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Asymptotics of QCD Factorization in Exclusive Hadronic Decays of B Mesons

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Abstract

Using the renormalon calculus, we study the asymptotic behavior of the perturbative expansion of the hard-scattering kernels entering the QCD factorization formula for the nonleptonic weak decays $\bar{B}^0 \to D^{(*)+} M^-$, where M is a light meson. In the "large-β_0 limit", the kernels are infrared finite and free of endpoint singularities to all orders of perturbation theory. The leading infrared renormalon singularity corresponding to a power correction of order $\Lambda_{\rm QCD}/m_b$ vanishes if the light meson has a symmetric light-cone distribution amplitude. We calculate the Borel transforms and the corresponding momentum distribution functions of the hard-scattering kernels, and resum the series of $O(\beta_0^{n-1}\alpha_s^n)$ corrections to explore the numerical significance of higher-order perturbative and power corrections. We also derive explicit expressions for the $O(\beta_0\,\alpha_s^2)$ contributions to the kernels, and for the renormalon singularities corresponding to power corrections of order $(\Lambda_{\rm QCD}/m_b)^2$. Finally, we study the limit $m_c \to 0$ relevant to charmless hadronic decays such as $B \to \pi\pi$.

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1 Introduction

The theoretical understanding of nonleptonic weak decays of hadrons is complicated by the intricate effects of strong interactions. Gluon exchange between the quarks is characterized by a multitude of relevant mass scales, ranging from the electroweak scale $\mu \sim M_W$ down to the confinement region $\mu \sim \Lambda_{\rm QCD}$, where perturbative methods fail. Recently, however, it was shown that this picture simplifies drastically for most two-body hadronic decays of B mesons into final states containing at least one fast, light meson [1, 2]. In the heavy-quark limit, the decay amplitudes for these processes factorize into a semileptonic form factor and a meson decay constant. So-called "nonfactorizable" corrections are predominantly perturbative and taken into account by convolutions of hard-scattering kernels with light-cone distribution amplitudes of the mesons. Corrections to this limit are suppressed by powers of $\Lambda_{\rm QCD}/m_b$.

Factorization as established in [1, 2] is a nontrivial property of the decay amplitudes that holds true to all orders of perturbation theory and to leading power in $\Lambda_{\rm QCD}/m_b$. For practical applications, it is important to obtain an estimate of the leading, power-suppressed corrections. Naive dimensional analysis suggests that such corrections are of order $\Lambda_{\rm QCD}/m_b \approx 10\%$; however, it is difficult to quantify this statement. The main obstruction is that, in contrast to simpler applications of the heavy-quark expansion to exclusive semileptonic decays or inclusive processes [3], the power corrections to factorization cannot be organized using a local operator product expansion. Whereas certain types of potentially large corrections were identified and estimated in [1], it is not (yet) possible to write down the complete set of leading power corrections in terms of field-theoretic objects.

The goal of the present work is to gain some insight into the structure of power corrections arising from soft, "nonfactorizable" gluon exchange. We use the renormalon calculus [4] to investigate the asymptotic behavior of the perturbation series for the hard-scattering kernels. These series are divergent and require power corrections of a certain pattern in order to be consistently defined. The Borel transforms of the perturbation series have singularities in the complex plane, whose structure indicates the power (via their position) and strength (via their residues) of these corrections. We study renormalon singularities by adopting the "large-β_0 limit" (corresponding to the exchange of a single renormalon chain), in which nontrivial partial resummations of perturbation series become possible. This method provides the correct location of the renormalon singularities, but only approximately accounts for their residues.

To illustrate our approach, we recall the example of the Adler D-function, i.e. the Euclidean correlator of two vector currents. For this quantity, the leading infrared (IR) renormalon singularity arising from soft gluon exchange indicates a power correction of order $\Lambda_{\rm QCD}^4/Q^4$, which has the same momentum dependence as the contribution of the gluon condensate. A consistent definition of the perturbation series with an accuracy of order $1/Q^4$ or better therefore requires inclusion of the gluon condensate, and the value of the condensate depends on the resummation prescription (e.g., principal-value Borel summation, truncation at the minimal term, etc.) adopted to define the divergent perturbation series [5]. Although the presence of a $1/Q^4$ power correction proportional to the gluon condensate can be inferred from the operator product expansion, it is remarkable that the perturbation expansion itself signals the existence of this nonperturbative effect through its divergent large-order behavior. The correspondence of renormalon singularities and power corrections of (some) higher-dimensional operators in the operator product expansion has also been verified with several explicit examples in the context of the heavy-quark expansion [6, 7, 8].

Building on this experience, the renormalon calculus has been applied to observables that do not admit an expansion in local operators. Examples are event-shape variables in e^+e^- annihilation [9, 10, 11, 12, 13], Drell–Yan production [13, 14], fragmentation functions [15, 16], and structure functions in deep-inelastic scattering [17, 18]. In these cases, there is no systematic framework that would allow us to classify the power corrections in terms of operator matrix elements. However, the pattern of renormalon singularities determines at least a minimal set of power corrections that must be included for a consistent field-theoretic description. Although, in general, these are not the only sources of power-suppressed effects, the inclusion of the corrections corresponding to the leading IR renormalons significantly improves the phenomenological predictions. Interesting attempts to formalize this method include the dispersive approach developed by Dokshitzer, Marchesini and Webber [19], and the non-local operator method of Korchemsky and Sterman [20].

Our analysis will be similar in spirit to these approaches in that we will use the renormalon calculus to obtain a minimal model of power corrections to factorization in hadronic B decays. We will also derive explicit results for the presumably dominant part of the two-loop perturbative contributions to the hard-scattering kernels, i.e. the contributions of order $\beta_0\,\alpha_s^2$. For simplicity, we focus mainly on the class-1 B decays into a heavy–light final state such as $\bar{B}^0 \to D^{(*)+}M^-$, where $M = \pi, \rho, \dots$ is a light meson. For these processes, the factorization formula takes its simplest form and is best established theoretically.

The effective weak Hamiltonian is (considering $b \to c\bar{u}d$ transitions for concreteness)

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}}\,V_{cb}V_{ud}^*\left[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)\right], \tag{1}$$

where

$$\begin{aligned} O_1 &= \bar{d}_\alpha\gamma_\mu(1-\gamma_5)u_\alpha\,\bar{c}_\beta\gamma^\mu(1-\gamma_5)b_\beta\,, \\ O_2 &= \bar{d}_\alpha\gamma_\mu(1-\gamma_5)u_\beta\,\bar{c}_\beta\gamma^\mu(1-\gamma_5)b_\alpha \end{aligned} \tag{2}$$

are local four-quark operators, summed over color indices α, β. The Wilson coefficients $C_i(\mu)$ contain short-distance corrections arising from gluon exchange with virtualities between the electroweak scale M_W and the renormalization scale $\mu \sim m_b$. The hadronic matrix elements of the operators $O_i(\mu)$ greatly simplify in the heavy-quark limit $m_b \gg \Lambda_{\rm QCD}$. To leading power in $\Lambda_{\rm QCD}/m_b$, and to all orders of perturbation theory, they can be expressed in the factorized form [1]

$$
\begin{aligned}
&\langle M^- D^{(*)+} |\, O_i(\mu)\, | \bar{B}^0 \rangle \\
&= \int_0^1 dx\, \Phi_M(x,\mu) \Big[T_{iL}(x,\mu)\, \langle M^- |\, \bar{d}\gamma_\mu(1-\gamma_5)u \,|\, 0 \,\rangle \langle D^{(*)+} |\, \bar{c}\gamma^\mu(1-\gamma_5)b \,| \bar{B}^0 \rangle \\
&\qquad + T_{iR}(x,\mu)\, \langle M^- |\, \bar{d}\gamma_\mu(1-\gamma_5)u \,|\, 0 \,\rangle \langle D^{(*)+} |\, \bar{c}\gamma^\mu(1+\gamma_5)b \,| \bar{B}^0 \rangle \Big] . \qquad (3)
\end{aligned}
$$

$\Phi_M(x,\mu)$ is the leading-twist light-cone distribution amplitude of the light meson M, normalized such that $\int_0^1 dx\, \Phi_M(x,\mu) = 1$, and x is the longitudinal momentum fraction of the d quark in the meson. The current matrix elements on the right-hand side of the factorization formula can be expressed in terms of $\bar{B} \to D^{(*)}$ semileptonic form factors and the decay constant f_M of the meson M. In the heavy-quark limit, "nonfactorizable" strong-interaction effects are dominated by hard gluon exchange and included in the hard-scattering kernels $T_i(x,\mu)$. In [1], these kernels were calculated explicitly at one-loop order and were shown to be free of IR divergences at the two-loop order.

The goal of the present work is to estimate the higher-order corrections in the loop expansion of the hard-scattering kernels and, at the same time, to gain some insight into the structure of power corrections to factorization. In Section 2 we derive expressions for the Borel transforms of the hard-scattering kernels using the single renormalon-chain approximation. We present resummation formulae for the kernels in the large-β_0 limit and establish several important results: the systematic cancellation of IR divergences to all orders of perturbation theory, the absence of endpoint singularities in the kernels, and the vanishing of the leading renormalon singularity (corresponding to a first-order power correction) in the case of a symmetric light-cone distribution amplitude. In Section 3 we study higher-order contributions in the perturbative expansion of the kernels. We derive exact results for the terms of order $\beta_0\, \alpha_s^2$, and give all-order expressions for the anomalous dimensions of the four-quark operators in the large-β_0 limit. We also investigate numerically the asymptotic behavior of the perturbation series for the hard-scattering kernels. A systematic study of IR renormalons and power corrections is performed in Section 4. We present explicit results for the first two IR renormalon singularities corresponding to power corrections of order $\Lambda_{\rm QCD}/m_b$ and $(\Lambda_{\rm QCD}/m_b)^2$. In Section 5 we study the limit $m_c \to 0$ and derive results valid for massless quarks in the final-state. They are relevant to charmless decays such as $B \to \pi\pi$.

2 Borel transforms and distribution functions

The perturbation series for the hard-scattering kernels $T_i(x,\mu)$ in the factorization formula (3) can be arranged as

$$
T_i(x,\mu) = t_i^{(0)} + \sum_{\ell=1}^{\infty} \sum_{n=0}^{\ell-1} t_i^{(n,\ell)}(x,\mu)\, \beta_0^n \alpha_s^\ell(\mu) \,, \qquad (4)
$$

where ℓ is the number of loops, and $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ is the first coefficient of the β-function for an SU(N_c) gauge theory ($C_A = N_c$ and $T_F = \frac{1}{2}$) with n_f light quark flavors. The tree-level coefficients are $t_{1L}^{(0)} = 1$, $t_{2L}^{(0)} = 1/N_c$, and $t_{iR}^{(0)} = 0$. Our goal is to sum the terms of order $\beta_0^{\ell-1}\alpha_s^\ell$ to all orders of perturbation theory. In other words, we consider the limit of large β_0 for fixed $\beta_0\,\alpha_s$ and calculate the kernels $T_i(x,\mu)$ to order $1/\beta_0$, neglecting terms of order $1/\beta_0^2$ and higher. Strictly speaking, there is no sensible limit of QCD in which β_0 may be considered a large parameter (in particular, this is not the large-N_c limit) – except, perhaps, taking $n_f \to -\infty$. Nevertheless, retaining only the leading terms in $1/\beta_0$ often gives a good approximation to exact multi-loop results (see, e.g., [21]), in particular in cases when there is a nearby IR renormalon [22].

2.1 Borel summation in the large-β_0 limit

The coefficients $t_i^{(\ell-1,\ell)}$ of the terms with the highest degree of β_0 in (4) are determined by diagrams with $(\ell-1)$ light-quark loops, which are rather straightforward to calculate. We work in dimensional regularization with $4-2\varepsilon$ space-time dimensions and adopt the $\overline{\rm MS}$ subtraction scheme. At first order in $1/\beta_0$, coupling-constant renormalization is accomplished by ($\bar\mu^2 = \mu^2 e^\gamma/4\pi$)

$$\frac{\beta_0\,g_s^2}{(4\pi)^2} = \frac{\bar\mu^{2\varepsilon}\,b(\mu)}{1+b(\mu)/\varepsilon}\,, \qquad b(\mu) = \frac{\beta_0\,\alpha_s(\mu^2)}{4\pi} = \frac{1}{\ln(\mu^2/\Lambda_{\overline{\rm MS}}^2)}\,. \tag{5}$$

To leading order in the large-β_0 limit the "nonfactorizable" vertex corrections to the operator O_1 in (2) vanish when projected onto color-singlet meson states, i.e.

$$T_{1L}(x,\mu) = 1 + O(1/\beta_0^2)\,, \qquad T_{1R}(x,\mu) = O(1/\beta_0^2)\,. \tag{6}$$

The perturbation series for the kernels corresponding to the operator O_2 can be written as

$$\begin{aligned} T_{2L}(x,\mu) &= \frac{1}{N_c}\left[1 + \frac{2C_F}{\beta_0}\sum_{\ell=1}^\infty \frac{F_L(\varepsilon,\ell\varepsilon)}{\ell}\Big(\frac{b}{\varepsilon+b}\Big)^\ell - (\text{UV subtractions}) + O(1/\beta_0^2)\right], \\ T_{2R}(x,\mu) &= \frac{1}{N_c}\left[\frac{2C_F}{\beta_0}\sum_{\ell=1}^\infty \frac{F_R(\varepsilon,\ell\varepsilon)}{\ell}\Big(\frac{b}{\varepsilon+b}\Big)^\ell + O(1/\beta_0^2)\right], \end{aligned} \tag{7}$$

where $b = b(\mu)$. The kernel T_{2R} corresponding to the right-handed (i.e. chirality-flipped) heavy-quark current in (3) is ultraviolet (UV) finite in the large-β_0 limit and does not require $\overline{\rm MS}$ subtractions.

The functions $F_i(\varepsilon,u)$ are regular at $\varepsilon = u = 0$. Expanding $F_i(\varepsilon,u)$ in powers of ε and u, and $[b/(\varepsilon+b)]^\ell$ in powers of b/ε, gives a quadruple sum in (7). Combinatoric identities relate the $1/\varepsilon$ terms, and hence the anomalous dimensions of the local four-quark operators O_1 and O_2, to the Taylor coefficients of $F_L(\varepsilon,0)$ [23]. Using the well-known fact that the linear combinations $O_\pm = O_1 \pm O_2$ are multiplicatively renormalized,

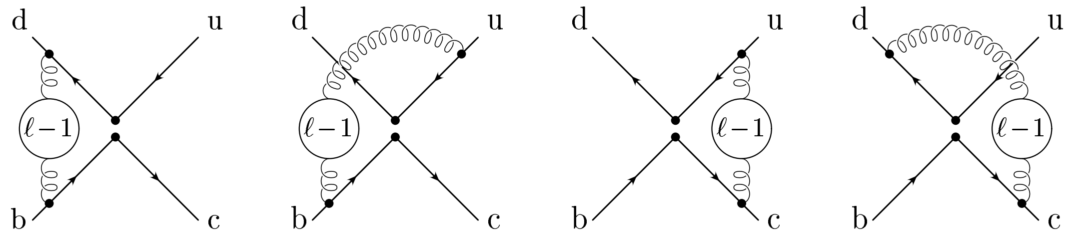

Figure 1: "Nonfactorizable" vertex diagrams contributing to the hard-scattering kernels in the factorization formula. The split vertex represents an insertion of one of the four-quark operators O_i. The circles represent $(\ell-1)$ insertions of light-quark loops on the gluon propagators.

we find that the corresponding anomalous dimensions $\gamma_\pm$ are given by

$$\gamma_\pm = \pm\frac{N_c \mp 1}{N_c}\,\frac{(-2b)}{\beta_0}\,F_L(-b,0) + O(1/\beta_0^2)\,. \tag{8}$$

The finite terms, which determine the kernels themselves, receive contributions from the Taylor coefficients of $F_L(\varepsilon,0)$ and $F_i(0,u)$ [24]. We obtain

$$\begin{aligned}
N_c\,T_{2L}(x,\mu) &= 1 + \frac{2C_F}{\beta_0}\int\limits_{-b(\mu)}^{0} d\varepsilon\,\frac{F_L(\varepsilon,0)-F_L(0,0)}{\varepsilon} \\
&\quad + \frac{2C_F}{\beta_0}\int\limits_0^\infty du\,e^{-u/b(\mu)}\,\frac{F_L(0,u)-F_L(0,0)}{u} + O(1/\beta_0^2)\,, \\
N_c\,T_{2R}(x,\mu) &= \frac{2C_F}{\beta_0}\int\limits_0^\infty du\,e^{-u/b(\mu)}\,\frac{F_R(0,u)}{u} + O(1/\beta_0^2)\,.
\end{aligned} \tag{9}$$

If they were unambiguously defined, these expressions would determine the Borel-resummed perturbation series for the hard-scattering kernels in the large-β_0 limit. However, the presence of singularities located on the integration contour renders the integrals over u ambiguous. These (renormalon) singularities will provide us with information about power corrections to the factorization formula (3).

The functions $F_L(\varepsilon,u)$ and $F_R(\varepsilon,u)$ are obtained by computing the "nonfactorizable" vertex corrections shown in Figure 1. It is convenient to introduce the abbreviations

$$z = \frac{m_c}{m_b}\,, \qquad \delta = (1-z^2)\,x\,, \qquad \bar\delta = (1-z^2)(1-x)\,. \tag{10}$$

The variable δ ($\bar\delta$) appears in the calculation of the first (second) diagram in Figure 1. The third and fourth diagrams are obtained using crossing symmetry and require the substitutions

$$m_b \to m_c\,, \quad z \to \frac{1}{z}\,, \quad x \to (1-x)\,, \quad \delta \to -\frac{\bar\delta}{z^2} - i\epsilon\,, \quad \bar\delta \to -\frac{\delta}{z^2} - i\epsilon\,. \tag{11}$$

These contributions, to which we will refer as "crossed terms", contain imaginary parts (specified by the $i\epsilon$ prescriptions), which determine the strong-interaction phases of the decay amplitudes. From now on we will omit the $i\epsilon$'s; they can be reinstated by recalling that $z^2 \equiv z^2 - i\epsilon$. The results are

$$\begin{aligned} F_L(\varepsilon,u) &= \left(\frac{\mu}{m_b}\right)^{2u} e^{\gamma\varepsilon}\,[D(\varepsilon)]^{u/\varepsilon-1}\Big[A_1(\delta,\varepsilon,u) - A_2(\bar{\delta},\varepsilon,u)\Big] + \text{crossed terms}\,, \\ F_R(\varepsilon,u) &= \left(\frac{\mu}{m_b}\right)^{2u} e^{\gamma\varepsilon}\,[D(\varepsilon)]^{u/\varepsilon-1}\,z\,B_1(\delta,\varepsilon,u) + \text{crossed terms}\,, \end{aligned} \tag{12}$$

where

$$\begin{aligned} A_1(r,\varepsilon,u) &= (1-\varepsilon)^2\,(r\,g_{11} + g_{20}) + u\,(g_{10} - g_{20}) \\ &\quad - ur\,(g_{00} - g_{01} - g_{10} + g_{11})\,, \\ A_2(r,\varepsilon,u) &= (4-\varepsilon-\varepsilon^2)(r\,g_{11} + g_{20}) + u\,(g_{10} - 2g_{20}) \\ &\quad - ur\,(g_{00} - g_{01} - g_{10} + 2g_{11})\,, \\ B_1(r,\varepsilon,u) &= -u\,g_{20}\,, \end{aligned} \tag{13}$$

and

$$g_{mn} \equiv g_{mn}(r,\varepsilon,u) = \frac{\Gamma(m+n-2u)\,\Gamma(1+u)}{\Gamma(1+m+n-u-\varepsilon)} \int_0^1 dy\, \frac{y^m(1-y)^n}{[y(r+(1-r)y)]^{1+u}}\,. \tag{14}$$

The integral can be expressed in terms of incomplete Euler β-functions. The notation in (12) is such that the functions A_1 and B_1 correspond to the contributions of the first diagram, whereas A_2 is obtained from the second diagram. The linear factor of z in front of B_1 comes from the chirality flip on the charm-quark line. The contributions from the third and fourth diagrams are obtained by performing the substitutions shown in (11). Finally, the function

$$D(\varepsilon) = 6e^{\gamma\varepsilon}\,\frac{\Gamma(1+\varepsilon)\,\Gamma^2(2-\varepsilon)}{\Gamma(4-2\varepsilon)} = 1 + \frac{5}{3}\,\varepsilon + O(\varepsilon^2) \tag{15}$$

is related to the contribution of a light-quark loop to the gluon self-energy.

In the limit $u \to 0$ most of the terms in (13) vanish. However, the functions g_{00} and g_{01} have poles at $u = 0$, corresponding to soft and collinear IR singularities of the individual diagrams. Working out the residues we find

$$F_L(\varepsilon,0) = e^{\gamma\varepsilon}\,[D(\varepsilon)]^{-1}\left[-\frac{(3-2\varepsilon)(1+\varepsilon)}{\Gamma(3-\varepsilon)} + \frac{1}{\Gamma(1-\varepsilon)}\,\ln\frac{x}{1-x}\right] + \text{crossed terms}\,. \tag{16}$$

The logarithm corresponds to a remaining collinear singularity in the sum of the first two diagrams, and cancels when adding the crossed terms [1]. As a consequence, only the first term in the square brackets, which comes from the first term in the expressions for A_1 and A_2 in (13), remains. This term depends on the renormalization scheme. Specifically, the ε-dependent factors in (13) follow from the Dirac identities

$$\gamma_\mu\gamma_\nu\gamma_\alpha(1-\gamma_5)\otimes\gamma^\alpha\gamma^\nu\gamma^\mu(1-\gamma_5) = 4(1-\varepsilon)^2\,\gamma_\rho(1-\gamma_5)\otimes\gamma^\rho(1-\gamma_5)\,,$$
$$\gamma_\mu\gamma_\nu\gamma_\alpha(1-\gamma_5)\otimes\gamma^\mu\gamma^\nu\gamma^\alpha(1-\gamma_5) = 4(4-\varepsilon-\varepsilon^2)\,\gamma_\rho(1-\gamma_5)\otimes\gamma^\rho(1-\gamma_5)\,, \qquad (17)$$

which are valid in the so-called "naive dimensional regularization" (NDR) scheme with anticommuting γ_5, and with a projection of evanescent operators as specified in eq. (4.3) of [25]. The scheme-dependence cancels against that of the Wilson coefficients in the effective weak Hamiltonian. Using the expression for $D(\varepsilon)$ from (15), we obtain

$$F_L(\varepsilon,0) = -\frac{(3-2\varepsilon)(1+\varepsilon)\,\Gamma(4-2\varepsilon)}{3\Gamma(1+\varepsilon)\,\Gamma^2(2-\varepsilon)\,\Gamma(3-\varepsilon)}\,, \qquad F_L(0,0)=-3\,, \qquad (18)$$

and $F_R(\varepsilon,0)=0$. In the opposite limit $\varepsilon\to 0$ we find

$$F_L(0,u) = \left(\frac{\mu}{m_b}\right)^{2u} e^{5u/3}\left[A_1(\delta,0,u)-A_2(\bar\delta,0,u)\right] + \text{crossed terms}\,,$$
$$F_R(0,u) = \left(\frac{\mu}{m_b}\right)^{2u} e^{5u/3}\,z\,B_1(\delta,0,u) + \text{crossed term}\,. \qquad (19)$$

We now rewrite the resummed expressions (9) for the kernels in a more convenient form. The μ-dependent factor in the expressions for $F_i(0,u)$ in (19) combines with the exponential $e^{-u/b(\mu)}$ in (9) into the renormalization-scheme invariant combination

$$e^{-u/b(\mu)}\left(\frac{\mu}{m_b}\right)^{2u} e^{5u/3} = \left(\frac{\Lambda_{\overline{\rm MS}}}{m_b}\right)^{2u} e^{5u/3} \equiv \left(\frac{\Lambda_{\rm V}}{m_b}\right)^{2u} = e^{-u/b_V(m_b)}\,, \qquad (20)$$

where $\Lambda_{\rm V}=e^{5/6}\Lambda_{\overline{\rm MS}}$ is the QCD scale parameter in the so-called "V scheme" [26], and

$$b_V(m_b) = \frac{1}{\ln(m_b^2/\Lambda_{\rm V}^2)} = \frac{\beta_0\,\alpha_s^{(V)}(m_b^2)}{4\pi} = \frac{\beta_0\,\alpha_s(e^{-5/3}m_b^2)}{4\pi} \qquad (21)$$

is proportional the running coupling constant in that scheme. Here, as always, $\alpha_s(\mu^2)$ without a label "V" is the coupling constant in the $\overline{\rm MS}$ scheme. We define the Borel transform of the hard-scattering kernel T_{2R} as

$$S_R(u,x) \equiv e^{-5u/3}\,\frac{F_R(0,u)}{u}\bigg|_{\mu=m_b} = z\,\frac{B_1(\delta,0,u)}{u} + z^{-1-2u}\,\frac{B_1(-\bar\delta/z^2,0,u)}{u}\,. \qquad (22)$$

The factor z^{-2u} in front of the crossed term comes from the fact that the scale factor associated with this term is $(\mu/m_c)^{2u} = z^{-2u}\,(\mu/m_b)^{2u}$. With this definition, the Borel integral representation for the kernel becomes

$$N_c\,T_{2R}(x,\mu) = \frac{2C_F}{\beta_0}\int_0^\infty du\,e^{-u/b_V(m_b)}\,S_R(u,x) + O(1/\beta_0^2)\,. \qquad (23)$$

For the kernel T_{2L}, we define the Borel transform as

$$S_L(u,x) \equiv \frac{e^{-5u/3}}{u}\left[F_L(0,u)\Big|_{\mu=m_b} - F_L^{(12)}(0,0) - z^{-2u}\,F_L^{(34)}(0,0)\right] \tag{24}$$

$$= \frac{1}{u}\left[A_1(\delta,0,u) - A_2(\bar\delta,0,u) + e^{-5u/3}\left(\frac{3}{2} - \ln\frac{x}{1-x}\right) + z^{-2u}\times(\text{crossed terms})\right].$$

Here $F_L^{(12)}(0,0) = -\frac{3}{2} + \ln\frac{x}{1-x}$ and $F_L^{(34)}(0,0) = -\frac{3}{2} - \ln\frac{x}{1-x}$ correspond to the contributions of the only first two and the last two diagrams in Figure 1, respectively, as can be seen by taking the limit $\varepsilon \to 0$ in (16). With the above definition, the two parts of the Borel transform corresponding to each pair of diagrams are separately free of UV and IR singularities at $u = 0$, and the result for the kernel takes the form

$$N_c\,T_{2L}(x,\mu) = 1 + \frac{2C_F}{\beta_0}\int\limits_{-b(\mu)}^{0}\frac{d\varepsilon}{\varepsilon}\left[3 - \frac{(3-2\varepsilon)(1+\varepsilon)\Gamma(4-2\varepsilon)}{3\Gamma(1+\varepsilon)\,\Gamma^2(2-\varepsilon)\,\Gamma(3-\varepsilon)}\right]$$
$$+\frac{2C_F}{\beta_0}\left[3\ln\frac{b(\mu)}{b(m_b)} - \left(\frac{3}{2} + \ln\frac{x}{1-x}\right)\ln\frac{b(m_c)}{b(m_b)}\right]$$
$$+\frac{2C_F}{\beta_0}\int\limits_0^\infty du\,e^{-u/b_V(m_b)}\,S_L(u,x) + O(1/\beta_0^2)\,. \tag{25}$$

In the above expressions for the hard-scattering kernels, the Borel integrals (the integrals over u) contain all nontrivial information about the asymptotic divergence of the perturbation expansions. They are defined in a renormalization-scheme invariant way. So-called IR renormalon singularities located on the integration contour along the positive real u-axis render these integrals ill-defined, and hence the perturbation series are not Borel summable [27, 28, 29]. In the large-β_0 limit, renormalon singularities show up as poles at half-integer values of u, as seen from the explicit form of the functions g_{mn} in (14). A pole singularity located at $u = k/2$ corresponds to a factorial growth $t_i^{(\ell-1,\ell)} \sim (k/2)^{-\ell}\,(\ell-1)!$ of the perturbative expansion coefficients in (4), and leads to an irreducible ambiguity of order $(\Lambda_{\rm QCD}/m_b)^k$ in the definition of the resummed series. From the expression for g_{mn} it follows that for $n = 0$ these functions have single poles at $u = \frac{1}{2}$. Working out the residues we obtain

$$S_L(u,x) \stackrel{u\to\frac{1}{2}}{=} -\frac{4}{1-z}\left(\frac{1}{x} - \frac{1}{1-x}\right)\frac{1}{1-2u} + \text{regular terms}\,, \tag{26}$$

whereas $S_R(u,x)$ is regular at $u = \frac{1}{2}$. A pole at $u = \frac{1}{2}$ corresponds to a power correction of first order in $\Lambda_{\rm QCD}/m_b$, which could be large for realistic heavy-quark masses. However, because the residue of the pole in (26) is antisymmetric under exchange of $x \leftrightarrow (1-x)$, this contribution vanishes for the case of a symmetric light-cone distribution amplitude. Therefore, the renormalon analysis does not indicate a first-order power correction for decays such as $\bar B^0 \to D^{(*)+}\pi^-$ or $\bar B^0 \to D^{(*)+}\rho^-$. A more detailed investigation of power corrections will be presented in Section 4.

2.2 Momentum distribution functions

The analysis of higher-order renormalon singularities and the numerical evaluation of the Borel integrals in the form of (23) and (25) are difficult because of the complicated structure of the Borel transforms $S_i(u,x)$. It is advantageous to rewrite the Borel integrals as integrals over a running coupling constant multiplied by weight functions, using the formalism developed in [22]:

$$\frac{1}{\beta_0}\int\limits_0^\infty du\, e^{-u/b_V(m_b)}\, S_i(u,x) = \int\limits_0^\infty \frac{d\tau}{\tau}\, w_i(\tau,x)\, \frac{\alpha_s(\tau\, e^{-5/3}\, m_b^2)}{4\pi}\,. \qquad (27)$$

This elucidates the connection between IR (UV) renormalons and small-momentum (large-momentum) contributions in Feynman diagrams. The functions $w_i(\tau,x)$ determine the distribution of the gluon virtualities inside the vertex-correction diagrams in Figure 1. Technically, they are the inverse Mellin transforms of the Borel images $S_i(u,x)$. The running coupling constant under the integral on the right-hand side is nothing but the coupling $\alpha_s^{(V)}(\tau\, m_b^2)$ in the V scheme, as defined in (21). Equation (27) has been established in the literature as an integral representation of the Borel sum for quantities defined in Euclidean kinematics [22]. It is applied here for the first time to a situation in which the quantity of interest has a more complicated analytic structure and a non-vanishing dispersive part. We have checked numerically that the two sides in (27) agree even if the imaginary part is nonzero.

The distribution functions $w_i(\tau,x)$ can be computed in terms of Feynman parameter integrals starting from the relation [22]

$$S_i(u,x) = \int\limits_0^\infty \frac{d\tau}{\tau}\, w_i(\tau,x)\, \tau^{-u}\,, \qquad (28)$$

which is valid for $-1 < \mathrm{Re}\, u < \frac{1}{2}$. The result is

$$\begin{aligned} w_L(\tau,x) &= f_L(\tau,\delta,\bar\delta) + f_L(\tau/z^2, -\bar\delta/z^2, -\delta/z^2)\,, \\ w_R(\tau,x) &= z\, f_R(\tau,\delta) + \frac{1}{z}\, f_R(\tau/z^2, -\bar\delta/z^2)\,, \end{aligned} \qquad (29)$$

where the first (second) terms on the right-hand side correspond to the first (last) two diagrams in Figure 1. We find

$$\begin{aligned} f_L(\tau,\delta,\bar\delta) &= \tau\Big(\frac{1+\eta}{2\delta} - \frac{\eta}{\delta^2}\Big) - \frac{\tau}{\delta}\Big(2 - \tau\,\frac{1-\delta}{\delta^2}\Big) \ln\Big(1 + \frac{\eta\,\delta}{\tau}\Big) + \frac{3}{2}\,\theta(\tau\, e^{-5/3} - 1) \\ &\quad + \bigg[-\frac{\eta}{\delta} - \Big(1 - \frac{\tau}{\delta^2}\Big) \ln\Big(1 + \frac{\eta\,\delta}{\tau}\Big) + \ln\delta\cdot\theta(1 - \tau\, e^{-5/3}) - \{\delta\to\bar\delta\}\bigg]\,, \\ f_R(\tau,\delta) &= \tau\bigg[-\frac{1-\eta}{2\delta} + \frac{\eta}{\delta^2} - \frac{\tau}{\delta^3}\ln\Big(1 + \frac{\eta\,\delta}{\tau}\Big)\bigg]\,, \end{aligned} \qquad (30)$$

where $\theta(x)$ is the step function, and

$$\eta = \frac{\tau}{2}\left(\sqrt{1+\frac{4}{\tau}}-1\right). \tag{31}$$

Note that the terms inside the square brackets in the expression for f_L are antisymmetric in $x \leftrightarrow (1-x)$ and thus vanish for the case of a symmetric light-cone distribution amplitude.

The above expressions for the distribution functions $w_i(\tau,x)$ are a central result of this work. They will allow us to extract information about renormalon singularities (and hence the structure of power corrections) as well as about terms of arbitrarily high order in perturbation theory. Let us now stress some important general properties of these functions.

IR cancellations: Expanding the distribution functions for small τ shows that they vanish at least as fast as $\sqrt{\tau}$ as $\tau \to 0$, and therefore the integrals in (27) are convergent in the IR region. As a consequence, the resummed expressions for the hard-scattering kernels in (23) and (25) are IR finite. This is a nontrivial result of our analysis, which demonstrates the IR finiteness of the hard-scattering kernels (and hence factorization) to all orders of perturbation theory in the large-β_0 limit. (After $\overline{\mathrm{MS}}$ subtractions, accomplished by the term proportional to $\theta(\tau\, e^{5/3}-1)$ in the expression for $w_L(\tau,x)$, the distribution functions vanish like $1/\tau$ for large τ, and hence the integrals are also UV convergent.)

Endpoint behavior: The proof of IR finiteness of the hard-scattering kernels alone does not establish factorization; in addition, one must show that the convolutions of the hard-scattering kernels with the light-cone distribution amplitude of the meson M are convergent. In [1], an explicit calculation showed that at one-loop order the hard-scattering kernels tend to a constant (modulo integrable endpoint logarithms) for $x \to 0$ or $x \to 1$. We find that this property persists for the resummed perturbation series in the large-β_0 limit. Therefore, the integrals over the light-cone distribution amplitude $\Phi_M(x)$ are convergent at the endpoints. (Endpoint singularities at least as strong as $1/x^2$ or $1/(1-x)^2$ would be required to spoil factorization.) We define

$$\begin{aligned}
W_L(\tau) &= \int_0^1 dx\, \Phi_M(x)\, w_L(\tau,x) = F_L(\tau, 1-z^2) + F_L(\tau/z^2, 1-1/z^2)\,, \\
W_R(\tau) &= \int_0^1 dx\, \Phi_M(x)\, w_R(\tau,x) = z\, F_R(\tau, 1-z^2) + \frac{1}{z}\, F_R(\tau/z^2, 1-1/z^2)\,,
\end{aligned} \tag{32}$$

where the functions F_i are the convolutions of f_i with the light-cone distribution amplitude. As an example, we perform these convolutions adopting the asymptotic form

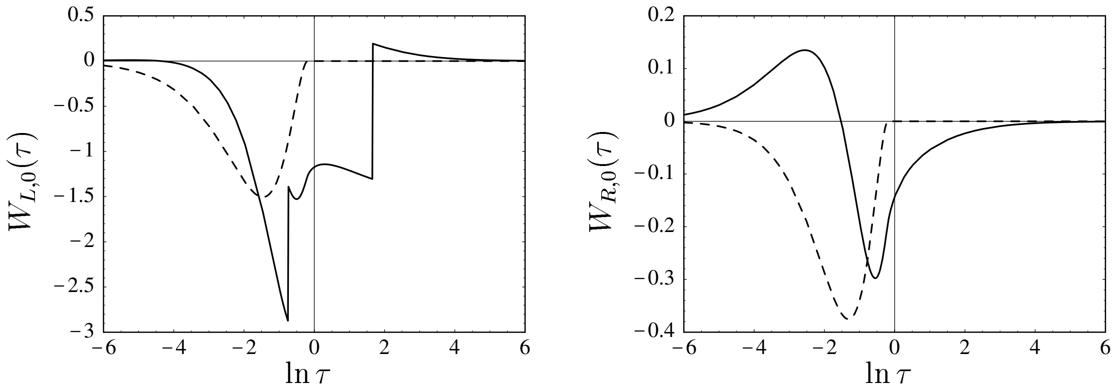

Figure 2: Integrated distribution functions $W_{i,0}(\tau)$ obtained using the asymptotic light-cone distribution amplitude. Solid (dashed) lines show the real (imaginary) parts.

$\Phi_0(x) = 6x(1-x)$ of the light-cone distribution amplitude, valid for light pseudoscalar and vector mesons in the limit $\mu \to \infty$. This yields

$$\begin{aligned} F_{L,0}(\tau,d) &= \frac{3\tau}{2d}\Big(7+\eta+\frac{12\eta}{d}\Big) - \frac{6\tau}{d}\Big(1+\frac{\tau(d+2)}{d^2}+\frac{3\eta}{d}\Big)\ln\Big(1+\frac{\eta\,d}{\tau}\Big) \\ &\quad + \frac{6\tau^2(1+d)}{d^3}\,\mathrm{Li}_2\Big(-\frac{\eta\,d}{\tau}\Big) + \frac{3}{2}\,\theta(\tau\,e^{-5/3}-1)\,, \\ F_{R,0}(\tau,d) &= \frac{3\tau}{2d}\Big(-1+\eta-\frac{8\eta}{d}\Big) + \frac{6\tau^2}{d^3}\Big(1+\frac{d\eta}{\tau}\Big)\ln\Big(1+\frac{\eta\,d}{\tau}\Big) - \frac{6\tau^2}{d^3}\,\mathrm{Li}_2\Big(-\frac{\eta\,d}{\tau}\Big)\,, \end{aligned} \tag{33}$$

where $\mathrm{Li}_2(x) = -\int_0^x (dt/t)\ln(1-t)$ is the dilogarithm, and the subscript "0" refers to the asymptotic distribution amplitude. In Figure 2, we show the real and imaginary parts of the functions $W_{i,0}(\tau)$ for the case where $z = m_c/m_b = 0.3$. Note that the steps in the real part of the left-handed kernel are an artifact of the $\overline{\mathrm{MS}}$ subtractions applied to the Borel transforms.

Strong-interaction phases: The hard-scattering kernels contain imaginary parts due to gluon exchange among the final-state quarks in the third and fourth diagrams in Figure 1. These imaginary parts determine the strong-interaction phases of the decay amplitudes in the heavy-quark limit. They are obtained from the branch cuts of the logarithms in the crossed terms in (29). Imaginary parts exist for $\tau < \delta^2/(z^2+\delta)$ or $\tau < \bar\delta^2/(z^2+\bar\delta)$, which in any case implies that $\tau < (1-z^2)^2$.

Perturbative expansion: If the running coupling constant under the integral in (27) is expanded in powers of a fixed coupling constant using

$$\alpha_s(\tau\,e^{-5/3}\,m_b^2) = \alpha_s(\mu^2)\sum_{n=0}^{\infty}\Big(\frac{5}{3}-\log\frac{m_b^2}{\mu^2}-\ln\tau\Big)^n\Big(\frac{\beta_0\,\alpha_s(\mu)}{4\pi}\Big)^n\,, \tag{34}$$

the integral reproduces the perturbative expansion in powers of $\alpha_s(\mu^2)$. Computing the first n terms of the perturbation series in the large-β_0 limit requires evaluating the integrals $\int_0^\infty (d\tau/\tau)(\ln\tau)^{n-1} w_i(\tau,x)$ over the distribution functions. To perform these integrals analytically, it is convenient to change variables from τ to η. Below, we derive explicit expressions for the hard-scattering kernels at order $\beta_0\,\alpha_s^2$.

The shapes of the distribution functions determine the momentum regions from which there arise important contributions to the Borel integrals. As a first indicator, one may consider the average value of $\ln\tau$, which determines the so-called BLM scale [26] through $\mu_{\rm BLM}^2 = e^{\langle\ln\tau\rangle}\,m_b^2$. (This relation holds in the V scheme. In the $\overline{\rm MS}$ scheme, the BLM scale $\mu_{\rm BLM}$ is reduced by a factor $e^{-5/6}$.) As explained in [22], a BLM scale is meaningful only in cases where the quantity of interest is renormalization-group invariant and has a distribution function of definite sign. In our case, this means that we can introduce BLM scales only for the imaginary parts of the kernels. The dashed curves in Figure 2 show that these scales are significantly smaller than m_b. Specifically, we find $\mu_{\rm BLM}\approx 0.33 m_b$ ($0.39 m_b$) for the imaginary part of the left-handed (right-handed) kernel. When transformed into the $\overline{\rm MS}$ scheme, the corresponding scales are of order 0.6–0.7 GeV. This suggests that the imaginary parts of the kernels, and hence the strong-interaction phases of the decay amplitudes, are susceptible to nonperturbative physics.

Renormalon ambiguities and power corrections: The integrals over the distribution functions in (27) are ambiguous because of the Landau pole in the running coupling constant,

$$\frac{\beta_0\,\alpha_s(\tau\,e^{-5/3}m_b^2)}{4\pi} = \frac{1}{\ln\tau + \ln(m_b^2/\Lambda_{\rm V}^2)}\,, \tag{35}$$

which is located at $\ln\tau = \ln\tau_L \equiv \ln(\Lambda_{\rm V}^2/m_b^2)$. The principal values of these integrals exactly reproduce the principal values of the original Borel integrals [22]. The residue of the Landau pole therefore provides a measure of the renormalon ambiguity, which we define as

$$\Delta^{\rm ren}\left[N_c\,T_{2i}(x)\right] = \frac{2C_F}{\beta_0}\,w_i(\tau_L,x)\,; \qquad \tau_L = \frac{\Lambda_{\rm V}^2}{m_b^2}\,. \tag{36}$$

Since $\tau_L \ll 1$, the leading contributions to the ambiguity are given by the first few terms in the Taylor expansion of the distribution functions for small τ.

3 Large-order behavior of perturbation theory

In this section we study in more detail the significance of higher-order perturbative corrections to the hard-scattering kernels, using the all-order results derived in the large-β_0 limit. We start by focusing on the anomalous dimensions of the four-quark operators O_1 and O_2, whose perturbation series are convergent and can be summed exactly in the large-β_0 limit. We then consider higher-order contributions to the hard-scattering kernels.

3.1 All-order results for the anomalous dimensions

Using the explicit result for the function $F_L(\varepsilon, 0)$ in (18), we obtain from (8)

$$\begin{aligned}\gamma_\pm &= \pm\frac{N_c \mp 1}{N_c}\,\frac{b}{\beta_0}\,\frac{2(3+2b)(1-b)\,\Gamma(4+2b)}{3\Gamma(1-b)\,\Gamma^2(2+b)\,\Gamma(3+b)} + O(1/\beta_0^2)\\ &= \pm\frac{N_c \mp 1}{N_c}\left[6\cdot\frac{\alpha_s}{4\pi} - \beta_0\left(\frac{\alpha_s}{4\pi}\right)^2 - \frac{65\beta_0^2}{6}\left(\frac{\alpha_s}{4\pi}\right)^3 + \dots\right]. \end{aligned} \tag{37}$$

The radius of convergence of the perturbation series is $\beta_0|\alpha_s| < 4\pi$. The all-order results in the large-β_0 limit may be compared with the exact two-loop expressions [30]

$$\gamma_\pm = \pm\frac{N_c \mp 1}{N_c}\left[6\cdot\frac{\alpha_s}{4\pi} + \left(-\beta_0 + \frac{N_c}{2} + \frac{57}{2N_c} \mp \frac{21}{2}\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right]. \tag{38}$$

Numerically, keeping only the term proportional to $\beta_0\,\alpha_s^2$ is an excellent approximation for γ_+ but not for γ_-. However, in both cases the two-loop coefficients are small, so there is no reason to expect a dominance of the β_0 terms. (These terms are often dominant in cases where the series is divergent and the expansion coefficients are large.)

3.2 Partial two-loop results for the hard-scattering kernels

Perturbative expansions of the hard-scattering kernels can be obtained by expanding the running coupling constant under the integral in (27), as well as the couplings $\alpha_s(m_b^2)$ and $\alpha_s(m_c^2)$ in (25), in powers of $\alpha_s(\mu^2)$. The kernels are then written as

$$T_{2i}(x,\mu) = \frac{1}{N_c}\,\delta_{iL} + \frac{2C_F}{N_c}\left[\frac{\alpha_s(\mu)}{4\pi}\,t_i^{(1)}(x,\mu) + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\beta_0\,t_i^{(2)}(x,\mu) + \dots\right], \tag{39}$$

where $i = L, R$, and $\delta_{iL} = 1$ if $i = L$ and zero otherwise. The results can be expressed in a compact form by introducing the functions

$$\begin{aligned} h_L^{(1)}(\delta,\bar\delta) &= -\frac{3\delta\ln\delta}{2(1-\delta)} + \left[\frac{\ln\delta}{1-\delta} - \ln^2\delta - \mathrm{Li}_2(1-\delta) - \{\delta\to\bar\delta\}\right],\\ h_R^{(1)}(\delta) &= -\frac{1}{2(1-\delta)} - \frac{\delta\ln\delta}{2(1-\delta)^2}\,,\\ h_L^{(2)}(\delta,\bar\delta) &= \frac{\delta}{4(1-\delta)}\left[7\ln\delta - 6\ln^2\delta - 6\mathrm{Li}_2(1-\delta)\right]\\ &\quad + \left[\frac{\ln^2\delta - \ln\delta}{1-\delta} + \ln(1-\delta)\ln^2\delta - \frac{2}{3}\ln^3\delta + \frac{\mathrm{Li}_2(1-\delta)}{1-\delta}\right.\\ &\qquad \left. + \mathrm{Li}_3(1-\delta) + 2\mathrm{Li}_3(\delta) - \{\delta\to\bar\delta\}\right],\\ h_R^{(2)}(\delta) &= \frac{3}{4(1-\delta)} + \frac{\delta}{4(1-\delta)^2}\left[\ln\delta - 2\ln^2\delta - 2\mathrm{Li}_2(1-\delta)\right]. \end{aligned} \tag{40}$$

Here $\mathrm{Li}_3(x) = \int_0^x (dt/t)\,\mathrm{Li}_2(t)$ is the trilogarithm function. At one-loop order we reproduce the expressions obtained in [1], i.e.

$$
\begin{aligned}
t_L^{(1)}(x,\mu) &= -9 + 6\ln\frac{m_b}{\mu} + \left(3 + 2\ln\frac{x}{1-x}\right)\ln z + h_L^{(1)}(\delta,\bar\delta) + h_L^{(1)}(-\bar\delta/z^2,-\delta/z^2)\,, \\
t_R^{(1)}(x,\mu) &= z\,h_R^{(1)}(\delta) + \frac{1}{z}\,h_R^{(1)}(-\bar\delta/z^2)\,.
\end{aligned}
\tag{41}
$$

For the corrections of order $\beta_0\,\alpha_s^2$ we find

$$
\begin{aligned}
t_L^{(2)}(x,\mu) &= -\frac{395}{24} - \pi^2 - 6\ln^2\!\left(\frac{m_b}{\mu}\right) + 17\ln\frac{m_b}{\mu} \\
&\quad - \left(3 + 2\ln\frac{x}{1-x}\right)\left(\ln^2 z + 2\ln z\,\ln\frac{m_b}{\mu}\right) + \left(\frac{17}{2} + \frac{10}{3}\,\ln\frac{x}{1-x}\right)\ln z \\
&\quad + \left(\frac{5}{3} - 2\ln\frac{m_b}{\mu}\right)h_L^{(1)}(\delta,\bar\delta) + \left(\frac{5}{3} - 2\ln\frac{m_b}{\mu} - 2\ln z\right)h_L^{(1)}(-\bar\delta/z^2,-\delta/z^2) \\
&\quad - \left[h_L^{(2)}(\delta,\bar\delta) + h_L^{(2)}(-\bar\delta/z^2,-\delta/z^2)\right], \\
t_R^{(2)}(x,\mu) &= \left(\frac{5}{3} - 2\ln\frac{m_b}{\mu}\right)z\,h_R^{(1)}(\delta) + \left(\frac{5}{3} - 2\ln\frac{m_b}{\mu} - 2\ln z\right)\frac{1}{z}\,h_R^{(1)}(-\bar\delta/z^2) \\
&\quad - \left[z\,h_R^{(2)}(\delta) + \frac{1}{z}\,h_R^{(2)}(-\bar\delta/z^2)\right].
\end{aligned}
\tag{42}
$$

The constants -9 and $-\frac{395}{24}$ in the expressions for $t_L^{(1)}(x,\mu)$ and $t_L^{(2)}(x,\mu)$ are scheme dependent and specific to the NDR scheme.

In Figure 3 we show the one-and two-loop kernels for $z = 0.3$ and two values of the renormalization scale. In order to suppress the integrable, logarithmic endpoint singularities we have multiplied the kernels with the asymptotic distribution amplitude $\Phi_0(x) = 6x(1-x)$. The two-loop contributions are sizable. They are suppressed relative to the one-loop contributions by a factor $\beta_0\alpha_s/4\pi \approx 0.15$ (for $\mu \approx m_b$), but have coefficients that are typically larger by about a factor 3.

3.3 Numerical results

The results derived in Section 2 allow us to study the significance of higher-order perturbative contributions to the hard-scattering kernels in the large-β_0 limit, which may be considered as a toy model for the asymptotic behavior of perturbation theory. In Table 1, we show the results for the kernels obtained at fixed order in perturbation theory and compare them with the principal values of the Borel-resummed series computed from (23), (25) and (27). For simplicity, the kernels are integrated over x with the asymptotic light-cone distribution amplitude. As a measure of the irreducible uncertainty in the value of the Borel integral we quote the renormalon ambiguity defined in (36). As input

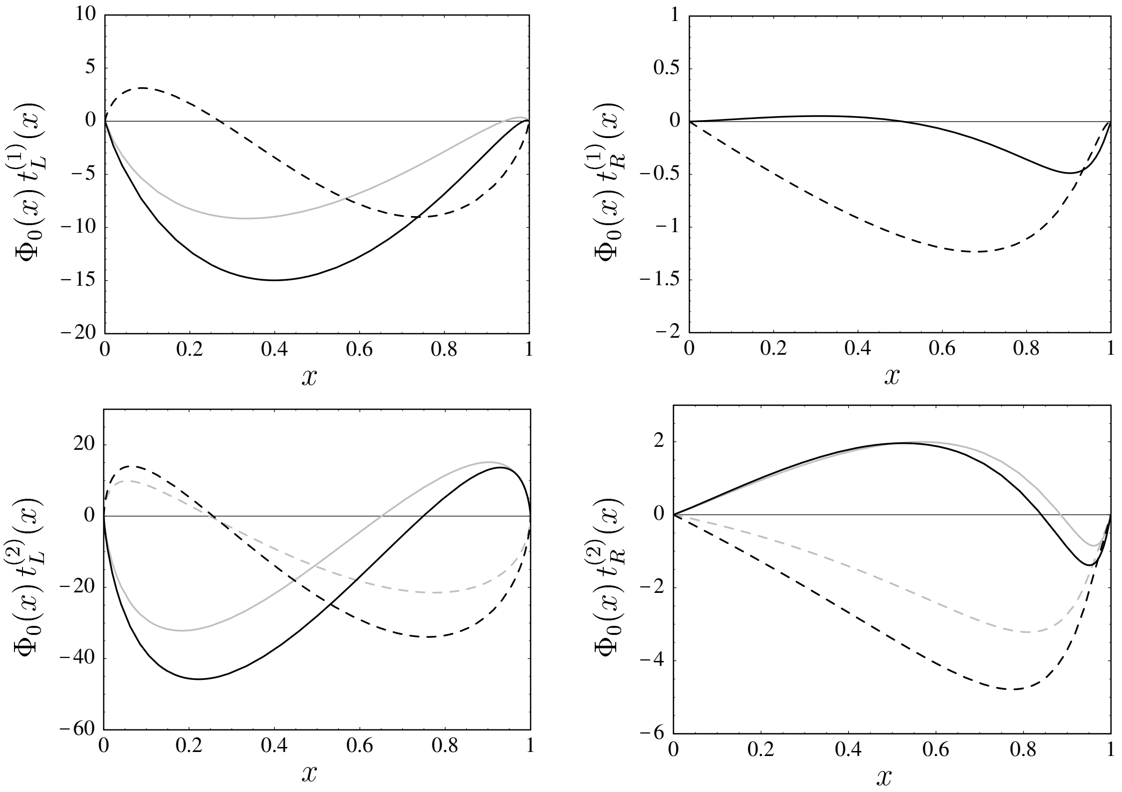

Figure 3: One-loop (upper plots) and partial two-loop (lower plots) contributions to the hard-scattering kernels for $\mu = m_b$ (black curves) and, if different, $\mu = m_b/2$ (gray curves). Results are multiplied by the asymptotic light-cone distribution amplitude. Solid (dashed) lines show the real (imaginary) parts.

parameters we take $m_b = 4.2\,\text{GeV}$, $z = m_c/m_b = 0.3$, and $n_f = 4$. We use the one-loop running coupling constant (as is appropriate in the large-β_0 limit) with $\Lambda_{\overline{\text{MS}}} = 150\,\text{MeV}$, which gives $\alpha_s(m_b^2) = 0.226$. The corresponding scale parameter in the V scheme is $\Lambda_\text{V} = 345\,\text{MeV}$.

The results in the table show that the large-order behavior of the series is improved by lowering the renormalization scale. For $\mu = m_b/2$, the two- and three-loop results ($N = 2$ or 3) are reasonably close to the Borel-resummed value of the entire series. (We stress that these resummed values must be considered as estimates only. As a rule of thumb, the "true" resummed values lie somewhere between the two-loop results and the resummed values obtained in the large-β_0 limit.) An exception is the real part of the right-handed kernel, which suffers from large cancellations in the integral over the weight function. In this case, higher-order contributions in both α_s and $1/\beta_0$ are potentially more important. Note that the imaginary parts of the kernels approach their asymptotic values slower than the real parts. This is a reflection of the low BLM scales obtained earlier. Physically, it means that the strong-interaction phases of the decay amplitudes are sensitive to nonperturbative hadronization effects.

Table 1: Fixed-order $O(\alpha_s^N)$ perturbative approximations to the hard-scattering kernels (convoluted with the asymptotic distribution amplitude) and the corresponding principal values of the Borel-resummed series, for two choices of the renormalization scale. Results for the right-handed kernel are multiplied by 10.

	$\int_0^1 dx\,\Phi_0(x)\,N_c\,T_{2L}(x,\mu)$		$10\times\int_0^1 dx\,\Phi_0(x)\,N_c\,T_{2R}(x,\mu)$	
N	$\mu=m_b$	$\mu=m_b/2$	$\mu=m_b$	$\mu=m_b/2$
0	1	1	0	0
1	$0.53-0.18i$	$0.66-0.23i$	$-0.05-0.37i$	$-0.06-0.47i$
2	$0.39-0.29i$	$0.54-0.34i$	$0.02-0.57i$	$0.06-0.66i$
3	$0.34-0.35i$	$0.52-0.40i$	$0.09-0.68i$	$0.15-0.76i$
4	$0.32-0.40i$	$0.52-0.45i$	$0.17-0.75i$	$0.25-0.81i$
5	$0.31-0.44i$	$0.52-0.49i$	$0.24-0.80i$	$0.33-0.85i$
6	$0.31-0.48i$	$0.52-0.53i$	$0.31-0.83i$	$0.43-0.88i$
7	$0.31-0.51i$	$0.53-0.58i$	$0.39-0.86i$	$0.55-0.90i$
8	$0.32-0.55i$	$0.54-0.65i$	$0.49-0.88i$	$0.73-0.93i$
9	$0.33-0.60i$	$0.58-0.76i$	$0.61-0.91i$	$0.99-0.96i$
10	$0.35-0.67i$	$0.62-0.94i$	$0.78-0.93i$	$1.46-0.99i$
Borel Sum	$0.29-0.47i$	$0.49-0.47i$	$0.26-0.90i$	$0.26-0.90i$
Ambiguity	$0.003-0.04i$	$0.003-0.04i$	$0.10-0.03i$	$0.10-0.03i$

Another important observation from the table is that the renormalon ambiguity, given in the last row, is always much smaller than the contributions of the first few terms in the perturbation series. (Note that the very small ambiguity for the real part of the left-handed kernel results from cancellations between various terms in the Taylor expansion of the function $W_L(\tau_L)$. The leading power contribution alone is 0.017.) The series for the kernels are well-behaved in the sense that their divergent behavior sets in only in high orders ($N\sim 6$). This suggests that power corrections to factorization (at least those connected with "nonfactorizable" soft gluon exchange) are smaller than perturbative contributions due to hard gluon exchange. For this reason, it would be useful to have complete two-loop expressions for the hard-scattering kernels.

As a final remark, we note that for the real part of the left-handed kernel the perturbation series never comes close (to within an amount of order the renormalon ambiguity) to the principal value of the Borel integral. This is because the kernels relevant to $b\to c\bar{u}d$ transitions are the sum of two perturbation series, one with a "natural scale" of order m_b (diagrams 1 and 2), and the other with a "natural scale" of order m_c (diagrams 3 and 4). These two series reach their minimal term at different values of N. When their sum is truncated at fixed N, one of the two series can be close to its minimal term, but not both. This is a general feature of multi-scale problems in quantum field theory. If the scales m_b and m_c were widely separated, one could disentangle the two series by constructing an effective field theory. However, this does not help if, as in the real world, the two scales are relatively close to each other.

4 IR renormalons and power corrections

After the study of higher-order perturbative contributions in the previous section, we now turn to the discussion of power-suppressed effects. We stress, from the outset, that the power corrections inferred from the renormalon analysis arise from soft "nonfactorizable" gluon exchange of the type shown in Figure 1. These are not the only sources of power corrections to the factorization formula (3). For instance, different decay topologies such as weak annihilation or gluon exchange with the spectator quark in the B meson are known to contribute at order $\Lambda_{\rm QCD}/m_b$ and have been estimated in [1]. For the case of $\bar{B}^0 \to D^{(*)+}M^-$ decays, their effects were found to be less than 10%, as suggested by naive power counting. (For the decay $B \to \pi\pi$, the leading power corrections to the factorization formula have recently been estimated using light-cone QCD sum rules. The result is, again, a moderate correction of less than 10% [31].)

As mentioned earlier, due to the presence of IR renormalon singularities the resummed perturbation series for the hard-scattering kernels can only be defined up to an irreducible ambiguity, which in the large-β_0 limit can be estimated from (36). We will now analyze the structure of the leading power-suppressed effects in more detail, and obtain a minimal model of power corrections that is consistent with the renormalon analysis. It follows from (36) that the renormalon ambiguity is given in terms of the distribution functions $w_i(\tau,x)$ evaluated at the small value $\tau=\tau_L=(\Lambda_{\rm V}/m_b)^2 \ll 1$. Terms of order $\tau^{k/2}$ in the Taylor expansions of these functions correspond to power corrections of order $(\Lambda_{\rm QCD}/m_b)^k$. We focus first on the functions f_i in (30) and note that the limits $\tau\to 0$ and $x\to 0,1$ do not commute. Naive expansions of these functions for $\tau\to 0$ would yield more and more powers of $1/x$ and $1/(1-x)$, thereby introducing endpoint singularities. However, we have seen earlier that the kernels can be integrated over x without encountering such singularities. Therefore, the expansions for small τ must be written in terms of distributions. For simplicity, we shall assume that the light-cone distribution amplitude $\Phi_M(x)$ vanishes linearly at the endpoints. This is true for the asymptotic distribution amplitude and, more generally, whenever the amplitude can be approximated by a finite expansion in Gegenbauer polynomials, i.e.

$$\Phi_M(x,\mu) = 6x(1-x)\Big[1+\sum_{n\ge 1}^{N} a_n^M(\mu)\, C_n^{(3/2)}(2x-1)\Big]\,. \tag{43}$$

The Gegenbauer moments $a_n^M(\mu)$ are multiplicatively renormalized and vanish for $\mu\to\infty$, so that $\Phi_M(x,\mu)$ tends to its asymptotic form. Here we adopt the Gegenbauer expansion for convenience only; our analysis of the endpoint behavior in Section 2 showed that the kernels can be integrated with any normalizable and smooth function $\Phi_M(x)$.

In our analysis we keep only the leading terms of order $\sqrt{\tau}$ or τ, neglecting higher-order contributions corresponding to power corrections of order $(\Lambda_{\rm QCD}/m_b)^3$ and higher. We obtain

$$\begin{aligned} f_L(\tau,\delta,\bar\delta) &= \frac{\tau}{d\,x}\left(2\ln\frac{\sqrt{\tau}}{d\,x}+\frac12\right) \\ &\quad + \left[-\frac{2\sqrt{\tau}}{d\,x}+\frac{\tau}{d^2x}\,\Delta(x,\sqrt{\tau}/d)-\{x\to(1-x)\}\right]+O(\tau^{3/2})\,, \\ f_R(\tau,\delta) &= -\frac{\tau}{2d\,x}+O(\tau^{3/2})\,, \end{aligned} \tag{44}$$

where $d=1-z^2$, and we have introduced the distribution

$$\Delta(x,a)=\left[\frac{1}{x}\left(\ln x-\ln a+\frac12\right)\right]_+ + \frac{\delta(x)}{2}\left(\ln^2 a-\ln a+\frac{\pi^2}{3}+\frac12\right). \tag{45}$$

Its integral with the light-cone distribution amplitude is defined as

$$\begin{aligned} \int_0^1 dx\,\frac{\Phi_M(x)}{x}\,\Delta(x,a) &= \int_0^1 dx\left[\frac{\Phi_M(x)}{x}-\Phi_M'(0)\right]\frac{1}{x}\left(\ln x-\ln a+\frac12\right) \\ &\quad + \frac{\Phi_M'(0)}{2}\left(\ln^2 a-\ln a+\frac{\pi^2}{3}+\frac12\right), \end{aligned} \tag{46}$$

where

$$\Phi_M'(0)=\lim_{x\to0}\frac{\Phi_M(x)}{x}=6\left[1+\sum_{n\ge1}(-1)^n\,\frac{(n+1)(n+2)}{2}\,a_n^M(\mu)\right]. \tag{47}$$

The "+"-distribution for $x\to1$ is defined in a similar way. Its evaluation involves the derivative $\Phi_M'(1)$, which up to an overall sign is given by the same sum over Gegenbauer moments, but without the factor $(-1)^n$.

To get the corresponding expansions of the functions $w_i(\tau,x)$ we must add the crossed terms, for which $\tau\to\tau/z^2$ and $d\to -d/z^2$. This yields

$$\begin{aligned} w_L(\tau,x) &= -\frac{2\tau}{d\,(1-x)}\,(\ln z+i\pi) \\ &\quad + \Bigg[-\frac{2\sqrt{\tau}}{(1-z)\,x}+\frac{\tau}{d^2x}\left[\Delta(x,\sqrt{\tau}/d)-z^2\,\Delta(x,-z\sqrt{\tau}/d)\right] \\ &\qquad + \frac{\tau}{d\,x}\left(2\ln\frac{\sqrt{\tau}}{d\,x}+\frac12\right)-\{x\to(1-x)\}\Bigg]+O(\tau^{3/2})\,, \\ w_R(\tau,x) &= -\frac{\tau}{2d}\left(\frac{z}{x}-\frac{1}{z(1-x)}\right)+O(\tau^{3/2})\,. \end{aligned} \tag{48}$$

The asymmetric part of the function w_L has a small-τ behavior corresponding to a first-order power correction in $\Lambda_{\rm QCD}/m_b$, in accordance with the result (26) for the leading renormalon pole. When integrated with a symmetric light-cone distribution amplitude this term vanishes, and the leading power corrections are of order $(\Lambda_{\rm QCD}/m_b)^2$.

In the limit of isospin symmetry, the light-cone distribution amplitude for a pion or a ρ meson is symmetric in $x \leftrightarrow (1-x)$. We then obtain

$$\begin{aligned} W_L(\tau_L) &= \frac{6\Lambda_{\rm V}^2}{m_b^2-m_c^2}\left(\ln\frac{m_b}{m_c}-i\pi\right)\left(1+\sum_{n={\rm even}} a_n^M\right)+O[(\Lambda_{\rm QCD}/m_b)^3]\,, \\ W_R(\tau_L) &= \frac{3\Lambda_{\rm V}^2}{2m_b m_c}\left(1+\sum_{n={\rm even}} a_n^M\right)+O[(\Lambda_{\rm QCD}/m_b)^3]\,, \end{aligned} \tag{49}$$

where we have used that $\int_0^1(dx/x)\,\Phi_M(x) = 3(1+a_2^M+a_4^M+\dots)$ for a symmetric distribution amplitude. For decays involving strange particles in the final state, such as $\bar B_0 \to D^+K^-$, the function $\Phi_M(x)$ is no longer expected to be symmetric. In such a case, the function $W_L(\tau)$ receives a first-order power correction given by

$$W_L(\tau_L) = \frac{12\Lambda_{\rm V}}{m_b-m_c}\left(\sum_{n={\rm odd}} a_n^M\right)+O[(\Lambda_{\rm QCD}/m_b)^2]\,. \tag{50}$$

A comment is in order concerning the peculiar dependence of the power corrections in (49) and (50) on the heavy-quark masses, which prevents us from taking the limits $m_c\to m_b$ or $m_c\to 0$. The limit where the charm-quark mass tends to zero is actually not a singular one, but has to be dealt with carefully. We will come back to this in the following section. The fact that some power corrections blow up for $m_c\to m_b$ is physical. There are three relevant mass scales in this problem: the mass of the decaying b quark, the mass of the charm quark, and the energy $E_M\approx(m_b^2-m_c^2)/(2m_b)$ of the light final-state meson. The factorization properties of the decay amplitude in the heavy-quark limit rely crucially on the fact that $E_M\gg\Lambda_{\rm QCD}$, since only then can the color transparency argument [32, 33] be employed to demonstrate the cancellation of soft IR contributions [1]. This condition no longer holds in the limit $m_c\to m_b$.

Finally, we can use the above results to write down a minimal model for the power corrections due to soft gluon exchange in hadronic B decays. The decay amplitudes for the class-1 decays $\bar B^0\to D^{(*)+}M^-$ are conveniently parameterized in terms of quantities $a_1(D^{(*)}M)$, which contain the QCD corrections to the results obtained using "naive" factorization. The factorization formula (3) implies that

$$a_1 = \sum_{i=1,2} C_i(\mu)\int_0^1 dx\,\Phi_M(x,\mu)\,\Big[T_{iL}(x,\mu)\pm T_{iR}(x,\mu)\Big]+O(\Lambda_{\rm QCD}/m_b)\,, \tag{51}$$

where the upper (lower) sign refers to the case with a D (D^*) meson in the final state. Inserting the Borel-resummed expressions for the hard-scattering kernels, we find that the renormalon ambiguity in a_1 is given by

$$\Delta^{\rm ren}\,a_1 = \frac{C_2(\mu)}{N_c}\,\frac{2C_F}{\beta_0}\Big[W_L(\tau_L)\pm W_R(\tau_L)\Big]+O(1/\beta_0^2)\,. \tag{52}$$

To obtain a model for the leading power corrections, we insert here expressions (49) or (50), depending on whether or not the distribution amplitude is symmetric, and replace $\Lambda_{\rm V}$ by nonperturbative hadronic parameters $\Lambda_{L,R}$. Ideally, these parameters would be determined from a fit to experimental data. However, for $\Lambda_{L,R}\sim 0.5\,\text{GeV}$ and a symmetric light-cone distribution amplitude, we find power corrections of order few times 10^{-3}, which are insignificantly small. This is in part due to the smallness of the Wilson coefficient C_2. The power corrections could be much larger for other decays, such as $B\to\pi\pi$. For decays with a strange meson in the final state, such as $\bar B^0\to D^+K^-$, our model predicts first-order power corrections as large as a few percent. Unfortunately, these modes are Cabibbo suppressed, and no experimental data has been reported yet.

5 Results for charmless hadronic decays

Our discussion so far has referred to the simplest application of the QCD factorization formula, i.e. to class-1 B decays into a heavy–light final state. However, factorization in the heavy-quark limit also occurs for the phenomenologically more interesting, rare hadronic decays into two light mesons. Examples are the charmless decays $B\to\pi\pi$ and $B\to\pi K$, which might provide information about the CP-violating phase of the quark mixing matrix [2]. The effective weak Hamiltonian for these processes contains many penguin operators besides the current–current operators O_1 and O_2, and several different decay topologies must be considered in addition to the diagrams shown in Figure 1. A complete renormalon analysis for these processes is beyond the scope of this paper; however, the "nonfactorizable" vertex corrections investigated here are an important part of such an analysis. It is thus interesting to apply our results to the case where the charm quark in the final state is replaced by a massless u quark.

The limit $z\to 0$ is smooth but must be taken carefully. For instance, the power corrections in (49) cannot be extrapolated to $m_c\to 0$, but we obtain regular expressions by first computing the distribution functions in the limit $z\to 0$, and then expanding the results for small τ. We now collect the most important formulae valid for $z=0$.

5.1 Borel representation and distribution function

In the massless limit the chirality of the external quark states is preserved, and hence the right-handed kernel T_R vanishes for $z\to 0$. This can also be seen explicitly by taking the limit $z\to 0$ in our results for this kernel. The Borel-resummed expression (25) for the left-handed kernel simplifies to

$$N_c\,T_{2L}(x,\mu) = 1+\frac{2C_F}{\beta_0}\left(\,\int\limits_{-b(\mu)}^{0}\frac{d\varepsilon}{\varepsilon}\left[3-\frac{(3-2\varepsilon)(1+\varepsilon)\Gamma(4-2\varepsilon)}{3\Gamma(1+\varepsilon)\,\Gamma^2(2-\varepsilon)\,\Gamma(3-\varepsilon)}\right]+3\,\ln\frac{b(\mu)}{b(m_b)}\right)$$

$$+2C_F\int\limits_0^\infty\frac{d\tau}{\tau}\,w_L(\tau,x)\,\frac{\alpha_s(\tau\,e^{-5/3}\,m_b^2)}{4\pi}+O(1/\beta_0^2)\,, \qquad (53)$$

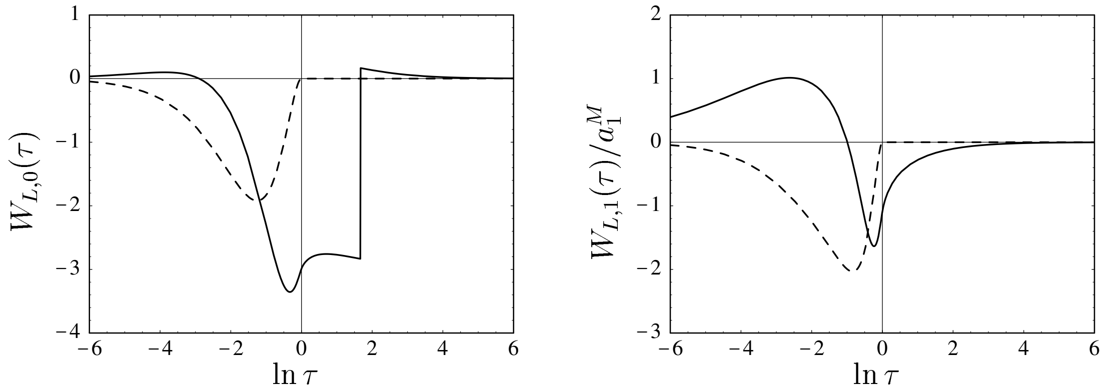

Figure 4: Integrated distribution function $W_L(\tau)$ for $m_c = 0$. The left-hand plot shows the asymptotic result, and the right-hand one the contribution proportional to the first Gegenbauer moment.

where

$$
\begin{aligned}
w_L(\tau,x) &= -\tau\left(\frac{1-\eta}{2x}+\frac{\eta}{x^2}\right)-\frac{\tau\,[2x^2-\tau(1-x)]}{x^3}\,\ln\left(1+\frac{\eta\,x}{\tau}\right)\\
&\quad+\frac{\tau(2x-\tau)}{x^2}\,\ln\left(1-\frac{x}{\tau}-i\epsilon\right)+3\,\theta(\tau\,e^{-5/3}-1)\\
&\quad+\Bigg[\frac{\tau-\eta}{x}-\left(1-\frac{\tau}{x^2}\right)\ln\left(1+\frac{\eta\,x}{\tau}\right)+\left(1-\frac{\tau}{x}\right)^2\ln\left(1-\frac{x}{\tau}-i\epsilon\right)\\
&\qquad-\{x\to(1-x)\}\Bigg]. \qquad (54)
\end{aligned}
$$

As before, the kernel is well-behaved in the endpoint regions $x \to 0$ and $x \to 1$. In particular, the integral of $w_L(\tau, x)$ with the asymptotic light-cone distribution amplitude yields

$$
\begin{aligned}
W_{L,0}(\tau) &= -\frac{3\tau}{2}\,(1-13\eta)-6\tau(1+3\eta+3\tau)\,\ln\left(1+\frac{\eta}{\tau}\right)+6\tau(1-\tau)\,\ln\left(1-\frac{1}{\tau}-i\epsilon\right)\\
&\quad+12\tau^2\,\mathrm{Li}_2\left(-\frac{\eta}{\tau}\right)+6\tau^2\,\mathrm{Li}_2\left(\frac{1}{\tau}+i\epsilon\right)+3\,\theta(\tau\,e^{-5/3}-1)\,. \qquad (55)
\end{aligned}
$$

This function is shown in the left-hand plot in Figure 4. In the second plot, we show the contribution proportional to the first Gegenbauer moment, denoted by $W_{L,1}(\tau)$. It is evident that the real part of $W_{L,1}(\tau)$ receives large contributions from the region of very small τ, corresponding to low gluon virtualities. This is in accordance with the fact that the renormalon ambiguity for this contribution is of first order in $\Lambda_{\mathrm{QCD}}/m_b$.

5.2 Partial two-loop results for the hard-scattering kernel

The easiest way to obtain the one and two-loop coefficients of the kernel in the massless case is to take the limit $z \to 0$ in the expressions collected in Section 3.2. It is a nontrivial check of our results that all logarithms of the mass ratio z cancel in that limit. For the expansion coefficients in (39) we obtain

$$
\begin{aligned}
t_L^{(1)}(x,\mu) &= -9 + 6\ln\frac{m_b}{\mu} + \frac{3}{2}\Big(\frac{1-2x}{1-x}\,\ln x - i\pi\Big) \\
&\quad + \left[-\frac{\ln^2 x}{2} + \frac{\ln x}{1-x} + \mathrm{Li}_2(x) - \Big(\frac{3}{2} + i\pi\Big)\ln x - \{x \to (1-x)\}\right], \\
t_L^{(2)}(x,\mu) &= 6\ln^2\Big(\frac{m_b}{\mu}\Big) - 11\ln\frac{m_b}{\mu} - \frac{35}{24} + \Big(\frac{5}{3} - 2\ln\frac{m_b}{\mu}\Big)\,t_L^{(1)}(x,\mu) \\
&\quad - \frac{3(1-3x)}{4(1-x)}\,\ln^2 x + \frac{7(1-2x)}{4(1-x)}\,\ln x + \frac{3(1-x)}{2x}\,\mathrm{Li}_2(x) + i\pi\Big(\frac{3}{2}\,\ln x - \frac{7}{4}\Big) \\
&\quad + \left[\frac{\ln^3 x}{2} - \ln^2 x\ln(1-x) - \frac{1+3x}{4(1-x)}\,\ln^2 x + \Big(\pi^2 - \frac{7}{4} + \frac{1}{1-x}\Big)\ln x \right. \\
&\quad \left. - \frac{1-3x}{2x}\,\mathrm{Li}_2(x) - \mathrm{Li}_3(x) + \frac{i\pi}{2}\,(\ln^2 x - 3\ln x) - \{x \to 1-x\}\right].
\end{aligned} \tag{56}
$$

5.3 Numerical results

In Table 2, we show the results for the kernel obtained at fixed order in perturbation theory and compare them with the principal value of the Borel-resummed series. To illustrate the effect of the leading IR renormalon singularity, we include the contribution of the first Gegenbauer moment a_1^M. Then the light-cone distribution amplitude is no longer symmetric, and the leading renormalon ambiguity is of first order in $\Lambda_{\mathrm{QCD}}/m_b$.

The results obtained using the asymptotic distribution amplitude exhibit a similar behavior as in the case with a finite charm-quark mass. The convergence of the series is improved by using the smaller renormalization scale $\mu = m_b/2$. In this case, the two-loop result is already very close to the asymptotic value. As previously, the imaginary part of the kernel reaches its asymptotic value at larger N than the real part. The situation for the contribution proportional to the first Gegenbauer moment is different. Whereas the imaginary part shows a similar behavior as for the leading term, the series for the real part of $W_{L,1}(\tau)$ diverges already in low orders. Even the two-loop results exceed the asymptotic values, and starting at $N \sim 4$ the expansion coefficients exhibit a rapid factorial growth. This is a reflection of the leading renormalon pole at $u = \frac{1}{2}$, corresponding to a first-order power correction to the real part, which is absent in the case of a symmetric distribution amplitude. Accordingly, the renormalon ambiguity is an order of magnitude larger than in the symmetric case.

Finally, we note that using the one-loop expressions for the kernel, as is done in

Table 2: Fixed-order $O(\alpha_s^N)$ perturbative approximations to the hard-scattering kernel in the limit $m_c \to 0$ and the corresponding principal value of the Borel-resummed series, for two choices of the renormalization scale. Shown are the first two terms in the Gegenbauer expansion of the distribution amplitude.

	$\int_0^1 dx\, \Phi_M(x,\mu)\, N_c\, T_{2L}(x,\mu)$	
N	$\mu = m_b$	$\mu = m_b/2$
0	1	1
1	$(0.56-0.23i)+a_1^M(0.13-0.23i)$	$(0.69-0.29i)+a_1^M(0.17-0.29i)$
2	$(0.44-0.35i)+a_1^M(0.31-0.35i)$	$(0.61-0.41i)+a_1^M(0.41-0.40i)$
3	$(0.41-0.43i)+a_1^M(0.51-0.42i)$	$(0.61-0.48i)+a_1^M(0.67-0.46i)$
4	$(0.41-0.48i)+a_1^M(0.77-0.46i)$	$(0.62-0.53i)+a_1^M(1.06-0.51i)$
5	$(0.42-0.52i)+a_1^M(1.13-0.50i)$	$(0.65-0.57i)+a_1^M(1.71-0.54i)$
6	$(0.44-0.55i)+a_1^M(1.76-0.53i)$	$(0.67-0.61i)+a_1^M(3.05-0.58i)$
7	$(0.46-0.59i)+a_1^M(2.96-0.57i)$	$(0.72-0.66i)+a_1^M(6.28-0.63i)$
8	$(0.49-0.63i)+a_1^M(5.63-0.60i)$	$(0.78-0.72i)+a_1^M(15.2-0.69i)$
9	$(0.54-0.68i)+a_1^M(12.3-0.65i)$	$(0.91-0.82i)+a_1^M(42.7-0.79i)$
10	$(0.61-0.74i)+a_1^M(30.5-0.71i)$	$(1.15-0.99i)+a_1^M(138.-0.96i)$
Borel Sum	$(0.38-0.55i)+a_1^M(0.27-0.53i)$	$(0.59-0.55i)+a_1^M(0.27-0.53i)$
Ambiguity	$(0.02-0.04i)+a_1^M(0.19-0.04i)$	$(0.02-0.04i)+a_1^M(0.19-0.04i)$

all phenomenological applications of the QCD factorization approach to date, gives a reasonable approximation to the real part (for $\mu = m_b/2$), but underestimates the strong-interaction phase by almost a factor 2. This observation may be relevant for studies of CP asymmetries in rare hadronic B decays.

5.4 IR renormalons and power corrections

The pattern of IR renormalons in the massless limit is very similar to that for finite charm-quark mass. Following our analysis in Section 4, we first construct the expansion of the distribution function (54) for small values of τ. The result can be written as

$$\begin{aligned} w_L(\tau,x) &= -\frac{\tau}{1-x}\left(\ln\tau + \frac{1}{2} + 2i\pi\right) \\ &\quad + \left[-\frac{2\sqrt{\tau}}{x} + \frac{\tau}{x}\left(2\ln\frac{\sqrt{\tau}}{x} - \frac{1}{2}\right) + \frac{\tau}{x}\,\widetilde{\Delta}(x,\tau) - \{x \to (1-x)\}\right] + O(\tau^{3/2})\,, \end{aligned} \tag{57}$$

where

$$\tilde{\Delta}(x,\tau) = \left[\frac{1}{x} \left(\ln x - \frac{1}{2} \ln \tau + \frac{1}{2} \right) \right]_+ + \delta(x) \left(\frac{1}{8} \ln^2 \tau - \frac{1}{4} \ln \tau + \frac{\pi^2}{6} + \frac{1}{4} \right) . \tag{58}$$

After integration with the light-cone distribution amplitude, we find that for the case of a symmetric amplitude the leading power-suppressed contributions are given by

$$W_L(\tau_L) = \frac{6\Lambda_{\rm V}^2}{m_b^2} \left(\ln \frac{m_b}{\Lambda_{\rm V}} - \frac{1}{4} - i\pi \right) \left(1 + \sum_{n=\rm even} a_n \right) + O[(\Lambda_{\rm QCD}/m_b)^3] \,, \tag{59}$$

which may be compared with the corresponding result in (49). If the distribution amplitude is not symmetric, the function $W_L(\tau)$ receives a first-order power correction given by

$$W_L(\tau_L) = \frac{12\Lambda_{\rm V}}{m_b} \left(\sum_{n=\rm odd} a_n \right) + O[(\Lambda_{\rm QCD}/m_b)^2] \,. \tag{60}$$

As in Section 4, these results could be used to construct a simple model of power corrections to factorization in rare hadronic B decays. However, as mentioned earlier, this model would be incomplete without taking into account other decay topologies such as penguin contractions and interactions with the spectator quark in the B meson. We plan to analyze these contributions in a future publication.

6 Conclusions

We have used the renormalon calculus to study the asymptotic behavior of the hard-scattering kernels entering the QCD factorization formula for the nonleptonic weak decays $\bar{B}^0 \to D^{(*)+} M^-$. We have obtained explicit results for the Borel transforms and momentum distribution functions of the kernels in the approximation of retaining a single renormalon chain (the large-β_0 limit). This method estimates power corrections to the factorization formula, and allows us to investigate the (divergent) higher-order behavior of the perturbation series for the kernels. From the pattern of singularities in the Borel plane, we have derived a simple model of power corrections that is consistent with the renormalon analysis. This model accounts only for corrections due to soft, "nonfactorizable" gluon exchange. Other types of power-suppressed effects exist and have been estimated in the literature [1].

An unexpected result of our work is that the leading IR renormalon singularity, corresponding to a first-order power correction in $\Lambda_{\rm QCD}/m_b$, vanishes for mesons with a symmetric light-cone distribution amplitude. We have explicitly calculated the second-order correction and shown it to be numerically small. Higher-order perturbative effects are thus expected to be more important than power corrections. We have presented analytic results for the contributions of order $\beta_0 \, \alpha_s^2$ to the hard-scattering kernels, which presumably are the dominant part of the full two-loop contributions. We have also given numerical results for the terms of order $\beta_0^{n-1} \alpha_s^n$ in the perturbation series. We have shown that lowering the renormalization scale below m_b improves the rate of approach of the series to their asymptotic values. The BLM scales associated with the imaginary parts of the hard-scattering kernels are below 1 GeV (in the $\overline{\rm MS}$ scheme). This indicates that the strong-interaction phases of the decay amplitudes are not insensitive to nonperturbative physics. Finally, by showing that the kernels are free of IR divergences and power-divergent endpoint singularities, we have proven the factorization formula presented in [1] to all orders in perturbation theory in the large-β_0 limit.

In the future, it would be worthwhile to carry out a renormalon analysis for the phenomenologically more interesting charmless hadronic B decays. Power corrections in these processes are, in general, not suppressed by small Wilson coefficient functions. By giving explicit results valid in the limit $m_c \to 0$, we have accomplished the technically most challenging part of such an analysis. It remains to add the contributions from penguin contractions and gluon exchange with the spectator quark.

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Note added: While this paper was in writing, the work *Renormalon analysis of heavy–light exclusive B decays* appeared [34], in which the authors present expressions for the Borel transforms of the hard-scattering kernels that are equivalent to our eq. (19), and observe the vanishing of first-order power corrections for the case of a symmetric light-cone distribution amplitude, in accordance with our eq. (26).

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