability in general will cause other problems for the model. One may fix this problem by introducing a radion stabilizing potential for this mode [6]. For example, we may consider a potential with its minimum at $y = L$,

$$V(y) = -\Lambda + C(y - L)^2 + ... \tag{16}$$

where $C > 0$ so the probe brane is stable at $y = L$. Here $-\Lambda$ is simply the term that contributes to the bulk cosmological constant (6). In the scenario with more than one probe brane, the various branes tend to fall into the same position, causing brane collisions that originate the big bang. Inter-brane interactions may lead to small brane separations, so our visible brane may sit close to a hidden brane that contains some or all of the dark matter we see today. The situation in a more realistic scenario will be discussed elsewhere.

The above toy model does suggest a number of interesting scenarios. For example, suppose we have two probe branes. Typically they will collide when moving towards the stable point $y = L$. In this scenario, the collision is expected to release energy into radiation, and the two branes may fuse to form the visible brane after the collision [11]. The resulting visible brane may be relatively stationary.

IV. 3 BRANES AT LARGE SEPARATION

Let us consider a specific scenario, namely a model with three parallel 3-branes and see explicitly how the visible brane becomes a probe brane when its brane tension and brane charge vanish, that is, the metric dependence on its position drops out. We now calculate the Hubble constant directly from the 5-D gravitational equations of motion. Since the only source for the bulk stress energy tensor is the 5-D cosmological constant, we have Anti-deSitter (AdS) bulk spacetime and we assume the branes lie along deSitter (dS) foliations. For the case of three branes, with brane tensions σ_i and brane charges e_{ij}, $i = 0, 1, 2$, there are 2 background field strengths e_{Bj}, $j = 1, 2$, that are integration constants to be determined. Here we require $\Lambda_i > 0$ for AdS spaces between branes. This requires $-\Lambda$ to be negative enough. Depending on the charges of the branes, this will allow some number of bulk cosmological constants to be determined by the Einstein equation, or equivalently, by the boundary conditions at the branes. The model is shown in Figure 1, where we use the rescaled brane tension $q_i \equiv \kappa^2\sigma_i/3$ and $k_i^2 \equiv \kappa^2\Lambda_i/6$.

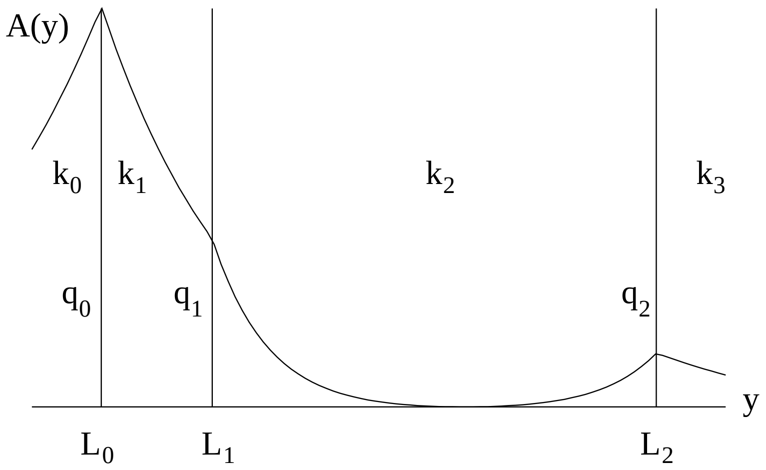

FIG. 1 Schematic arrangement of the branes and the warp factor $A(y)$.

In this model, the above probe brane scenario corresponds to the situation where only the σ_0 and the σ_2 branes couple to the 4-form potential A_2. There, the σ_0 brane is the Planck brane while the σ_1 brane is the visible brane. We shall also take both the visible brane charge, namely e_{11}, and its tension to zero, keeping the charge-to-tension ratio, $Q = e_{11}/\sigma_1$ fixed to a finite non-zero value. Before going to this limit, let us first consider the general case.

Integrating the G_{00} equation across the branes leads to the Israel jump condition [10]:

$$\lim_{y\to L_i+}\left(\frac{A'}{A}\right) - \lim_{y\to L_i-}\left(\frac{A'}{A}\right) = -\frac{2\kappa^2\sigma_i}{3} \equiv -2q_i, \tag{17}$$

in addition to which, the metric and hence the scale factor, $A(y)$, is required to be continuous across the branes.

With a $5^{\rm th}$ dimension of infinite extent, and three parallel 3-branes located at $y = L_0, L_1, L_2$, the jump and continuity conditions become

$$\begin{aligned}
&q_0 = k_0\coth[k_0(L_0-y_0)] - k_1\coth[k_1(L_0-y_1)],\\
&q_1 = k_1\coth[k_1(L_1-y_1)] - k_2\coth[k_2(L_1-y_2)],\\
&q_2 = k_2\coth[k_2(L_2-y_2)] - k_3\coth[k_3(L_2-y_3)],\\
&k_0^2(\coth^2[k_0(L_0-y_0)]-1) = k_1^2(\coth^2[k_1(L_0-y_1)]-1),\\
&k_1^2(\coth^2[k_1(L_1-y_1)]-1) = k_2^2(\coth^2[k_2(L_1-y_2)]-1),\\
&k_2^2(\coth^2[k_2(L_2-y_2)]-1) = k_3^2(\coth^2[k_3(L_2-y_3)]-1).
\end{aligned} \tag{18}$$

This is the setup of the problem. In this case, we like to consider two 4-form potentials. So, of the four k_j's, two of them, or two combinations of them are to be treated as integration constants. Now we can solve for H and G_N and so $\Lambda_{\rm eff}$. Since we are interested mainly in the small $\Lambda_{\rm eff}$ case, we shall solve this set of equations in the limit of large $k_2(L_2 - L_1)$ and $k_1(L_1 - L_0)$. Writing

$$
\begin{aligned}
t_1 \equiv \tanh[k_1(L_1 - L_0)] &\simeq 1 - \epsilon_1, \quad \text{where } \epsilon_1 \equiv 2e^{-2k_1(L_1-L_0)}, \\
t_2 \equiv \tanh[k_2(L_2 - L_1)] &\simeq 1 - \epsilon_2, \quad \text{where } \epsilon_2 \equiv 2e^{-2k_2(L_2-L_1)}, \\
\coth[k_0(L_0 - y_0)] &\simeq 1 + \eta_0, \\
\coth[k_1(L_1 - y_1)] &\simeq -1 - \eta_1.
\end{aligned} \tag{19}
$$

in which η_0 and η_1 are expected to be exponentially small in the interbrane separations. To $\mathcal{O}(\epsilon^0)$, the jump conditions give

$$
\begin{aligned}
q_0 &= k_0 + k_1, \\
q_1 &= k_2 - k_1,
\end{aligned} \tag{20}
$$

and to $\mathcal{O}(\epsilon^1)$, we have for $\eta_{0,1}$

$$
\begin{aligned}
\eta_0 &= \frac{1}{2}\left(\frac{q_0 + q_1 - k_0}{k_0}\right)^2 \left[\frac{\coth[k_2(L_2 - y_2)] + 1}{\coth[k_2(L_2 - y_2)] - 1}\right] \epsilon_1\epsilon_2, \\
\eta_1 &= \left(\frac{q_0 + q_1 - k_0}{q_0 - k_0}\right)^2 \left[\frac{\coth[k_2(L_2 - y_2)] + 1}{\coth[k_2(L_2 - y_2)] - 1}\right] \epsilon_2.
\end{aligned} \tag{21}
$$

The jump and continuity conditions (18) can be used to write

$$
\coth[k_2(L_2 - y_2)] = \frac{q_2^2 + k_2^2 - k_3^2}{2q_2k_2}. \tag{22}
$$

Given that this expression does not approach unity, the system can be solved as

$$
\begin{aligned}
H_0^2 \equiv \frac{H^2}{A(L_0)} &= 2k_0^2\eta_0 = (q_0 + q_1 - k_0)^2 \left[\frac{(q_0 + q_1 + q_2 - k_0)^2 - k_3^2}{(q_0 + q_1 - q_2 - k_0)^2 - k_3^2}\right] \epsilon_1\epsilon_2, \\
H_1^2 \equiv \frac{H^2}{A(L_1)} &= 2k_1^2\eta_1 = (q_0 + q_1 - k_0)^2 \left[\frac{(q_0 + q_1 + q_2 - k_0)^2 - k_3^2}{(q_0 + q_1 - q_2 - k_0)^2 - k_3^2}\right] \epsilon_2,
\end{aligned} \tag{23}
$$

where ϵ_1 and ϵ_2 are given in Eq(19). Here, k_0 and k_3 are treated as undetermined by the boundary and jump conditions. For given brane charges, k_0 and k_3 are related to k_1 and k_2.

The important behavior of this solution lies in the exponentially small terms. There is an exponentially large hierarchy between mass scales on the branes at L_0 and L_1, such that, as in Ref. [3] (also see section V),

$$
\left(\frac{M_{Higgs}}{M_{Pl}}\right)^2 \sim \frac{A(L_1)}{\int dy A(y)} \simeq \frac{A(L_1)}{A(L_0)} \simeq e^{-2k_1(L_1-L_0)}, \tag{24}
$$

giving Planck masses exponentially greater than Higgs' masses with the exponent related to the distance scale between the Planck and visible brane. Similarly, the exponential scale for the smallness of the cosmological constant with respect to the Planck mass arises from the distance between the L_1 and L_2 branes. For large brane separations, the cosmological constant behaves like

$$
\frac{\Lambda_{\text{eff}}}{M_{Pl}^4} \simeq \frac{H_1^2}{M_{Pl}^2} \simeq \frac{\epsilon_2 A(L_1)}{A(L_0)} \simeq e^{-2[k_2(L_2-L_1)+k_1(L_1-L_0)]} \tag{25}
$$

This behavior is generic of these such models, with the two different length scales in the 5D spacetime leading to the two different exponentially large physical scale diffenences on the visible brane.

This result (23) can be applied to a variety of scenarios. Suppose that the q_0 and q_2 branes have no brane charges, i.e., $e_{0j} = e_{2j} = 0$, $j = 1, 2$. Then $k_0 = k_1$ and $k_3 = k_2$, and (23) becomes

$$H_0^2 = 4\left(\frac{q_0}{2} + q_1\right)^2 \left[\frac{q_2 + 2q_1 + q_0}{q_2 - 2q_1 - q_0}\right] e^{-q_0(L_1 - L_0) - (q_0 + 2q_1)(L_2 - L_1)},$$
$$H_1^2 = 2\left(\frac{q_0}{2} + q_1\right)^2 \left[\frac{q_2 + 2q_1 + q_0}{q_2 - 2q_1 - q_0}\right] e^{-(q_0 + 2q_1)(L_2 - L_1)}. \tag{26}$$

We may also apply the result (23) to a model with a compactified 5$^{\text{th}}$ dimension. Here, $\coth[k_2(L_2 - y_2)] \to 1$ in (21). and we identify the branes at L_0 and L_2, such that $q_2 \equiv q_0$, $e_{0j} \equiv e_{2j}$ ($j = 1, 2$), $k_0 \equiv k_2$ and $k_3 \equiv k_1$. Because the 5$^{\text{th}}$ dimension is compactified, charge conservation requires that $e_{0j} + e_{1j} = 0$, $j = 1, 2$. Writing

$$\coth[k_2(L_2 - y_2)] \equiv 1 + \eta_2,$$

we can solve for η_2 using (18) to be

$$\eta_2 = \sqrt{\epsilon_1 \epsilon_2},$$

which when substituted into (21) gives the following expressions for $H_{0,1}^2$:

$$H_0^2 = (q_0 + q_1)^2 e^{-\frac{1}{2}[(q_0 + q_1)(L_2 - L_1) + (q_0 - q_1)(L_1 - L_0)]},$$
$$H_1^2 = (q_0 + q_1)^2 e^{-\frac{1}{2}[(q_0 + q_1)(L_2 - L_1) - (q_0 - q_1)(L_1 - L_0)]}. \tag{27}$$

The result is simply the compactified model with two branes discussed in Ref [3]. The probe brane case simply requires both $e_{11} \to 0$, and $\sigma_1 \to 0$, while keeping $Q = e_{11}/\sigma_1$ fixed to a finite non-zero value.

Let us consider the overall picture of the three brane model, where the visible brane is sitting between the Planck brane and a third brane, which is beyond the particle horizon. As the third brane and the Planck brane slowly move apart, the effective 4-D cosmological constant decreases. We can use this motion to accommodate the quintessence picture. As the visible brane moves away from the Planck brane, the effective electroweak scale decreases, so there is a constraint on such motion. An analysis of this scenario will be interesting.

V. Λ_{eff} FROM 4-D EFFECTIVE ACTION

It is instructive to consider the derivation of Λ_{eff} starting from the 5-D action $\mathbf{S}^{(5)}$:

$$\mathbf{S}^{(5)} = \int dy d^4x \sqrt{-g}\left[\frac{R^{(5)}}{2\kappa^2} + \Lambda + \sum_i \delta(y - L_i)\sqrt{g^{55}}(-\sigma_i + \mathbf{L}_i)\right] + \int_\Sigma \frac{K\sqrt{-\gamma}}{\kappa^2} d^4x, \tag{28}$$

with $\mathbf{L}_i$ being the i^{th} brane mode Lagrangian. The last term is the Hartle-Hawking boundary term. In obtaining the same results as found by algebraic means, the sources of the various cancelling contributions, and their geometric interpretations, will become clear.

We use the 5-D metric ansatz,

$$ds^2 = A\left(y\right)\hat{\gamma}_{\mu\nu}\left(x\right)dx^\mu dx^\nu + dy^2, \tag{29}$$

and the decomposition of the 5-D Ricci tensor, $R^{(5)}_{\mu\nu}(g_{ab})$, into its 4-D counterpart, $R^{(4)}_{\mu\nu}(\hat{\gamma}_{\mu\nu})$, and the extrinsic curvature $K_{\mu\nu}$ ($K \equiv g^{\mu\nu}K_{\mu\nu}$):

$$R^{(5)}_{\mu\nu} = R^{(4)}_{\mu\nu} - g^{55}\partial_y K_{\mu\nu} - g^{55}K_{\mu\nu}K + 2g^{55}K^{\lambda}_{\mu}K_{\lambda\nu}. \tag{30}$$

For our metric, $K_{\mu\nu} = g_{\mu\nu}A'/2A$ and $K = 2A'/A$. Integrating over y, we obtain the 4-D effective action:

$$\mathbf{S}^{(4)} = \int d^4x\sqrt{-\hat{\gamma}}\left[\frac{R^{(4)}}{2\kappa_N^2} - \Lambda_{\text{eff}} + \sum_i \mathbf{L}_i\right] \tag{31}$$

where the 4-D gravitational coupling is

$$\frac{1}{2\kappa_N^2} = \frac{1}{2\kappa^2}\int A(y)dy, \tag{32}$$

and the effective 4-D cosmological constant is:

$$\Lambda_{\text{eff}} = \left[-\int \Lambda A(y)^2\,dy\right] + \left[\sum_i A(L_i)^2\sigma_i\right] + \left[\frac{1}{\kappa^2}\int\left(AA'' + (A')^2\right)\,dy\right] \tag{33}$$

$$+\left[\frac{1}{\kappa^2}\int\left(AA'' - \frac{1}{2}(A')^2\right)\,dy\right] + \left[-\frac{2}{\kappa^2}AA'|_{boundaries}\right]. \tag{34}$$

The five separate contributions to Λ_{eff} come from the bulk cosmological constant, Λ; the brane tension σ; the extrinsic curvature, $K_{\mu\nu}$, and its derivatives; the diagonal component of the Ricci tensor in the extra dimension, $R^{(5)}_{55}$; and the Hartle-Hawking term, for spaces with boundary. An integration by parts simplifies the last three terms, leaving:

$$\Lambda_{\text{eff}} = \left[-\int \Lambda A(y)^2\,dy\right] + \left[\sum_i A(L_i)^2\sigma_i\right] + \left[-\frac{3}{2\kappa^2}\int A'(y)^2\,dy\right] \tag{35}$$

$$\equiv \delta_\Lambda + \delta_\sigma + \delta_R. \tag{36}$$

For spaces without boundary, the contribution from extrinsic curvature vanishes, and the final term, δ_R, arises solely from $R^{(5)}_{55}$.

When $A(y)$ satisfies the Einstein equation (4), we can use Eq.(32) and the relation $H^2 = \kappa_N^2\Lambda_{\text{eff}}/3$ to obtain:

$$\begin{aligned}\delta_R &= -\frac{3}{2\kappa^2}\int A'(y)^2\,dy = -\frac{6H^2}{\kappa^2}\int A(y)\,dy - \int \Lambda A(y)^2\,dy \\ &= -\frac{6H^2}{\kappa_N^2} - \int \Lambda A(y)^2\,dy = -2\Lambda_{\text{eff}} - \int \Lambda A(y)^2\,dy.\end{aligned} \tag{37}$$

Following from Eqs.(35),(37), we see immediately that if Λ_{eff} is small, then δ_Λ and δ_R are approximately equal and of the same sign, while δ_σ is twice as large and of the opposite sign. If we now use the definitions $q_i \equiv \kappa^2\sigma_i/3$, $k_i^2 \equiv \kappa^2\Lambda_i/6$, and introduce:

$$X \equiv \sum_i q_i A(L_i)^2, \qquad Y \equiv \int (2k)^2 A(y)^2 \, dy \tag{38}$$

it then follows that the 4-D effective cosmological constant is $\kappa^2 \Lambda_{\rm eff} = X - Y$, and the separate contributions are:

$$\kappa^2 \delta_\Lambda = -\frac{3}{2} Y, \qquad \kappa^2 \delta_\sigma = 3X, \qquad \kappa^2 \delta_R = -2X + \frac{1}{2} Y. \tag{39}$$

As a simple illustration, consider the case shown in Fig.1, but where $q_1 = 0$ and $k_0 = k_1 = k_2 = k_3$ (i.e., zero brane charges). (Here we set $L_0 = 0$ and $L_2 - L_0 \equiv L$.) Using the bulk solution for the scale factor $A(y)$ (5), for $-y_0 < y < y_0$, with particle horizons at $\pm y_0$, the jump conditions are ($\xi \equiv e^{-2ky_0}$):

$$\frac{2k}{q_0} = \coth[k y_0] = \frac{1-\xi}{1+\xi} \tag{40}$$

$$\frac{2k}{q_2} = \coth[k(L - y_0)] = \frac{\xi - e^{-2kL}}{\xi + e^{-2kL}}. \tag{41}$$

For large brane separation, $q_0 L \gg 1$, these conditions imply that

$$k = \frac{q_0}{2} \left(1 - 2\xi + \mathcal{O}(\xi^2)\right) \tag{42}$$

$$\xi = \frac{q_0 + q_2}{q_2 - q_0} e^{-q_0 L} + \mathcal{O}(\xi^2). \tag{43}$$

We next need to calculate the quantities X and Y defined in Eq.(38):

$$X = q_0 A(0)^2 \tag{44}$$

$$Y = (2k)^2 A(0)^2 \int_{-y_0}^{y_0} \frac{\sinh^4[k(|y| - y_0)]}{\sinh^4[k y_0]} \, dy \tag{45}$$

$$= 2k A(0)^2 \left[1 - 4\xi + \mathcal{O}(\xi^2)\right] \tag{46}$$

$$= q_0 A(0)^2 \left[1 - 6 \frac{q_0 + q_2}{q_2 - q_0} e^{-q_0 L} + \cdots \right] \tag{47}$$

Note that the y integration is taken over the region between particle horizons which contains the q_0 brane (i.e. $-y_0 < y < y_0$). We can now list the various contributions to $\Lambda_{\rm eff}$:

$$\kappa^2 \delta_\Lambda = -\frac{3}{2} Y = A(0)^2 \left[-\frac{3}{2} q_0 + 9 q_0 \frac{q_0 + q_2}{q_2 - q_0} e^{-q_0 L} + \cdots \right] \tag{48}$$

$$\kappa^2 \delta_\sigma = 3X = A(0)^2 \left[3 q_0\right] \tag{49}$$

$$\kappa^2 \delta_R = -2X + \frac{1}{2} Y = A(0)^2 \left[-\frac{3}{2} q_0 - 3 q_0 \frac{q_0 + q_2}{q_2 - q_0} e^{-q_0 L} + \cdots \right] \tag{50}$$

$$\kappa^2 \Lambda_{\rm eff} = \kappa^2 \delta_\Lambda + \kappa^2 \delta_\sigma + \kappa^2 \delta_R = A(0)^2 \left[6 q_0 \frac{q_0 + q_2}{q_2 - q_0} e^{-q_0 L} + \cdots \right]. \tag{51}$$

To relate $\Lambda_{\rm eff}$ to H^2, we need:

$$\frac{\kappa^2}{\kappa_N^2} = \int A(y) = A(0)\int_{-y_0}^{y_0} \frac{\sinh^2[k(|y|-y_0)]}{\sinh^2[ky_0]}\,dy = \frac{A(0)}{k} + \cdots \tag{52}$$

Then $H^2 = (\kappa_N^2/\kappa^2)(\kappa^2\Lambda_{\rm eff})/3$, or:

$$\frac{H^2}{A(0)} = q_0^2 \frac{q_0+q_2}{q_2-q_0} e^{-q_0 L}. \tag{53}$$

This agrees with the appropriate limit of the first of Eqs.(23). This approach, derived from the 4-D effective action, can be used similarly in other brane configurations to obtain the 4-D effective cosmological constant.

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