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String Webs from Field Theory

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Abstract: The spectrum of stable electrically and magnetically charged supersymmetric particles can change discontinuously as one changes the vacuum on the Coulomb branch of gauge theories with extended supersymmetry in four dimensions. We show that this decay process can be understood and is well described by semiclassical field configurations purely in terms of the low energy effective action on the Coulomb branch even when it occurs at strong coupling. The resulting picture of the stable supersymmetric spectrum is a generalization of the "string web" picture of these states found in string constructions for certain theories.

Contents

1. Introduction

Four dimensional gauge theories with at least eight supersymmetries have a Coulomb branch of inequivalent vacua in which the low energy effective theory generically has an unbroken $\mathrm{U}(1)^n$ gauge invariance. These theories also have a spectrum of massive charged particles with various electric and magnetic charges under the U(1)'s, lying in supersymmetry multiplets. Those lying in short multiplets of the supersymmetry algebra (the BPS states) leave some fraction of the supercharges unbroken, and their masses are related to their charges by the supersymmetry algebra [1]. The spectrum of the possible BPS masses can then be determined using supersymmetry selections rules [2, 3]. This, however, leaves open the question of the existence and multiplicity of these states. Furthermore, even if such a state exists in some region of the Coulomb branch (for some values of the vevs), they may be unstable to decay at curves of marginal stability (CMS) on the Coulomb branch [4, 2]. In this paper we propose a solution to the question of the multiplicity of BPS states for $\mathcal{N}$=2 and 4 theories in four dimensions just in terms of the low energy effective $\mathrm{U}(1)^n$ action on the Coulomb branch.

The form of the answer we get coincides with the "string web" picture of BPS states [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] developed in the context of the D3-brane construction of $\mathcal{N}$=4 SU(n) superYang-Mills (SYM) theory [15], and the F theory solution to $\mathcal{N}$=2 SU(2) gauge theory with fundamental matter [16, 17]. (In general, it is possible to realize certain classes of gauge theories as worldvolume field theories on probe D3-branes in specific string theory backgrounds in the limit of vanishing string length, $\ell_s \to 0$—see [18] for a review of these constructions.) BPS states carrying electric and magnetic charges (p, q) in the low energy theory (which is the effective action on the brane probe) are realized in these constructions as webs of (p, q) strings meeting at 3-string junctions and ending on the probe D3-brane as well as various "sources" in the string theory background space time.

Our solution coincides with the string web constructions in the cases mentioned above, and generalizes these constructions to arbitrary field theory data (gauge groups, matter representations, couplings and masses). The resulting picture is quite simple: BPS states are represented by string webs on the Coulomb branch of the theory with one end at the point corresponding to the vacuum in question (the analog of the 3-brane probe in the F theory picture) and the other ends lying on the complex codimension 1 singularities on the Coulomb branch (the analogs of the (p, q) 7-branes of the F theory picture). The strands of the string web lie along geodesics in the Coulomb branch metric. Each strand carries electric and magnetic charges under the $\mathrm{U}(1)^n$ low energy gauge group: the total charge flowing into the vacuum point determines the total charge of the BPS state, while only multiples of the charges determined by the $\mathrm{Sp}(2n, \mathbb{Z})$ monodromies around the codimension 1 singularities are allowed to flow into those ends of the web; see figure 1. Finally, three string junctions obey a tension-balancing constraint, where the tension of the strings is given by the usual formula in terms of the electric and magnetic charges they carry.

Perhaps the most surprising thing about our solution is that it describes the stability of the monopole and dyon BPS spectrum wholly in terms of the $\mathrm{U}(1)^n$ low energy effective action. This is possible because the distance ΔX from a given vacuum on the

Coulomb branch to a CMS acts as a new low energy scale which can be made arbitrarily small compared to the strong coupling scale Λ of the nonabelian gauge theory as we approach the CMS. In particular, we will show that, as we approach the CMS, the classical low energy field configuration describing a state which decays across the CMS will develop two or more widely separated charge centers (which appear as singularities in the low energy solution); the distance between these centers varies inversely with ΔX. Thus, in this limit the details of the microscopic physics become irrelevant for the decay of a BPS state across a CMS, which is described by a low energy field configuration with charge centers becoming infinitely separated.

In general, classical BPS field configurations in the low energy effective field theory can be interpreted as a static deformation of a 3-brane worldvolume known as a brane spike [19], or as a "brane prong" in the case where the field configuration has multiple sources. In the limit that the cutoff scale is removed (*e.g.*, $\Lambda \to \infty$ or $\ell_s \to 0$) such a brane spike or prong approaches a (p,q) string or string web ending on the 3-brane. Thus, in the limit that we approach the CMS this picture becomes arbitrarily accurate in the $\mathrm{U}(1)^n$ effective theory; it can then be extended to the rest of the Coulomb branch essentially by analyticity.

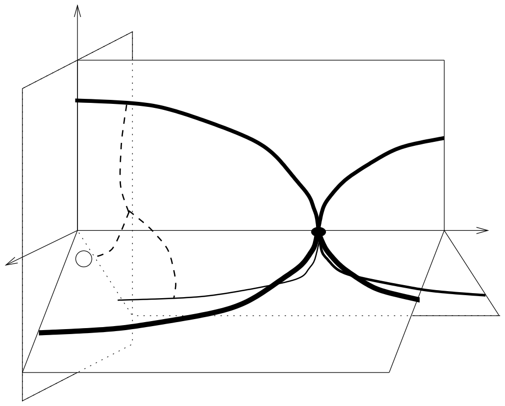


Figure 1: A slice of the Coulomb branch of $SU(3)$ $\mathcal{N}$=2 SYM showing some complex co-dimension one curves of singularities (dark lines), and a BPS state represented as a string web (dashed lines) joining a given vacuum (the open circle).

The brane prong picture arises from thinking of the classical BPS field configuration of the scalar fields in the low energy supersymmetric $\mathrm{U}(1)^n$ gauge theory on the Coulomb branch as maps from space-time (the worldvolume of the 3-brane) to the Coulomb branch (the background geometry that the 3-branes live in). Such a field configuration will have one or more singularities or sources where scalar field gradients and $\mathrm{U}(1)$ field strengths diverge. Near these points the low energy description breaks down and should be cut off by boundary conditions reflecting the matching onto the microscopic physics of the nonabelian gauge theory. We will show that these boundary conditions are essentially determined by the BPS condition: all other details of the choice of boundary conditions do not affect the behavior of the prong solution in the limit that the cut off length scale vanishes relative to the low energy length scale. In particular, since the ratio of the cut off length scale to the relevant low energy length scale (the distance between the separating sources) vanishes at the CMS, the way the BPS spectrum jumps across the

CMS is independent of the the details of the way the BPS charges are regularized.

This paper is organized as follows. In the next section we present a simple physical argument for why BPS states near a CMS develop widely separated charge centers and therefore have a good semiclassical description in the low energy theory. In section 3 we solve for the semiclassical field configurations and show the decay of the relevant BPS states in a simple $\mathrm{U}(1)$ toy model. In particular, the separation of the charge centers near the CMS can be simply and explicitly demonstrated in this example. In section 4 we generalize our analysis to the $\mathrm{U}(1)^n$ low energy theory of $\mathcal{N}$=4 SYM, and show how the string web picture of 1/2 and 1/4 BPS states is recovered in the $\mathrm{SU}(N)$ case. Section 5 discusses the generalization to $\mathcal{N}$=2 theories. Section 6 closes with some open questions and directions for future research: the most pressing open question has to do with the derivation of the "s-rule" [20, 9, 11, 12] in $\mathcal{N}$=2 theories, which is not apparent in our solutions; interesting extensions of the arguments of this paper apply to similar phenomena in other dimensions and in gravitational theories.

The work in this paper overlaps that of a number of other papers which have appeared in the last few years. The main new contribution of this paper is the understanding that a string web picture of decaying BPS states follows directly from the low energy effective theory in the vicinity of a CMS and relies on approximate boundary conditions which become exact in the limit of approaching the CMS. More specifically, the semiclassical description of BPS states in low energy $\mathrm{U}(1)^n$ effective theories has been explored in many contexts, especially [21] whose discussion of brane prong solutions was a starting point for this paper, and [22] whose discussion of how brane prong solutions approximate string webs near the CMS overlaps with our discussion in section 3. A related discussion of brane spikes in an $\mathcal{N}$=2 F theory background appears in [23]. Our discussion of brane spike and prong solutions differs mainly in our treatment of the boundary conditions at the charge sources as well as in our generalization of these solutions to arbitrary $\mathcal{N}$=4 or $\mathcal{N}$=2 supersymmetric field theory data. Also, the basic phenomenon of separating charge centers (or at least the growth in overall size) of BPS states near CMS has been noted repeatedly in the context of semiclassical dyon solutions in SYM theories [24, 25, 26, 27, 28, 29, 30, 31]. In particular the general picture of loosely bound composite BPS states in [32] overlaps with our discussion in section 2; we add to it the observation that the separation of the charge centers is generic and persists in strong coupling regions of the Coulomb branch. Finally, the discussion in the last section of [33] amounts to a gravitational version of our discussion in section 5; in addition to our different treatment of the boundary conditions following from the regime of validity of the low energy effective theory, we add the observation that the image of the low energy solution in moduli space more and more closely approximates a string web configuration as we approach the CMS.

2. BPS states near CMS

Since the total electric and magnetic charges (Q_E^i, Q_{Bi}) with respect to the unbroken $U(1)^n$ gauge group on the Coulomb branch are conserved, we can restrict our attention to a single charge sector of the theory. Our question is whether at a given vacuum there is or is not a one-particle BPS state in that charge sector. The mass of a BPS state is determined by the superalgebra to be the absolute value of the central charge, and the central charge is the sum of terms proportional to the charges

$$Z = Q_E^i a_i + Q_{Bi} a_D^i \tag{2.1}$$

where the coefficients a_i and a_D^i depend only on the vacuum in question and not on the charges. This mass, $|Z|$, is the minimum mass of any state (BPS or not, single particle or not) in this charge sector and so, in particular, the spectrum in this charge sector is gapped.

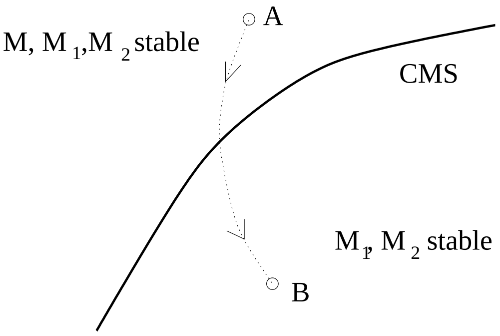


Figure 2: A generic decay across a CMS. The dotted line denotes a path on the Coulomb branch.

A single BPS particle in this charge sector would have a mass $M = |Z|$. It is stable or at worst marginally stable against decay into two (or any number of) constituent particles, since by charge conservation $Z = Z_1 + Z_2$, so by the triangle inequality

$$M \leq M_1 + M_2. \tag{2.2}$$

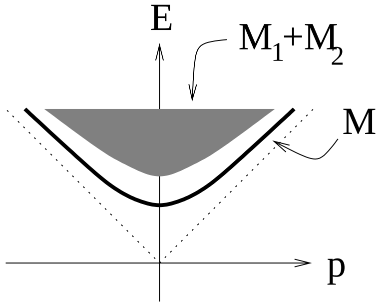
The CMS are submanifolds of the Coulomb branch where the inequality in (2.2) is saturated. As one adiabatically changes the order parameter on the Coulomb branch from a vacuum A on one side of a CMS to a vacuum B on the other, the one particle state M in our charge sector will become more and more nearly degenerate with the two particle state $M_1 + M_2$. Supposing M does decay across the CMS, then only the two particle state $M_1 + M_2$ will remain in the spectrum once we have crossed the CMS. This is illustrated in figure 2. By assumption M_1 and M_2 are in the stable spectrum everywhere on the B side of the CMS. It follows from the BPS mass formula that they are also generically stable on the A side, since possible decays like $M_1 \to M + (-M_2)$ (where $-M_2$ is the charge conjugate of M_2) are not allowed since $M_1 < M + M_2$ on the CMS. Also, it follows from (2.2) that the two particle state $M_1 + M_2$ is not BPS, even for zero relative momentum, except precisely at the CMS.

In terms of the density of states, on the A side of the CMS where M is stable, we have a discrete one particle state lying below the threshold to the M_1+M_2 two particle state continuum. Right at the CMS the one particle state just coincides with the two particle threshold. On the B side of the CMS, where M is no longer in the spectrum, there is just the two particle continuum; see figure 3.

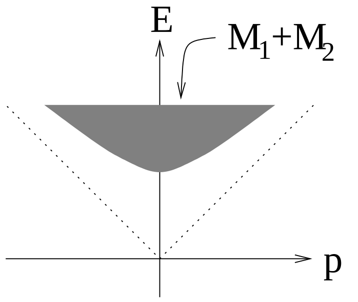
Since the transition takes place right at threshold, M will "decay" into the two particle state with zero relative momentum. Now, it follows from (2.2) that the two particle state M_1+M_2 is not BPS, even for zero relative momentum, except precisely at the CMS. Thus there will generically be no BPS force cancelation between particles M_1 and M_2, so the zero relative momentum two particle state is classically approximated by two spatially infinitely separated one particle states (to have a static configuration).

The transition across the CMS of a one particle state to a widely separated, zero momentum, two particle state should go by way of field configurations with large spatial overlap, as follows heuristically from locality and the adiabatic theorem. In particular, this implies that a state decaying just where its mass reaches the two-particle threshold of its decay products (zero "phase space") should have a diverging spatial extent as it approaches the transition. This is despite the fact that precisely at the CMS the two particle state is BPS and so may be spatially small: the relevant field configurations are the ones with widely separated centers which have large overlap with the two particle decay state away from the CMS.

The large spatial extent of a state decaying at a CMS means that the decay of such a state should be visible semiclassically in the low energy effective action. We will show in a simple U(1) example in the next section that this is, indeed, the case. This argument gives the basic physics underlying our approach to decays across CMS: *if a BPS state does decay across a CMS, that decay will be visible semiclassically in the low energy effective action even if it takes place at strong coupling from a microscopic point of view.*



A vacuum



B vacuum

Figure 3: Generic density of states in the charge sector of M at the A and B vacua.

We can analyze this picture further to predict in more detail what we should see in the classical BPS field configurations of the low energy $\mathrm{U}(1)^n$ effective action on the Coulomb branch. Since the two particle zero momentum state is classically given by two static charges at very large spatial separation, its corresponding classical low energy field configuration is just the long-range response of the massless fields to the charge sources. Thus we expect the decaying one particle state to be a static dumbbell-like configuration of the massless fields, *i.e.* one with two charge centers whose relative separation $|\vec{x}_1 - \vec{x}_2| \sim 1/\Delta X$ diverges as we approach the CMS: $\Delta X \to 0$. Here X represents the vevs of the scalar fields parameterizing the Coulomb branch.

The charge centers themselves are singularities in the low energy description. They should be regularized at an appropriate microscopic length scale, $1/\Lambda$, where Λ is some strong coupling scale of the underlying gauge theory. This cut off scale and the boundary conditions for the massless fields near the charge cores will be discussed in detail in the next section. Since the mass gap between the two particle threshold and the one particle state decreases as we approach the CMS, the two charge cores must be more and more loosely bound in this limit. Indeed, the low lying field theory spectrum in the given charge sector should be approximated by the spectrum of excitations around our static "soliton" configuration. Quantizing the collective motions of the soliton will reproduce the diminishing mass gap and the two particle continuum in the limit as we approach the CMS: the extra three (non-relativistic) degrees of freedom of the two particle continuum come from the two rotational and one vibrational mode of the dumbbell configuration. The increasing separation of the charge (and mass) centers and the weakening of their binding implies that the energy level spacing and gap of this rotator and oscillator indeed vanishes at the CMS.[1]

The resulting picture is quite intuitive: a BPS state decaying across a CMS does so by becoming an ever larger, more loosely bound state of its eventual decay products. Once the CMS is crossed, the bound state ceases to exist, and so, in particular, there will be no static BPS solutions to the low energy equations of motion and boundary conditions in this region of the Coulomb branch.

3. A U(1) toy example

In this section we will consider the low energy U(1) effective action with two real scalars X_r, $r = 1, 2$:

$$S = -\int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu X_1 \partial^\mu X_1 + \frac{1}{2} \partial_\mu X_2 \partial^\mu X_2 \right) \tag{3.1}$$

This can be thought of either as the bosonic sector of a U(1) $\mathcal{N}$=4 SYM theory with four of the six (real) scalars suppressed, or as that of a U(1) $\mathcal{N}$=2 SYM theory with

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[1] This point is made more concretely (and supersymmetrically) in [32] in semiclassical computations at weak coupling.

a flat Coulomb branch. The gauge coupling has been set to unity for simplicity and a theta angle term does not affect the energy functional which is really all we will be working with, so it has not been included in this expression. We will be looking for static solutions to this theory, and will denote spatial vectors by arrows, *e.g.* $\vec{x}$, and vectors on the X_1–X_2 plane by boldface letters, *e.g.* $\mathbf{X}$. The static sourceless equations of motion are $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \times \vec{E} = \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \times \vec{B} = 0$ and $\nabla^2 \mathbf{X} = 0$, where the electric $\vec{E}$ and magnetic $\vec{B}$ fields have been defined in the usual way. If a region contains a charge core with electric and magnetic charges (Q_E, Q_B), then by Gauss' law we have

$$\oint_{S^2_\Lambda} \vec{E} \cdot d\vec{a} = Q_E, \qquad \oint_{S^2_\Lambda} \vec{B} \cdot d\vec{a} = Q_B, \tag{3.2}$$

where the integral is over a sphere S^2_Λ of radius r_Λ enclosing the charged core.

The toy theory (3.1) by itself is too simple to be interesting, but all we need to add to it to capture the essential physics of the decay of BPS states across CMS are the presence of singularities in the vacuum manifold. In this case the vacuum manifold (Coulomb branch) is the $\mathbf{X}$ plane. In actual $\mathcal{N}$=2 or $\mathcal{N}$=4 SYM theories there are complex submanifolds of the Coulomb branch which are singularities in the effective action since they correspond to vacua at which charged states become massless, and so must be included in the effective action. In $\mathcal{N}$=2 theories these appear as curvature singularities in the low energy $\mathrm{U}(1)^n$ effective action sigma model metric on the Coulomb branch; in $\mathcal{N}$=4 theories the Coulomb branch is flat and the singularities appear as orbifold fixed points (*i.e.* curvature delta functions). For convenience and concreteness, we will include such singularities in our toy model (3.1) by simply positing that there are two singularities at points on the Coulomb branch with coordinates

$$\mathbf{X} = \mathbf{X}^E \equiv (L, 0) \tag{3.3}$$

where a particle with electric charge

$$(Q_E, Q_B) = (1, 0) \tag{3.4}$$

becomes massless, and

$$\mathbf{X} = \mathbf{X}^B \equiv (0, L) \tag{3.5}$$

where a particle with magnetic charge

$$(Q_E, Q_B) = (0, 1) \tag{3.6}$$

becomes massless.

In section 4 we generalize the calculations of this section to the full $\mathrm{U}(1)^n$ $\mathcal{N}$=4 effective action with $6n$ free real scalar fields, and in section 5 to the $\mathrm{U}(1)^n$ $\mathcal{N}$=2 effective action with $2n$ real scalar fields with curved sigma model metric. In both cases the core of the physics will be seen to be the same as that of the toy model of eqns. (3.1) and (3.3–3.6), though the combinatorics and some details of the analysis will be more complicated.

3.1 Boundary conditions and BPS bounds

As a low energy effective action, (3.1) should be considered as the first terms in a derivative expansion. Suppressing all the higher derivative terms is a good approximation as long as we do not probe the physics describing the core of the BPS state. Near such a core the $\mathrm{U}(1)$ field strength and, by supersymmetry, the values of the scalar fields $\mathbf{X}$ will grow large. Indeed, in the presence of a static source at spatial position $\vec{x}_0$ the electric and magnetic fields and the scalar fields diverge as

$$|\vec{E}|,\ |\vec{B}|,\ |\vec{\nabla}\mathbf{X}|\ \sim\ \frac{1}{r^2}, \tag{3.7}$$

where $r = |\vec{x} - \vec{x}_0|$. With an ultraviolet scale Λ suppressing derivative corrections to the $\mathrm{U}(1)$ effective action by powers of Λ, for example $\Lambda^{-2}|\partial\mathbf{X}|^4$, we see that when $|\vec{\nabla}\mathbf{X}| \sim \Lambda^2$, or at a typical distance

$$r \sim r_\Lambda \equiv \frac{1}{\Lambda} \tag{3.8}$$

from the core, the low energy $\mathrm{U}(1)$ solution ceases to be valid. Also, when expanding about a given vacuum $\mathbf{X}^0$ on the Coulomb branch there will also be irrelevant terms with polynomial coefficients, *e.g.* $\Lambda^{-6}|\mathbf{X} - \mathbf{X}^0|^2|\partial\mathbf{X}|^4$, implying the breakdown of the low energy solution when $|\mathbf{X}-\mathbf{X}^0| \gtrsim \Lambda^3 r^2$ where r is the spatial distance from a charge core. Such terms imply that the value $\mathbf{X}(\vec{x})$ of the scalars at any given spatial position $\vec{x}$ is itself reliable only up to some accuracy

$$|\delta\mathbf{X}(\vec{x})| \simeq \widehat{\Lambda}. \tag{3.9}$$

This fuzziness in the solution $\widehat{\Lambda}$ is not easy to estimate directly from the action, for it varies with $\vec{x}$ as well as with the choice of vacuum $\mathbf{X}^0$ and charges (Q_E, Q_B). We will see below how it can be determined self-consistently from static solutions.

Our strategy will be to use the low energy $\mathrm{U}(1)$ solution away from the cores ($r > r_\Lambda$) and impose appropriate boundary conditions in the vicinity of the charge cores ($r \sim r_\Lambda$). The qualitative features of these boundary conditions are easy to deduce, as we will discuss momentarily; we will see in the course of this section that only these qualitative features are important for the physics in the vicinity of a CMS—other details of the boundary conditions do not affect the results.

The basic boundary condition (3.2) for the U(1) gauge field follows from charge conservation. The boundary conditions for the scalar fields $\mathbf{X}$ are more subtle. Suppose we are looking for a solution with only electric charge $(Q_E, Q_B) = (1,0)$. The solutions to the equation of motion near a point-like charge source together with supersymmetry imply that $|\mathbf{X}|$ diverges as one approaches the source. On the cutoff sphere S^2_Λ around the source $\mathbf{X}$ takes some finite values. Suppose they are approximately constant, $\mathbf{X} = \mathbf{X}_\Lambda$ on S^2_Λ. The solution interior to S^2_Λ is then approximately that of a state with charge $(1,0)$ in the vacuum $\mathbf{X}_\Lambda$. The minimum mass of this state is given by its BPS mass $M_{(1,0)}$ which is some function of the vacuum $\mathbf{X}_\Lambda$. This mass is minimized at the singularities $\mathbf{X} = \mathbf{X}^E$ where the $(1,0)$ charged BPS states become massless by assumption. Thus energy minimization implies that the scalar fields satisfy the approximate Dirichlet boundary condition

$$\mathbf{X} \simeq \mathbf{X}^E \quad \text{within} \quad B^3_\Lambda \quad \text{for electric charges,} \tag{3.10}$$

and a similar argument implies

$$\mathbf{X} \simeq \mathbf{X}^B \quad \text{within} \quad B^3_\Lambda \quad \text{for magnetic charges,} \tag{3.11}$$

where B^3_Λ is a ball of approximate radius r_Λ around each charge core and $\mathbf{X} \simeq \mathbf{X}_E$ means only that $\mathbf{X}$ pass within distance $\widehat{\Lambda}$ to $\mathbf{X}_E$ on the Coulomb branch. We will call these conditions on the scalar fields "fuzzy ball" boundary conditions, a kind of weak form of Dirichlet boundary conditions. Figure 5 below gives an illustration of these fuzzy ball boundary conditions on the Coulomb branch in our toy example.

So far we have only argued that the fuzzy ball boundary conditions express a *tendency* for the scalar field to approach those values. The key step in our analysis of BPS states near CMS is to *assume* the above fuzzy ball boundary conditions on the scalar fields. The idea is that the fuzzy ball boundary condition is all we can physically demand of the low energy solution since it is not accurate on spatial resolutions less than r_Λ nor for field value resolutions less than $\widehat{\Lambda}$. We then use them to solve for BPS configurations of charge (Q_E, Q_B) by positing some number of electric sources of total charge Q_E and magnetic sources of total charge Q_B, minimizing their energies, and checking that the fuzzy ball boundary conditions are self consistent, *i.e.* that for a spatial cutoff length scale r_Λ consistent with (3.8) the static solution has $\widehat{\Lambda} \ll |\mathbf{X}^E - \mathbf{X}^B|$.

In the limit as the vacuum approaches a CMS, as we have described qualitatively in the last section and will see explicitly below, the size of the field configuration grows, and so the relative size of the cutoff region r_Λ to the field configuration shrinks. Thus in this limit one expects that our fuzzy ball boundary condition to become a simple

Dirichlet boundary condition, *i.e.* $\widehat{\Lambda} \to 0$. Indeed, as shown in [21] for spherically symmetric 1/2 BPS states in $\mathcal{N}$=4 SYM theory, a simple (spherical) Dirichlet boundary condition exactly reproduces the BPS bound; near the CMS a decaying 1/4 BPS state will become to arbitrary accuracy a very loosely bound state of two 1/2 BPS states, and so the fuzzy ball boundary conditions should approach Dirichlet boundary conditions to the same accuracy.

Furthermore, far enough away from the CMS (at distances greater than or on the order of Λ on the Coulomb branch) the boundary conditions (3.10) and (3.11) will break down entirely, and cannot be satisfied even approximately. Thus, far out on the Coulomb branch, where monopole states are well-described by semiclassical nonabelian field configurations, our low energy $\mathrm{U}(1)$ description breaks down entirely.[2] In this sense our description of BPS states is complementary to the semiclassical one.

Now let us use these boundary conditions to derive the low energy field equations satisfied by our static BPS solutions. We do this by the familiar method of minimizing the energy of the field configurations. From the action (3.1), an energy density functional for static configurations can be constructed by the usual canonical methods

$$\mathcal{E} = \frac{1}{2}\left[\vec{E}^2 + \vec{B}^2 + (\vec{\nabla}\mathbf{X})^2\right]. \tag{3.12}$$

This can be rewritten in the following fashion

$$\begin{aligned}\mathcal{E} = &\frac{1}{2}\left[(\vec{E} - \cos\alpha\vec{\nabla}X_1 + \sin\alpha\vec{\nabla}X_2)^2 + (\vec{B} - \sin\alpha\vec{\nabla}X_1 - \cos\alpha\vec{\nabla}X_2)^2\right] \\ &+ \cos\alpha(\vec{E}\cdot\vec{\nabla}X_1 + \vec{B}\cdot\vec{\nabla}X_2) + \sin\alpha(\vec{B}\cdot\vec{\nabla}X_1 - \vec{E}\cdot\vec{\nabla}X_2).\end{aligned} \tag{3.13}$$

Integrating this expression over three dimensional space gives the mass of the configuration as

$$\begin{aligned}M = &\int d^3\vec{x}\,\frac{1}{2}\left[(\vec{E} - \cos\alpha\vec{\nabla}X_1 + \sin\alpha\vec{\nabla}X_2)^2 + (\vec{B} - \sin\alpha\vec{\nabla}X_1 - \cos\alpha\vec{\nabla}X_2)^2\right] \\ &+ \sum_{I=0}^{n}\left\{\cos\alpha\oint_{S_I^2}(X_1\vec{E} + X_2\vec{B})\cdot d\vec{a} + \sin\alpha\oint_{S_I^2}(X_1\vec{B} - X_2\vec{E})\cdot d\vec{a}\right\}\end{aligned} \tag{3.14}$$

where the boundaries S_I^2 are spheres around each charge source and one sphere at infinity. We have used the divergence-free equation of motion for the electric and magnetic fields $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ away from the sources. If we label the boundaries so that the $I = 0$ boundary is the one at infinity and the $I = i \neq 0$ are the ones around the

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[2] More precisely, it becomes equivalent to the Dirac monopole solution which carries no information about the structure of the state.

charge sources, then our approximate Dirichlet boundary conditions (3.10) and (3.11) imply that at the ith boundary the scalars $\mathbf{X}$ take the constant values $\mathbf{X}^i = \mathbf{X}^E$ or $\mathbf{X}^B$ while at infinity they take their asymptotic values $\mathbf{X}^0$, the Coulomb branch coordinates of the vacuum. Since these are constants they can be taken outside the surface integrals which then give by (3.2) the charges (Q_E^I, Q_B^I) enclosed by each sphere, so that

$$\begin{aligned} M = & \int d^3\vec{x}\, \frac{1}{2} \left[(\vec{E} - \cos\alpha \vec{\nabla} X_1 + \sin\alpha \vec{\nabla} X_2)^2 + (\vec{B} - \sin\alpha \vec{\nabla} X_1 - \cos\alpha \vec{\nabla} X_2)^2 \right] \\ & + \sum_{I=0}^{n} \left[\cos\alpha (X_1^I Q_E^I + X_2^I Q_B^I) + \sin\alpha (X_1^I Q_B^I - X_2^I Q_E^I) \right]. \end{aligned} \tag{3.15}$$

Here $(Q_E^0, Q_B^0) = (Q_E, Q_B)$ is the total charge of the configuration. By charge conservation

$$(Q_E, Q_B) = -\sum_{i=1}^{n} (Q_E^i, Q_B^i) \tag{3.16}$$

so, defining the position vectors of the singularities on the Coulomb branch relative to the vacuum by

$$\boldsymbol{\xi}^i \equiv \mathbf{X}^i - \mathbf{X}^0, \tag{3.17}$$

and since the first line in (3.15) is positive definite we get the bound

$$M \geq \cos\alpha (\xi_1^i Q_E^i + \xi_2^i Q_B^i) + \sin\alpha (\xi_1^i Q_B^i - \xi_2^i Q_E^i), \tag{3.18}$$

where the sum on i over the charge sources is implied. The tightest bound on M comes from maximizing the right hand side with respect to α, giving the BPS bound

$$M \geq \sqrt{(\xi_1^i Q_E^i + \xi_2^i Q_B^i)^2 + (\xi_1^i Q_B^i - \xi_2^i Q_E^i)^2}. \tag{3.19}$$

Thus the minimal energy configurations saturating the inequality are solutions of the BPS equations

$$\begin{aligned} \vec{E} &= \cos\alpha \vec{\nabla} X_1 - \sin\alpha \vec{\nabla} X_2, \\ \vec{B} &= \sin\alpha \vec{\nabla} X_1 + \cos\alpha \vec{\nabla} X_2 \end{aligned} \tag{3.20}$$

where

$$\tan\alpha = \frac{\xi_1^i Q_B^i - \xi_2^i Q_E^i}{\xi_1^i Q_E^i + \xi_2^i Q_B^i}. \tag{3.21}$$

The rest of this section is devoted to analyzing the solutions to (3.20) subject to our boundary conditions (3.10) and (3.11) minimizing the BPS mass (3.19).

In finding the solutions to the BPS equations (3.20) we have the discrete choice of charges at the sources, *i.e.* the set of integer charges (Q^i_E, Q^i_B), which are constrained by charge conservation (3.16). In our simple example, though, since only electric charges can flow into the $\mathbf{X}^E = (L,0)$ singularity and magnetic into the $\mathbf{X}^B = (0,L)$ one, we see that there are only two possible values for the $\boldsymbol{\xi}^i$:

$$\boldsymbol{\xi}^E \equiv \mathbf{X}^E - \mathbf{X}^0 \tag{3.22}$$

for sources with only electric charges ($Q^i_B = 0$), or

$$\boldsymbol{\xi}^B \equiv \mathbf{X}^B - \mathbf{X}^0 \tag{3.23}$$

for sources with only magnetic charges ($Q^i_E = 0$); see figure 4.

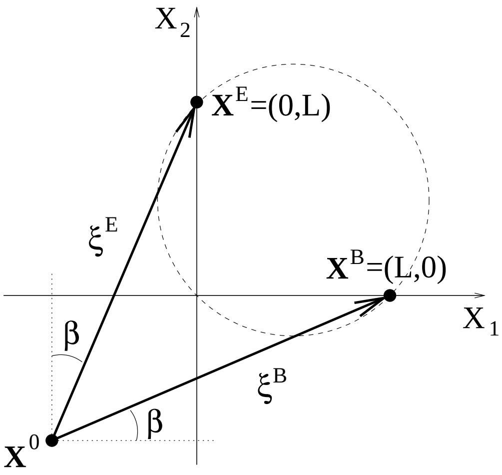

Figure 4: Coordinates on the Coulomb branch of the U(1) toy model. $\mathbf{X} = \mathbf{X}^0$ denotes the vacuum, while $\boldsymbol{\xi}^E$ and $\boldsymbol{\xi}^B$ are vectors from the vacuum to the electric and magnetic singularities, respectively. The dashed line is the CMS.

This leads to a simplification of the BPS mass formula (3.19) to

$$M(Q_E, Q_B) = \sqrt{(Q_E)^2 \boldsymbol{\xi}^E \cdot \boldsymbol{\xi}^E + (Q_B)^2 \boldsymbol{\xi}^B \cdot \boldsymbol{\xi}^B + 2Q_E Q_B \boldsymbol{\xi}^E \times \boldsymbol{\xi}^B}. \tag{3.24}$$

In the case of purely electric or purely magnetic total charges, the bound further simplifies to

$$\begin{aligned} M(Q_E, 0) &= |Q_E||\boldsymbol{\xi}^E|, \\ M(0, Q_B) &= |Q_B||\boldsymbol{\xi}^B|, \end{aligned} \tag{3.25}$$

i.e. the charge times the distance to relevant singularity on the Coulomb branch.

The CMS for the decay of a (Q_E, Q_B) state to a $(Q_E, 0)$ plus a $(0, Q_B)$ state is then given by the curve on the Coulomb branch for which $M(Q_E, Q_B) = M(Q_E, 0) + M(0, Q_B)$. From (3.24) and (3.25) this implies that $|\boldsymbol{\xi}^E||\boldsymbol{\xi}^B| = |\boldsymbol{\xi}^E \times \boldsymbol{\xi}^B|$, or that $\boldsymbol{\xi}^E$ be perpendicular to $\boldsymbol{\xi}^B$. This describes a circle on the Coulomb branch, shown as the dashed curve in figure 4.

3.2 Spike and prong solutions

We will now illustrate some simple solutions to the BPS equations and boundary conditions in our toy model. To simplify the algebra and make the basic points clear we choose the vacuum to be symmetrically placed at the point $\mathbf{X}^0 = (X^0, X^0)$ on the Coulomb branch.

The simplest case is where there is a single charge source of purely electric or magnetic charge. For example, suppose $Q_B = 0$. By (3.20) and since $\vec{B} = 0$ (because all the magnetic charges in the problem vanish) and $\tan\alpha = -\xi_2^E/\xi_1^E$, the component of $\mathbf{X}$ perpendicular to $\boldsymbol{\xi}^E$ on the Coulomb branch must be constant, while its component parallel to $\boldsymbol{\xi}^E$ satisfies Laplace's equation with a source of total charge Q_E. The solution is thus simply

$$X_1 - X^0 = \frac{Q_E \cos\beta}{4\pi|\vec{x} - \vec{x}_E|}, \qquad X_2 - X^0 = \frac{Q_E \sin\beta}{4\pi|\vec{x} - \vec{x}_E|}, \tag{3.26}$$

which implies electric and magnetic fields

$$\vec{E} = \frac{Q_E}{4\pi} \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^3}, \qquad \vec{B} = 0. \tag{3.27}$$

Here β is the angle on the Coulomb branch shown in figure 4, and $\vec{x}_E$ is the spatial location of the electric charge source. Since both α and $\alpha + \pi$ satisfy (3.21) the sign of Q_E in (3.26) is undetermined; we will determine it below.

Following the brane picture of [19] we will call this a "spike" solution. In the brane picture, the space-time coordinates are identified with the worldvolume coordinates of a D3-brane, and the $\mathbf{X}$ scalars take values in the dimensions transverse to the brane. Thus the solution (3.26) can be thought of as describing a semi-infinite spike-like deformation of the brane in this enlarged space. As described in [19] such a solution can be identified with a fundamental string ending on the D3-brane. Likewise, the solution for a purely magnetically charged BPS state is given by

$$X_1 - X^0 = \frac{Q_B \sin\beta}{4\pi|\vec{x} - \vec{x}_B|}, \qquad X_2 - X^0 = \frac{Q_B \cos\beta}{4\pi|\vec{x} - \vec{x}_B|}, \tag{3.28}$$

which can be interpreted as a spike with mass per unit length equal to that of a D-string attached to the D3-brane.

We still need to apply our fuzzy ball boundary conditions (3.10) and (3.11) to these solutions. These boundary conditions fix the undetermined signs of Q_E and Q_B in (3.26) and (3.28). The signs of these solutions are appropriate for

$$X^0 \leq L/2 \tag{3.29}$$

or, equivalently, for $\beta > -\pi/4$, which we will assume from now on. The boundary conditions also determine the radii of the cutoff spheres S^2_Λ to be

$$r_\Lambda = \frac{|Q_E|}{4\pi\ell} \tag{3.30}$$

implying a cutoff energy scale of $\Lambda \sim \ell$, where $\ell = |\boldsymbol{\xi}^E| = |\boldsymbol{\xi}^B|$ is the distance from the vacuum to either Coulomb branch singularity. This is indeed the appropriate cutoff scale for the low energy effective theory since there are new light charged particles with masses (3.25) proportional to this Coulomb branch distance.

The simple spike solutions found above do not have any interesting structure on the Coulomb branch: they exist for any vacuum and obey exactly Dirichlet boundary conditions with spherical boundaries which we expected to be only approximately satisfied in general. The situation becomes more interesting when we turn to states with two or more charge sources.

First consider a purely electrically charged two center solution with $Q_E = Q_E^1 + Q_E^2$. It is easy to find solutions with the fuzzy ball boundary conditions (3.10) and (3.11). Indeed,

$$\begin{aligned} X_1 - X^0 &= \frac{Q_E^1 \cos\beta}{4\pi|\vec{x} - \vec{x}_{E,1}|} + \frac{Q_E^2 \cos\beta}{4\pi|\vec{x} - \vec{x}_{E,2}|}, \\ X_2 - X^0 &= \frac{Q_E^1 \sin\beta}{4\pi|\vec{x} - \vec{x}_{E,1}|} + \frac{Q_E^2 \sin\beta}{4\pi|\vec{x} - \vec{x}_{E,2}|}, \end{aligned} \tag{3.31}$$

does the job as long as the distance

$$r_{12} = |\vec{x}_{E,1} - \vec{x}_{E,2}| \tag{3.32}$$

between the two sources satisfies

$$r_{12} > \frac{1}{4\pi\ell}. \tag{3.33}$$

That is to say, as long as sources are sufficiently far apart one can enclose the sources in disjoint spheres within which $\mathbf{X}$ take the value $(L, 0)$. This behavior is sensible: since a static configuration of two charged BPS states with commensurate charge vectors is itself BPS, there should exist a static low energy configuration for any separation of the sources. The restriction (3.33) that they not be too close just reflects the fact that the low energy description of their interaction breaks down on scales $r \lesssim \ell^{-1}$.

Now focus on a dyonic state with both electric and magnetic charges. For simplicity we take $(Q_E, Q_B) = (1, 1)$. By (3.21) we see that $\alpha = \pi/2$, so the BPS equations (3.20) reduce to

$$\vec{E} = \vec{\nabla} X_1, \qquad \vec{B} = \vec{\nabla} X_2. \tag{3.34}$$

From the linearity of the BPS equations and since $\mathbf{X}$ are harmonic functions, we expect the solution to be given at least approximately by a solution of the form

$$\begin{aligned} X_1 - X^0 &= \frac{\cos\gamma}{4\pi|\vec{x} - \vec{x}_E|} + \frac{\sin\delta}{4\pi|\vec{x} - \vec{x}_B|}, \\ X_2 - X^0 &= \frac{\sin\gamma}{4\pi|\vec{x} - \vec{x}_E|} + \frac{\cos\delta}{4\pi|\vec{x} - \vec{x}_B|}. \end{aligned} \tag{3.35}$$

Compatibility with Gauss' law and the BPS equations then imply that $\gamma = \delta = 0$ so

$$\begin{aligned} X_1 - X^0 &= \frac{1}{4\pi|\vec{x} - \vec{x}_E|}, \\ X_2 - X^0 &= \frac{1}{4\pi|\vec{x} - \vec{x}_B|}. \end{aligned} \tag{3.36}$$

We want to check whether there are values of the electric and magnetic charge source centers, $\vec{x}_E$ and $\vec{x}_B$, for which this solution satisfies our fuzzy ball boundary conditions (3.10) and (3.11). What this asks is that the $\mathbf{X}$ values taken by this solution enter (and go through) a small ball around $\mathbf{X}^E = (L, 0)$ as $\vec{x} \to \vec{x}_E$ and a small ball around $\mathbf{X}^B = (0, L)$ as $\vec{x} \to \vec{x}_B$. As $\vec{x} \to \vec{x}_E$ the above solution approaches

$$\begin{aligned} X_1 - X^0 &= \frac{1}{4\pi r_\Lambda}, \\ X_2 - X^0 &\simeq \frac{1}{4\pi r_{EB}}, \end{aligned} \tag{3.37}$$

where

$$r_\Lambda = |\vec{x} - \vec{x}_E|, \qquad r_{EB} = |\vec{x}_E - \vec{x}_B|. \tag{3.38}$$

The $X_1 = L$ fuzzy boundary condition is achieved if the spatial cutoff length scale around the electric source is taken to be

$$r_\Lambda = \frac{1}{4\pi(L - X^0)}, \tag{3.39}$$

while the $X_2 = 0$ boundary condition fixes the separation of the electric and magnetic charge sources to be [22]

$$r_{EB} = \frac{1}{4\pi(-X^0)}. \tag{3.40}$$

Furthermore, on the sphere of radius r_Λ around the electric source at $\vec{x}_E$, by (3.36) X_2 takes values in a range $|X_2| < \widehat{\Lambda}$ with

$$\widehat{\Lambda} = \frac{(X^0)^2 L}{1 - 2(X^0/L)}. \tag{3.41}$$

The boundary conditions at the magnetic source give the same results.

The conditions for our solution to be consistent are thus that

$$X^0 < 0 \tag{3.42}$$

since the distance r_{EB} is necessarily positive, and that $\widehat{\Lambda}$ is less than the distance $\sim L$ between the electric and magnetic singularities on the Coulomb branch, which implies

$$|X^0| \lesssim L. \tag{3.43}$$

These conditions illustrate the qualitative picture of decaying BPS states that we have argued for in the previous section. Referring to figure 4 we see that $X^0 = 0$ is the location of the CMS. So (3.42) implies that a solution exists on one side of the CMS, and ceases to exist once one crosses it. Furthermore, the region of self-consistency of the fuzzy ball boundary conditions includes the CMS; indeed, the fuzzy ball boundary conditions become more and more accurately simple Dirichlet boundary conditions as one approaches the CMS since (3.41) implies that

$$\widehat{\Lambda} \sim (X^0)^2/L \tag{3.44}$$

vanishes as $X^0 \to 0$. Finally, (3.40) shows that the spatial distance between the charge cores diverges as we approach the CMS. Figure 5 illustrates the qualitative behavior of these solutions for different values of X^0. These figures show the projection of our low energy solutions on the Coulomb branch. Note that this projection suppresses the spatial extent of the field configuration; in particular note that as the vacuum approaches the CMS, though the projection on the Coulomb branch degenerates to a string web configuration, in space-time it expands as the positions of the charge centers diverge. Figure 5(a) shows how the fuzzy ball boundary conditions break down too far from the CMS, while figure 5(b) illustrates how the image of the solution in the Coulomb branch more closely approximates a string web configuration (shown as heavy lines) in the limit as the vacuum approaches the CMS.

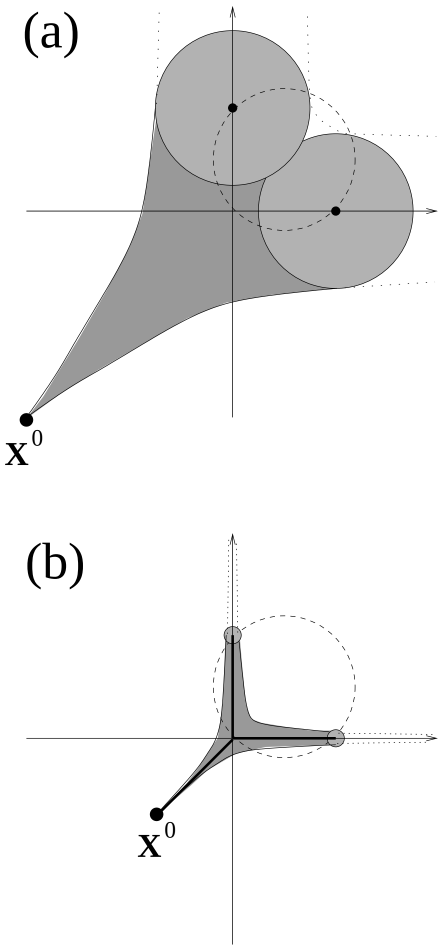

Figure 5: The shaded regions are the images of the low energy dyon solution in the Coulomb branch for various values of the vacuum $\mathbf{X}^0$. They are only valid outside the lighter circles about the singularities which denote the $\widehat{\Lambda}$ fuzzy ball boundary conditions.

We will refer to these multi-center solutions as brane prongs, again in analogy to the brane picture where the Coulomb branch is realized as additional transverse spatial dimensions. The breakdown of the low energy solution when X^0 is large compared to L so that (3.43) is not satisfied simply reflects the fact that in this case the uncertainties in the locations of $\vec{x}_E$ and $\vec{x}_B$ are of the order of the separation r_{EB} itself, so now one cannot really tell if there are two centers of charge in the low energy theory or one. Thus this configuration effectively looks like one brane spike from a $(1,1)$ dyon.

Finally, it is straightforward to generalize the calculations of this section to the case of arbitrary position of the $\mathbf{X}^0$ of the vacuum, arbitrary (Q_E, Q_B) charge sector, and arbitrary value for the low energy $\mathrm{U}(1)$ coupling g. The result is a static solution for the (Q_E, Q_B) dyon outside the CMS for any relatively prime Q_E and Q_B, which decays into Q_E charge $(1,0)$ plus Q_B charge $(0,1)$ particles inside the CMS. A two particle state with mutually local charges (*i.e.* commensurate charge vectors) is BPS and so a static solution exists for any spatial separation. Thus our low energy method is not able to determine whether there is a one particle bound state at threshold for electric and magnetic charges which are not mutually prime.

3.3 Recovery of a string junction picture

We have thus explicitly verified all the qualitative features expected of a BPS state decaying across a CMS. The last thing to show is how this information extracted from the low energy effective action on the Coulomb branch is equivalent to a string web picture of the BPS states.

The string web picture is recovered in two stages. First, in the limit as the vacuum approaches the CMS, the image of the low energy field configuration in the Coulomb branch degenerates to a collection of curves on that space. These curves are the images of brane spike solutions which carry precisely the energy per unit length of the corresponding (p,q) string [19]. This is easy to see in our example by taking $X^0 \to 0$ in (3.36) and comparing to our electric and magnetic spike solutions (3.26) and (3.28) (with $\beta = 0$). Thus arbitrarily close to the CMS the projection onto the Coulomb branch of all the (Q_E, Q_B) charged states look like strings of appropriate tension stretched across the Coulomb branch.

In string theory, 3-pronged string states that stretch between D-branes satisfy charge conservation and tension balance. For example, the above configuration corresponds to a $(1,0)$ (fundamental string), a $(0,1)$ (D-string)and a $(1,1)$ string stretched between three D3-branes. The three prongs or string legs meet at a common point, the junction, which remains point-like even arbitrarily close to the CMS. In the field theory picture discussed in this section, we have seen that the leg corresponding to the decaying $(1,1)$ dyon grows in spatial size (*i.e.*, along the D3-brane worldvolume) near the CMS. Arbitrarily close to the CMS, the configuration resembles two separate strings that end on the D3-brane corresponding to the decaying $U(1)$. Thus there does not appear to be a point-like junction as the decaying configuration approaches the CMS in the field theory picture.

The difference between the two perspectives is the result of different orders of limits. Consider the Dirac-Born-Infeld action for the D3-brane corresponding to the decaying dyon, treating it as a probe in a background of other D3-branes. The brane prong corresponding to the decaying $(1,1)$ dyon then ends on other D3-branes that are treated as a fixed background. Looking at the quadratic terms and comparing with the low energy effective action we have used above, we can see that the scalars X in the field theory with mass dimension unity and the coordinates in the brane transverse space, x, are related by $X = x/\alpha'$, where α' is the string length squared. The separation between the charge centers in the D-brane worldvolume is

$$r_{EB} \sim 1/(-X_0) = \alpha'/(-x_0). \tag{3.45}$$

The low energy field theory is a good approximation in the $\alpha' \to 0$ limit holding the scalar vevs, including X_0, fixed. On the other hand, in the string junction/geodesic picture in string theory, the coordinates x_0 are what are held fixed: taking $\alpha' \to 0$ to suppress higher stringy corrections gives a vanishing separation r_{EB} between the charge centers, *i.e.* a pointlike junction. For $x_0 = 0$, the separation is indeterminate, which corresponds to the two strings ending anywhere on the D-brane. This recovers the string theory result and it further illustrates that the point-like junction is not visible within the field theory approximation.

The second stage is to extend this picture to the rest of the Coulomb branch by continuity and matching to the BPS mass formula. This can be done uniquely since the given tension of a (Q_E, Q_B) string (namely $T(Q_E, Q_B) = \sqrt{Q_E^2 + Q_B^2}$ in our example where we have set the coupling $g = 1$) fixes the path away from the CMS it must be extended in in order to give the contribution to the state's mass required to match its BPS mass. That this path is a geodesic on the Coulomb branch follows from the form of the BPS mass formula [34]. In the case of our toy example, the Coulomb branch is flat so the geodesics are straight lines and it is clear that the string web picture of the BPS states extends without obstruction over the whole Coulomb branch.

The result is therefore the prediction in our toy model that outside the CMS the spectrum consists of all (Q_E, Q_B)-charged dyons with Q_E and Q_B relatively prime, while inside the CMS only the $(\pm 1, 0)$ and $(0, \pm 1)$ states are in the spectrum. Of course, this is only a toy example. We will perform essentially the same analysis in the next section for the $\mathcal{N}$=4 SYM theories.

4. BPS states in $\mathcal{N}$=4 SU(N) theories

In this section we turn to an analysis of the static BPS field configurations of given total electric and magnetic charges in the low energy $\mathrm{U}(1)^{N-1}$ theory on the moduli space of an $\mathcal{N}$=4 SU(N) SYM theory. The $\mathcal{N}$=4 superalgebra allows both 1/2 and 1/4 BPS particle states. From the expression for the BPS mass, there are no CMS for 1/2 BPS states, but there are CMS for 1/4 BPS states to decay.

In section 4.1 below we will effectively rederive the $\mathcal{N}$=4 BPS mass formula, as well as the BPS equations in the low energy theory. We then find solutions to the BPS equations subject to our fuzzy ball boundary conditions and show how they reproduce a string web picture on the moduli space of the $\mathcal{N}$=4 theory.

However this is not the usual string web picture of BPS states in $\mathcal{N}$=4 theories derived from string theory. In this picture [15] the $\mathrm{U}(N)$ SYM theory is realized by open strings and D1-branes ending on a set of N parallel D3-branes in a flat 10-dimensional IIB string theory background. The relative positions of the N D3-branes in the transverse 6-dimensional space form the $6N$-dimensional moduli space of the $\mathcal{N}$=4 theory. BPS states are described by webs of (p, q) strings stretching between the D3-branes [6]; a web has charge (Q_E, Q_B) with respect to a given low energy $\mathrm{U}(1)$ factor if it ends on the associated D3-brane with a (Q_E, Q_B) string. These string webs thus live in a 6-dimensional space stretched between N point sources, which is different from the picture we derive below as string webs stretched in the $6N$-dimensional moduli space and ending on $6(N-1)$-dimensional singular submanifolds. In section 4.2 below we show how the 6-dimensional string theory webs are obtained from our $6N$-dimensional webs by a simple mapping. The basic reason this works is that the $6N$-dimensional moduli space $\mathcal{M}$ of the $\mathrm{U}(N)$ theory is $\mathcal{M} = (\mathbb{R}^6)^N/S_N$ where the permutation group S_N interchanging the $\mathbb{R}^6$ factors is the Weyl group of $\mathrm{U}(N)$. Because of this simple relation between the transverse $\mathbb{R}^6$ of the D3-branes and $\mathcal{M}$, webs on $\mathcal{M}$ can be uniquely mapped onto webs in $\mathbb{R}^6$, recovering the string picture.

A similar construction presumably also works for the SO and Sp $\mathcal{N}$=4 SYM theories, since their Weyl groups are also fairly simple, differing from the action of the SU Weyl group by the addition of $\mathbb{Z}_2$ identifications on each $\mathbb{R}^6$. This will fold webs on $\mathcal{M}$ down to webs on $\mathbb{R}^6/\mathbb{Z}_2$, thus realizing the string web picture of BPS states for these theories which are found by placing D3-branes in the background of an appropriate orientifold O3-plane. The action of the Weyl groups of the exceptional groups is more complicated, and it is not clear whether any folding of our webs on $\mathcal{M}$ to a 6-dimensional space can be performed. This may “explain” why there are no D3 brane constructions of the $\mathcal{N}$=4 SYM theories with exceptional gauge groups.

4.1 BPS states as webs on the moduli space

The generic point on the moduli space of an $\mathcal{N}$=4 SYM theory with gauge group $\mathrm{SU}(N)$ is described by a low energy $\mathrm{U}(1)^{N-1}$ $\mathcal{N}$=4 theory. A convenient trick to simplify the algebra will be look at the $\mathrm{U}(N)$ theory instead, which has a $\mathrm{U}(1)^N$ effective description, but restrict ourselves to states which are neutral under the extra $\mathrm{U}(1)$.

Now, each $\mathrm{U}(1)$ $\mathcal{N}$=4 multiplet has six real scalar fields, so the moduli space is locally coordinatized by the values of the $6N$ scalars X_{ra} where $r = 1, \ldots, N$ and

$a = 1, \ldots, 6$. The other bosonic massless fields are the electric and magnetic fields $\vec{E}_r$ and $\vec{B}_r$ for each U(1) factor. We are interested in static low energy solutions carrying electric and magnetic charges (Q_{Er}, Q_{Br}) with respect to these fields. We normalize the fields so that the low energy effective action for the bosonic fields is

$$S = \int d^4x \sum_{r=1}^{N} \left(\frac{1}{4g^2} F_{r\,\mu\nu} F_r^{\mu\nu} + \frac{\vartheta}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_r^{\mu\nu} F_r^{\rho\sigma} + \frac{1}{4g^2} \sum_{a=1}^{6} \partial_\mu X_{ra} \partial^\mu X_{ra} \right), \tag{4.1}$$

with the $\vec{E}_r$ and $\vec{B}_r$ related to $F_r^{\mu\nu}$ in the usual way. The fact that all the U(1)'s have the same coupling and no cross-couplings reflects the specific basis of $\mathrm{U}(1)^N$ we have chosen. The fact that the coupling is independent of the X_{ra} reflects the flatness of the metric on $\mathcal{M}$ for $\mathcal{N}$=4 theories. We normalize the charges so that

$$\begin{aligned} \oint_{S^2} \vec{E}_r \cdot d\vec{a} &= g^2 Q_{Er}, \\ \oint_{S^2} \vec{B}_r \cdot d\vec{a} &= g^2 Q_{Br}. \end{aligned} \tag{4.2}$$

Thus, the Dirac quantization condition plus the effect of the theta angle imply that the charges are quantized as

$$Q_{Er} = n_{Er} + \frac{\vartheta}{2\pi} n_{Br} \qquad \text{and} \qquad Q_{Br} = \frac{4\pi}{g^2} n_{Br} \tag{4.3}$$

for some integers n_{Er} and n_{Br}.

4.1.1 Moduli space and boundary conditions

Globally, the moduli space is

$$\mathcal{M} = \mathbb{R}^{6N}/S_N \tag{4.4}$$

where S_N acts by permuting the N $\mathbb{R}^6$ factors. Suppose we have some ordering of points in $\mathbb{R}^6$ (say "alphabetical" ordering on their six coordinates in some given basis) denoted by $X_{1a} \geq X_{2a}$ where X_1 and X_2 are two points in $\mathbb{R}^6$. Then we can take as a fundamental domain of S_N on $\mathbb{R}^{6N}$ the set of all N 6-vectors satisfying

$$X_{1a} \geq X_{2a} \geq \cdots \geq X_{Na}. \tag{4.5}$$

The boundaries of this wedge-like convex domain are identified by the S_N action.

The locus of fixed points of the action of S_N are orbifold singularities of $\mathcal{M}$ where BPS states of certain charges become massless. The typical such singularity is the fixed line under interchange of, say, X_{1a} with X_{2a}. The result is a $6(N-1)$-dimensional manifold of $\mathbb{Z}_2$ singularities at $X_{1a} = X_{2a}$, with all the other X_{ra} arbitrary for $r > 2$. At this singularity states with charges $Q_{E1} = -Q_{E2}$ and $Q_{B1} = -Q_{B2}$ arbitrary and $Q_{Er} = Q_{Br} = 0$ for $r > 2$ become massless. Actually all such $6(N-1)$-dimensional manifolds of $\mathbb{Z}_2$ singularities are identified by the S_N orbifolding. However, which charged states become massless there depends on the direction the singularity is approached from. Thus it is convenient to treat $\mathcal{M}$ as the wedge (4.5) with different charge images of the $\mathbb{Z}_2$ singularity on its boundary.

In addition, these images of the $\mathbb{Z}_2$ singularities may intersect one another along submanifolds of smaller dimension where states charged under three or more $\mathrm{U}(1)$ factors become massless. In fact, on the 6-dimensional submanifold of $\mathcal{M}$ where all the X_{ra}, $r = 1, \ldots, N$ are equal, BPS states of arbitrary charge become massless. Since we are only considering states which are neutral with respect to the diagonal $\mathrm{U}(1)$ factor (to decouple the extra $\mathrm{U}(1)$ of the $\mathrm{U}(N)$ gauge group), all configurations will be invariant under translations in $\mathcal{M}$ by any such vector of equal X_r's.

With this description of the moduli space and its singularities in hand, we are ready to solve for static low energy field configurations in a given charge sector subject to our fuzzy ball boundary conditions. Since we are looking for BPS configurations which preserve at least 1/4 of the supersymmetries, we can restrict ourselves to configurations which lie in some fixed two-dimensional subspace of each $\mathbb{R}^6$ factor, which we will take to be that given by the X_{ra} with $a = 1, 2$. Furthermore, the residual ($\mathcal{N} = 1$) supersymmetry implies that the BPS configurations must depend holomorphically on the $X_{r1} + iX_{r2}$ complex coordinates. It will thus be convenient to introduce a notation in which all quantities are assembled into N component complex vectors defined by

$$\begin{aligned} X_r &\equiv X_{r1} + iX_{r2}, \\ \vec{F}_r &\equiv \vec{E}_r + i\vec{B}_r, \\ Q_r &\equiv Q_{Er} + iQ_{Br}. \end{aligned} \tag{4.6}$$

Denote the total charge of our configuration by Q_r^0. Suppose our configuration has M charge centers with (approximate) spatial positions $\vec{x} = \vec{x}_i$, with charges labeled by Q_r^i, $i = 1, \ldots, M$. (We will mainly concentrate on the case of prong solutions with $M = 2$ below.) Then charge conservation implies

$$\sum_{I=0}^{M} Q_r^I = 0. \tag{4.7}$$

Label the boundaries of space by the spheres S_I^2 where S_0^2 refers to the sphere at infinity, while the S_i^2 for $i = 1, \ldots, M$ are small spheres around the M charge sources at $\vec{x} = \vec{x}_i$.

The Gauss' law implies

$$\oint_{S_I^2} d\vec{a}\cdot\vec{F}_r = g^2 Q_r^I. \tag{4.8}$$

The scalars approach

$$X_r = X_r^0 \quad \text{on } S_0^2, \tag{4.9}$$

where X_r^0 are the moduli space coordinates of the vacuum. Our fuzzy ball boundary conditions imply that the X_r will approach (at least approximately) constant values on the S_i^2 boundaries, which we will denote

$$X_r \simeq X_r^i \quad \text{on } S_i^2. \tag{4.10}$$

Furthermore, the X_r^i are constrained to lie in only those singular submanifolds of $\mathcal{M}$ where states of charge Q_r^i become massless. From the above description of $\mathcal{M}$ the X_r^i must live on the appropriate singular submanifold where a BPS state with its associated charges becomes massless.

4.1.2 BPS bound

The mass of our configuration is computed by integrating the field energy density (with an implicit sum over repeated indices)

$$M = \frac{1}{2g^2}\int d^3\vec{x}\,(\vec{F}_r^*\cdot\vec{F}_r + \vec{\nabla}X_r^*\cdot\vec{\nabla}X_r). \tag{4.11}$$

This can be rewritten as

$$M = \frac{1}{2g^2}\int d^3\vec{x}\,|\vec{F}_r - A_{rs}\vec{\nabla}X_s|^2 + \frac{1}{2g^2}\,\mathrm{Re}\int d^3\vec{x}\,A_{rs}\vec{F}_r^*\cdot\vec{\nabla}X_s \tag{4.12}$$

if A_{rs} is an $U(N)$ matrix

$$A_{rs}^* A_{rt} = \delta_{st}. \tag{4.13}$$

Using $\vec{\nabla}\cdot\vec{F}_r = 0$ away from the sources, the second term in (4.12) becomes a surface term which can be evaluated using (4.8)–(4.10) to

$$M = \frac{1}{2g^2}\int d^3\vec{x}\,|\vec{F}_r - A_{rs}\vec{\nabla}X_s|^2 + \mathrm{Re}\sum_I Q_r^{I*}A_{rs}X_s^I. \tag{4.14}$$

Since the first term on the right hand side is positive definite, we get the bound

$$M \geq \mathrm{Re}\{Q_r^{I*}A_{rs}X_s^I\}. \tag{4.15}$$

The BPS bound arises from maximizing the right hand side of this expression subject to (4.13)

$$M_{BPS} = \max_{A \in \mathrm{U}(N)} \mathrm{Re}\{Q_r^{I*} A_{rs} X_s^I\}. \tag{4.16}$$

Finally, this BPS bound is saturated when the BPS equations

$$\vec{F}_r = A_{rs} \vec{\nabla} X_s, \qquad \nabla^2 X_s = 0, \tag{4.17}$$

are satisfied for the A_{rs} which maximizes (4.16), where the last equation is a result of the divergence free property of $\vec{F}_r$.

The expression for M_{BPS} can be simplified slightly using charge conservation. Define by

$$\xi_s^i \equiv X_s^i - X_s^0 \tag{4.18}$$

the vector pointing from the vacuum X_s^0 to the source (singularity) at X_s^i. Then, by (4.7),

$$Q_r^{I*} A_{rs} X_s^I = Q_r^{i*} A_{rs} \xi_s^i, \tag{4.19}$$

and the BPS bound becomes

$$M_{BPS} = \max_{A \in \mathrm{U}(N)} \mathrm{Re}\{Q_r^{i*} A_{rs} \xi_s^i\}. \tag{4.20}$$

Now, as described above, the singularities of $\mathcal{M}$ are whole submanifolds, not isolated points, so the vectors ξ_s^i pointing to the singularities can vary continuously. We must therefore vary our boundary conditions to find the lightest (potentially) BPS state in the given charge sector. This has two consequences. First, this implies that generically the charge sources will lie along the the $\mathbb{Z}_2$ singularities in $\mathcal{M}$, *i.e.* that only with some special fine tuning will a source charged under three or more U(1)'s not split into several sources each with (opposite) charges under only two U(1)'s. This follows simply because the multi-charged sources are constrained to lie in submanifolds at the intersection of the higher-dimensional $\mathbb{Z}_2$ singularities. The second consequence is that the true BPS bound is given by the the minimization of (4.20) with respect to the set of vectors ξ_r^i ending on the $\mathbb{Z}_2$ singularities

$$M_{BPS} = \min_{\xi \in \text{sings.}} \max_{A \in \mathrm{U}(N)} \mathrm{Re}\{Q_r^{i*} A_{rs} \xi_s^i\}. \tag{4.21}$$

It is important to note that the maximization with respect to the rotation A_{rs} be done for each ξ_r^i *before* the minimizing with respect to ξ_r^i. If the space of singularities over which the ξ^i can vary were compact and smooth, then (since the space of A's is compact) there would be no issue over the order in which the extremizations were performed. But the space of singularities over which the ξ^i can vary themselves have singularities, and so some care must be taken in case the solution lies not at a saddle point of $\mathrm{Re}\, Q^{*i} \cdot A \cdot \xi^i$, but at one of these singularities. We have found, however, that for generic Q^i's and X^0 this does not happen so the order of extremization can be reversed. This greatly simplifies the problem of finding M_{BPS} in $\mathcal{N}$=4 theories because the singularities lie along linear subspaces of the flat moduli space. Thus minimization with respect to the ξ^i by itself gives M independent complex conditions on A, essentially fixing it and M_{BPS} entirely in the cases of one or two charge sources ($M = 1$ or 2) each charged under only two U(1) factors. This minimization is carried out in the Appendix for two charge sources.

Even in this case one must still do the maximization with respect the A to determine the values of the ξ^i in terms of given source charges Q^i_s and vacuum X^0_s. Solving this minimization and maximization problem for the ξ^i and M_{BPS} with an arbitrary number of given charge sources Q^i_r is difficult in general. However, as we will explain below, it is sufficient to solve it for just two sources each charged under only two U(1) factors. This is also done explicitly in the Appendix for two charge sources. We see there that the ξ^i are not completely determined by the extremization of the BPS bound, but are ambiguous up to a single undetermined real parameter. The final condition needed to fix the boundary conditions (the ξ^i) comes from demanding that solutions to the BPS equations exist.

4.1.3 BPS solutions

So, suppose we fix the ξ^i, and thus the boundary values of the scalars. We will now solve the low energy BPS equations (4.17) subject to the charge boundary conditions (4.8), the boundary conditions at infinity (4.9), as well as our fuzzy ball boundary conditions centered around the above determined boundary values. In doing so we will determine necessary conditions for a solution to exist given these boundary conditions. These conditions can phrased as the conditions that a certain matrix α_{ij} be real, symmetric, and have only positive entries. The reality and symmetry condition will provide the extra condition needed to determine the ξ^i for two or more charge sources. Finally, the positivity condition is satisfied only on one side of the CMS, and so it is this condition which determines the stability of BPS states.

The BPS equations can be solved as in the toy model of section 3 by a superposition of single source solutions:

$$\xi_s(\vec{x}) = \sum_{i=1}^{M} \frac{Q^i_r A^*_{rs}}{4\pi|\vec{x} - \vec{x}_i|}, \tag{4.22}$$

where we have defined

$$\xi_s(\vec{x}) \equiv X_s(\vec{x}) - X_s^0. \tag{4.23}$$

The numerators on the right hand side are determined by (4.17) and (4.8). Now, the fuzzy ball boundary conditions are that ξ_s goes through a small ball around ξ_s^i as $\vec{x} \to \vec{x}_i$. In this limit $\xi(\vec{x})$ approaches

$$\lim_{\vec{x}\to\vec{x}_i} \xi_s \simeq \frac{A^*_{rs}Q^i_r}{\epsilon} + \sum_{j\neq i}^{M} \frac{A^*_{rs}Q^j_s}{r_{ij}} \to \infty \tag{4.24}$$

where

$$\epsilon \equiv 4\pi|\vec{x}-\vec{x}_j|, \quad \text{and} \quad r_{ij} \equiv 4\pi|\vec{x}_i - \vec{x}_j|. \tag{4.25}$$

Thus as $\vec{x} \to \vec{x}_j$ our solutions go to infinity asymptoting a line along the $A^*_{rs}Q^j_s$ direction and with intercept $\delta^i_r = \sum_{j\neq i} \frac{A^*_{rs}Q^j_s}{r_{ij}}$. So a necessary condition for the fuzzy ball boundary conditions to be satisfied is that the approximate boundary value ξ^i at the ith source lies on this asymptote:

$$\xi^i_r = \alpha A^*_{rs}Q^i_s + \sum_{j\neq i} \frac{A^*_{rs}Q^j_s}{r_{ij}}, \qquad \text{for some } \alpha > 0. \tag{4.26}$$

In particular, the fact that $r_{ij} > 0$ means that the condition (4.26) has the form

$$\xi^i = \sum_j \alpha_{ij} A^*_{rs} Q^j_s \tag{4.27}$$

for α_{ij} a real symmetric matrix of positive numbers. Multiplying this equation on both sides by the unitary matrix A_{rt} implies therefore that a necessary condition for there to exist a solution to the BPS equations with our fuzzy ball boundary conditions is that *the $A_{rs}\xi^i_s$ must be in the real, symmetric, and positive span of the Q^k_r:*

$$A_{rs}\xi^i_s = \sum_k \alpha_{ij} Q^j_r, \qquad \alpha_{ij} = \alpha_{ji} \geq 0. \tag{4.28}$$

4.1.4 Spike solutions (1/2 BPS states)

It is straightforward to find single source brane spike solutions representing 1/2 BPS states in the low energy theory. These correspond to solutions with a single charge source charged under one pair of U(1) factors (recall that the total charge under the diagonal U(1) is assumed zero to decouple it). Thus, without loss of generality, we can restrict ourselves to just two U(1) factors, with everything neutral under the diagonal U(1). We will take the associated complex charge vector to be

$$Q = \begin{pmatrix} q \\ -q \end{pmatrix}. \tag{4.29}$$

We also choose the vacuum to be at the complex scalar vevs

$$X^0 = \begin{pmatrix} a \\ -a \end{pmatrix}. \tag{4.30}$$

From our earlier discussion, the singular submanifold of the (relevant one-complex-dimensional submanifold of) moduli space where states of charge Q can end is only the origin

$$X^{\text{sing}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.31}$$

This follows because the singular manifolds are those points with equal complex coordinates under the two U(1)'s; but the decoupling of the diagonal U(1) restricts us to the submanifold where the two coordinates sum to zero, as in (4.30).

Since we are effectively restricted to a one-complex-dimensional space, it is convenient to change basis on the moduli space by means of a simple unitary transformation

$$\begin{pmatrix} a \\ -a \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ -a \end{pmatrix} = \begin{pmatrix} \sqrt{2}a \\ 0 \end{pmatrix}. \tag{4.32}$$

In this basis, where we can ignore the last component which will always vanish by the tracelessness condition, the charge and vacuum are simply the complex numbers

$$Q = \sqrt{2}q, \qquad X^0 = \sqrt{2}a. \tag{4.33}$$

Our general expression derived above for the BPS mass (4.21) simplifies to

$$M_{BPS} = \max_{\phi} \mathrm{Re}\{Q^* e^{i\phi}(-X^0)\} = \max_{\phi} \mathrm{Re}\{-2q^* a e^{i\phi}\} = 2|q|\,|a|. \tag{4.34}$$

Here we have written the U(1) "matrix" A as the phase $e^{i\phi}$. The maximization determines ϕ to be the phase of $-qa^*$.

The solution to the BPS equations (4.22) also simplifies to

$$X(\vec{x}) - a = \frac{q e^{-i\phi}}{4\pi|\vec{x} - \vec{x}_0|} \tag{4.35}$$

where, $\vec{x}_0$ is the arbitrary spatial location of the charge source. This solution clearly exists everywhere on the moduli space (*i.e.* for all a): it has no structure and satisfies Dirichlet boundary conditions $X = 0$ on the sphere around $\vec{x}_0$:

$$|\vec{x} - \vec{x}_0| = -\frac{q e^{-i\phi}}{4\pi a} = \frac{|q|}{4\pi|a|}, \tag{4.36}$$

where the last equality follows from the solution for ϕ in (4.34). These properties are just as in the spike solutions in our toy example of section 3.

4.1.5 Prong solutions (1/4 BPS states)

Now let us specialize to the case $M = 2$, that is, two sources, each charged under one pair of U(1) factors (recall that the total charge under the diagonal U(1) is assumed zero to decouple it). Thus, without loss of generality, we can restrict ourselves to just three U(1) factors, with everything neutral under the diagonal U(1). We will take the associated complex charge vectors to be

$$Q^1 = \begin{pmatrix} q_1 \\ -q_1 \\ 0 \end{pmatrix}, \qquad Q^2 = \begin{pmatrix} 0 \\ q_2 \\ -q_2 \end{pmatrix}. \tag{4.37}$$

We also choose the vacuum to be at the complex scalar vevs

$$X^0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad a + b + c = 0. \tag{4.38}$$

From our earlier discussion, the singular submanifold of the moduli space where states of charge Q^1 and Q^2 can end can have coordinates

$$X^1 = \begin{pmatrix} x \\ x \\ -2x \end{pmatrix}, \qquad X^2 = \begin{pmatrix} -2y \\ y \\ y \end{pmatrix}, \tag{4.39}$$

respectively, for arbitrary complex x and y.

In the Appendix we analyze the conditions on x and y that result from extremizing the BPS bound and demanding reality and symmetry of the α_{ij}. These conditions fix x and y completely.

The result can be expressed as follows. First we set up a convenient notation. Define θ_1 and θ_2 to be the phases of q_1 and q_2:

$$q_j = |q_j| e^{i\theta_j}. \tag{4.40}$$

Then decompose each of the complex numbers a, b, c, x, y in the (non-orthogonal) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ defined by

$$\mathbf{e}_j \equiv e^{i(\theta_j - \phi)}, \tag{4.41}$$

where ϕ is a phase to be defined below. Thus, for example, we write

$$a = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \tag{4.42}$$

for unique real numbers a_j; define similarly the real numbers b_j, c_j, x_j,and y_j, for $j = 1, 2$.

Finally, ϕ is defined by

$$q_1(b^* - a^*) + q_2(c^* - b^*) = M_{BPS} e^{i\phi}. \tag{4.43}$$

In other words, ϕ is the phase of the left hand side, and M_{BPS}, the BPS mass, is the norm:

$$M_{BPS} = |q_1^*(a - b) + q_2^*(b - c)|. \tag{4.44}$$

Then the result for x and y is given by

$$\begin{aligned} x &= -\frac{1}{2} c_1 \mathbf{e}_1 + a_2 \mathbf{e}_2, \\ y &= c_1 \mathbf{e}_1 - \frac{1}{2} a_2 \mathbf{e}_2. \end{aligned} \tag{4.45}$$

The final condition for the existence of a prong type solution in the $\mathcal{N}$=4 effective action with our boundary conditions is that

$$\alpha_{ij} \geq 0, \tag{4.46}$$

where the α_{ij} are defined in (4.27). From the expressions for the α_{ij} derived in the Appendix it is easy to see that (4.46) is equivalent to the four conditions

$$\begin{aligned} b_1 &\geq a_1, \qquad & b_1 &\geq c_1, \\ c_2 &\geq b_2, \qquad & a_2 &\geq b_2. \end{aligned} \tag{4.47}$$

Note that (since $a + b + c = 0$) the only way *both* conditions in either line can be saturated is if $a_j = b_j = c_j = 0$, a singular vacuum. Thus the non-singular way these conditions can be saturated is if only one inequality in each line of (4.47) is saturated, *e.g.*,

$$\begin{aligned} b_1 &> a_1, \qquad & b_1 &= c_1, \\ c_2 &> b_2, \qquad & a_2 &= b_2. \end{aligned} \tag{4.48}$$

A check of our picture is that this boundary at which prong solutions cease to exist corresponds to being on a CMS.

We can check that this is indeed the case as follows. Now, (4.43) and (4.44) imply that the BPS mass can be written as

$$M_{BPS} = \mathrm{Re}\left\{ e^{i\phi} \left[q_1^*(b - a) + q_2^*(c - b) \right] \right\}, \tag{4.49}$$

which when expanded in the $\mathbf{e}_i$ basis gives

$$M_{BPS} = |q_1| \left[(b_1 - a_1) + (b_2 - a_2) \cos(\theta_1 - \theta_2) \right] + |q_2| \left[(c_2 - b_2) + (c_1 - b_1) \cos(\theta_1 - \theta_2) \right]. \tag{4.50}$$

If, on the other hand, there is just one charge source of charge Q^1 then its BPS mass is given by (4.44) with $q_2 = 0$, implying

$$M_1 = |q_1|\,|b - a| = |q_1|\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + 2(b_1 - a_1)(b_2 - a_2)\cos(\theta_1 - \theta_2)}. \tag{4.51}$$

Similarly, the BPS mass for the charge Q^2 one-source state

$$M_2 = |q_2|\sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2 + 2(c_1 - b_1)(c_2 - b_2)\cos(\theta_1 - \theta_2)}. \tag{4.52}$$

The condition to be on a CMS is, by definition,

$$M_{BPS} = M_1 + M_2, \tag{4.53}$$

which by (4.50), (4.51), and (4.52), precisely corresponds to (4.48) as required.

Note that if the phases of the charge vectors are identical, *i.e.* $\theta_1 = \theta_2$, then the vector space spanned by the e_i basis degenerates and there are no 1/4 BPS states, only 1/2 BPS states, as expected.

It will be useful to note that in the vicinity of the CMS and on the side where prong solutions exist, *i.e.* where (4.47) is satisfied, the following inequalities hold:

$$b_1 > c_1 > 0 > a_1, \qquad \text{and} \qquad c_2 > 0 > a_2 > b_2. \tag{4.54}$$

These follow from $a + b + c = 0$, (4.47), and the condition that we are near the CMS, *i.e.* $b_1 \simeq c_1$ and $a_2 \simeq b_2$.

Putting all the pieces of this calculation together we have thus shown that in the $\mathcal{N}$=4 low energy effective theory, two source "prong" solutions (4.22) always exist on one side of the relevant CMS. Furthermore, the CMS condition (4.48) is equivalent to $\alpha_{12} = \alpha_{21} = 0$ in (4.27), which, comparing to (4.26), implies $r_{12} = \infty$ where r_{12} is the spatial source separation (4.25). Thus we learn that in the prong solutions the sources have definite spatial separation which diverges as we approach the CMS, in accord with the qualitative picture described in section 2.

Finally, in the limit as we approach the CMS the prong solutions obey Dirichlet boundary conditions (as opposed to the more general fuzzy ball boundary conditions) to ever-greater accuracy, just as in the discussion in the toy example in section 3. Hence there will always be some distance from the CMS on the moduli space within which our low energy fuzzy ball boundary conditions are self-consistent.

4.1.6 Webs on moduli space

The final step in deriving a string web picture of BPS states on the moduli space of the $\mathcal{N}$=4 theory is to project the solutions we have constructed above onto that moduli space.

We start with the spike (or single source) solutions given in (4.35). Since the spatial dependence only enters in the positive factor $|\vec{x}-\vec{x}_0|^{-1}$, the image of $X(\vec{x})$ on the moduli space is simply a straight line segment, starting at $X^0 = \sqrt{2}a$ (at $\vec{x} = \infty$) and ending at the origin $X = 0$ on the sphere of (4.36). This, together with its mass (4.34), is thus consistent with its interpretation as a string of tension $|Q| = |\sqrt{2}q|$ stretched a length $|\sqrt{2}a|$ on the moduli space.

A similar picture applies to the brane prong (or two source) solutions as well. As the vacuum approaches the CMS, the image of the solution (4.22) in the moduli space approaches that of a web of line segments, just as discussed of section 3. One segment leads from the vacuum to the CMS, where two more segments emanate, leading to the singularities X^j. From the form of the solution (4.22), the two segments leading towards the singularities point along the direction

$$Q^j_r A^*_{rs} = e^{-i\phi} Q^j_s \tag{4.55}$$

in the moduli space. Recalling our definitions (4.37) of the complex charge vectors Q^j, it follows that the directions of the line segments going to X^1 and X^2 can be written as

$$|q_1| \begin{pmatrix} \mathbf{e}_1 \\ -\mathbf{e}_1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad |q_2| \begin{pmatrix} 0 \\ \mathbf{e}_2 \\ -\mathbf{e}_2 \end{pmatrix}, \tag{4.56}$$

respectively. Then from our solution for the X^j it follows immediately that the three line segments that the prong solution degenerates to are given by

$$\begin{aligned}
X^0 \to \text{CMS}: &\qquad X(t) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t \begin{pmatrix} (b_1 - c_1)\mathbf{e}_1 \\ (c_1 - b_1)\mathbf{e}_1 + (a_2 - b_2)\mathbf{e}_2 \\ (b_2 - a_2)\mathbf{e}_2 \end{pmatrix}, \\
X^1 \to \text{CMS}: &\qquad X(t) = \begin{pmatrix} -\frac{1}{2}c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\ -\frac{1}{2}c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\ +c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2 \end{pmatrix} + t \begin{pmatrix} -\frac{3}{2}c_1\mathbf{e}_1 \\ +\frac{3}{2}c_1\mathbf{e}_1 \\ 0 \end{pmatrix}, \\
X^2 \to \text{CMS}: &\qquad X(t) = \begin{pmatrix} -2c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\ +c_1\mathbf{e}_1 - \frac{1}{2}a_2\mathbf{e}_2 \\ +c_1\mathbf{e}_1 - \frac{1}{2}a_2\mathbf{e}_2 \end{pmatrix} + t \begin{pmatrix} 0 \\ +\frac{3}{2}a_2\mathbf{e}_2 \\ -\frac{3}{2}a_2\mathbf{e}_2 \end{pmatrix},
\end{aligned} \tag{4.57}$$

where in all cases $t \in [0, 1]$. Notice that the common $t = 1$ endpoint of each of these segments,

$$\begin{pmatrix} -2c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\ +c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\ +c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2 \end{pmatrix}, \tag{4.58}$$

satisfies (4.48) so is indeed on the CMS.

Finally, we check that this three-string junction on the moduli space of the $\mathcal{N}$=4 theory indeed gives the correct BPS mass just by adding the lengths of its segments weighted by the tension of each string (which is the norm of its charge vector in our conventions). Showing this shows that this string-junction picture can be continued to all vacua, and not just those close to the CMS.

The charges for the $X^j \to$ CMS segments are Q^j, $j = 1, 2$ respectively, thus the tension of these strings are $|Q^j| = \sqrt{2}|q_j|$. The multiplied by the lengths of their respective segments gives their masses as

$$\begin{aligned} M(X^1 \to \text{CMS}) &= 3|q_1|c_1, \\ M(X^2 \to \text{CMS}) &= -3|q_1|a_2, \end{aligned} \tag{4.59}$$

where we have used the fact that c_1 is positive and a_2 negative from (4.54). The charge of the $X^0 \to$ CMS segment is the total charge which can be written

$$Q^1 + Q^2 = e^{i\phi} \begin{pmatrix} +|q_1|\mathbf{e}_1 \\ -|q_1|\mathbf{e}_1 + |q_2|\mathbf{e}_2 \\ -|q_2|\mathbf{e}_2 \end{pmatrix}. \tag{4.60}$$

Now, as shown in the Appendix the condition determining ϕ implies eq. (A.19) which can be written as

$$(a_2 - b_2) = \beta|q_2|, \qquad (b_1 - c_1) = \beta|q_1|, \qquad \text{for some real } \beta > 0, \tag{4.61}$$

where we have again used the fact that $a_2 - b_2$ and $b_1 - c_1$ are positive from (4.54). This implies that $e^{-i\phi}$ times the charge vector and the vector on moduli space for the $X^0 \to$ CMS line segment are proportional with positive real proportionality factor β. Thus the product of the lengths of these two vectors is the same as their inner product, giving the mass of this string segment as

$$M(X^0 \to \text{CMS}) = (b_1 - c_1)\,[2|q_1| - |q_2|\cos(\theta_1 - \theta_2)] + (a_2 - b_2)\,[2|q_2| - |q_1|\cos(\theta_1 - \theta_2)]\,. \tag{4.62}$$

Adding the masses of the segments (4.59), (4.62), indeed gives the BPS mass (4.50).

This completes our construction of our representation of BPS states as three-string junctions on the moduli space of $\mathcal{N}$=4 theories.

So far we have only dealt with one and two source solutions. The general BPS state will be charged under more than just three U(1) factors, and so will generically break up into more that just two charge sources. However, our low energy method can deal with such cases only in an indirect manner. The reason is that our approximate boundary conditions are only valid when the vacuum is close to a CMS. But a generic solution with three or more charge sources generically has two or more distinct CMS corresponding in the string web picture to two or more separate three-string junctions. Thus, except for exceptional states (represented by a single n-string junction in the string picture), we can generically only tune the vacuum close to one CMS. Thus our approximate boundary conditions will generically not be valid for the whole low energy solution since the other junctions need not be close to their CMS.

However, having deduced the 3-string junction picture of the two source states allows us to bootstrap our way to the string web picture of a general n-source state. To do this, group the n sources in pairs (in all possible ways). Then solve for all 3-string junction states for each of those pairs of charges to find the manifold of possible "vacua" where such 3-string junctions could originate. We can then use these manifolds of 3-string junction vacua as the values for a new set of fuzzy ball boundary conditions, and thus determine the existence of a new 3-string junction connecting pairs of the previously calculated 3-string junctions, and so on, thereby building up many-branched trees of 3-string junctions. This procedure thus constructs a picture of the general BPS state as a string web on the moduli space of $\mathcal{N}$=4 theories.

4.2 Projecting BPS prongs to string webs

So far we have developed a picture of $\mathcal{N}$=4 BPS states as string webs on the moduli space of vacua. However, the actual string web construction of BPS states in string theory does *not* in general involve strings on the moduli space of the theory. For example, the SU(N) $\mathcal{N}$=4 theory can be realized as open string and D1-brane degrees of freedom propagating on a stack of parallel D3-branes in IIB string theory. BPS states are then represented as webs of these (p,q) strings ending on the D3-branes and extending in the transverse $\mathbb{R}^6$ spatial dimensions. This $\mathbb{R}^6$ is not the moduli space of the $\mathcal{N}$=4 theory, which is a $6N$-dimensional space $\mathcal{M} = \mathbb{R}^{6N}/S_N$.

However, there is a simple map relating points in the full moduli space to arrangements of branes in the transverse $\mathbb{R}^6$. We will now show how this map can be used to map our string webs on the moduli space of $\mathcal{N}$=4 theories to the string webs in the $\mathbb{R}^6$ transverse to D3-branes in IIB theory. First, just as for our low energy solutions we could restrict ourselves to a $\mathbb{C}^N/S_N$ subspace of $\mathcal{M}$, the condition that 1/4 of the supersymmetry be left unbroken in the IIB string construction allows one to confine oneself to string webs living in a one-complex-dimensional subspace $\mathbb{C}$ of the transverse $\mathbb{R}^6$ space. A general point X^0 in $\mathbb{C}^N/S_N$ is given as a complex vector with N unordered components, *e.g.* $X^0 = (a,b,c)$ in the case of $N = 3$. This is mapped to a configuration of N D3-branes at positions in the complex transverse plane corresponding to the complex coordinates, *e.g.* for $N = 3$, X^0 corresponds to three D3-branes at positions a, b, and c in $\mathbb{C}$. (Our freedom to set $a+b+c=0$ upon decoupling the diagonal U(1) gauge factor corresponds in the brane picture to the freedom to shift the center of mass position of the D3-branes to the origin.)

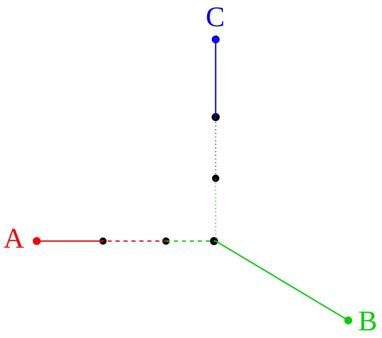

Figure 6: Image of a 3-string junction on moduli space in a transverse plane to three D3-branes in IIB string theory. The images of the $X^0 \to$ CMS segment are solid lines, while those of $X^1 \to$ CMS are dashed, and those of $X^2 \to$ CMS are dotted. The segments traced out by the A brane are shown in red, the B brane in green, and the C brane in blue.

This map can also be used to map string webs on the moduli space to paths on the transverse $\mathbb{C}$. As each point in moduli space translates to an arrangement of D3-branes in $\mathbb{C}$, the image of a path in moduli space will be a set of N paths in $\mathbb{C}$ of each D3-brane. Thus, in the case of our 3-string junction (4.57) on moduli space, each segment will translate into three segments in $\mathbb{C}$ traced out by the corresponding D3-branes. Since there are a total of three straight segments on the moduli space, generically the image in $\mathbb{C}$ will consist of nine segments. We will now show that our BPS solution is just such that these nine segments line up to form a 3-string junction on $\mathbb{C}$, thus recovering the usual IIB string web.

To see this, simply trace out the path of each D3-brane. Starting at the vacuum, $X^0 = (a, b, c)$, we label the three D3-branes corresponding to the points a, b, and $c \in \mathbb{C}$, as "A", "B", and "C", respectively. Let us first follow the image of the B D3-brane in $\mathbb{C}$. Since along the $X^0 \to$ CMS segment the second coordinate on the moduli space traverses $b \to c_1\mathbf{e}_1 + a_2\mathbf{e}_2$, so therefore does the B brane in $\mathbb{C}$. We can similarly trace out its image path in $\mathbb{C}$ of the other segments. Doing this for the A and C branes as well we get the nine segments

	$X^0 \to$ CMS	CMS $\to X^1$	CMS $\to X^2$
A	$a \to -2c_1\mathbf{e}_1 + a_2\mathbf{e}_2$	$-2c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \to -\frac{1}{2}c_1\mathbf{e}_1 + a_2\mathbf{e}_2$	$-2c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \to -2c_1\mathbf{e}_1 + a_2\mathbf{e}_2$
B	$b \to +c_1\mathbf{e}_1 + a_2\mathbf{e}_2$	$+c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \to -\frac{1}{2}c_1\mathbf{e}_1 + a_2\mathbf{e}_2$	$+c_1\mathbf{e}_1 + a_2\mathbf{e}_2 \to +c_1\mathbf{e}_1 - \frac{1}{2}a_2\mathbf{e}_2$
C	$c \to +c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2$	$+c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2 \to +c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2$	$+c_1\mathbf{e}_1 - 2a_2\mathbf{e}_2 \to +c_1\mathbf{e}_1 - \frac{1}{2}a_2\mathbf{e}_2$

(4.63)

Notice first that the A: CMS $\to X^2$ segment and the C: CMS $\to X^1$ segment are both just points in $\mathbb{C}$. Second, notice that the A: $X^0 \to$ CMS segment plus the A: CMS $\to X^1$ segment plus the B: $X^1 \to$ CMS segment form a straight line $a \to c_1\mathbf{e}_1 + a_2\mathbf{e}_2$. (Recall from (4.54) and the fact that $a_1 + b_1 + c_1 = 0$ that $c_1 > -\frac{1}{2}c_1 > -2c_1 > a_1$, so the segments are traced out consecutively.) Similarly, the C: $X^0 \to$ CMS segment plus the C: CMS $\to X^2$ segment plus the B: $X^2 \to$ CMS segment form a straight line $c \to c_1\mathbf{e}_1 + a_2\mathbf{e}_2$. Finally, the B: $X^0 \to$ CMS segment forms a third line to the same point. Thus the image in $\mathbb{C}$ of the 3-string junction on moduli space is again a three string junction leading from the vacuum positions a, b, c, of the D3-branes to a common central point; see figure 6.

We have thus recovered the IIB string web picture of 1/4 BPS states of the $\mathcal{N}=4$ SU(N) superYang-Mills theory.

5. BPS states in $\mathcal{N}=2$ theories

A similar picture of 1/2 BPS states as string webs on the Coulomb branch of $\mathcal{N}=2$ theories can be derived in close analogy to our discussion of $\mathcal{N}=4$ theories in the last section. The generic point on the Coulomb branch of an $\mathcal{N}=2$ gauge theory with gauge group G is described by a low energy $\mathrm{U}(1)^N$ $\mathcal{N}=2$ theory where $N = \mathrm{rank}\, G$. The theory may have N_f "matter" hypermultiplets in various representations; they are massive at the generic Coulomb branch vacuum. Each U(1) $\mathcal{N}=2$ multiplet has one complex scalar field, ϕ_r $r = 1, \ldots, N$, whose vevs parameterize the Coulomb branch. The other bosonic massless fields are the electric and magnetic fields $\vec{E}_r$ and $\vec{B}_r$ for each U(1) factor. We are interested in static low energy solutions carrying electric and magnetic charges (n_E^r, n_{Br}) with respect to these fields. We normalize the fields so that the low energy effective action for the bosonic fields is[3]

$$S = \int d^4x \left\{ \frac{\mathrm{Im}\, \tau^{rs}}{16\pi} F_{r\,\mu\nu} F_s^{\mu\nu} + \frac{\mathrm{Re}\, \tau^{rs}}{32\pi} \epsilon_{\mu\nu\rho\sigma} F_r^{\mu\nu} F_s^{\rho\sigma} + \frac{\mathrm{Im}\, \tau^{rs}}{16\pi} \partial_\mu \phi_r \partial^\mu \phi_s^* \right\}, \tag{5.1}$$

with an implicit sum over repeated indices, and where $\vec{E}_r$ and $\vec{B}_r$ are related to $F_r^{\mu\nu}$ in the usual way, and

$$\tau^{rs}(\phi) \equiv \frac{\vartheta^{rs}(\phi)}{2\pi} + i \frac{4\pi}{g_{rs}^2(\phi)} \tag{5.2}$$

is the matrix of effective U(1) couplings and theta angles which depend holomorphically on the ϕ_r. Thus the metric on the Coulomb branch is given by the line element

$$ds^2 = \mathrm{Im}\, \tau^{rs}(\phi)\, d\phi_r d\phi_s^*. \tag{5.3}$$

Note that τ^{rs} is a symmetric matrix by definition, and that $\mathrm{Im}\, \tau^{rs}$ is positive definite by unitarity. The rigid special Kahler geometry of the $\mathcal{N}=2$ Coulomb branch implies that

$$\tau^{rs}(\phi) = \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi_r \partial \phi_s} \tag{5.4}$$

---

[3] We have chosen the normalization of the scalars ϕ_r to differ from their canonical normalization by a factor of $\sqrt{2}$ to simplify later formulas.

for some prepotential $\mathcal{F}$.

It is useful to introduce the "dual scalars"

$$\phi_D^r \equiv \frac{\partial \mathcal{F}}{\partial \phi_r}, \tag{5.5}$$

the complex field strength

$$\vec{F}_r \equiv \vec{B}_r + i\vec{E}_r \tag{5.6}$$

which appears in the same supermultiplet as ϕ_r, as well as its dual

$$\vec{F}_D^r \equiv \tau^{rs}\vec{F}_s, \tag{5.7}$$

which is in the same multiplet as ϕ_D^r.

The Bianchi identities as well as the static equations of motion in vacuum (without sources) for the electric and magnetic fields that follow from (5.1) are

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \mathrm{Re}\,\vec{F}_D^r, \\ 0 &= \vec{\nabla} \cdot \mathrm{Re}\,\vec{F}_r, \\ 0 &= \vec{\nabla} \times \mathrm{Im}\,\vec{F}_D^r, \\ 0 &= \vec{\nabla} \times \mathrm{Im}\,\vec{F}_r, \end{aligned} \tag{5.8}$$

implying, together with the Dirac quantization condition, that the electric and magnetic charges are quantized as

$$\begin{aligned} \oint_{S^2} \mathrm{Re}\,\vec{F}_D^r \cdot d\vec{a} &= -4\pi n_E^r, \\ \oint_{S^2} \mathrm{Re}\,\vec{F}_r \cdot d\vec{a} &= +4\pi n_{Br}, \end{aligned} \tag{5.9}$$

for some integers n_E^r and n_{Br}. (The magnetic field contribution in the first line includes the effect of the theta angle on electric charge quantization.)

Just as in our discussion in section 3 the scalar fields should obey fuzzy ball boundary conditions at the charge sources, *i.e.* their values in the vicinity of a charge source approaches those of the coordinates of a singularity on the Coulomb branch where a state of that charge becomes massless. The generic singularities on an $\mathcal{N}$=2 Coulomb branch are complex codimension one submanifolds, which we denote $\mathcal{S}^{(i)}$, at which a state of given charges $(n_E^{(i)r}, n_{Br}^{(i)})$ becomes massless. In a solution with M charge sources, we denote by $(n_E^{(i)r}, n_{Br}^{(i)})$ the charges of each source, $i = 1, \ldots, M$, and by $(n_E^{(0)r}, n_{Br}^{(0)})$ the total charge, so that

$$(n_E^{(0)r}, n_{Br}^{(0)}) + \sum_{i=1}^{M} (n_E^{(i)r}, n_{Br}^{(i)}) = 0. \tag{5.10}$$

The relation between the charges of the states which become massless at $\mathcal{S}^{(i)}$ and the coordinate values of the ϕ_r on $\mathcal{S}^{(i)}$ follows from the monodromies that the ϕ_r experience upon traversing a path encircling $\mathcal{S}^{(i)}$. To see this, recall [3] that the monodromy in the $\mathrm{U}(1)^N$ effective theory is an element of $\mathrm{Sp}(2n,\mathbb{Z}) \ltimes \mathbb{Z}^{N_f}$ which acts on the scalar fields and their duals, as well as the electric, magnetic, and hypermultiplet number charges. The latter are the integer charges n_Q^p, $p = 1,\ldots,N_f$, under the $\mathrm{U}(1)^{N_f}$ global flavor symmetry group. If the (bare) masses of the hypermultiplets are denoted by m_p, then the monodromy acts on the scalar fields as

$$\begin{pmatrix} \phi_D^r \\ \phi_r \end{pmatrix} \to \mathbf{S} \begin{pmatrix} \phi_D^r \\ \phi_r \end{pmatrix} + \mathbf{T} \begin{pmatrix} m_1 \\ \vdots \\ m_{N_f} \end{pmatrix}, \tag{5.11}$$

where $\mathbf{S} \in \mathrm{Sp}(2N,\mathbb{Z})$ and $\mathbf{T}$ is a $2N \times N_f$ integer matrix. The monodromy matrices around a submanifold $\mathcal{S}$ where a state of charges (n_E^r, n_{Br}, n_Q^p) is massless is then [36]

$$\begin{aligned} \mathbf{S} &= \begin{pmatrix} \delta_s^r + n_E^r n_{Bs} & n_E^r n_E^s \\ -n_{Br} n_{Bs} & \delta_r^s - n_{Br} n_E^s \end{pmatrix}, \\ \mathbf{T} &= \begin{pmatrix} n_Q^p n_E^r \\ -n_Q^p n_{Br} \end{pmatrix}. \end{aligned} \tag{5.12}$$

The coordinates ϕ_r or ϕ_D^r of the singularity $\mathcal{S}$ are characterized by the condition that they be invariant under this monodromy. This condition together with (5.11) and (5.12) then imply that

$$n_E^r \phi_r + n_{Br} \phi_D^r + n_Q^p m_p = 0. \tag{5.13}$$

This one complex equation (locally) determines the complex-codimension one singular manifold $\mathcal{S}$, and will play an important role in what follows.

(There is a global issue concerning the identification of the charges of the state which becomes massless. As we have discussed extensively in earlier sections, the static low energy BPS solutions can be thought of as a map from three-space into the Coulomb branch interpolating between the vacuum (at $\vec{x} = \infty$) and various singularities $\mathcal{S}^{(i)}$ (at points $\vec{x} = \vec{x}_i$). Since τ^{rs} can undergo $\mathrm{Sp}(2N,\mathbb{Z})$ monodromies upon traversing paths which loop around singularities in the Coulomb branch, the charges of the state which becomes massless at $\mathcal{S}^{(i)}$ must be redefined by the monodromy corresponding to the path on the Coulomb branch traced out by the low energy solution as it interpolates between $\vec{x} = \infty$ and $\vec{x}_i$. Thus the integers $(n_E^{(i)r}, n_{Br}^{(i)})$ will refer to the charges referred back in this way to $\vec{x} = \infty$.)

With this description of the moduli space and its singularities in hand, we are ready to solve for static low energy field configurations in a given charge sector subject to our fuzzy ball boundary conditions. The mass of our configuration is computed by integrating the field energy density (with an implicit sum over repeated indices)

$$M = \frac{1}{8\pi} \int d^3\vec{x}\ \mathrm{Im}(\tau^{rs})(\vec{F}_r \cdot \vec{F}_s^* + \vec{\nabla}\phi_r \cdot \vec{\nabla}\phi_s^*). \tag{5.14}$$

This can be rewritten [35] by the usual trick of completing the square with surface terms:

$$\begin{aligned} M &= \frac{1}{8\pi} \int d^3\vec{x}\ \mathrm{Im}(\tau^{rs})(\vec{F}_r + e^{i\alpha}\vec{\nabla}\phi_r) \cdot (\vec{F}_s^* + e^{-i\alpha}\vec{\nabla}\phi_s^*) \\ &\quad - \frac{1}{4\pi} \int d^3\vec{x}\ \mathrm{Im}(\tau^{rs}) \left\{ \vec{B}_r \cdot \mathrm{Re}(e^{i\alpha}\vec{\nabla}\phi_s) + \vec{E}_r \cdot \mathrm{Im}(e^{i\alpha}\vec{\nabla}\phi_s) \right\} \\ &\geq -\frac{1}{4\pi} \int d^3\vec{x} \left\{ \mathrm{Re}(\vec{F}_r) \cdot \mathrm{Im}(e^{i\alpha}\vec{\nabla}\phi_D^r) - \mathrm{Re}(\vec{F}_D^r) \cdot \mathrm{Im}(e^{i\alpha}\vec{\nabla}\phi_r) \right\} \\ &= \mathrm{Im} \sum_{I=0}^{M} e^{i\alpha} \left\{ n_{Br}^{(I)} \phi_D^{(I)r} + n_E^{(I)r} \phi_r^{(I)} \right\}. \end{aligned} \tag{5.15}$$

Here α is a constant phase to be determined below. The third line comes from integrating by parts and using (5.9). The inequality arises because the first term in the first line is positive, so that the inequality is saturated only if the BPS equations

$$\vec{F}_r + e^{i\alpha}\vec{\nabla}\phi_r = 0 \tag{5.16}$$

are satisfied. It is useful to note, using (5.5) and (5.7) that this can be rewritten as

$$\vec{F}_D^r + e^{i\alpha}\vec{\nabla}\phi_D^r = 0. \tag{5.17}$$

It follows from these equations and (5.8) that away from the sources

$$\nabla^2\, \mathrm{Re}(e^{i\alpha}\phi_r) = \nabla^2\, \mathrm{Re}(e^{i\alpha}\phi_D^r) = 0. \tag{5.18}$$

The BPS bound arises from maximizing the right hand side of (5.15). Define

$$Z^{(I)} \equiv n_{Br}^{(I)} \phi_D^{(I)r} + n_E^{(I)r} \phi_r^{(I)}, \tag{5.19}$$

the central charge associated with the Ith source, and denote by

$$Z \equiv \sum_{I=0}^{M} Z^{(I)} \tag{5.20}$$

their sum. Then

$$M_{BPS} = \max_{\alpha} \mathrm{Im}\{e^{i\alpha} Z\} = |Z|, \tag{5.21}$$

implying that

$$e^{i\alpha} = iZ^*/|Z|. \tag{5.22}$$

This expression for the BPS mass should coincide with the usual $\mathcal{N}$=2 BPS mass formula [3]

$$M_{BPS} = |n_E^{(0)r}\phi_r^{(0)} + n_{Br}^{(0)}\phi_D^{(0)r} + n_Q^{(0)p}m_p|, \tag{5.23}$$

where $n_Q^{(0)p}$ are the total "quark" numbers of the state in question. For (5.23) and (5.21) to be the same requires that

$$n_Q^p m_p = \sum_{i=1}^{M} Z^{(i)}. \tag{5.24}$$

But this follows automatically from the condition (5.13) characterizing the singularities on the Coulomb branch, which reads

$$Z^{(i)} = -n_Q^{(i)p}m_p \tag{5.25}$$

where $n_Q^{(i)p}$ are the "quark" quantum numbers of the ith charge source. Conservation of quark number implies

$$\sum_{I=0}^{M} n_Q^{(I)p} = 0, \tag{5.26}$$

which, together with (5.25) implies (5.24).

Now, as described above, the singularities of the Coulomb branch are whole complex codimension one submanifolds $\mathcal{S}^{(i)}$, not isolated points, so the coordinates $\phi_r^{(i)}$ of the singularities can vary continuously. We have just demonstrated that the BPS mass bound is independent of the precise values of the $\phi_r^{(i)}$. Therefore, unlike the $\mathcal{N}$=4 case, no minimization of the BPS bound over the $\mathcal{S}^{(i)}$ needs to be performed. But, as in the $\mathcal{N}$=4 case, the extra conditions needed to fix the boundary conditions (the $\phi^{(i)}$) come from demanding that solutions to the BPS equations exist.

So, suppose we fix the $\phi^{(i)}$, and thus the boundary values of the scalars. We will now solve the low energy BPS equations (5.18) subject to the charge and fuzzy ball boundary conditions. In doing so we will determine necessary conditions for a solution to exist given these boundary conditions. As in the $\mathcal{N}$=4 case it is sufficient to solve the equations with just one or two charge sources, as solutions with more sources can be built up from these simpler ones. Also just as in the $\mathcal{N}$=4 case we will show below that the condition that these solutions exist can be phrased as the conditions that a certain matrix α_{ij} be real, symmetric, and have only positive entries. The reality and symmetry conditions will provide the extra conditions needed to determine the $\phi_r^{(i)}$ for one or two charge sources. Finally, the positivity condition is satisfied only on one side of the CMS, and so it is this condition which determines the stability of BPS states.

The harmonic BPS equations (5.18) can be solved as in the toy model of section 3 by a superposition of single source solutions:

$$\begin{aligned}\mathrm{Re}\left(e^{i\alpha}\phi_r(\vec{x})\right) &= \sum_{i=1}^{M} \frac{n_{Br}^{(i)}}{|\vec{x}-\vec{x}_i|} + \mathrm{Re}\left(e^{i\alpha}\phi_r^{(0)}(\vec{x})\right), \\ \mathrm{Re}\left(e^{i\alpha}\phi_D^r(\vec{x})\right) &= -\sum_{i=1}^{M} \frac{n_E^{(i)r}}{|\vec{x}-\vec{x}_i|} + \mathrm{Re}\left(e^{i\alpha}\phi_D^{(0)r}(\vec{x})\right). \end{aligned} \tag{5.27}$$

The numerators on the right hand side are determined by (5.16) and (5.9). Now, the fuzzy ball boundary conditions are that ϕ_r goes through a small ball around $\phi_r^{(i)}$ as $\vec{x}\to\vec{x}_i$. In this limit we have

$$\begin{aligned}\lim_{\vec{x}\to\vec{x}_i} \mathrm{Re}\left(e^{i\alpha}\phi_r(\vec{x})\right) &\simeq \frac{n_{Br}^{(i)}}{\epsilon_i} + \sum_{j\neq i}^{M} \frac{n_{Br}^{(j)}}{r_{ij}} + \mathrm{Re}\left(e^{i\alpha}\phi_r^{(0)}\right), \\ \lim_{\vec{x}\to\vec{x}_i} \mathrm{Re}\left(e^{i\alpha}\phi_D^r(\vec{x})\right) &\simeq -\frac{n_E^{(i)r}}{\epsilon_i} - \sum_{j\neq i}^{M} \frac{n_E^{(j)r}}{r_{ij}} + \mathrm{Re}\left(e^{i\alpha}\phi_D^{(0)r}\right), \end{aligned} \tag{5.28}$$

where

$$\epsilon_i \equiv |\vec{x}-\vec{x}_i|, \quad \text{and} \quad r_{ij} \equiv |\vec{x}_i-\vec{x}_j|. \tag{5.29}$$

Thus as $\vec{x}\to\vec{x}_i$ our solutions go to infinity asymptoting a line in the $\mathrm{Re}\, e^{i\alpha}\phi_r$–$\mathrm{Re}\, e^{i\alpha}\phi_D^r$ plane along the $(n_{Br}^{(i)}, n_E^{(i)r})$ direction and with some intercept. So a necessary condition for the fuzzy ball boundary conditions to be satisfied is that the approximate boundary value $\phi_r^{(i)}$ at the ith source lies on this asymptote:

$$\begin{aligned}\mathrm{Re}\left(e^{i\alpha}\phi_r^{(i)}\right) &= \sum_j \alpha_{ij} n_{Br}^{(j)} + \mathrm{Re}\left(e^{i\alpha}\phi_r^{(0)}\right), \\ \mathrm{Re}\left(e^{i\alpha}\phi_D^{(i)r}\right) &= -\sum_j \alpha_{ij} n_E^{(j)r} + \mathrm{Re}\left(e^{i\alpha}\phi_D^{(0)r}\right), \end{aligned} \tag{5.30}$$

for some α_{ij} a real symmetric matrix of positive numbers,

$$\alpha_{ij} = \alpha_{ji} > 0, \tag{5.31}$$

which follows because $\alpha_{ij} = 1/r_{ij}$ for $i \neq j$ are the spatial source separations, while $\alpha_{ii} = 1/\epsilon_i$ is the spatial cutoff around the ith source.

The conditions (5.30) subject to (5.31) are $2MN$ real equations for $2MN - M(3-M)/2$ real unknowns. (There are $M(M+1)/2$ α_{ij}'s and $2N-2$ real components of

$\phi_r^{(i)}$ in each of the M $\mathcal{S}^{(i)}$ singular submanifolds.) In particular, for the one source $M=1$ ("spike") and two source $M=2$ ("prong") solutions in which we are interested, we have one more equation than unknown, which would imply that generically there are no solutions to the BPS equations and our boundary conditions. However, there is one combination of these equations which is automatically satisfied by virtue of the condition (5.13) defining the singular manifolds $\mathcal{S}^{(i)}$. In particular, the sum over i and r of $n_E^{(i)r}$ times the first equation in (5.30) plus that of $n_{Br}^{(i)}$ times the second gives an identity using (5.13) as well as charge conservation (5.10), (5.26), and the definition (5.22) of the phase $e^{i\alpha}$. Therefore there is generically a unique spike or prong solution to the BPS equations and our boundary conditions which uniquely fixes the boundary values of the $\phi_r^{(i)}$. Just as in the $\mathcal{N}$=4 case the positivity condition on the α_{ij} will prevent some solutions from existing; places where a solution ceases to exist because one of the α_{ij} changes sign will correspond to the decay of a state across a CMS.

It now only remains to show that the projection of these spike and prong solutions onto the Coulomb branch leads to a string junction picture of the states, as least in the limit as the vacuum approaches the appropriate CMS. But this follows with almost no work from what we have done so far.

In particular, a one source (spike) solution is rotationally invariant about its spatial center $\vec{x}_1$; therefore the projection of the solution (5.27) on the Coulomb branch is just a one real dimensional path leading from the vacuum to the singular submanifold. In the case of a one complex dimensional Coulomb branch ($N=1$), taking the sum of n_E times the first equation in (5.27) plus n_B times the second gives the condition (satisfied everywhere along the solution)

$$\mathrm{Re}\left(e^{i\alpha}[n_E\phi + n_B\phi_D + n_Q\cdot m]\right) = 0. \tag{5.32}$$

But this is precisely the condition determining the path of an (n_E, n_B) string stretched on the SU(2) Coulomb branch found in its F-theory representation [5, 9, 12], and implies in particular that they lie along geodesics in the Coulomb branch metric. We thus recover the string picture for the spike solutions in the case of a one complex dimensional Coulomb branch. Furthermore, in the cases where F theory describes a higher-dimensional Coulomb branch (*i.e.* with multiple D3-brane probes of a 7-brane background corresponding to the Sp($2N$) theory with four fundamental and one massless antisymmetric hypermultiplet [37, 38] and the higher rank generalizations of the exceptional theories with E_n global symmetry groups [39, 40]), the Coulomb branch is just the tensor product of one complex dimensional Coulomb branches (modulo permutations) just as with the $\mathcal{N}$=4 moduli space. In these cases the F theory string picture is found by a similar mapping of the "string" found by projecting the spike solution onto the multi-dimensional Coulomb branch onto the one dimensional space.

Note, however, that our string picture persists in theories with higher-rank Coulomb branches even when they do *not* have a direct product geometry.

Finally, it remains to show that the projection of the brane prong (two source) solutions onto the Coulomb branch recovers the string junction picture of these states in the vicinity of the CMS. We will illustrate this in a simple example below, but the main point follows from our general discussion so far: the diverging spatial separation of the charge centers as we approach the CMS implies that any prong state will more and more accurately approach the sum of two spike solutions, whose Coulomb branch projections we have just seen are (n_E, n_B) strings lying along geodesics. Furthermore, in the limit as the vacuum approaches the CMS, the analysis of the solution in a neighborhood of the vacuum point will reproduce precisely the string junction conditions with the junction lying on the CMS. This follows because in this limit we can ignore the curvature of the Coulomb branch metric in a small enough neighborhood of the vacuum point, thus taking τ^{rs} to be locally constant, so that $\phi_D^r \sim \tau^{rs}\phi_s$. Then the $\mathcal{N}$=2 solutions (5.27) become formally the same as the $\mathcal{N}$=4 solutions (4.22), and so inherit their local properties.

This can also be checked explicitly using some detailed properties of a given Coulomb branch geometry. We will illustrate this in the simplest example, namely the decay of the W boson in SU(2) $\mathcal{N}$=2 superYang-Mills across its CMS. We follow the conventions of [8] (our ϕ and ϕ_D correspond to their a and a_D). The massive W boson is a BPS state with charges $(n_E^{(0)}, n_B^{(0)}) = (2, 0)$ and so in a vacuum with coordinates ϕ or ϕ_D, has central charge

$$Z = 2\phi, \tag{5.33}$$

implying that

$$e^{i\alpha} = i\frac{\phi^*}{|\phi|}. \tag{5.34}$$

(We have dropped the (0) superscripts on the ϕ and ϕ_D coordinates for the vacuum.) The CMS is given by a curve satisfying the real condition

$$\mathrm{Im}\left(\frac{\phi_D}{\phi}\right) = 0. \tag{5.35}$$

The Coulomb branch has two singularities, a "magnetic" one with coordinates

$$\phi^{(1)} = \frac{2}{\pi}, \qquad \phi_D^{(1)} = 0 \tag{5.36}$$

where a state of charge

$$(n_E^{(1)}, n_B^{(1)}) = (0, 1) \tag{5.37}$$

becomes massless; and a "dyonic" one with coordinates

$$\phi^{(2)} = \frac{2i}{\pi}, \qquad \phi_D^{(2)} = -\frac{4i}{\pi} \tag{5.38}$$

where a state of charge

$$(n_E^{(2)}, n_B^{(2)}) = (-2, -1) \tag{5.39}$$

becomes massless. Plugging all this into the conditions (5.30) gives three independent equations for the α_{ij} which can be solved to give

$$\begin{aligned}
\alpha_{12} &= \frac{|\phi|}{2} \operatorname{Im}\left(\frac{\phi_D}{\phi}\right), \\
\alpha_{11} &= \frac{|\phi|}{2} \operatorname{Im}\left(\frac{\phi_D}{\phi}\right) + \frac{2}{\pi}\frac{\operatorname{Im}\phi}{|\phi|}, \\
\alpha_{22} &= \frac{|\phi|}{2} \operatorname{Im}\left(\frac{\phi_D}{\phi}\right) + \frac{2}{\pi}\frac{\operatorname{Re}\phi}{|\phi|}.
\end{aligned} \tag{5.40}$$

The positivity condition for α_{12} then implies

$$\operatorname{Im}\left(\frac{\phi_D}{\phi}\right) > 0, \tag{5.41}$$

which is precisely the condition to be outside the CMS. Furthermore, as we approach the CMS, $\alpha_{12} \to 0$, which, since $\alpha_{12} = 1/r_{12}$, implies that the charge sources are becoming infinitely spatially separated in the limit.

We have thus seen how to recover the string web picture of $\mathcal{N}$=2 BPS states from the low energy $\mathrm{U}(1)^N$ effective action on the Coulomb branch. We have checked that it matches precisely with the string webs found in the F theory realization of certain $\mathcal{N}$=2 theories as D3-brane probes of 7-brane backgrounds. But we emphasize that our string web picture has much greater generality: we have shown that BPS states can be represented by string webs on the Coulomb branch of an *arbitrary* $\mathcal{N}$=2 theory, irrespective of whether a string construction of that theory is known.

6. Open questions and future directions

In this paper we have shown that a string junction picture of decaying BPS states follows directly from the low energy effective theory in the vicinity of a CMS and relies on approximate boundary conditions which become exact in the limit of approaching the CMS. Furthermore this construction works for arbitrary field theory data (gauge group, matter representations, couplings and masses).

There are, however, a number of questions arising in this framework which we have not addressed. The most pressing of these is the "s-rule" [20, 9, 11, 12] in $\mathcal{N}$=2 theories, which is not apparent in our solutions. The s-rule is a selection rule on string junction configurations in 7-brane backgrounds which rules out some string junctions as not being supersymmetric even though they satisfy the charge conservation and tension-balancing constraints. Our generalized string webs reproduce these constraints but not the s-rule constraints. The s-rule presumably arises from the low energy point of view as an extra condition on our $\mathcal{N}$=2 solutions (5.27) to be BPS. In fact, such an extra condition may already be present in our discussion of the $\mathcal{N}$=2 BPS states, for we really solved only the harmonic equations (5.18) following from the BPS equations (5.16) and the Gauss constraints (5.9). To rederive the BPS equations from the harmonic equations requires an extra condition, namely that ϕ_r and ϕ_D^r be related by the special geometry relation (5.5).[4] Perhaps this extra condition gives rise to the s-rule for our low energy configurations. A low energy understanding of the s-rule is also needed to examine proposed stable non-BPS states in $\mathcal{N}$=2 theories [41].

There are also a number of other issues which can be examined in our framework. One is to use our string web picture to compute the spectrum of stable BPS states at strong coupling and conformal points in higher rank $\mathcal{N}$=2 theories; in some cases such spectra have been proposed using other methods [42, 43, 44]. Another open question which should be addressable in our framework is the question of the multiplicity of stable BPS states, for which only partial information is known from semiclassical techniques [45, 10, 46, 47].

It is an open question whether the string T-dual version of the string web picture of BPS states—namely BPS states as curves on the Seiberg-Witten Riemann surface with a certain metric [48, 49, 50, 51, 42, 43, 44]—has a low energy field theory explanation, and if so, what its relation is to the string web picture we have derived here.

Finally, our string web picture can be generalized in a number of directions. Our basic physical picture for the existence of a string web picture of BPS states near a CMS was very general and did not depend on the dimension of space-time. It would be interesting to apply these arguments to CMS in two dimensional [4], three dimensional [20], and five dimensional [52, 53] theories, especially as string constructions in the latter two cases already provide a string web picture of BPS states in certain cases. Another interesting generalization is to effective gravitational theories [33] and the associated question of the connection to formulations of BPS stability in string theory compactifications [54, 55].

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A. Appendix

This appendix solves the conditions coming from the extremization of the BPS bound, (4.21), and the reality and symmetry of the α_{ij} in (4.27) in the case of two charge sources. The setup is as described in section 4.1.5, so points on the SU(3) moduli space are parameterized by complex 3-vectors (a, b, c) satisfying $a+b+c=0$. The low energy $\mathrm{U}(1)^2$ charges are also described by complex 3-vectors whose components sum to zero. Vectors of this form can be transformed into complex 2-component vectors by a convenient unitary transformation

$$\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{3}a \\ b-c \\ 0 \end{pmatrix} \tag{A.1}$$

which rotates the third component to zero. Then the charges become

$$Q_1 = \frac{1}{\sqrt{2}} q_1 \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}, \qquad Q_2 = \frac{1}{\sqrt{2}} q_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \tag{A.2}$$

the manifolds of singularities become

$$X^1 = \frac{1}{\sqrt{2}} x \begin{pmatrix} \sqrt{3} \\ 3 \end{pmatrix}, \qquad X^2 = \frac{1}{\sqrt{2}} y \begin{pmatrix} -2\sqrt{3} \\ 0 \end{pmatrix}, \tag{A.3}$$

and the vacuum becomes

$$X^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}a \\ b-c \end{pmatrix}. \tag{A.4}$$

The vectors pointing from the vacuum to the singularities are thus

$$\xi^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}x - \sqrt{3}a \\ 3x + c - b \end{pmatrix}, \qquad \xi^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -2\sqrt{3}y - \sqrt{3}a \\ c - b \end{pmatrix}. \tag{A.5}$$

The BPS bound to be extremized is

$$\begin{aligned} M &= \mathrm{Re}\{\xi^1 \cdot A \cdot Q_1^* + \xi^2 \cdot A \cdot Q_2^*\} \qquad\qquad (\mathrm{A.6}) \\ &= \frac{1}{2}\,\mathrm{Re}\Big\{ q_1^* \Big[x(3A_{11} - \sqrt{3}A_{12} + 3\sqrt{3}A_{21} - 3A_{22}) - 3aA_{11} + \sqrt{3}aA_{12} \\ &\qquad + (c-b)(\sqrt{3}A_{21} - A_{22}) \Big] - q_2^* \Big[4\sqrt{3}yA_{12} + 2\sqrt{3}aA_{12} + 2(b-c)A_{22} \Big] \Big\} \end{aligned}$$

where A_{ij} is a general $\mathrm{U}(2)$ matrix. Extremizing with respect to x and y first implies that $A_{12} = A_{21} = 0$ and $A_{11} = A_{22} = e^{i\phi}$ with ϕ undetermined. Maximizing with respect to ϕ gives the BPS mass as

$$\begin{aligned} M_{BPS} &= \max_{\phi} \mathrm{Re}\left\{e^{i\phi}[q_1^*(b-a) + q_2^*(c-b)]\right\} \\ &= |q_1^*(b-a) + q_2^*(c-b)|. \end{aligned} \tag{A.7}$$

The maximization in the first line implies that ϕ is the phase of $q_1(b^* - a^*) + q_2(c^* - b^*)$, or equivalently, that

$$0 = \mathrm{Im}\left\{e^{i\phi}[q_1^*(b-a) + q_2^*(c-b)]\right\}. \tag{A.8}$$

To determine x and y we must perform the extremization with respect to the $\mathrm{U}(2)$ matrix A before doing the x, y extremization. Expand A about its extremal value as

$$A = e^{i\phi}\begin{pmatrix} 1 + iA & iB - C \\ iB + C & 1 + iD \end{pmatrix} \tag{A.9}$$

for small real A, B, C and D. Inserting this in the BPS bound (A.6) and extremizing with respect to A, B, C and D gives the four conditions

$$\begin{aligned} 0 &= \mathrm{Im}\left\{e^{i\phi} q_1^*(x-a)\right\}, \\ 0 &= \mathrm{Im}\left\{e^{i\phi}[q_1^*(3x+c-b) + 2q_2^*(b-c)]\right\}, \\ 0 &= \mathrm{Im}\left\{e^{i\phi}[q_1^*(x-b) - q_2^*(2y+a)]\right\}, \\ 0 &= \mathrm{Re}\left\{e^{i\phi}[q_1^*(2x+c) + q_2^*(2y+a)]\right\}. \end{aligned} \tag{A.10}$$

Subtracting three times the first equation from the second gives back the condition (A.8) which determines ϕ. Thus a convenient set of independent conditions on x and y can be taken to be

$$\begin{aligned} 0 &= \mathrm{Im}\left\{e^{i\phi} q_1^*(x-a)\right\}, \\ 0 &= \mathrm{Im}\left\{e^{i\phi} q_2^*(y-c)\right\}, \\ 0 &= \mathrm{Re}\left\{e^{i\phi}[q_1^*(2x+c) + q_2^*(2y+a)]\right\}, \end{aligned} \tag{A.11}$$

where the second condition is a linear combination of the first three equations of (A.10). These are three real equations for the two complex variables x and y.

The extra condition needed to determine x and y completely comes from the reality and symmetry of the α_{ij} coefficients defined in (4.28). Inserting our two-component vector values for the ξ^i and Q^i gives the equations

$$\begin{aligned} e^{i\phi}\begin{pmatrix} \sqrt{3}x - \sqrt{3}a \\ 3x + c - b \end{pmatrix} &= \alpha_{11} q_1 \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} + \alpha_{12} q_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \\ e^{i\phi}\begin{pmatrix} -2\sqrt{3}y - \sqrt{3}a \\ c - b \end{pmatrix} &= \alpha_{21} q_1 \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} + \alpha_{22} q_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \end{aligned} \tag{A.12}$$

which can be inverted to give

$$\begin{aligned}\alpha_{11} &= e^{i\phi}(x-a)/q_1,\\ \alpha_{21} &= e^{i\phi}(-2y-a)/q_1,\\ \alpha_{12} &= e^{i\phi}(2x+c)/q_2,\\ \alpha_{22} &= e^{i\phi}(-y+c)/q_2. \end{aligned} \tag{A.13}$$

The reality and symmetry conditions on the α_{ij} give five more conditions: $\operatorname{Im}\alpha_{ij} = 0$ and $\operatorname{Re}(\alpha_{12} - \alpha_{21}) = 0$. In fact only one of these conditions is independent of the three conditions in (A.11) and the condition (A.8) determining ϕ.

To see this and to solve these conditions, it is convenient to choose a non-orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for the complex numbers defined by

$$\mathbf{e}_j \equiv e^{i(\theta_j - \phi)}, \tag{A.14}$$

where θ_1 and θ_2 are defined to be the phases of q_1 and q_2:

$$q_j = |q_j| e^{i\theta_j}. \tag{A.15}$$

We then write, for example,

$$a = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \tag{A.16}$$

for unique real numbers a_j; define similarly the real numbers b_j, c_j, x_j,and y_j, for $j = 1, 2$.

In this basis the conditions (A.11) become

$$\begin{aligned} 0 &= x_2 - a_2,\\ 0 &= y_1 - c_1, \end{aligned} \tag{A.17}$$

and

$$0 = [2x_1 + c_1 + (2x_2 + c_2)\cos(\theta_1 - \theta_2)]|q_1| + [2y_2 + a_2 + (2y_1 + a_1)\cos(\theta_1 - \theta_2)]|q_2|, \tag{A.18}$$

and

$$(a_2 - b_2)|q_1| = (b_1 - c_1)|q_2|, \tag{A.19}$$

where this last condition is the translation into this basis of the condition (A.8) determining ϕ. The conditions that $\operatorname{Im}\alpha_{11} = \operatorname{Im}\alpha_{22} = 0$ are easily seen to give the first two conditions in (A.17) again, and so are not independent. The other two α_{ij} reality conditions give

$$\begin{aligned} \operatorname{Im}\alpha_{12} &= 0 = 2y_2 + a_2,\\ \operatorname{Im}\alpha_{21} &= 0 = 2x_1 + c_1. \end{aligned} \tag{A.20}$$

These two conditions, together with (A.17) and (A.19), imply (A.18). Thus (A.17) and (A.20) are an independent set of conditions determining x and y to be

$$\begin{aligned} x &= -\frac{1}{2}c_1\mathbf{e}_1 + a_2\mathbf{e}_2, \\ y &= +c_1\mathbf{e}_1 - \frac{1}{2}a_2\mathbf{e}_2. \end{aligned} \tag{A.21}$$

Finally, we must show that the symmetry condition on the α_{ij}, namely $\mathrm{Re}(\alpha_{12} - \alpha_{21}) = 0$, is not an independent condition. This condition reads in our basis

$$0 = |q_1|[2x_2 + c_2 + (2x_1 + c_1)\cos(\theta_1 - \theta_2)] + |q_2|[2y_1 + a_1 + (2y_2 + a_2)\cos(\theta_1 - \theta_2)]; \tag{A.22}$$

plugging in the values for x and y then gives precisely (A.19), which shows that it is indeed not an independent condition.

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STRING WEBS AND THE DECAY OF SUPERSYMMETRIC PARTICLES

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The spectrum of stable electrically and magnetically charged supersymmetric particles can change discontinuously due to the decay of these particles as the vacuum on the Coulomb branch is varied. We show that this decay process is well described by semi-classical field configurations purely in terms of the low energy effective action on the Coulomb branch even when it occurs at strong coupling. The resulting picture of the stable supersymmetric spectrum is a generalization of the "string web" picture of these states found in string constructions of certain theories.

Four dimensional gauge theories with at least eight supersymmetries have a Coulomb branch of inequivalent vacua in which the low energy theory has unbroken $U(1)$ gauge factors. These theories also have a spectrum of massive charged particles with various electric and magnetic charges under the $U(1)$'s, lying in supersymmetry multiplets. The masses of those in short multiplets—the BPS states—are related to their charges by the supersymmetry algebra.[1] The spectrum of possible BPS masses can then be determined using supersymmetry selections rules.[2] This, however, leaves open the question of the existence and multiplicity of these states. Furthermore, even if such a state exists in some region of the Coulomb branch, it may be unstable to decay at curves of marginal stability (CMS) on the Coulomb branch.[3] We propose[4] a solution to the question of the multiplicity of BPS states for $N=2$ supersymmetric theories in four dimensions just in terms of the low energy effective $U(1)^n$ action on the Coulomb branch.

The form of the answer we get coincides with the "string web" picture of BPS states developed in the context of the D3-brane construction of $N=4$ $SU(n)$ superYang-Mills theory and the F-theory solution to $N=2$ $SU(2)$ gauge theory with fundamental matter[5]. Moreover, our solution generalizes these constructions to arbitrary field theory data (gauge groups, matter representations, couplings and masses). The resulting picture is quite simple: BPS states are represented by string webs on the Coulomb branch of the theory with one end at the point corresponding to the vacuum in question (the analog of the 3-brane probe in the F-theory picture) and the other ends lying on the complex codimension 1 singularities on the Coulomb branch (the analogs of the (p,q) 7-branes of the F-theory picture). The strands of the string web lie along geodesics in the Coulomb branch metric. Each strand carries electric and magnetic charges under the $U(1)^n$ low energy gauge group: the total charge flowing into the vacuum point determines the total charge of the BPS state, while only multiples of the charges determined by the $Sp(2n,Z)$ monodromies around the codimension 1 singularities are allowed to flow into those ends of the web.

Perhaps the most surprising thing about our solution is that it describes the stability of the monopole and dyon BPS spectrum wholly in terms of the $U(1)^n$ low energy effective action. This is possible because the distance ΔX from a given vacuum on the Coulomb branch to a CMS acts as a new low energy scale which can be made arbitrarily small compared to the strong coupling scale Λ of the non-Abelian gauge theory as we approach the CMS. In particular, we show that as we approach the CMS the classical low energy field configuration describing a state which decays across the CMS will develop two or more widely separated charge centers (which appear as singularities in the low energy solution); the distance between these centers varies inversely with ΔX. In this limit the details of the microscopic physics become irrelevant for the decay of a BPS state across a CMS, which is described by a low energy field configuration with charge centers becoming infinitely separated.

The classical BPS field configuration of the scalar fields in the low energy $U(1)^n$ theory will have one or more singularities or sources where scalar field gradients and $U(1)$ field strengths diverge. Near these points the low energy description breaks down and should be cut off by boundary conditions reflecting the matching onto the microscopic physics of the non-Abelian gauge theory. We show that these boundary conditions are determined by the BPS condition.

Several previous discussions[6,7,8] of the semi-classical description of BPS states in supersymmetric theories were the starting point for this work.

1. BPS states near CMS

We restrict our attention to a single charge sector of the low energy $U(1)^n$ theory on the Coulomb branch. The question is whether at a given vacuum there is or is not a one-particle BPS state in that charge sector. The mass of a BPS state is $M = |Z|$ where the central charge Z is the sum of terms proportional to the charges. This mass is the minimum mass of any state (BPS or not, single particle or not) in this charge sector. A single BPS particle M in this charge sector is stable or at worst marginally stable against decay into two (or more) constituent particles, since by charge conservation $Z = Z_1 + Z_2$, so by the triangle inequality $M \leq M_1 + M_2$. The CMS are submanifolds of the Coulomb branch where this inequality is saturated. As one adiabatically changes the order parameter on the Coulomb branch from a vacuum on one side of a CMS to a vacuum on the other, the one particle state M becomes more and more nearly degenerate with the two particle state $M_1 + M_2$. Suppose M does decay across the CMS, so only the two particle state $M_1 + M_2$ remains in the spectrum once we have crossed the CMS. Since the transition takes place right at threshold, M will decay into the two particle state with zero relative momentum.

Now, it follows from the triangle inequality that the two particle state $M_1 + M_2$ is not BPS, even for zero relative momentum, except precisely at the CMS. Thus there will generically be no BPS force cancelation between particles M_1 and M_2, so the zero relative momentum two particle state is classically approximated by two spatially infinitely separated one-particle states (to have a static configuration). By locality and the adiabatic theorem, the transition across the CMS of a one particle state to a widely separated, zero momentum, two particle state should go by way of field configurations with large spatial overlap, implying that a state decaying just where its mass reaches the two-particle threshold of its decay products should have a diverging spatial extent as it approaches the transition. This argument gives the basic physics underlying our approach to decays across CMS: *if a BPS state does decay across a CMS, that decay will be visible semiclassically in the low energy effective action even if it takes place at strong coupling from a microscopic point of view.* The resulting picture is quite intuitive: a BPS state decaying across a CMS does so by becoming an ever-larger, more loosely bound state of its eventual decay products. Once the CMS is crossed, the bound state ceases to exist, and so, in particular, there will be no static BPS solutions to the low energy equations of motion and boundary conditions in this region of the Coulomb branch.

2. A $U(1)$ toy example

We illustrate these properties in a simple model which captures the main physics of the CMS. Consider the $U(1)$ effective action with two real scalars X, Y:

$$S = -\int d^4x \left(\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu X \partial^\mu X + \frac{1}{2}\partial_\mu Y \partial^\mu Y \right) \tag{1}$$

This theory by itself is too simple to be interesting, but all we need to add to it to capture the essential physics of the decay of BPS states across CMS are the presence of singularities in the vacuum manifold. In this case the vacuum manifold (Coulomb branch) is the X-Y plane. For concreteness we posit that there are two singularities at points on the Coulomb branch with coordinates $(X, Y) = (L, 0)$ where a particle with electric charge $(Q_E, Q_B) = (1, 0)$ becomes massless, and $(X, Y) = (0, L)$ where a particle with magnetic charge $(Q_E, Q_B) = (0, 1)$ becomes massless.

The low energy effective action (1) includes only the first terms in a derivative expansion with higher terms suppressed by powers of a cutoff energy scale Λ, or distance scale $r_\Lambda \equiv \frac{1}{\Lambda}$. Our strategy will be to use the low energy $U(1)$ solution away from the charge cores ($r > r_\Lambda$) and impose appropriate boundary conditions in the vicinity of the cores ($r \sim r_\Lambda$). The qualitative features of these boundary conditions are easy to deduce. The basic boundary condition on the electric and magnetic fields follow from Gauss' law, $\oint \mathbf{E} \cdot d\mathbf{a} = 4\pi Q_E$, and $\oint \mathbf{B} \cdot d\mathbf{a} = 4\pi Q_B$, where the integrals are over spheres enclosing the charge cores.

The X, Y scalar fields tend to approach their values at Coulomb branch singularities near the charge cores. Indeed, such field configurations are energetically favored because they approach those of the charged BPS states which become massless at the singularities $(X, Y) = (0, L)$ or $(L, 0)$. Thus the scalar fields satisfy the approximate Dirichlet boundary conditions

$$(X,Y) \simeq (0,L) \text{ or } (L,0) \quad \text{within} \quad B^3_\Lambda \tag{2}$$

where B^3_Λ is a ball of approximate radius r_Λ around each charge core and $(X,Y) \simeq (L,0)$ means that (X,Y) pass within a distance Λ of $(L,0)$ on the Coulomb branch. This "fuzzy ball" boundary condition is all we can physically demand of the low energy solution since it is not accurate on spatial resolutions less than r_Λ nor for field value resolutions less than Λ. In the limit as the vacuum approaches a CMS, as we have described qualitatively in the last section and will see explicitly below, the size of the field configuration grows, and so the relative size of the cutoff region r_Λ to the field configuration shrinks. In this limit the fuzzy ball Dirichlet boundary conditions become precise.

To simplify the algebra and make the basic points clear we choose the vacuum to be symmetrically placed at the point (X_0, X_0) on the Coulomb branch and choose the state to have electric and magnetic charges $(Q_E, Q_B) = (1,1)$. There will then be separate electric and magnetic charge cores. The usual BPS equations for a static field configuration following from (1) are then simply $\nabla^2 X = \nabla^2 Y = 0$ while the electric and magnetic fields are determined in terms of X and Y by $\mathbf{E} = \boldsymbol{\nabla} X$, and $\mathbf{B} = \boldsymbol{\nabla} Y$. The Gauss constraint implies a solution of the approximate form

$$X - X_0 = \frac{1}{|\mathbf{x} - \mathbf{x}_E|}, \qquad Y - X_0 = \frac{1}{|\mathbf{x} - \mathbf{x}_B|} \tag{3}$$

so that the electric and magnetic charge cores are located at $\mathbf{x}_E$ and $\mathbf{x}_B$ respectively. We want to check whether there are values of $\mathbf{x}_E$ and $\mathbf{x}_B$ for which this solution satisfies the fuzzy ball Dirichlet boundary condition (2), which demands that the (X,Y) values taken by a solution enters a small ball around $(L,0)$ as $\mathbf{x} \to \mathbf{x}_E$ (and a small ball around $(0,L)$ as $\mathbf{x} \to \mathbf{x}_B$).

Now for $|\mathbf{x} - \mathbf{x}_E| \simeq (L - X_0)^{-1}$ it follows that $X \simeq L$ as desired near the electric singularity. But this then implies that $Y \simeq X_0 + r_{EB}^{-1}$ where $r_{EB} \equiv |\mathbf{x}_E - \mathbf{x}_B|$ is the spatial distance between the charge centers. The fuzzy Dirichlet boundary condition $Y \simeq 0$ then implies that the separation of the charge centers must be fixed to be

$$r_{EB} = -\frac{1}{X_0}. \tag{4}$$

There are two striking things about this conclusion. First, there is only a solution for $X_0 < 0$. This inequality is equivalent to saying that there is only a solution on one side of the CMS for our dyonic $(Q_E, Q_B) = (1,1)$ state (as can be checked from the explicit BPS mass formula). Second, as $X_0 \to 0^-$ (*i.e.* as we approach the CMS) the spatial separation of the electric and magnetic charge centers goes to infinity. This confirms the qualitative description of the decay of a supersymmetric state near the CMS given above.

As we describe in detail elsewhere[4], in the limit as the vacuum approaches the CMS, the projection on the Coulomb branch of the above low energy solution more and more accurately approximates a string web, *i.e.* it degenerates to a collection of segments of geodesics—"strings"—on the Coulomb branch which can meet in 3-string junctions or end at the vacuum or singularities on the Coulomb branch. For example, in the $U(n)$ $N=4$ SYM theory these "string" webs are thus stretched in the $6n$-dimensional moduli space and end on $6(n-1)$-dimensional singular submanifolds. Note that this is different from the string web picture found in the string construction of this theory, where the strings are stretched in the 6-dimensional space transverse to N D3-branes. It can be shown[4] that the 6-dimensional string theory webs are obtained from our $6n$-dimensional webs by a simple mapping. The basic reason this works is that the $6N$-dimensional moduli space $\mathcal{M}$ of the $U(n)$ theory is $\mathcal{M} = (R^6)^n/S_n$ where the permutation group S_n interchanging the R^6 factors is the Weyl group of $U(n)$.

Another property of the string webs found in string constructions is that 3-pronged string states that stretch between D-branes satisfy charge conservation and tension balance. These properties are also reproduced[4] in the webs from our low energy solutions. However, the low energy solutions have an extended spatial structure which is not seen in the string constructions. For example, the configuration discussed above corresponds to a $(1,0)$ (fundamental string), a $(0,1)$ (D-string)and a $(1,1)$ string stretched between three D3-branes. The three prongs or string legs meet at a common point, the junction, which remains point-like even arbitrarily close to the CMS. In the field theory picture discussed here, we have seen that the leg corresponding to the decaying $(1,1)$ dyon grows in spatial size (*i.e.*, along the D3-brane worldvolume) near the CMS. Arbitrarily close to the CMS, the configuration resembles two separate strings that end on the D3-brane corresponding to the decaying $U(1)$. Thus there does not appear to be a point-like junction as the decaying configuration approaches the CMS in the field theory picture.

The difference between the two perspectives is the result of different orders of limits. Consider the Dirac-Born-Infeld action for the D3-brane corresponding to the decaying dyon, treating it as a probe in a background of other D3-branes. The brane prong corresponding to the decaying $(1,1)$ dyon then ends on other D3-branes that are treated as a fixed background. Looking at the quadratic terms and comparing with the low energy effective action we have used above, we can see that the scalars X in the field theory with mass dimension unity and the coordinates in the brane transverse space, x, are related by $X = x/\alpha'$, where α' is the string length squared. The separation between the charge centers in the D-brane worldvolume is

$$r_{EB} \sim 1/(-X_0) = \alpha'/(-x_0). \tag{5}$$

The low energy field theory is a good approximation in the $\alpha' \to 0$ limit holding the scalar vevs, including X_0, fixed. On the other hand, in the string junction/geodesic picture in string theory, the coordinates x_0 are what are held fixed: taking $\alpha' \to 0$ to suppress higher stringy corrections gives a vanishing separation r_{EB} between the charge centers, *i.e.* a pointlike junction. For $x_0 = 0$, the separation is indeterminate, which corresponds to the two strings ending anywhere on the D-brane. This recovers the string theory result and it further illustrates that a *spatially* point-like junction is not visible within the field theory approximation.

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