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Anomaly Cancelations in Orientifolds with Quantized B Flux

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Abstract

We consider anomaly cancelations in Type IIB orientifolds on $T^4/\mathbb{Z}_N$ with quantized NS-NS sector background B-flux. For a rank b B-flux on T^4 (b is always even) and when N is even, the cancelation requires a $2^{b/2}$ multiplicity of states in the 59-open string sector. We identify the twisted sector R-R scalars and tensor multiplets which are involved in the Green-Schwarz mechanism. We give more details of the construction of these models and argue that consistency with the $2^{b/2}$ multiplicity of 59-sector states requires a modification of the relation between the open string 1-loop channel modulus and the closed string tree channel modulus in the 59-cylinder amplitudes.

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I. INTRODUCTION

Probably the most attractive feature of string theory is that it incorporates both quantum gravity and gauge interactions in a consistent framework. A consistent string model should therefore have the appropriate particle content and interactions such that all apparent field-theoretic gauge and/or gravitational anomalies are absent. In four dimensional string models, there can be $U(1)_A$ gauge symmetries which are pseudo-anomalous. However, the Green-Schwarz mechanism [1] ensures that counterterms are generated from the exchange of pseudoscalar fields and any field-theoretic $U(1)_A$ triangle anomaly is canceled by the corresponding non-trivial $U(1)_A$ transformation of the pseudoscalars.

The analysis of anomalous $U(1)$ have mainly be focused on perturbative heterotic string theory [2] where many semi-realistic string models can be easily constructed. In recent years, however, tremendous progress has been made in constructing four-dimensional $\mathcal{N}=1$ Type I string models [3]. Moreover, it was recently suggested that if the Standard Model particles are localized on a set of p-branes where $3<p<9$ whereas gravity lives in higher dimensional spacetime, then the string scale can be as low as a few TeV [7–9] . Type I string theory provides a natural framework to realize this "brane world" scenario [8–10]. In view of these developments, it is therefore important to carry out similar analysis for Type I string vacua.

Anomaly cancelations in some four-dimensional $\mathcal{N}=1$ Type I string models were studied in some recent papers [12]. Anomaly cancelations in $D=6$ Type IIB $\mathbb{Z}_N$ orientifolds without the B-flux background were studied in [13,14]. These simple examples illustrate some new features in Type I string vacua that are distinct from that of the perturbative heterotic string. First, the R-R scalars which participate in the Green-Schwarz mechanism come from the twisted sectors which can couple differently to different gauge factors. Therefore, in contrast to perturbative heterotic string, there can be more than one anomalous $U(1)$ in these Type I vacua. Moreover, the Fayet-Iliopoulos term already appears at tree-level and is dependent on the blowing-up modes of the orbifolds.

In this paper, we consider orientifold models with a non-vanishing NS-NS sector B-field background [15–17]. Although the fluctuations of the NS-NS B-field are projected out of the orientifold spectrum (since it is odd under worldsheet parity reversal), a quantized vacuum expectation value of this B-field is allowed. Furthermore, the presence of this B-field gives rise to some novel features. In particular, the rank of the gauge group is reduced by $2^{b/2}$ [15–17] where b is the rank of the background B-field. Moreover, in models where there are both $D9$ and $D5$-branes, the 59 sector states come with a multiplicity of $2^{b/2}$ [17]. These reduced rank orientifold models are dual to the CHL strings [19] in heterotic string theory. For models with both $D9$ and $D5$-branes, the corresponding heterotic duals contain NS fivebranes. Therefore, the analysis of these orientifolds may shed some lights on the non-perturbative properties of the CHL strings.

Following the work of Sen [21], we also expect a close relation between orientifolds and F theory vacua. In Ref [22], some eight-dimensional F theory vacua with non-zero background B-flux were studied. Naively, the presence of this NS-NS sector B-field background is incompatible with the $SL(2,\mathbb{Z})$ symmetry (the S-duality) of Type IIB theory. However, the presence of this B-flux freezes 8 of the moduli of the elliptic $K3$, leaving only a 10-dimensional subspace. The monodromy group on this subspace is reduced from $SL(2,\mathbb{Z})$ to the congruence group $\Gamma_0(2)$ which is the largest subgroup of $SL(2,\mathbb{Z})$ that keeps the

B-flux invariant. In this paper, we consider models with both $D9$ and $D5$-branes. They can be viewed as generalizations of F theory compactifications of this kind to lower dimensions when there are more than one type of 7-branes.

On the other hand, this class of models is closely related to the setup considered in [32]. According to [32], the positions of the branes cease to commute in the presence of a B-field background. It is possible that our results here can be understood from the point of view of noncommutative geometry.

Type IIB orientifolds with quantized background B-field are also interesting from the phenomenological point of view. In the absence of B-field, the residual gauge symmetries are typically too large for the models to be phenomenologically interesting. Since the rank of the gauge group is reduced by $2^{b/2}$ in the presence of B-field, it is possible to construct string models containing the Standard Model with fewer additional gauge symmetries. In fact, the three-family Pati-Salam like model in Ref. [9] was constructed by turning on a non-zero background B-field with $b = 2$ in a $\mathbb{Z}_6$ orientifold. There are both $D9$ and $D5$-branes in this model. Under T-duality, they become 2 sets of $D5$-branes: $D5_1$ and $D5_2$ whose intersection is our four-dimensional non-compact spacetime. The Standard Model $SU(3)$ lives on the $D5_1$-branes and the three chiral families of fermions in the Standard Model come from open strings with at least one end attached to the $D5_1$-branes. One of the families comes from $5_1 5_1$ open strings whereas the other two families are $5_1 5_2$ open string states. The fact that there are two families in the $5_1 5_2$ sector depends crucially on the multiplicity of $2^{b/2} = 2$ in the $5_1 5_2$ sector.

Since the main new features in models with non-zero B-field that we discuss here already appear in six dimensions, we will focus our attention to six-dimensional models in this paper. Moreover, as we will see, anomaly cancelations in six dimensions provide rather stringent constraints on the consistencies of these models. In particular, we will show that the multiplicity of $2^{b/2}$ in the 59 sector is crucial for the cancelation of the leading anomalies (*i.e.*, the $\mathrm{tr}\, F^4$ and $\mathrm{tr}\, R^4$ terms). Moreover, the remaining anomalies are properly factorized which can then be canceled by the Green-Schwarz mechanism. The analysis of four-dimensional $\mathcal{N} = 1$ orientifold models with background B-field will appear in a separate publication [31].

This paper is organized as follows. In Section II, we describe in detail the construction of some six-dimensional orientifold models with non-zero background B-field considered in Ref. [17]. In Section III, we discuss anomaly cancelations in these models. We end with some discussion in Section IV. Some of the technical details are relegated to the appendices.

II. SIX DIMENSIONAL EXAMPLES

In this section, we discuss in detail the construction of six-dimensional Type IIB orientifolds with background B-flux [17]. In particular, we consider the orbifold limits of K3: $T^4/\mathbb{Z}_N$ where $N = 2, 3, 4, 6$.[1] Since we are interested in models with both $D9$ and $D5$-branes, we only consider the cases $N = 2, 4, 6$. However, turning on the NS-NS antisymmetric two-

[1] These orientifolds without background B-flux have been constructed in [4,5].

form background seems to render the $T^4/\mathbb{Z}_6$ Type IIB perturbative orientifold inconsistent[2], so we will concentrate on the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ models. Type I vacua on smooth K3 with non-zero B-field have been studied in [20].

Toroidal compactification of Type IIB string theory on a four dimensional torus T^4 gives rise to a six dimensional model with $\mathcal{N}=4$ supersymmetry. By gauging the world-sheet parity Ω ($X_L \leftrightarrow X_R$) of Type IIB strings reduces by half the number of supersymmetries. One can further reduce the number of supersymmetries to $\mathcal{N}=1$ by orbifolding. Specifically, the $\mathbb{Z}_N$ orbifold action is realized by powers of the twist generator θ ($\theta^N=1$) which can be written in the form

$$\theta = \exp(2i\pi(v_1 J_{67} + v_2 J_{89})), \tag{1}$$

where J_{mn} are $SO(4)$ Cartan generators and $v \equiv (v_1, v_2) = \frac{1}{N}(1,-1)$ represents the twist. In terms of the complex bosonic coordinates $Y_1 = X_6 + iX_7$, $Y_2 = X_8 + iX_9$ that parametrize the torus, θ acts diagonally as

$$\theta^k Y_i = \mathrm{e}^{2i\pi k v_i} Y_i. \tag{2}$$

Similarly, we define complex fermionic fields ψ^i as $\psi^1 = \psi^6 + i\psi^7$ and $\psi^2 = \psi^8 + i\psi^9$.

To derive the massless spectra of the orientifolds, we will work in light-cone gauge. For example, in the closed untwisted sector the NS massless states are $\psi^\mu_{-\frac{1}{2}}|0\rangle$ which are invariant under θ, and $\psi^i_{-\frac{1}{2}}|0\rangle$ which transforms as

$$\theta^k \psi^i_{-\frac{1}{2}}|0\rangle = \mathrm{e}^{2i\pi k v_i} \psi^i_{-\frac{1}{2}}|0\rangle. \tag{3}$$

Complex conjugates $\psi^{\bar{i}}_{-\frac{1}{2}}$ transform with a phase $\mathrm{e}^{-2i\pi k v_i}$. The untwisted massless Ramond states are of the form $|s_0 s_1 s_2 s_3\rangle$ with $s_0, s_i = \pm\frac{1}{2}$. To implement the GSO projection we retain only the states with $s_0 + s_1 + s_2 + s_3 = 0,\ \mathrm{mod}\ 2$. These states transform as

$$\theta^k |s_0 s_1 s_2 s_3\rangle = \mathrm{e}^{2i\pi k v\cdot s} |s_0 s_1 s_2 s_3\rangle. \tag{4}$$

The close string spectrum is obtained by retaining only those states in the untwisted sector which are invariant under the orientifold group action and by including twisted sector states. This will be discussed in more detail in Sec. II B.

Although Type IIB theory is a theory of closed strings, the orientifold projection requires both closed and open string sectors for consistency. The Klein bottle amplitude (which is present due to the orientifold projection) in the closed string sector generically gives rise to tadpole divergences. These divergences are canceled by the new contributions from the open string sector [29]. Alternatively, orientifold fixed planes are sources for the Ramond-Ramond (R-R) $(p+1)$-forms. Charge cancelation can be generically achieved by including

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[2] $T^4/\mathbb{Z}_6$ Type IIB orientifold with rank two B-flux does not contain states $(\mathbf{4},\mathbf{4};\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{4})$ given in Table II of [17]. As a result, the $\mathrm{tr}\,R^4$ and $\mathrm{tr}\,F^4$ anomalies do not cancel. However, it is consistent [31] to turn on B-field in the $T^6/\mathbb{Z}_6$ orientifold in Ref. [9] since the corresponding $\mathbb{Z}_3$ twist is different.

the right number of Dp-branes, carrying opposite charge with respect to these forms [6]. The endpoints of an open string are labeled by a and b which lie on a D_p and D_q-brane respectively (the corresponding excitations are pq sector states). The models we discuss here have both D9 and D5-branes. The open string sector of the models will be discussed in Sec. II A.

A. Open String Sector

The open string spectrum of Type IIB orientifolds on $T^4/\mathbb{Z}_N$ are determined by the type and the number of D-branes necessary to cancel the tadpole divergences in the closed string Klein bottle amplitude. The worldsheet parity element Ω of the orientifold group produces the tadpole which must be canceled by the 9-branes. For even N, the closed string sector tadpole from $\Omega\theta^{N/2}$ orientifold element is canceled by introducing 5-branes, whose worldvolume spans the uncompactified $D = 6$ space-time.

Let Ψ be a world-sheet excitation and a, b represent Chan-Paton indices associated with the string endpoints on Dp-brane and Dq-brane ($p, q = \{5, 9\}$). These Chan-Paton indices must be contracted with a hermitian matrix λ_{ab}. The action of the group elements on the open string state $|\Psi, ab\rangle$ is given by [4]

$$\begin{aligned} \theta^k :\ & |\Psi, ab\rangle \to (\gamma_{k,p})_{aa'} |\theta^k.\Psi, a'b'\rangle (\gamma_{k,q})^{-1}_{b'b}, \\ \Omega\theta^k :\ & |\Psi, ab\rangle \to (\gamma_{\Omega_k,p})_{aa'} |\Omega\theta^k.\Psi, b'a'\rangle (\gamma_{\Omega_k,q})^{-1}_{b'b}, \end{aligned} \tag{5}$$

where $\gamma_{k,p}$ and $\gamma_{\Omega_k,p}$ are unitary matrices associated with θ^k. In order to be consistent with group multiplication (5), we can choose

$$\gamma_{\Omega_k,p} = \gamma_{k,p} \cdot \gamma_{\Omega,p}, \tag{6}$$

and

$$\gamma_{k,p} = \gamma^k_{1,p}. \tag{7}$$

The absence of pure gauge twists requires $\gamma_{0,p} = 1$ [4]. Because $\theta^N = 1$,

$$\gamma^N_{1,p} = \pm 1. \tag{8}$$

Otherwise $\gamma_{N,p}$ would be a pure gauge twist. Similarly, from $(\Omega\theta^k)^2 = \theta^{2k}$,

$$\gamma_{\Omega_k,p} = \pm \gamma_{2k,p}\, \gamma^T_{\Omega_k,p}. \tag{9}$$

Finally, using eqs. (6), (7), (9) and the unitarity of the γ matrices we obtain

$$\gamma^*_{k,p} = \pm \gamma^*_{\Omega,p}\, \gamma_{k,p} \gamma_{\Omega,p}. \tag{10}$$

Since different types of branes are present it is also necessary to consider the action of $(\Omega\theta^k)^2$ on pq states. In Ref. [4], it was argued that $\Omega^2 = -1$ on 95 states. This implies that in eq. (9) there are opposite signs for 9 and 5-branes. Since a simultaneous T-duality on $X_{6,7,8,9}$ interchanges 9 and 5- branes along with Ω_0 and $\Omega_{N/2}$,

$$\gamma_{\Omega_0,p} = \pm\gamma^T_{\Omega_0,p} \qquad \Rightarrow \qquad \gamma_{\Omega_{N/2},p} = \mp\gamma^T_{\Omega_{N/2},p}, \tag{11}$$

where the various signs are now correlated. From eqs. (9)-(11) it further follows that

$$\gamma^N_{1,p} = -1. \tag{12}$$

for both 9 and 5- branes.

Consider first the 99 open string states. The massless NS states include gauge bosons $\psi^\mu_{-\frac{1}{2}}|0,ab\rangle\,\lambda^{(0)}_{ab}$ and matter scalars $\psi^i_{-\frac{1}{2}}|0,ab\rangle\,\lambda^{(i)}_{ab}$. The Chan-Paton matrices must be such that the full states are invariant under the action of the orientifold group. Hence,

$$\begin{aligned}
\lambda^{(0)} &= \gamma_{1,9}\lambda^{(0)}\gamma^{-1}_{1,9} \qquad ; \qquad \lambda^{(0)} = -\gamma_{\Omega,9}\lambda^{(0)^T}\gamma^{-1}_{\Omega,9},\\
\lambda^{(i)} &= \mathrm{e}^{2\pi i v_i}\gamma_{1,9}\lambda^{(i)}\gamma^{-1}_{1,9} \qquad ; \qquad \lambda^{(i)} = -\gamma_{\Omega,9}\lambda^{(i)^T}\gamma^{-1}_{\Omega,9}.
\end{aligned} \tag{13}$$

The massless NS states of the 55 open strings also include gauge bosons $\psi^\mu_{-\frac{1}{2}}|0,ab\rangle\,\lambda^{(0)}_{ab}$ and scalars $\psi^i_{-\frac{1}{2}}|0,ab\rangle\,\lambda^{(i)}_{ab}$. We consider models where all 5-branes sit at the origin of the orientifold. The Chan-Paton matrices then satisfy

$$\begin{aligned}
\lambda^{(0)} &= \gamma_{1,5}\lambda^{(0)}\gamma^{-1}_{1,5} \qquad ; \qquad \lambda^{(0)} = -\gamma_{\Omega,5}\lambda^{(0)^T}\gamma^{-1}_{\Omega,5},\\
\lambda^{(i)} &= e^{2\pi i v_i}\gamma_{1,5}\lambda^{(i)}\gamma^{-1}_{1,5} \qquad ; \qquad \lambda^{(i)} = \gamma_{\Omega,5}\lambda^{(i)^T}\gamma^{-1}_{\Omega,5}.
\end{aligned} \tag{14}$$

The sign change in the Ω projection is due to the DD boundary conditions on the directions transverse to the 5-branes.

Finally, consider 59 open string states. For 5-branes, the compact coordinates obey mixed DN boundary conditions and have expansions with half-integer modded creation operators. By world-sheet supersymmetry their fermionic partners in the NS sector are integer modded. Their zero modes form a representation of a Clifford algebra and can be labeled as $|s_i, s_j\rangle$, $i,j = \{2,3\}$, with $s_i, s_j = \pm\frac{1}{2}$. The GSO projection further imposes $s_i = s_j$. Since $|s_i, s_j\rangle$ are invariant under θ, the orientifold projection on these states implies

$$\lambda = \gamma_{1,5}\lambda\gamma^{-1}_{1,9}. \tag{15}$$

Notice that here the index a (b) lies on a 5-brane (9-brane). Ω relates 59 with 95 sectors and does not impose extra constraints on λ.

The perturbative spectrum of Type IIB theory contains two massless antisymmetric tensor fields: one coming from the NS-NS sector, and another coming from the R-R sector. Under the world-sheet parity reversal, the NS-NS two-form $B_{\mu\nu}$ is projected out, while the R-R two-form $B'_{\mu\nu}$ is kept. Although the fluctuations of B_{ij} (the components of $B_{\mu\nu}$ in the compactified dimensions) are projected out of the perturbative unoriented closed string spectrum, a quantized vacuum expectation value of B_{ij} is allowed. To see this, consider the left- and right-moving momenta in the 4 dimensions compactified on a torus T^4:

$$P_{L,R} = \frac{\tilde{e}^i}{R_i}(n_i - b_{ij}m^j) \pm \frac{e_i m^i R_i}{\alpha'}\ , \tag{16}$$

where m_i and n^i are integers, e_i are constant vielbeins such that $e_i \cdot e_j = G_{ij}$ is the constant background metric on T^4, R_i are compactification radii of the T^4, and $e_i \cdot \tilde{e}^j = \delta_i{}^j$. In (16) $b_{ij} = B_{ij} R_i R_j / \alpha'$. Note that the components of b_{ij} are defined up to a shift $b_{ij} \to b_{ij} + 1$ (which can be absorbed by redefining n_i). With this normalization, only the values $b_{ij} = 0$ and $1/2$ are invariant under Ω, and hence the vacuum expectation values of b_{ij} are quantized. Let $b = rank(B_{ij})$ ($i, j \in \{6, 7, 8, 9\}$) be the rank of NS-NS antisymmetric tensor, it is clear that $b \in 2\mathbb{Z}$.

The unitary matrices $\gamma_{k,p}$ and $\gamma_{\Omega_k,p}$ must be chosen so that NS-NS tadpoles from the Möbius strip (MS) and the cylinder (C) of the open string sector cancel the NS-NS tadpole of the closed string sector coming from the Klein bottle (KB). The specific form of $\gamma_{k,p}$ and $\gamma_{\Omega_k,p}$, and so the massless open string sector of the orientifold, is sensitive to the NS-NS antisymmetric tensor background.

Consider first the case of zero B-flux, $b = 0$. The 1-loop vacuum amplitudes of the models are given in Appendix A. The $t \to 0$ divergences of the KB, C and MS (c.f., Eqs.(A1),(A7) and (A15)) produce tadpoles for the untwisted sector R-R 6- and 10-form as well as the twisted sector R-R 6-form. Relating the tree channel modulus ℓ to the KB, MS and C loop moduli through

$$\mathrm{KB}: t_{\mathcal{K}} = \frac{1}{4\ell}, \qquad \mathrm{MS}: t_{\mathcal{M}} = \frac{1}{8\ell}, \qquad \mathrm{C}: t_{\mathcal{C}} = \frac{1}{2\ell}, \tag{17}$$

all the tadpoles vanish provided[3]

$$\begin{aligned}
T^4/\mathbb{Z}_2: \quad & \mathrm{Tr}\,\gamma_{0,9} = \mathrm{Tr}\,\gamma_{0,5} = 32, \\
& \gamma_{\Omega_0,9} = \gamma^T_{\Omega_0,9}\,, \qquad \gamma_{\Omega_0,5} = -\gamma^T_{\Omega_0,5}, \\
& \mathrm{Tr}\,\gamma_{1,9} = \mathrm{Tr}\,\gamma_{1,5} = 0, \\
\\
T^4/\mathbb{Z}_4: \quad & \mathrm{Tr}\,\gamma_{0,9} = \mathrm{Tr}\,\gamma_{0,5} = 32, \\
& \gamma_{\Omega_0,9} = \gamma^T_{\Omega_0,9}\,, \qquad \gamma_{\Omega_0,5} = -\gamma^T_{\Omega_0,5}, \\
& \mathrm{Tr}\,\gamma_{1,9} = \mathrm{Tr}\,\gamma_{2,9} = \mathrm{Tr}\,\gamma_{1,5} = \mathrm{Tr}\,\gamma_{2,5} = 0.
\end{aligned} \tag{18}$$

The solutions are given by

$$\begin{aligned}
T^4/\mathbb{Z}_2: \quad & \gamma_{1,9} = \gamma_{1,5} = \Big(iI_{16}, -iI_{16}\Big), \\
T^4/\mathbb{Z}_4: \quad & \gamma_{1,9} = \gamma_{1,5} = \Big(\beta I_8, \beta^3 I_8, \overline{\beta} I_8, \overline{\beta}^3 I_8\Big), \qquad \beta = e^{\pi i/4}.
\end{aligned} \tag{19}$$

where I_n denotes an identity matrix of rank n. Furthermore, we can choose $\gamma_{\Omega_0,9}$ to be purely real and $\gamma_{\Omega_0,5}$ purely imaginary. Eqs. (18) then imply

$$\gamma_{\Omega_0,9} = \begin{pmatrix} 0_{16} & I_{16} \\ I_{16} & 0_{16} \end{pmatrix}, \qquad \gamma_{\Omega_0,5} = \begin{pmatrix} 0_{16} & iI_{16} \\ -iI_{16} & 0_{16} \end{pmatrix}. \tag{20}$$

The tadpole cancelation (18) requires the introduction of 32 9-branes and 32 5-branes. In principle, one can directly solve the constraints (13)-(15) on the 32×32 hermitian Chan

[3]We consider models with all 5-branes at the origin.

Paton matrices to determine the open string spectrum of the model. Alternatively, when the λ matrices are written in a Cartan-Weyl basis, the constraints on the Chan Paton matrices become restrictions on the weight vectors. In this formalism [11], the 99-sector gauge bosons correspond to both the Cartan generators which trivially satisfy the $\lambda^{(0)}$ constraint, together with charged generators belonging to a subset of $SO(32)$ root vectors selected by

$$\rho^a \cdot V_{(99)} = 0 \bmod \mathbb{Z}, \tag{21}$$

where $V_{(99)}$ is the "shift" vector of the gauge twist $\gamma_{1,9}$

$$\gamma_{1,9} = e^{-2i\pi V_{(99)} \cdot H}, \tag{22}$$

and $\rho^a = \underline{(\pm 1, \pm 1, 0, \cdots, 0)}$, $a = 1, \cdots, 480$ are $SO(32)$ roots. (Here, underlining indicates that all permutations must be considered). In eq. (22) H is a vector of $SO(32)$ Cartan generators $\{H_I\}$, $I = 1, \cdots, 16$ and $V_{(99)}$ is determined from (19) to be

$$\begin{aligned} T^4/\mathbb{Z}_2 : \qquad & V_{(99)} = \frac{1}{4}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), \\ T^4/\mathbb{Z}_4 : \qquad & V_{(99)} = \frac{1}{8}(1,1,1,1,1,1,1,1,3,3,3,3,3,3,3,3). \end{aligned} \tag{23}$$

Similarly, from the equation for $\lambda^{(i)}$ in (13) it follows that matter states correspond to charged generators with

$$\rho^a \cdot V_{(99)} = v_i \bmod \mathbb{Z}. \tag{24}$$

Since $\gamma_{\Omega_0,5}$ is antisymmetric, Ω projection in the 55-sector constraints gauge bosons to both the Cartan generators plus charged generators of $Sp(32)$ satisfying

$$\tilde{\rho}^a \cdot V_{(55)} = 0 \bmod \mathbb{Z}, \tag{25}$$

where $\tilde{\rho}^a$ include all the $SO(32)$ roots plus long roots $\underline{(\pm 2, 0, \cdots, 0)}$ and $V_{(55)} = V_{(99)}$. Matter fields in the 55-sector correspond to charged generators with

$$\rho^a \cdot V_{(55)} = v_i \bmod \mathbb{Z}. \tag{26}$$

Note that only a subset of short roots is used in determining the matter content of the 55-sector. This is due to the extra minus sign in (14) compare to the corresponding projection in (13). Finally, the 95 sector is handled using an auxiliary $SO(64) \supset SO(32)_{(99)} \otimes SO(32)_{(55)}$ algebra. Since we have generators acting simultaneously on both 9-branes and 5-branes, only roots of the form

$$W_{(95)} = W_{(9)} \otimes W_{(5)} = (\underline{\pm 1, 0, \cdots, 0}; \underline{\pm 1, 0, \cdots, 0}) \tag{27}$$

must be considered. Here the first (second) 16 components transform under $SO(32)_{(99)}$ ($SO(32)_{(55)}$). The shift in this sector is defined to be $V_{95} = V_{(99)} \otimes V_{(55)}$. From (15) we learn that massless states correspond to $W_{(95)}$ roots satisfying

$$W_{(95)} \cdot V_{(95)} = 0 \bmod \mathbb{Z}. \tag{28}$$

The open string spectrum of the $T^4/\mathbb{Z}_N$ orientifolds for $b=0$ is included in Table I. Our convention for the $U(1)$ charges, as explained in Appendix D, differs from that in [4,5].

Consider now the tadpoles of $T^4/\mathbb{Z}_N$ orientifolds with non-zero B-flux, $b \neq 0$. Tadpoles with a volume factor $V_6 V_{T^4}$, correspond to the untwisted R-R 10-form exchange in the closed string tree channel. All three surfaces KB, MS and C contribute to the tadpole. The KB contribution comes from the $k=0$ untwisted closed string sector. Since the parity projection Ω in this sector requires all the windings to be zero, from (16) we conclude that the compact momentum sum does not change, and the overall tadpole is the same as in the absence of the B_{ij}:

$$\mathcal{K}_u^{10-\text{form}} = (1-1)i \int_0^\infty \frac{dt}{t^2} \Big(\frac{32}{N} \frac{V_6 V_{T^4}}{(4\pi^2\alpha')^5} \Big). \tag{29}$$

The C contribution comes from the $k=0$ 99-string sector. The partition function in this sector involves the summation over the 99-string momentum modes. Upon Poisson resummation, which is necessary in order to extract the tadpole, the momentum sum in the open string loop channel is reinterpreted as a sum over the closed string winding modes in the tree channel. Unlike the momentum states which remain unaffected by the B_{ij} background, only winding states with even winding numbers are allowed along the compact directions with $B_{ij} \neq 0$. Indeed, closed string winding states satisfy

$$P_L = -P_R, \tag{30}$$

which with the quantization (16) and for a generic choice of metric on T^4 implies

$$n_i - b_{ij} m^j = 0. \tag{31}$$

Eq. (31) requires $m^j \in 2\mathbb{Z}$ for compact directions j such that $b_{ij} \neq 0$. Even winding numbers in the closed tree channel further imply an effective reduction by half the basis momentum lattice vectors along these directions compare to the $b=0$ case [15,16]. In the nontrivial B_{ij} background, the C tadpole is thus increased by 2^b

$$\mathcal{C}_u^{10-\text{form}} = i(1-1) \int_0^\infty \frac{dt}{t^2} \Big(\frac{2^b}{16N} \frac{V_6 V_{T^4}}{(4\pi^2\alpha')^5} \, (\text{Tr}\, \gamma_{0,9})^2 \Big). \tag{32}$$

Alternatively, the 2^b factor of the C tadpole can be interpreted as a sum over 2^b different sub-lattices within the $b=0$ case unit volume, shifted with respect to each other by half the generating vectors. A priori, the MS tadpole can have a different Ω projection $\gamma_{\Omega_0,9} = \pm\gamma^T_{\Omega_0,9}$, depending on a specific sub-lattice. In fact, the tadpole factorization requires that the MS contribution increases by $2^{b/2}$, which is achieved by choosing the same Ω projection for $(2^{b-1} + 2^{b/2-1})$ sub-lattices and the opposite projection for the other $(2^{b-1} - 2^{b/2-1})$ sub-lattices

$$\mathcal{M}_u^{10-\text{form}} = i(1-1) \int_0^\infty \frac{dt}{t^2} \Big(\mp \frac{2^{b/2}}{N} \frac{V_6 V_{T^4}}{(4\pi^2\alpha')^5} \, (\text{Tr}\, \gamma_{0,9}) \Big), \tag{33}$$

where a choice of the sign correlates with the choice of sign in Ω projection for the larger set of momentum sub-lattices. Using the loop/tree channel moduli translation (17), the full tadpole of the untwisted 10-form becomes

$$[\mathcal{K}+\mathcal{C}+\mathcal{M}]_u^{10-\text{form}} = i(1-1)\int_0^\infty d\ell \frac{1}{8N}\frac{V_6 V_{T^4}}{(4\pi^2\alpha')^5}\Big[32^2 \mp 2\cdot 2^{b/2}\cdot 32 \ \mathrm{Tr}\,\gamma_{0,9} + 2^b\cdot (\mathrm{Tr}\,\gamma_{0,9})^2\Big]. \tag{34}$$

Tadpoles with a volume factor V_6/V_{T_4} represent the untwisted R-R 6-form exchange in the closed string tree channel. The KB contribution comes from the $k=N/2$ untwisted closed string sector. In this sector the parity projection requires all the momenta to be zero and the NS-NS background allows only even windings along the compact directions with $b_{ij}\neq 0$. The net effect of the nontrivial background is thus a reduction of the corresponding $b=0$ tadpole by 2^b

$$\mathcal{K}_u^{6-\text{form}} = (1-1)i\int_0^\infty \frac{dt}{t^2}\Big(\frac{32}{N\cdot 2^b}\,\frac{V_6}{4\pi^2\alpha' V_{T^4}}\Big). \tag{35}$$

Since $\mathcal{K}_u^{6-\text{form}}$ tadpole measures the square of the total R-R charge of the sixteen orientifold O_5-planes, its reduction compare to the zero NS-NS background tadpole implies that $2^{4-b}\cdot(2^{b-1}+2^{b/2-1})$ O_5-planes have the same charge and the other $2^{4-b}\cdot(2^{b-1}-2^{b/2-1})$ O_5-planes have the opposite R-R charge [16]. The C contribution comes from the $k=0$ 55-string sector. Here, the partition function sums over the 55-string winding modes. The 55-string winding modes in the open string loop channel are reinterpreted upon the Poisson resummation as momentum modes in the closed string tree channel. Since the latter are insensitive to the NS-NS tensor background, the C tadpole is unchanged

$$\mathcal{C}_u^{6-\text{form}} = i(1-1)\int_0^\infty \frac{dt}{t^2}\Big(\frac{1}{16N}\,\frac{V_6}{4\pi^2\alpha' V_{T^4}}\,(\mathrm{Tr}\,\gamma_{0,5})^2\Big). \tag{36}$$

The MS amplitude relevant to the 6-form exchange in the tree channel is interpreted as a closed string exchange between 5-branes and sixteen orientifold 5-planes, and so is proportional to the total R-R charge of the O_5-planes. It is suggested by (35) that the absolute value of the R-R charge of the O_5-planes is reduced by $2^{b/2}$, so

$$\mathcal{M}_u^{6-\text{form}} = i(1-1)\int_0^\infty \frac{dt}{t^2}\Big(\mp\frac{1}{N\cdot 2^{b/2}}\,\frac{V_6}{4\pi^2\alpha' V_{T^4}}\,(\mathrm{Tr}\,\gamma_{0,5})\Big), \tag{37}$$

where a choice of a sign correlates with the R-R sign of the larger set of O_5-planes. The full tadpole of the untwisted 6-form in terms of the tree channel cylinder modulus ℓ (17) becomes

$$[\mathcal{K}+\mathcal{C}+\mathcal{M}]_u^{6-\text{form}} = i(1-1)\int_0^\infty d\ell\frac{1}{8N}\,\frac{V_6}{4\pi^2\alpha' V_{T^4}}\Big[\frac{32^2}{2^b}\mp\frac{2\cdot 32}{2^{b/2}}\,\mathrm{Tr}\,\gamma_{0,5}+(\mathrm{Tr}\,\gamma_{0,5})^2\Big]. \tag{38}$$

The untwisted tadpole cancelation from (34) and (38) requires

$$\mathrm{Tr}\,\gamma_{0,9} = \mathrm{Tr}\,\gamma_{0,5} = \frac{32}{2^{b/2}}, \tag{39}$$

and the same Ω projection for the larger set of the momentum sub-lattices and orientifold O_5-planes as in the $b=0$ case:

$$\gamma_{\Omega_0,9} = \gamma^T_{\Omega_0,9}\ , \qquad \gamma_{\Omega_0,5} = -\gamma^T_{\Omega_0,5}. \tag{40}$$

Since traces of $\gamma_{0,p}$ count the number of D-branes, (39) implies the rank reduction of a gauge group in the open string sector when the B-flux is turned on [15–17].

In addition to the untwisted R-R 6- and 10-form tadpoles — which are canceled as discussed above, the orientifolds of interest have potential tadpoles corresponding to the twisted sector R-R 6-form exchange in the tree channel. These tadpoles come from the cylinder vacuum amplitudes $C_{pq},\ p,q = \{5,9\}$, (A7) with $k \neq 0$ and so do not get contributions from momentum and winding modes on T^4. Since the nonzero NS-NS tensor background effects only the quantization of the momentum and winding modes on T^4, one would expect the twisted R-R 6-form tadpole to be the same as in the $b = 0$ case. As we will discuss in detail in Sec. III, anomaly cancelations in six dimensions requires the $2^{b/2}$ multiplicity of massless states in the 59 open string sector. In Type IIB orientifolds, gauge degrees of freedom are carried by the Chan-Paton indices, so the multiplicity of massless states is extended to the multiplicity of Chan-Paton indices in the 59-sector and thus the multiplicity of massive states in this sector as well. The latter implies that the cylinder vacuum amplitudes $\mathcal{C}_{99}$, $\mathcal{C}_{55}$ are unaffected by the B-flux while $\mathcal{C}_{95}$ and $\mathcal{C}_{59}$ get an extra factor of $2^{b/2}$. This raises a question: once the relative factor of $\mathcal{C}_{99}$ and $\mathcal{C}_{95}$ vacuum amplitudes is changed in the nonzero NS-NS background, how one maintains the factorization of the twisted R-R 6-form tadpoles? Recall that going to the tree channel interpretation in the KB vacuum amplitude we related (for $b = 0$) the tree channel modulus ℓ and the loop channel modulus $t_{\mathcal{K}}$ through $\ell = 1/(4t_{\mathcal{K}})$. This KB tree channel "cylinder" has length ℓ and two crosscaps. The C and MS vacuum amplitudes of the open string sector in the tree channel picture were represented by an identical length cylinder and a cylinder with a crosscap. The fact that we used the same tree channel modulus for C and MS implicitly assumed the ability to relate KB, C and MS amplitudes to each other by adding/removing crosscaps. This is possible only for $\mathcal{C}_{99}$ and $\mathcal{C}_{55}$ since only they have the same type boundaries, which allows putting the crosscaps. The above arguments do not fix the relation between ℓ and $t_{\mathcal{C}_{59}}$. We claim that this relation must be fixed by demanding anomaly cancelation along with factorization of twisted R-R 6-form tadpoles. Thus we require

$$\ell = \frac{2^{b-1}}{t_{\mathcal{C}_{59}}}. \tag{41}$$

The twisted R-R 6-form tadpoles vanish provided

$$\begin{array}{ll} T^4/\mathbb{Z}_2: & \mathrm{Tr}\,\gamma_{1,9} = \mathrm{Tr}\,\gamma_{1,5} = 0, \\ T^4/\mathbb{Z}_4: & \mathrm{Tr}\,\gamma_{1,9} = \mathrm{Tr}\,\gamma_{2,9} = \mathrm{Tr}\,\gamma_{1,5} = \mathrm{Tr}\,\gamma_{2,5} = 0. \end{array} \tag{42}$$

Eqs. (39) and (42) are solved by

$$\begin{array}{ll} T^4/\mathbb{Z}_2: & \gamma_{1,9} = \gamma_{1,5} = \Big(iI_{2^{4-b}}, -iI_{2^{4-b}}\Big), \\ T^4/\mathbb{Z}_4: & \gamma_{1,9} = \gamma_{1,5} = \Big(\beta I_{2^{3-b}}, \beta^3 I_{2^{3-b}}, \overline{\beta} I_{2^{3-b}}, \overline{\beta}^3 I_{2^{3-b}}\Big), \qquad \beta = e^{\pi i/4}. \end{array} \tag{43}$$

The "shift" vectors corresponding to the above gauge twists are

$$
\begin{aligned}
T^4/\mathbb{Z}_2: \qquad & V_{(99)} = \frac{1}{4}(\overbrace{1,\cdots,1}^{2^{4-b}\ \text{times}}), \\
T^4/\mathbb{Z}_4: \qquad & V_{(99)} = \frac{1}{8}(\underbrace{1,\cdots,1}_{2^{3-b}\ \text{times}},\ \underbrace{3,\cdots,3}_{2^{3-b}\ \text{times}}).
\end{aligned} \tag{44}
$$

with $V_{(55)} = V_{(99)}$ and $V_{(95)} = V_{(99)} \otimes V_{(55)}$. As for the $b=0$ case, we can choose $\gamma_{\Omega_0,9}$ purely real and $\gamma_{\Omega_0,5}$ to be purely imaginary. Eqs. (40) then imply

$$
\gamma_{\Omega_0,9} = \begin{pmatrix} 0_{2^{4-b}} & I_{2^{4-b}} \\ I_{2^{4-b}} & 0_{2^{4-b}} \end{pmatrix}, \qquad \gamma_{\Omega_0,5} = \begin{pmatrix} 0_{2^{4-b}} & iI_{2^{4-b}} \\ -iI_{2^{4-b}} & 0_{2^{4-b}} \end{pmatrix}. \tag{45}
$$

Given (43) and (45), the massless open string states can be determined as in the $b=0$ case. The open string spectra are given in Table I.

B. Closed String Sector

In this section we discuss the closed string sector of Type IIB orientifolds on $T^4/\mathbb{Z}_N$, $(N=2,4)$, in the presence of B-flux.

First consider the $\mathbb{Z}_2$ orientifold models. The untwisted sector of the models is constructed by keeping states of Type IIB orbifold on $T^4/\mathbb{Z}_2$ invariant under the world sheet parity Ω. This gives rise to a $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and four hypermultiplets.

The twisted sectors will produce additional multiplets. The bosonic content of a hypermultiplet is four scalars transforming as $4(\mathbf{1},\mathbf{1})$ under the six dimensional little group $SO(4) \approx SU(2) \otimes SU(2)$, while that of a tensor multiplet as $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$. There are overall 16 fixed points in the $\mathbb{Z}_2$ orbifold with an O_5 orientifold plane sitting at each fixed point. As we explained in Sec. II A, out of 16 O_5-planes, each one of $2^{4-b} \cdot (2^{b-1} + 2^{b/2-1})$ O_5-planes carries (-2) units of R-R charge while the other $2^{4-b} \cdot (2^{b-1} - 2^{b/2-1})$ O_5-planes have opposite R-R charge. This implies that precisely $2^{4-b} \cdot (2^{b-1} + 2^{b/2-1})$ $\mathbb{Z}_2$ fixed points are even under the Ω projection while the other $2^{4-b} \cdot (2^{b-1} - 2^{b/2-1})$ $\mathbb{Z}_2$ fixed points pick up a minus sign under the Ω projection. At each fixed point of the twisted sector, the NS and R sector massless states transform as

$$
\begin{array}{ll}
\text{Sector} & SU(2) \otimes SU(2) \text{ rep.} \\
\text{NS} & \qquad 2(\mathbf{1},\mathbf{1}) \ , \\
\text{R} & \qquad (\mathbf{1},\mathbf{2}) \ .
\end{array} \tag{46}
$$

The twisted sector spectrum is obtained by taking products of states from the left- and right-moving sectors. The orientifold projection Ω keeps symmetric (antisymmetric) combinations in the NS-NS sector and antisymmetric (symmetric) combinations in the R-R sector for each Ω-even (Ω-odd) $\mathbb{Z}_2$ fixed point. This gives the bosonic content of a hypermultiplet for each Ω-even $\mathbb{Z}_2$ fixed point and the bosonic content of a tensor multiplet for each Ω-odd $\mathbb{Z}_2$ fixed point. Overall, the twisted sectors of $\mathbb{Z}_2$ orientifold contribute $2^{4-b} \cdot (2^{b-1} + 2^{b/2-1})$ hypermultiplets and $2^{4-b} \cdot (2^{b-1} - 2^{b/2-1})$ extra tensor multiplet (see Table I).

The untwisted sector of $T^4/\mathbb{Z}_4$ orientifold contains $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and two hypermultiplets.

The four $\mathbb{Z}_4$ invariant fixed points of $T^4/\mathbb{Z}_4$ orientifold give four hypermultiplets and four tensor multiplet. They are also $\mathbb{Z}_2$ fixed points and so supply an additional 4 hypermultiplets. The other 12 $\mathbb{Z}_2$ fixed points form 6 $\mathbb{Z}_4$ invariant pairs. Out of 12 $\mathbb{Z}_2$ fixed points which are not $\mathbb{Z}_4$ invariant, $2^{4-b} \cdot (2^{b-1} - 2^{b/2-1})$ fixed points are odd under the Ω projection and the other $12 - 2^{4-b} \cdot (2^{b-1} - 2^{b/2-1})$ are even. Thus they contribute $2^{3-b} \cdot (2^{b-1} - 2^{b/2-1})$ extra tensor multiplets and $6 - 2^{3-b} \cdot (2^{b-1} - 2^{b/2-1})$ hypermultiplets. Overall, twisted sectors of $\mathbb{Z}_4$ orientifold contribute $4 + 2^{3-b} \cdot (2^{b-1} - 2^{b/2-1})$ extra tensor multiplets and $14 - 2^{3-b} \cdot (2^{b-1} - 2^{b/2-1})$ hypermultiplets (see Table I).

III. ANOMALY CANCELATIONS

An important check on the consistency of the $\mathcal{N} = 1$ six dimensional vacua constructed in the previous sections is provided by the cancelation of all gauge, gravitational, and mixed anomalies. Cancelation of tadpoles in Type IIB orientifolds on $T^4/\mathbb{Z}_N$ $(N = 2, 4)$ discussed in Sec. II A guarantees 1-loop consistency of the equations of motion for the untwisted R-R 6- and 10-form as well as the twisted R-R 6-form. Recall that the tadpoles were extracted from the KB, MS and C vacuum amplitudes. Each of these amplitudes involves a summation over four spin structures (*i.e.*, summation over $\{\alpha, \beta\} = \{0, \frac{1}{2}\}$ in Appendix A). Actually, the R-R spin structure, $\{\alpha, \beta\} = \{\frac{1}{2}, \frac{1}{2}\}$ in Appendix A, does not contribute to the 1-loop vacuum amplitudes because of the fermionic zero modes. On the other hand, it will contribute to the 2-loop vacuum amplitudes, as well as to various scattering processes. In fact, being odd under the spacetime parity, it is the only spin structure (at 1-loop level) that would generate potential chiral anomalies in the low-energy effective field theory. In this sense, the cancelation of six dimensional chiral anomalies follows from the absence of tadpoles coming from the R-R spin structure. In this section we show that $T^4/\mathbb{Z}_N$ $(N = 2, 4)$ Type IIB orientifolds with B-flux are free from chiral anomalies. First, we explain anomaly cancelation in the framework of the six dimensional effective field theory. Then we discuss how the Green-Schwarz terms can be computed from the corresponding non-planar diagrams in string theory. In the latter approach we confirm the necessity of $2^{b/2}$ multiplicity of 59-cylinder diagrams. Some useful facts on six dimensional chiral anomalies are given in Appendix C.

A. $\mathbb{Z}_2$ Orientifolds

We start with the $T^4/\mathbb{Z}_2$ orientifold models. From the spectrum of the models given in Table I it is straightforward to check using the results of Appendix C that the leading anomalies, *i.e.*, the $\mathrm{tr}\, R^4$ and $\mathrm{tr}\, F^4$ terms vanish. To see this explicitly, notice that the leading term in the pure gravitational anomalies is proportional to

$$(n_H - n_V + 29 n_T - 273)\ \mathrm{tr}\, R^4, \tag{47}$$

where n_H is the number of hypermultiplets from both open and closed string sectors. From Section II B, $n_T - 1 = 8 - 8/2^{b/2}$ and the number of closed string hypermultiplets is $n_H^c = 12 + 8/2^{b/2}$. The gauge group is $U(N) \times U(N)$ where $N = 16/2^{b/2}$, and hence $n_V = 2N^2$. The number of open string hypermultiplets is $n_H^o = 2N(N-1) + 2^{b/2} N^2$. It is easy to see that the above term (47) vanishes. Notice that it is crucial for the cancelation of the $\mathrm{tr}\, R^4$ gravitational anomaly that the multiplicity of 59-open string matter states is $2^{b/2}$. In addition, the hypermultiplet representations plus the $2^{b/2}$ multiplicity in the 59-sector ensure that the $\mathrm{tr}\, F^4$ term is absent. This can be seen from (C5) that the leading pure gauge non-abelian anomalies for each $SU(N)$ gauge group is proportional to

$$(2N - 2(N-8) - 2^{b/2}N)\ \mathrm{tr}\, F^4 \tag{48}$$

which vanishes since the multiplicity of 59 states is $2^{b/2}$ which is equal to $16/N$. The remaining gravitational + gauge anomaly may be written in the form

$$i(2\pi)^3 I = -\frac{1}{16 \cdot 2^{b/2}}\ X_4^{(9)} \wedge X_4^{(5)} + X_2 \wedge X_6^{(9)} - \tilde{X}_2 \wedge X_6^{(5)}, \tag{49}$$

where

$$\begin{aligned} X_4^{(a)} &= R^2 - 2 \cdot 2^{b/2}\ (F_a^2 + f_a^2), \\ X_6^{(a)} &= \frac{1}{6}\Big[R^2 f_a - 2^{b/2} f_a^3 - 3 \cdot 2^{b/2} F_a^2 f_a - 4 \cdot 2^{b/4} F_a^3\Big], \end{aligned} \tag{50}$$

and

$$\begin{aligned} X_2 &= \frac{4}{2^{b/2}} f_9 - f_5, \\ \tilde{X}_2 &= \frac{4}{2^{b/2}} f_5 - f_9. \end{aligned} \tag{51}$$

The subscript $a = 5, 9$ refers to the gauge group factors in the 55- and 99-open string sectors[4]. The anomaly polynomial (49) contains terms that involve only nonabelian field strength as well as terms that also involve abelian field strength. The former can be canceled by exchange of a two-form, which involves Green-Schwarz interactions of the form

$$\Gamma_{c.t.} = \frac{i}{16\ 2^{b/2}\ (2\pi)^3} \int B_2 \wedge X_4^{(5)}. \tag{52}$$

The B_2 field is the 2-form with gauge invariant field strength $H = dB_2 - X_3^{(9)}$ where $X_4^{(9)} = dX_3^{(9)}$. The counterterm (52) arises from the tree-level diagram which involves coupling of the B_2 field and a pair of gauge bosons. Such couplings can be computed from the disk diagram in string theory. When $b \neq 0$, a disk diagram involving two gauge bosons and a two-form potential of the twisted sector tensor multiplet is proportional to $\mathrm{tr}\,(\gamma_{1,p}(\lambda_p)^2)$, where $p = 5, 9$ refers to the 55- or 99-open string sector and λ_p denotes the Chan-Paton generator of the corresponding gauge boson. Using the results in Appendix D, this trace is zero. Thus we identify the 2-form B_2 as a combination of the two-form potential of the

[4] The anomalies of the $b = 0$ models were discussed previously in [13]. The difference between the corresponding anomaly polynomials of (51) and the one given in [13] is due to a different convention of the $U(1)$ charges. See Table I for our convention.

gravity multiplet with a self-dual field strength and the two-form potential with an anti-self-dual field strength of the untwisted sector tensor multiplet. The remaining terms of the anomaly polynomial (49) can be canceled by

$$\Gamma'_{c.t.} = -\frac{i}{(2\pi)^3}\int B_0^{(9)}\, X_6^{(9)} + \frac{i}{(2\pi)^3}\int B_0^{(5)}\, X_6^{(5)}, \tag{53}$$

if we assign the anomalous transformation laws:

$$\begin{aligned} B_0^{(9)} &\to B_0^{(9)} + \frac{4}{2^{b/2}}\epsilon_9 - \epsilon_5, \\ B_0^{(5)} &\to B_0^{(5)} + \frac{4}{2^{b/2}}\epsilon_5 - \epsilon_9, \end{aligned} \tag{54}$$

under the $U(1)_a$ gauge transformations $A_a \to A_a + d\epsilon_a$. As in (52), the counterterm (53) arises from a tree-level diagram. This diagram involves the coupling of the $B_0^{(a)}$ scalars with a single $U(1)_a$ gauge boson. From the disk diagram, the coupling of the untwisted sector R-R scalars to $U(1)_a$ gauge boson is proportional to $\mathrm{tr}\,(\lambda_a)$ which vanishes, see Appendix D. On the other hand, the corresponding coupling of the twisted sector R-R scalars is proportional to $\mathrm{tr}\,\gamma_{1,a}\lambda_a \neq 0$.

Consider first the $b = 0$ case. The number of $\mathbb{Z}_2$ twisted scalars ϕ_I is 16. All of them couple with the same relative strength to the 99-sector $U(1)$, because $\mathrm{tr}\,(\gamma_{1,9}\lambda_9)$ (which measures the coupling strength) is the same for all fixed points. Since all D5-branes sit at the origin, only the twisted scalar at the origin (which we denote by ϕ_1) will couple. This can be seen explicitly from the tadpole solution: the coupling of the scalar at a fixed point I to a 55-sector gauge boson is proportional to $\mathrm{tr}\,(\gamma_{1,5,I}\lambda_5)$, which is zero unless $I = 1$ is the origin, where all the D5-branes are located. Therefore,

$$\begin{aligned} B_0^{(9)} &= \alpha \sum_{I=1}^{16} \phi_I, \\ B_0^{(5)} &= \beta\ \phi_1, \end{aligned} \tag{55}$$

for some coefficients α and β to be determined. Since ϕ_1 couples both to 55-sector and 99-sector gauge fields, it transforms under $U(1)$ gauge transformations as

$$\phi_1 \to \phi_1 + a_1\epsilon_9 + a_2\epsilon_5, \tag{56}$$

for some coefficients a_1 and a_2 to be determined. The other scalars $\phi_I,\ I = 2, \cdots 16$, couple only to the 99-sector, but all in a same way, so

$$\phi_I \to \phi_I + b_1\epsilon_9. \tag{57}$$

Requiring that (55)-(57) generate (54) uniquely fixes all the coefficients: $\alpha, \beta, a_1, a_2, b_1$ up to field normalization. We have

$$B_0^{(9)} = \alpha \sum_{I=1}^{16} \phi_I,$$

$$B_0^{(5)} = -4\alpha\ \phi_1, \tag{58}$$

where

$$\begin{aligned} &\phi_1 \to \phi_1 + (\epsilon_9 - 4\epsilon_5)/(4\alpha), \\ &\phi_I \to \phi_I + \epsilon_9/(4\alpha), \qquad I = 2, \cdots, 16. \end{aligned} \tag{59}$$

Since the ϕ_I's are in linear multiplets, the choice of normalization α is irrelevant. The choice $\alpha = 1/4$ reproduces the result of [13]. One can always attain the canonical normalization of $B_0^{(9)}$ and $B_0^{(5)}$ with an appropriate change of the coupling constants in the counterterm (53). This is also generic to other orientifold models: a linear combination of twisted scalars that couple to the abelian gauge fields is unique, up to field normalization.

In the $b = 2$ case, the number of $\mathbb{Z}_2$ twisted scalars is 12. As in the previous case, one scalar is singled out as living at the fixed point where we put all D5-branes. Using the same logic as before, we find a unique solution

$$\begin{aligned} B_0^{(9)} &= \alpha \sum_{I=1}^{12} \phi_I, \\ B_0^{(5)} &= -2\alpha\ \phi_1, \end{aligned} \tag{60}$$

where

$$\begin{aligned} &\phi_1 \to \phi_1 + (\epsilon_9 - 2\epsilon_5)/(2\alpha), \\ &\phi_I \to \phi_I + 3\epsilon_9/(22\alpha), \qquad I = 2, \cdots, 12. \end{aligned} \tag{61}$$

In the $b = 4$ case, the number of $\mathbb{Z}_2$ twisted scalars is 10. Therefore, we find

$$\begin{aligned} B_0^{(9)} &= \alpha \sum_{I=1}^{10} \phi_I, \\ B_0^{(5)} &= -\alpha\ \phi_1, \end{aligned} \tag{62}$$

where

$$\begin{aligned} &\phi_1 \to \phi_1 + (\epsilon_9 - \epsilon_5)/\alpha, \\ &\phi_I \to \phi_I, \qquad I = 2, \cdots, 10. \end{aligned} \tag{63}$$

B. $\mathbb{Z}_4$ Orientifolds

We now consider $T^4/\mathbb{Z}_4$ orientifold models. From the open string spectrum of the models given in Table I and the closed string spectrum in Section II B, one can easily see that the $\mathrm{tr}\, R^4$ and all $\mathrm{tr}\, F^4$ terms vanish in the 8-form anomaly polynomial. The remaining gravitational + gauge anomaly may be written in the form

$$i(2\pi)^3 I = -\frac{1}{32 \cdot 2^{b/2}} X_4^{(9)} \wedge X_4^{(5)} - \frac{2^{b/2}}{8} \tilde{X}_4^{(9)} \wedge \tilde{X}_4^{(5)} + \frac{1}{8} \tilde{X}_4^{(5)} \wedge \tilde{X}_4^{(5)} + \frac{1}{8} \tilde{X}_4^{(9)} \wedge \tilde{X}_4^{(9)}$$

$$+X_2^{(1,9)} \wedge X_6^{(1,9)} + X_2^{(3,9)} \wedge X_6^{(3,9)} + X_2^{(1,5)} \wedge X_6^{(1,5)} + X_2^{(3,5)} \wedge X_6^{(3,5)}, \tag{64}$$

where

$$\begin{aligned}
X_4^{(a)} &= R^2 - 2 \cdot 2^{b/2}(F_{1,a}^2 + F_{3,a}^2 + f_{1,a}^2 + f_{3,a}^2), \\
\tilde{X}_4^{(a)} &= F_{1,a}^2 + f_{1,a}^2 - F_{3,a}^2 - f_{3,a}^2, \\
X_6^{(\alpha,a)} &= \frac{1}{12 \cdot 2^{b/2}} R^2 f_{\alpha,a} - \frac{1}{6} f_{\alpha,a}^3 - \frac{1}{2} F_{\alpha,a}^2 f_{\alpha,a} - \frac{2^{(2-b)/4}}{3} F_{\alpha,a}^3,
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
X_2^{(1,9)} &= 3f_{1,9} - f_{3,9} - 2^{b/2} f_{1,5}, \\
X_2^{(3,9)} &= 3f_{3,9} - f_{1,9} - 2^{b/2} f_{3,5}, \\
X_2^{(1,5)} &= 3f_{1,5} - f_{3,5} - 2^{b/2} f_{1,9}, \\
X_2^{(3,5)} &= 3f_{3,5} - f_{1,5} - 2^{b/2} f_{3,9}.
\end{aligned} \tag{66}$$

In the Z_4 orientifolds there are two nonabelian gauge factors in the 99-sector and 55-sector. One comes from the shift-vector 1/8 components and the other come from the shift-vector 3/8 components, (c.f. (44)). In (64), we use this to label different gauge factors. For example, the 99-sector nonabelian field strengths will be denoted by $F_{1,9}$ and $F_{3,9}$. The nonabelian factors in the 55-sector are denoted by $F_{1,5}$ and $F_{3,5}$. We use similar notation $f_{\alpha,a}$ for the abelian factors. The terms of the anomaly polynomial (64) that involve only nonabelian field strengths can be canceled by exchange of three two-forms with Green-Schwarz interactions

$$\Gamma_{c.t.} = \frac{i}{32\ 2^{b/2}\ (2\pi)^3} \int B_2^{(1)} \wedge X_4^{(5)} + \frac{i\ 2^{b/2}}{8\ (2\pi)^3} \int B_2^{(2)} \wedge \tilde{X}_4^{(5)} - \frac{i}{8\ (2\pi)^3} \int B_2^{(3)} \wedge \tilde{X}_4^{(9)}, \tag{67}$$

where $B_2^{(i)}$'s have gauge invariant field strengths

$$\begin{aligned}
H^{(1)} &= dB_2^{(1)} - X_3^{(9)}, & X_4^{(9)} &= dX_3^{(9)}, \\
H^{(2)} &= dB_2^{(2)} - X_3^{(9,5)}, & \tilde{X}_4^{(9)} - \frac{1}{2^{b/2}} \tilde{X}_4^{(5)} &= dX_3^{(9,5)}, \\
H^{(3)} &= dB_2^{(3)} - \tilde{X}_3^{(9)}, & \tilde{X}_4^{(9)} &= d\tilde{X}_3^{(9)}.
\end{aligned} \tag{68}$$

The counterterm (67) arises from a tree-level diagram which involves coupling of $B_2^{(i)}$ fields and a pair of gauge bosons. Recall that these models have a single tensor multiplet from the untwisted sector, 4 tensor multiplets from the $\mathbb{Z}_4$ twisted sector and $2^{3-b} \cdot (2^{b-1} - 2^{b/2-1})$ extra tensor multiplets coming from the $\mathbb{Z}_2$ twisted sector. By the same arguments as for the $T^4/\mathbb{Z}_2$ orientifolds, the $\mathbb{Z}_2$ twisted sector tensor multiplets do not couple to a pair of gauge bosons. On the contrary, the coupling of the $\mathbb{Z}_4$ twisted sector tensor multiplets from the disk diagram is nonzero. With all the D5-branes at the origin, only the $\mathbb{Z}_4$ twisted tensor multiplet at the origin will couple to the 55-sector gauge bosons. Thus we expect that $B_2^{(1)}$ and $B_2^{(3)}$ are combinations of the untwisted sector two-form potentials of the gravity and the tensor multiplets plus two-form potentials of all four $\mathbb{Z}_4$ twisted sector tensor multiplets. $B_2^{(2)}$ must contain the untwisted sector two-form potentials and a two-form potential from the $\mathbb{Z}_4$ twisted sector tensor multiplet at the origin. The remaining abelian terms of the anomaly polynomial (64) may be canceled by additional counterterms

$$\Gamma'_{c.t.} = -\frac{i}{(2\pi)^3}\int B_0^{(1,9)}\, X_6^{(1,9)} - \frac{i}{(2\pi)^3}\int B_0^{(3,9)}\, X_6^{(3,9)} - \frac{i}{(2\pi)^3}\int B_0^{(1,5)}\, X_6^{(1,5)} \\ -\frac{i}{(2\pi)^3}\int B_0^{(3,5)}\, X_6^{(3,5)}, \tag{69}$$

if we assign the anomalous transformation laws:

$$\begin{aligned} B_0^{(1,9)} &\to B_0^{(1,9)} + 3\epsilon_{1,9} - \epsilon_{3,9} - 2^{b/2}\epsilon_{1,5}, \\ B_0^{(3,9)} &\to B_0^{(3,9)} + 3\epsilon_{3,9} - \epsilon_{1,9} - 2^{b/2}\epsilon_{3,5}, \\ B_0^{(1,5)} &\to B_0^{(1,5)} + 3\epsilon_{1,5} - \epsilon_{3,5} - 2^{b/2}\epsilon_{1,9}, \\ B_0^{(3,5)} &\to B_0^{(3,5)} + 3\epsilon_{3,5} - \epsilon_{1,5} - 2^{b/2}\epsilon_{3,9}, \end{aligned} \tag{70}$$

under the $U(1)$ gauge transformations $A_{\alpha,a} \to A_{\alpha,a} + d\epsilon_{\alpha,a}$. We now proceed to identify the scalars $B_0^{(\alpha,a)}$ in terms of the twisted R-R states.

We start with the $b = 0$ case. There are in total 14 twisted scalars: 4 scalars (we denote them by $\phi_I^{(4)}$, $I = 1, \cdots 4$) coming from the $\mathbb{Z}_4$ twisted sector, the other 10 scalars (we denote them $\phi_J^{(2)}$, $J = 1, \cdots 10$) coming from the $\mathbb{Z}_2$ twisted sector. Out of 10 $\mathbb{Z}_2$ twisted fields, 4 come from the $\mathbb{Z}_2$ fixed points which are also $\mathbb{Z}_4$ fixed points, and the other 6 come from 6 pairs of the $\mathbb{Z}_2$ fixed points which are $\mathbb{Z}_4$ invariant. Let us analyze the coupling of these R-R twisted fields to the various $U(1)$ gauge bosons. All twisted fields couple to the 99-sector vector bosons. These couplings can be computed from the disk diagram: for the $\mathbb{Z}_4$ twisted fields the coupling is proportional to $\mathrm{tr}\,(\gamma_{1,9}\lambda_{\alpha,9})$ while for the $\mathbb{Z}_2$ twisted fields the coupling is proportional to $\mathrm{tr}\,(\gamma_{2,9}\lambda_{\alpha,9})$. Since

$$\begin{aligned} \frac{\mathrm{tr}\,(\gamma_{1,9}\lambda_{1,9})}{\mathrm{tr}\,(\gamma_{1,9}\lambda_{3,9})} &= 1, \\ \frac{\mathrm{tr}\,(\gamma_{2,9}\lambda_{1,9})}{\mathrm{tr}\,(\gamma_{2,9}\lambda_{3,9})} &= -1, \end{aligned} \tag{71}$$

the coupling of $\phi_I^{(4)}$ to the $A_{3,9}$ abelian gauge boson is identical to that of the $A_{1,9}$, while the coupling of $\phi_J^{(2)}$ to $A_{3,9}$ is opposite to its coupling to $A_{1,9}$. As in the case of $\mathbb{Z}_2$ orientifold, only twisted fields that live at the fixed point where the D5-branes are located will couple to the 55-sector gauge bosons. We call these fields $\phi_1^{(4)}$ and $\phi_1^{(2)}$. Their coupling to different $U(1)$ can be deduced as discussed above. Collecting the information about the coupling, we can easily generalize (55) to:

$$\begin{aligned} B_0^{(1,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} + \beta \sum_{J=1}^{10} \phi_J^{(2)}, \\ B_0^{(3,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} - \beta \sum_{J=1}^{10} \phi_J^{(2)}, \\ B_0^{(1,5)} &= \gamma\, \phi_1^{(4)} + \delta\, \phi_1^{(2)}, \end{aligned}$$

$$B_0^{(3,5)} = \gamma\ \phi_1^{(4)} - \delta\ \phi_1^{(2)}, \tag{72}$$

where $\alpha, \beta, \gamma, \delta$ are coefficients to be determined. Using the same logic as for the $\mathbb{Z}_2$ orientifolds, we assign the following gauge transformations to the twisted fields

$$\begin{aligned}
&\phi_1^{(4)} \to \phi_1^{(4)} + a_1\epsilon_{1,9} + a_2\epsilon_{3,9} + a_3\epsilon_{1,5} + a_4\epsilon_{3,5}, \\
&\phi_I^{(4)} \to \phi_I^{(4)} + b_1\epsilon_{1,9} + b_2\epsilon_{3,9}, \qquad\qquad I = 2, \cdots 4, \\
&\phi_1^{(2)} \to \phi_1^{(2)} + c_1\epsilon_{1,9} + c_2\epsilon_{3,9} + c_3\epsilon_{1,5} + c_4\epsilon_{3,5}, \\
&\phi_J^{(2)} \to \phi_J^{(2)} + d_1\epsilon_{1,9} + d_2\epsilon_{3,9}, \qquad\qquad J = 2, \cdots 10.
\end{aligned} \tag{73}$$

As we already mentioned, requiring that (72)-(73) satisfy (70), fixes uniquely all coefficients up to the field normalization. (The system is overconstraint and so generically would not have solution at all.) We end up with

$$\begin{aligned}
B_0^{(1,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} + \beta \sum_{J=1}^{10} \phi_J^{(2)}, \\
B_0^{(3,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} - \beta \sum_{J=1}^{10} \phi_J^{(2)}, \\
B_0^{(1,5)} &= -2\alpha\ \phi_1^{(4)} - 4\beta\ \phi_1^{(2)}, \\
B_0^{(3,5)} &= -2\alpha\ \phi_1^{(4)} + 4\beta\ \phi_1^{(2)},
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
&\phi_1^{(4)} \to \phi_1^{(4)} + (\epsilon_{1,9} + \epsilon_{3,9})/(4\alpha) - (\epsilon_{1,5} + \epsilon_{3,5})/(2\alpha), \\
&\phi_I^{(4)} \to \phi_I^{(4)} + (\epsilon_{1,9} + \epsilon_{3,9})/(4\alpha), \qquad I = 2, \cdots 4, \\
&\phi_1^{(2)} \to \phi_1^{(2)} + (\epsilon_{1,9} - \epsilon_{3,9})/(8\beta) - (\epsilon_{1,5} - \epsilon_{3,5})/(2\beta), \\
&\phi_J^{(2)} \to \phi_J^{(2)} + 5(\epsilon_{1,9} - \epsilon_{3,9})/(24\beta), \qquad J = 2, \cdots 10.
\end{aligned} \tag{75}$$

The appearance of two normalization constants in the final expression comes from an independent choice of normalizations for the $\mathbb{Z}_4$ and $\mathbb{Z}_2$ twisted fields.

Turning on rank two NS-NS two form field B_{ij} converts two of the $\mathbb{Z}_2$ twisted hypermultiplets into tensor multiples, so the total number of $\mathbb{Z}_2$ twisted scalars becomes 8. Repeating identical analysis as in the $b = 0$ case, we find a unique solution (up to field normalization)

$$\begin{aligned}
B_0^{(1,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} + \beta \sum_{J=1}^{8} \phi_J^{(2)}, \\
B_0^{(3,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} - \beta \sum_{J=1}^{8} \phi_J^{(2)}, \\
B_0^{(1,5)} &= -\alpha\ \phi_1^{(4)} - 2\beta\ \phi_1^{(2)}, \\
B_0^{(3,5)} &= -\alpha\ \phi_1^{(4)} + 2\beta\ \phi_1^{(2)},
\end{aligned} \tag{76}$$

where

$$\begin{aligned}
&\phi_1^{(4)} \to \phi_1^{(4)} + (\epsilon_{1,9} + \epsilon_{3,9})/\alpha - (\epsilon_{1,5} + \epsilon_{3,5})/\alpha, \\
&\phi_I^{(4)} \to \phi_I^{(4)}, \qquad I = 2, \cdots 4, \\
&\phi_1^{(2)} \to \phi_1^{(2)} + (\epsilon_{1,9} - \epsilon_{3,9})/(2\beta) - (\epsilon_{1,5} - \epsilon_{3,5})/\beta, \\
&\phi_J^{(2)} \to \phi_J^{(2)} + 3(\epsilon_{1,9} - \epsilon_{3,9})/(14\beta), \qquad J = 2, \cdots 8.
\end{aligned} \tag{77}$$

Finally, the rank four B-flux converts three of the $\mathbb{Z}_2$ twisted hypermultiplets into tensor multiples, so the total number of $\mathbb{Z}_2$ twisted scalars becomes 7. In this case we find

$$\begin{aligned}
B_0^{(1,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} + \beta \sum_{J=1}^{7} \phi_J^{(2)}, \\
B_0^{(3,9)} &= \alpha \sum_{I=1}^{4} \phi_I^{(4)} - \beta \sum_{J=1}^{7} \phi_J^{(2)}, \\
B_0^{(1,5)} &= -(\alpha/2)\ \phi_1^{(4)} - \beta\ \phi_1^{(2)}, \\
B_0^{(3,5)} &= -(\alpha/2)\phi_1^{(4)} + \beta\ \phi_1^{(2)},
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
&\phi_1^{(4)} \to \phi_1^{(4)} + 4(\epsilon_{1,9} + \epsilon_{3,9})/\alpha - 2(\epsilon_{1,5} + \epsilon_{3,5})/\alpha, \\
&\phi_I^{(4)} \to \phi_I^{(4)} - (\epsilon_{1,9} + \epsilon_{3,9})/\alpha, \qquad I = 2, \cdots 4, \\
&\phi_1^{(2)} \to \phi_1^{(2)} + 2(\epsilon_{1,9} - \epsilon_{3,9})/\beta - 2(\epsilon_{1,5} - \epsilon_{3,5})/\beta, \\
&\phi_J^{(2)} \to \phi_J^{(2)}, \qquad\qquad J = 2, \cdots 7.
\end{aligned} \tag{79}$$

C. Green-Schwarz Terms from Non-Planar Diagrams

In the field theory framework the cancelation procedure of the six dimensional chiral anomalies involves one-loop and tree-level diagrams. The former generates the anomaly polynomial, while the latter cancels it through the generalized Green-Schwarz mechanism. This cancelation has a simple interpretation from the string theory point of view. Consider 1-loop open string diagrams whose low-energy limit reproduces the 1-loop field theory anomalies. The factorized form of the anomaly polynomial comes from the appropriate non-planar diagrams in string theory. Let t denotes the standard 1-loop cylinder modulus, the field theory result comes from the $t \to +\infty$ boundary of the moduli space. Since the non-planar diagrams are finite [23], they can not generate chiral anomalies: the anomalous contribution of the 1-loop amplitude from the large t region of the moduli space is canceled exactly by the contribution from the small t region. Through the conformal transformation $t \to 1/\ell$, the $t \to 0$ region of the 1-loop open string amplitude is interpreted as $\ell \to \infty$ limit of a tree-level closed string exchange between two boundary states. This is precisely the string theory picture of the Green-Schwarz mechanism. In the rest of this section, we identify the string diagrams whose $t \to \infty$ limit produces purely gauge anomalies of the $T^4/\mathbb{Z}_2$ orientifold models. We consider the terms of the anomaly polynomial (49) which are sensitive to the $2^{b/2}$ multiplicity of states in the 59-open string sector. The analysis has straightforward extension to other types of anomalies and to $T^4/\mathbb{Z}_4$ orientifold models constructed in the previous sections.

From the anomaly polynomial (49), the ratio of the coefficients of $f_9 \wedge f_9^3$ and $f_5 \wedge f_9^3$ terms is $(-4/2^{b/2})$. This is the result we want to confirm by a string computation.

We start with the $f_9 \wedge f_9^3$ term. The 1-loop open string diagram that would generate this term is shown on Fig. 1. This amplitude is given by

$$A^{(99)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_9; f_9^3\Big] = \int \frac{dt}{2t}\,\Big[\int dx_1 \prod_{n=1}^{3} dy_n\Big] \\ \times \Big\langle \hat{V}_{\mu_1,9}(x_1)\hat{V}_{\nu_1,9}(y_1)\hat{V}_{\nu_2,9}(y_2)\hat{V}_{\nu_3,9}(y_3)\Big\rangle^{\mathcal{C}_{99}}_{\{\alpha,\beta\}=\{\frac{1}{2},\frac{1}{2}\}}, \tag{80}$$

where the 4 99-sector abelian gauge bosons are split into groups of three and one between the two cylinder boundaries. The subscript of the correlator indicates the relevant spin structure[5]. The correlation function in (80) decomposes into the non-compact part which depends on the vertex insertion coordinates x_1, y_n but is independent of the twists θ^k, and the compact part. Since the vertex insertions involve only the non-compact excitations, this compact part is identical to the compact part of the $\mathcal{C}_{99}$ tadpole with $\{\frac{1}{2},\frac{1}{2}\}$ spin structure (A9). Altogether, we can rewrite (80) as

$$\begin{aligned} A^{(99)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_9; f_9^3\Big] &= \frac{1}{3}\int \frac{dt}{2t}\, C_{(\mu_1;\nu_1\nu_2\nu_3)}(t) \\ &\times \Big\{ \sum_{k=0}^{1}\prod_{i=1}^{2} (-2\sin\pi k v_i)\ \mathrm{tr}\,(\gamma_{k,9}\cdot\lambda_9)\mathrm{tr}\,(\gamma^{-1}_{k,9}\cdot\lambda_9^3) \times S_{\mathcal{C}_{99}}(k)\Big\} \\ &= \frac{1}{3}\int \frac{dt}{2t}\, C_{(\mu_1;\nu_1\nu_2\nu_3)}(t)\ \Big\{\ -4\ \mathrm{tr}\,(\gamma_{1,9}\cdot\lambda_9)\mathrm{tr}\,(\gamma^{-1}_{1,9}\cdot\lambda_9^3)\Big\}, \end{aligned} \tag{81}$$

where λ_9 is the generator of the $U(1)$ in the 99-sector. A symmetry factor 1/3 in (81) accounts for the fact that all three gauge bosons on one of the boundaries of the cylinder come from the same $U(1)$.

The 1-loop open string diagram that generates $f_5 \wedge f_9^3$ term in the anomaly polynomial (49) is shown on Fig. 2. This amplitude is given by

$$A^{(59)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_5; f_9^3\Big] = 2^{b/2}\int \frac{dt}{2t}\,\Big[\int dx_1 \prod_{n=1}^{3} dy_n\Big] \\ \times \Big\langle \hat{V}_{\mu_1,5}(x_1)\hat{V}_{\nu_1,9}(y_1)\hat{V}_{\nu_2,9}(y_2)\hat{V}_{\nu_3,9}(y_3)\Big\rangle^{\mathcal{C}_{59}}_{\{\alpha,\beta\}=\{\frac{1}{2},\frac{1}{2}\}}, \tag{82}$$

where $2^{b/2}$ accounts for the multiplicity of 59-cylinders. As before, the correlation function in (82) decomposes into the non-compact part which depends on the vertex insertion coordinates x_1, y_n but is independent of the twists θ^k, and the compact part. The non-compact part

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[5] All the vertex operators in (80) are chosen to be in the (0)-picture. This can be achieved by inserting an appropriate number of worldsheet supercurrents.

of this correlation function is identical to the non-compact part of the correlation function in (80) since 99- and 59-cylinders have the same boundary conditions for the non-compact excitations. The compact part coincides with the compact part of the $\mathcal{C}_{59}$ tadpole with $\{\frac{1}{2},\frac{1}{2}\}$ spin structure (A14). Altogether, the can rewrite (82) as

$$\begin{aligned}
A^{(59)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_5;f_9^3\Big] &= \frac{2^{b/2}}{3}\int\frac{dt}{2t}\,C_{(\mu_1;\nu_1\nu_2\nu_3)}(t)\\
&\quad\times\Big\{\sum_{k=0}^{1}\prod_{i=1}^{2}\frac{\vartheta[{1\atop \frac{1}{2}+kv_i}]}{\vartheta[{0\atop \frac{1}{2}+kv_i}]}\,\mathrm{tr}\,(\gamma_{k,5}\cdot\lambda_5)\mathrm{tr}\,(\gamma^{-1}_{k,9}\cdot\lambda_9^3)\Big\}\\
&= \frac{2^{b/2}}{3}\int\frac{dt}{2t}\,C_{(\mu_1;\nu_1\nu_2\nu_3)}(t)\,\Big\{\ \mathrm{tr}\,(\gamma_{1,5}\cdot\lambda_5)\mathrm{tr}\,(\gamma^{-1}_{1,9}\cdot\lambda_9^3)\Big\},
\end{aligned}\tag{83}$$

where λ_5 is the generator of the $U(1)$ in the 55-sector. From (81) and (83)

$$\frac{A^{(99)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_9;f_9^3\Big]}{A^{(59)}_{(\mu_1;\nu_1\nu_2\nu_3)}\Big[f_5;f_9^3\Big]} = -\frac{4}{2^{b/2}},\tag{84}$$

which is the desired result.

IV. DISCUSSION

In this paper we have considered Type IIB orientifolds on $T^4/\mathbb{Z}_N$ $(N=2,4)$ with discrete B-flux, previously constructed in [17]. We have analyzed the gravitational, gauge and mixed anomalies of the resulting six dimensional vacua and showed that they all cancel. The cancelation required the $2^{b/2}$ multiplicity of states in the 59-open string sector. In a field theory framework, we have identified the twisted sector R-R scalars and tensor multiplets involved in the Green-Schwarz mechanism responsible for this cancelation.

We presented details of the construction of these models and argued that consistency with $2^{b/2}$ multiplicity of 59-sector states requires a modification of the relation between the open string 1-loop channel modulus and the closed string tree channel modulus for the 59- (95-) cylinder amplitudes. The latter should not be surprising, since only for cylinders with the same boundary conditions (99- or 55-cylinders) can one relate their loop moduli to the loop moduli of the Klein bottle and the Möbius strip by adding/removing crosscaps. In fact, we argued that it is precisely the anomaly cancelation condition that should be used to determine the relation between the 59-cylinder 1-loop and tree channel modulus. The reason is that chiral anomalies are sensitive to the R-R spin structure of 1-loop amplitudes, however this spin structure does not contribute to the 1-loop tadpoles. On the other hand, the R-R spin structure will contribute to 2-loop tadpoles, and hence the cancelation of anomalies probes the 2-loop tadpole consistency of the models.

We showed that the anomaly polynomial computed in field theory can be extracted from an appropriate limit of non-planar cylinder amplitudes. We identified certain 1-loop non-planar diagrams which are sensitive to the $2^{b/2}$ multiplicity of 59-sector states and showed that they correctly reproduce the field theory result, when the multiplicity of 59-cylinders is taken to be $2^{b/2}$ as required from the cancelation of the tadpoles.

Our results here can be extended to four dimensional orientifolds [31]. In [9] a $T^6/\mathbb{Z}_6$ orientifold with $b = 2$ B-flux was proposed as an explicit string realization of the "brane world" scenario. The model constructed there contains three chiral families and a Pati-Salam $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ gauge group. The fact that there are three chiral families depends crucially on the $2^{b/2}$ multiplicity of states in the 59-open string sector. In four dimensions, anomaly constraints are much weaker and so are not stringent enough to determine the doubling of the 59-sector states [31]. However, by compactifying the $b = 2$ $T^4/\mathbb{Z}_2$ orientifold considered in this paper on an extra T^2 and further orbifolding it by $\mathbb{Z}_3$, we obtain the $T^6/\mathbb{Z}_6$ orientifold. Before the $\mathbb{Z}_3$ orbifold projection, there is doubling of states in the 59 sector of the orientifold since it is connected to the six-dimensional $\mathbb{Z}_2$ orientifold discussed in this paper by decompactifying the T^2 [6]. This doublet of states can only transform as a singlet representation under $\mathbb{Z}_3$. Thus either all the 59-states of $T^6/\mathbb{Z}_6$ orientifolds are projected out, or all of them are kept. Since tadpole cancelation requires the presence of 59 open string sector states [9,31], this implies that the 59 sector states come with a multiplicity of 2.

The presence of the NS-NS sector B-field background is rather generic. For example, when one considers orientifolding an asymmetric orbifold [18], the background B-field is generically non-zero. The fact that that b_{ij} defined in Eq.(16) is invariant under $b_{ij} \to b_{ij} + 1$ and that it takes only quantized values of 0 or 1/2 suggests that it should be possible to associate with it a $\mathbb{Z}_2$ action. Indeed, one also finds a $2^{b/2}$ multiplicity of states in the 59-sector in this picture [18].

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APPENDIX A: ONE-LOOP VACUUM AMPLITUDES IN ORIENTIFOLDS WITHOUT NS-NS ANTISYMMETRIC TENSOR BACKGROUND.

In this section we review the computation of 1-loop vacuum amplitudes of the $T^4/\mathbb{Z}_N$ orientifolds [4,5]. The 1-loop vacuum amplitudes come from the Klein bottle $\mathcal{K}$, Möbius strip $\mathcal{M}$, and cylinder $\mathcal{C}$.

The Klein bottle amplitude is given by

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[6] For special radius of the T^2, it is possible that some momentum/winding states become massless. However, these states do not come from the 59 sector.

$$\mathcal{K} = \frac{iV_6}{8N} \sum_{n,k=0}^{N-1} \int_0^\infty \frac{dt}{t} \, (4\pi^2\alpha' t)^{-3} \, \mathcal{Z}_{\mathcal{K}}(\theta^n, \theta^k), \tag{A1}$$

where

$$\mathcal{Z}_{\mathcal{K}}(\theta^n, \theta^k) = \mathrm{Tr}\left\{(1 + (-1)^F)\Omega\, \theta^k \, \mathrm{e}^{-2\pi t [L_0(\theta^n) + \overline{L}_0(\theta^n)]}\right\}. \tag{A2}$$

The contribution of the uncompactified momenta is already extracted in (A1). Here, V_6 denotes the regularized space-time volume. Since Ω exchanges θ^n with θ^{N-n}, only $n = 0$ and $n = N/2$ terms survive the trace. The trace in $\mathcal{Z}_{\mathcal{K}}$ can be evaluated in a standard way using ϑ functions to write the contributions of complex bosons and fermions. Also, the GSO projection is implemented by summing over spin structures. Then, taking into account the insertion of Ω we find

$$\mathcal{Z}_{\mathcal{K}}(1, \theta^k) = \sum_{\alpha,\beta=0,\frac{1}{2}} \eta_{\alpha,\beta} \, \frac{\tilde{\vartheta}^2[{\alpha \atop \beta}]}{\tilde{\eta}^6} \prod_{i=1}^{2} (-2\sin 2\pi k v_i) \, \frac{\tilde{\vartheta}[{\alpha \atop \beta + 2kv_i}]}{\tilde{\vartheta}[{\frac{1}{2} \atop \frac{1}{2} + 2kv_i}]} \times S_{\mathcal{K}}(k), \tag{A3}$$

where $\eta_{0,0} = -\eta_{0,\frac{1}{2}} = -\eta_{\frac{1}{2},0} = 1$. The ϑ and η functions are defined in Appendix B. The tilde indicates that the argument is $\tilde{q} = q^2 = \mathrm{e}^{-4\pi t}$. Note that for $2kv_i$ = integer, (A3) has a well-defined limit. The factors $S_{\mathcal{K}}(k)$ account for the sums over quantized momenta (kv_i =integer) or winding (kv_i =half-integer) in the Y_i direction:

$$\begin{aligned} S_{\mathcal{K}}(k) &= 1, && k \neq \{0, N/2\}, \\ S_{\mathcal{K}}(k) &= \delta_{k,0} \sum_{p^2 \in \star\Gamma_4} q^{\alpha' p^2/2} + \delta_{k,N/2} \sum_{\omega^2 \in \Gamma_4} q^{\omega^2/(2\alpha')}, && k = \{0, N/2\}. \end{aligned} \tag{A4}$$

In (A4), Γ_4 and $\star\Gamma_4$ denote the winding and momentum lattice of T^4 correspondingly. The $t \to 0$ divergence of (A3) represents the exchange of the massless NS-NS (terms with $\{\alpha, \beta\} = \{0, 0\}$ and $\{\alpha, \beta\} = \{\frac{1}{2}, 0\}$ in (A3)) and R-R forms in the closed string tree channel.

The Z_2 twisted sector yields

$$\mathcal{Z}_{\mathcal{K}}(\theta^{N/2}, \theta^k) = \tilde{\chi}(\theta^{N/2}, \theta^k) \sum_{\alpha,\beta=0,\frac{1}{2}} \eta_{\alpha,\beta} \, \frac{\tilde{\vartheta}^2[{\alpha \atop \beta}]}{\tilde{\eta}^6} \prod_{i=1}^{2} \frac{\tilde{\vartheta}[{\alpha + \frac{1}{2} \atop \beta + 2kv_i}]}{\tilde{\vartheta}[{0 \atop \frac{1}{2} + 2kv_i}]}. \tag{A5}$$

In eq. (A5), $\tilde{\chi}(\theta^{N/2}, \theta^k)$ is a factor that takes into account the fixed point degeneracy [30], *i.e.*, it counts the number of fixed points of $\theta^{N/2}$ on T^4 invariant under θ^k:

$$\begin{aligned} \tilde{\chi}(\theta^{N/2}, \theta^k) &= 16, && k = \{0, N/2\}, \\ \tilde{\chi}(\theta^2, \theta^k) &= 4, && k = \{1, 3\}, \text{ and } N = 4. \end{aligned} \tag{A6}$$

As (A3), (A5) vanishes by virtue of supersymmetry. The NS-NS exchange in the closed string tree channel is represented by $\{\alpha, \beta\} = \{0, 0\}$ and $\{\alpha, \beta\} = \{\frac{1}{2}, 0\}$ terms in (A5).

The cylinder amplitudes are given by

$$\mathcal{C}_{pq} = \frac{iV_6}{8N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} \, (8\pi^2\alpha' t)^{-3} \, \mathcal{Z}_{pq}(\theta^k), \tag{A7}$$

where

$$\mathcal{Z}_{pq}(\theta^k) = \mathrm{Tr}_{\,pq}\{(1+(-1)^F)\theta^k\,\mathrm{e}^{-2\pi t L_0}\}. \tag{A8}$$

The trace is over open string states with boundary conditions according to the Dp and Dq-branes at the endpoints.

In $\mathcal{Z}_{99}$, boundary conditions are NN in all directions. Hence,

$$\mathcal{Z}_{99}(\theta^k) = \sum_{\alpha,\beta=0,\frac{1}{2}} \eta_{\alpha,\beta}\,\frac{\vartheta^2[{\alpha \atop \beta}]}{\eta^6}\,\prod_{i=1}^{2}(-2\sin\pi k v_i)\,\frac{\vartheta[{\alpha \atop \beta+kv_i}]}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+kv_i}]}\,(\mathrm{Tr}\,\gamma_{k,9})^2\ \times S_{\mathcal{C}_{99}}(k). \tag{A9}$$

The factors $S_{\mathcal{C}_{99}}(k)$ account for the sums over quantized momenta in Y_i when kv_i =integer :

$$\begin{aligned} S_{\mathcal{C}_{99}}(k) &= 1, \qquad\qquad k\neq 0,\\ S_{\mathcal{C}_{99}}(0) &= \sum_{p^2\in\star\Gamma_4} q^{\alpha' p^2}. \end{aligned} \tag{A10}$$

Eq. (A9) vanishes by supersymmetry. The terms $\{\alpha,\beta\} = \{0,0\}$ and $\{\alpha,\beta\} = \{\frac{1}{2},0\}$ represent NS-NS exchange in the closed string tree channel.

In the 55-sector there are DD boundary conditions in directions Y_1, Y_2 transverse to the 5-branes. The oscillator expansions with DD boundary conditions have integer modes but include windings instead of momenta. Then, $\mathcal{Z}_{55}$ has a form similar to (A9)

$$\mathcal{Z}_{55}(\theta^k) = \sum_{\alpha,\beta=0,\frac{1}{2}} \eta_{\alpha,\beta}\,\frac{\vartheta^2[{\alpha \atop \beta}]}{\eta^6}\,\prod_{i=1}^{2}(-2\sin\pi k v_i)\,\frac{\vartheta[{\alpha \atop \beta+kv_i}]}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+kv_i}]}\,\sum_{I=1}^{n(N,k)}(\mathrm{Tr}\,\gamma_{k,5,I})^2\ \times S_{\mathcal{C}_{55}}(k), \tag{A11}$$

where I refers to the fixed points of θ^k. Thus,

$$\begin{aligned} n(2,1) = n(4,2) = 16,\\ n(4,1) = n(4,3) = 4. \end{aligned} \tag{A12}$$

$S_{\mathcal{C}_{55}}(k)$ account for the sums over winding in Y_i when kv_i =integer :

$$\begin{aligned} S_{\mathcal{C}_{55}}(k) &= 1, \qquad\qquad k\neq 0,\\ S_{\mathcal{C}_{55}}(0) &= \sum_{\omega^2\in\Gamma_4} q^{\omega^2/\alpha'}. \end{aligned} \tag{A13}$$

In (A11) terms $\{\alpha,\beta\} = \{0,0\}$ and $\{\alpha,\beta\} = \{\frac{1}{2},0\}$ represent NS-NS exchange in the closed string tree channel.

In the 59-sector there are DN boundary conditions in coordinates Y_1, Y_2. Hence, their oscillator expansions include half-integer modes. For fermions, world-sheet supersymmetry requires that in Neveu-Schwarz (Ramond), the moddings are opposite (same) to that of the corresponding bosons. Hence,

$$\mathcal{Z}_{59}(\theta^k) = \sum_{\alpha,\beta=0,\frac{1}{2}} \eta_{\alpha,\beta} \frac{\vartheta^2[{\alpha \atop \beta}]}{\eta^6} \prod_{i=1}^{2} \frac{\vartheta[{\alpha+\frac{1}{2} \atop \beta+kv_i}]}{\vartheta[{0 \atop \frac{1}{2}+kv_i}]} \mathrm{Tr}\, \gamma_{k,9} \sum_{I}^{n(N,k)} \mathrm{Tr}\, \gamma_{k,5,I}. \tag{A14}$$

The $\{\alpha,\beta\} = \{0,0\}$ and $\{\alpha,\beta\} = \{\frac{1}{2},0\}$ contributions of (A14) represent the NS-NS exchange in the closed string tree channel.

The Möbius strip amplitudes are given by

$$\mathcal{M}_p = \frac{iV_6}{8N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} (8\pi^2\alpha' t)^{-3} \mathcal{Z}_p(\theta^k), \tag{A15}$$

where

$$\mathcal{Z}_p(\theta^k) = \mathrm{Tr}\,_p\{(1+(-1)^F)\,\Omega\,\theta^k\,\mathrm{e}^{-2\pi t L_0}\}. \tag{A16}$$

Here the trace is over open string states with boundary conditions according to the Dp-branes at both endpoints. The main difference between $\mathcal{Z}_p$ and $\mathcal{Z}_{pp}$ is the insertion of Ω that acts on the various bosonic and fermionic oscillators thereby introducing extra phases in the expansions in q. More precisely, Ω acts on oscillators as [4]

$$\alpha_r \to \pm \mathrm{e}^{i\pi r} \qquad ; \qquad \psi_r \to \pm \mathrm{e}^{i\pi r}. \tag{A17}$$

The upper (lower) sign is for NN (DD) boundary conditions. Furthermore, Ω acts as $\mathrm{e}^{-i\pi/2}$ on the NS vacuum. This ensures that $\Omega(\psi^\mu_{-\frac{1}{2}}|0\rangle_{NS}) = -\psi^\mu_{-\frac{1}{2}}|0\rangle_{NS}$ as needed for the orientifold projection on gauge vectors.

To derive the Möbius trace we can use (A17) and the results for 99-cylinders. After using ϑ identities we obtain

$$\begin{aligned}\mathcal{Z}_9(\theta^k) = -(1-1) \frac{\tilde\vartheta^2[{\frac{1}{2} \atop 0}]\,\tilde\vartheta^2[{0 \atop \frac{1}{2}}]}{\tilde\eta^6 \tilde\vartheta^2[{0 \atop 0}]} \prod_{i=1}^{2} \frac{-2\sin\pi k v_i\, \tilde\vartheta[{\frac{1}{2} \atop kv_i}]\,\tilde\vartheta[{0 \atop \frac{1}{2}+kv_i}]}{\tilde\vartheta[{\frac{1}{2} \atop \frac{1}{2}+kv_i}]\,\tilde\vartheta[{0 \atop kv_i}]} \mathrm{Tr}\,(\gamma^{-1}_{\Omega_k,9}\cdot\gamma^T_{\Omega_k,9}) \\ \times S_{\mathcal{M}_9}(k).\end{aligned} \tag{A18}$$

In (A18) we separate the NS-NS and R-R exchange in the closed string tree channel. $S_{\mathcal{M}_9}(k)$ account for a sum over quantized momentum in Y_i for kv_i =integer:

$$\begin{aligned} S_{\mathcal{M}_9}(k) &= 1, \qquad\qquad k \neq 0, \\ S_{\mathcal{M}_9}(0) &= \sum_{p^2\in\star\Gamma_4} q^{\alpha' p^2}. \end{aligned} \tag{A19}$$

For 5-branes, we find

$$\begin{aligned}\mathcal{Z}_5(\theta^k) = (1-1) \frac{\tilde\vartheta^2[{\frac{1}{2} \atop 0}]\,\tilde\vartheta^2[{0 \atop \frac{1}{2}}]}{\tilde\eta^6 \tilde\vartheta^2[{0 \atop 0}]} \prod_{i=1}^{2} \frac{2\cos\pi k v_i\, \tilde\vartheta[{\frac{1}{2} \atop \frac{1}{2}+kv_i}]\,\tilde\vartheta[{0 \atop kv_i}]}{\tilde\vartheta[{\frac{1}{2} \atop kv_i}]\,\tilde\vartheta[{0 \atop \frac{1}{2}+kv_i}]} \sum_I \mathrm{Tr}\,(\gamma^{-1}_{\Omega_k,5,I}\cdot\gamma^T_{\Omega_k,5,I}) \\ \times S_{\mathcal{M}_5}(k).\end{aligned} \tag{A20}$$

In (A20) we separate the NS-NS and R-R exchange in the closed string tree channel. $S_{\mathcal{M}_5}(k)$ account for a sum over winding in Y_i for kv_i =half-integer:

$$
\begin{aligned}
S_{\mathcal{M}_5}(k) &= 1, \qquad\qquad k \neq N/2, \\
S_{\mathcal{M}_5}(N/2) &= \sum_{\omega^2 \in \Gamma_4} q^{\omega^2/\alpha'} .
\end{aligned}
\tag{A21}
$$

APPENDIX B: SOME PROPERTIES OF THE ϑ FUNCTIONS

The ϑ function of rational characteristics δ and φ is given by

$$
\vartheta[{\delta \atop \varphi}](t) = \sum_n q^{\frac{1}{2}(n+\delta)^2} \, \mathrm{e}^{2i\pi(n+\delta)\varphi} . \tag{B1}
$$

Here the variable q is $q = \mathrm{e}^{-2\pi t}$. The ϑ function also has the product form

$$
\frac{\vartheta[{\delta \atop \varphi}]}{\eta} = \mathrm{e}^{2i\pi\delta\varphi} \, q^{\frac{1}{2}\delta^2 - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\delta-\frac{1}{2}} \mathrm{e}^{2i\pi\varphi}) \, (1 + q^{n-\delta-\frac{1}{2}} \mathrm{e}^{-2i\pi\varphi}), \tag{B2}
$$

where the Dedekind η function is

$$
\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) . \tag{B3}
$$

Notice that

$$
\lim_{\varphi \to 0} \frac{-2 \sin \pi\varphi}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+\varphi}]} = \frac{1}{\eta^3} . \tag{B4}
$$

The ϑ and η functions have the modular transformation properties

$$
\begin{aligned}
\vartheta[{\delta \atop \varphi}](t) &= \mathrm{e}^{2i\pi\delta\varphi} \, t^{-\frac{1}{2}} \, \vartheta[{-\varphi \atop \delta}](1/t), \\
\eta(t) &= t^{-\frac{1}{2}} \, \eta(1/t) .
\end{aligned}
\tag{B5}
$$

The ϑ's satisfy several Riemann identities [28]. In particular,

$$
\begin{aligned}
&\sum_{\alpha,\beta} \eta_{\alpha,\beta} \, \vartheta[{\alpha \atop \beta}] \prod_{i=1}^{3} \vartheta[{\alpha \atop \beta + u_i}] = 0, \\
&\sum_{\alpha,\beta} \eta_{\alpha,\beta} \, \vartheta[{\alpha \atop \beta}] \vartheta[{\alpha \atop \beta + u_3}] \prod_{i=1}^{2} \vartheta[{\alpha + \frac{1}{2} \atop \beta + u_i}] = 0,
\end{aligned}
\tag{B6}
$$

provided that $u_1 + u_2 + u_3 = 0$.

APPENDIX C: CHIRAL ANOMALIES IN SIX DIMENSIONS

In this section we collect some facts about anomalies in six dimensional $\mathcal{N}=1$ supergravity theories. Details can be found in Ref. [24–27].

$\mathcal{N}=1$ supersymmetry in six dimensions involves four types of massless multiplets. In terms of representations of the little group $SU(2)\otimes SU(2)$ (we label the $SU(2)$ representations by their multiplicities $(2J+1)$), the massless multiplets are

(i) gravity multiplet G: $(\mathbf{3},\mathbf{3})\oplus 2(\mathbf{3},\mathbf{2})\oplus(\mathbf{3},\mathbf{1})$,
(ii) tensor multiplet T: $(\mathbf{1},\mathbf{3})\oplus 2(\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{1})$,
(iii) vector multiplet V: $(\mathbf{2},\mathbf{2})\oplus 2(\mathbf{2},\mathbf{1})$,
(iv) hypermultiplet H: $2(\mathbf{1},\mathbf{2})\oplus 4(\mathbf{1},\mathbf{1})$.

In six dimensions chiral anomalies are characterized by an 8-form polynomial constructed from the gauge and gravitational field strengths. The basic diagram to be examined is the box diagram with an even number of external gravitons and gauge fields. The resulting anomalous diagrams can be classified as purely gravitational, purely gauge or mixed gauge and gravitational.

All four $\mathcal{N}=1$ supergravity multiplets contribute to purely gravitational anomaly. Their separate contributions to the anomaly polynomial are given by

$$\begin{aligned}
i(2\pi)^3 I_G^{grav.} &= \frac{1}{5760}\Big(273\ \mathrm{tr}\,R^4-\frac{5}{4}\ 51\ (\mathrm{tr}\,R^2)^2\Big),\\
i(2\pi)^3 I_T^{grav.} &= \frac{1}{5760}\Big(-29\ \mathrm{tr}\,R^4+\frac{5}{4}\ 7\ (\mathrm{tr}\,R^2)^2\Big),\\
i(2\pi)^3 I_V^{grav.} &= \frac{1}{5760}\Big(\mathrm{tr}\,R^4+\frac{5}{4}\ (\mathrm{tr}\,R^2)^2\Big),\\
i(2\pi)^3 I_H^{grav.} &= \frac{1}{5760}\Big(-\mathrm{tr}\,R^4-\frac{5}{4}\ (\mathrm{tr}\,R^2)^2\Big).
\end{aligned} \tag{C1}$$

where the trace over curvature matrices in R is in the vector representation of $SO(5,1)$. The cancellation of the leading term $\mathrm{tr}\,R^4$ in the pure gravitational anomalies implies

$$n_H-n_V=273-29n_T \tag{C2}$$

Mixed gauge and gravitational anomalies get contributions from gauginos and matter fermions:

$$\begin{aligned}
i(2\pi)^3 I^{mixed} = -\frac{1}{96}\ \mathrm{tr}\,R^2\Big(&\sum_A \mathrm{Tr}\,F_A^2-\sum_{I,A} s_A^I(\mathrm{tr}_{\,R^I}F_A^2)\\
&-\sum_{i,a} s_a^i\ q_{i,a}^2\ (\mathrm{tr}\,f_a^2)-2\sum_{ij,a<b} s_{ab}^{ij}\ q_{ij,a}\ q_{ij,b}\ (\mathrm{tr}\,f_a)\ (\mathrm{tr}\,f_b)\Big),
\end{aligned} \tag{C3}$$

where F_A represents the nonabelian field strength of a gauge group factor G_A and f_a represents the abelian field strength. s_A^I denotes the number of hypermultiplets transforming in representation R^I of the nonabelian factor G_A, while s_a^i counts the hypermultiplets with $q_{i,a}$ charge under $U(1)_a$. s_{ab}^{ij} is a multiplicity of states with charges $(q_{ij,a},q_{ij,b})$ under $U(1)_a\otimes U(1)_b$. Tr refers to the trace in the respective adjoint representation.

Finally, purely gauge anomaly is given by

$$
\begin{aligned}
i(2\pi)^3 I^{gauge} = \frac{1}{24} \Big(& \sum_A \mathrm{Tr}\, F_A^4 - \sum_{I,A} s_A^I (\mathrm{tr}_{\,R^I} F_A^4) - 6 \sum_{IJ,A<B} s_{AB}^{IJ}\ (\mathrm{tr}_{\,R^I} F_A^2)\ (\mathrm{tr}_{\,R^J} F_B^2) \\
& -4 \sum_{Ij,Aa} s_{Aa}^{Ij}\ q_{Ij,a}\ (\mathrm{tr}_{\,R^I} F_A^3)\ (\mathrm{tr}\, f_a) \\
& -6 \sum_{Ij,Aa} s_{Aa}^{Ij}\ q_{Ij,a}^2\ (\mathrm{tr}_{\,R^I} F_A^2)\ (\mathrm{tr}\, f_a)^2 \\
& -12 \sum_{Iij,A,a<b} s_{Aab}^{Iij}\ q_{Iij,a}\ q_{Iij,b}\ (\mathrm{tr}_{\,R^I} F_A^2)\ (\mathrm{tr}\, f_a)\ (\mathrm{tr}\, f_b) \\
& - \sum_{i,a} s_a^i\ q_{i,a}^4\ (\mathrm{tr}\, f_a)^4 - 4 \sum_{ij,ab} s_{ab}^{ij}\ q_{ij,a}^3\ q_{ij,b}\ (\mathrm{tr}\, f_a)^3\ (\mathrm{tr}\, f_b) \\
& -6 \sum_{ij,a<b} s_{ab}^{ij}\ q_{ij,a}^2\ q_{ij,b}^2\ (\mathrm{tr}\, f_a)^2\ (\mathrm{tr}\, f_b)^2 \Big),
\end{aligned} \tag{C4}
$$

where s_{AB}^{IJ} is the multiplicity of states in the (R^I, R^J) representation of $G_A \otimes G_B$, s_{Aa}^{Ij} is the multiplicity of states in the representation/charge $(R^I, q_{Ij,a})$ of $G_A \otimes U(1)_a$, and s_{Aab}^{Iij} is the multiplicity of states in the representation/charges $(R^I, q_{Iij,a}, q_{Iij,b})$ of $G_A \otimes U(1)_a \otimes U(1)_b$. In (C4) we included only hypermultiplet representations which are relevant to the orientifold models of interest (see Table I for the hypermultiplet spectrum).

It is convenient to express all traces over nonabelian field strengths in the fundamental representation of respective gauge groups. Here we list out the relations between traces in different representations for $SU(n)$ groups [26] (since they are the nonabelian gauge groups arise in the Type IIB orientifolds of interest in this paper). For $n \geq 3$,

$$
\begin{aligned}
\mathrm{Tr}\, F^4 &= 2n\ \mathrm{tr}\, F^4 + 6\ (\mathrm{tr}\, F^2)^2, \\
\mathrm{tr}_{\,a^{ij}} F^4 &= (n-8)\ \mathrm{tr}\, F^4 + 3\ (\mathrm{tr}\, F^2)^2, \\
\mathrm{tr}_{\,a^{ij}} F^3 &= (n-4)\ \mathrm{tr}\, F^3, \\
\mathrm{Tr}\, F^2 &= 2n\ \mathrm{tr}\, F^2, \\
\mathrm{tr}_{\,a^{ij}} F^2 &= (n-2)\ \mathrm{tr}\, F^2,
\end{aligned} \tag{C5}
$$

where a^{ij} denotes a second rank antisymmetric tensor representation. tr refers to the trace in the fundamental representation. Furthermore, for $SU(2)$:

$$
\mathrm{tr}\, F^4 = \frac{1}{2}\ (\mathrm{tr}\, F^2)^2. \tag{C6}
$$

APPENDIX D: TRACES OVER GAUGE GROUP GENERATORS.

In Sec. II A we discuss how the "shift" vector formalism of [11] can be used to represent gauge twists $\gamma_{1,a}$ and determine massless states in the open string sector of the Type IIB orientifold models. Here, we collect the formulas relevant to the computation of traces for the anomaly cancelation.

In Sec. II A we rewrite the Chan-Paton matrices λ_A^p and λ_a^p in a Cartan-Weyl basis. The subscript A refers to the nonabelian gauge group factor G_A in the $p = 5, 9$ open string sector, while a labels the generator of $U(1)_a$ in open string sector p. The Cartan generators are represented by tensor product of 2×2 σ_3 submatricies. We chose the normalization of the $SO(32)$ generators λ in such a way that $\mathrm{Tr}\,\lambda^2 = 1$. In all the models we discuss, there is a $U(1)$ factor for each of the $SU(n)$'s in the model. The generator λ_a^p of a given $U(1)_a$ is given by a linear combination

$$\lambda_a^p = Q_a^p \cdot H, \tag{D1}$$

where H is a vector of Cartan generators and Q_a^p is a 16-dimensional real vector of the form

$$Q_a^p = \frac{1}{\sqrt{2n_a}}(0, 0, \cdots, \underbrace{1, \cdots, 1,}_{n_a \text{ times}} 0, 0, \cdots, 0), \tag{D2}$$

where the non-zero entries sit at the positions where the corresponding $U(n_a)$ lives. With these convention and normalization

$$\begin{aligned}
&\mathrm{tr}\,(\gamma_{k,p}\ \lambda_a^p) = \mathrm{tr}\,(e^{-2i\pi k V_{(pp)}\cdot H}\ Q_a^p \cdot H) = (-i)\sqrt{2n_a}\ \sin 2\pi k V_{(pp)}^a, \\
&\mathrm{tr}\,(\gamma_{k,p}\ \lambda_A^p) = 0, \\
&\mathrm{tr}\,(\gamma_{k,p}\ (\lambda_a^p)^2) = \cos 2\pi k V_{(pp)}^a, \\
&\mathrm{tr}\,(\gamma_{k,p}\ (\lambda_A^p)^2) = \cos 2\pi k V_{(pp)}^A, \\
&\mathrm{tr}\,(\gamma_{k,p}\ (\lambda_a^p)^3) = (-i)\frac{1}{\sqrt{2n_a}}\ \sin 2\pi k V_{(pp)}^a, \\
&\mathrm{tr}\,(\gamma_{k,p}\ (\lambda_A^p)^3) = (-i)\frac{1}{\sqrt{2}}\ \sin 2\pi k V_{(pp)}^A, \\
&\mathrm{tr}\,(\gamma_{k,p}\ \lambda_a^p\ (\lambda_A^p)^2) = (-i)\frac{1}{\sqrt{2n_a}}\ \sin 2\pi k V_{(pp)}^A, \qquad \text{when } V_{(pp)}^a = V_{(pp)}^A.
\end{aligned} \tag{D3}$$

where n_a is the rank of the $U(n)$ group containing $U(1)_a$ and $V_{(pp)}^a$ is a component of the $V_{(pp)}$ shift vector along any of the overlapping entries with that $U(n)$, $V_{(pp)}^A$ is a component of $V_{(pp)}$ along any entry overlapping with the group G_A in the p-open string sector.

FIGURES

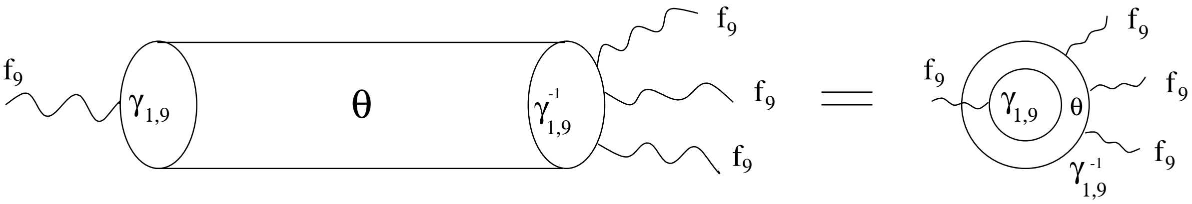

FIG. 1. This non-planar digram of the 99-open string sector generates the $f_9 \wedge f_9^3$ purely gauge anomaly in the low-energy limit. Four abelian gauge boson vertex operators (represented by f_9) are inserted at different boundaries of the cylinder. $\gamma_{1,9}$ represents a gauge twist associated with the insertion of an orbifold projector θ.

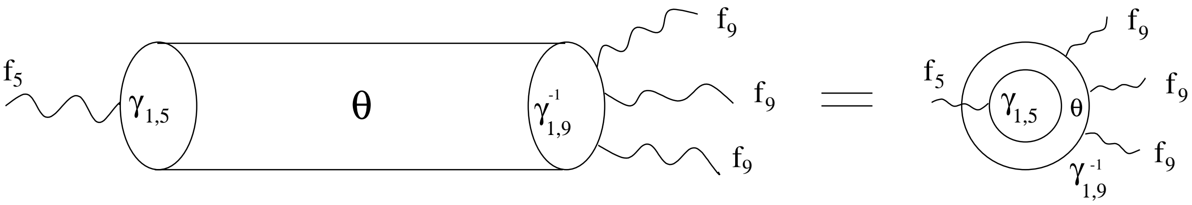

FIG. 2. This non-planar digram of the 59-open string sector generates the $f_5 \wedge f_9^3$ purely gauge anomaly in the low-energy limit. Four abelian gauge boson vertex operators (represented by f_5 and f_9) are inserted at different boundaries of the cylinder. $\gamma_{1,9}^{-1}$ and $\gamma_{1,5}$ represent gauge twists associated with the insertion of an orbifold projector θ.

TABLES

Model	b	Gauge Group	Charged Hypermultiplets	Neutral Hypermultiplets	Extra Tensor Multiplets
$\mathbb{Z}_2$	0	$[SU(16)\otimes U(1)]^2$	$2\times(\mathbf{120};\mathbf{1})(+1/2;0)$ $2\times(\mathbf{1};\mathbf{120})(0;+1/2)$ $(\mathbf{16};\overline{\mathbf{16}})(+1/4;-1/4)$	20	0
$\mathbb{Z}_2$	2	$[SU(8)\otimes U(1)]^2$	$2\times(\mathbf{28};\mathbf{1})(+1/\sqrt{2};0)$ $2\times(\mathbf{1};\mathbf{28})(0;+1/\sqrt{2})$ $2\times(\mathbf{8};\overline{\mathbf{8}})(+1/\sqrt{8};-1/\sqrt{8})$	16	4
$\mathbb{Z}_2$	4	$[SU(4)\otimes U(1)]^2$	$2\times(\mathbf{6};\mathbf{1})(+1;0)$ $2\times(\mathbf{1};\mathbf{6})(0;+1)$ $4\times(\mathbf{4};\overline{\mathbf{4}})(+1/2;-1/2)$	14	6
$\mathbb{Z}_4$	0	$[SU(8)\otimes SU(8)$ $\otimes U(1)^2]^2$	$(\mathbf{28},\mathbf{1};\mathbf{1},\mathbf{1})(+1/\sqrt{2},0;0,0)$ $(\mathbf{1},\overline{\mathbf{28}};\mathbf{1},\mathbf{1})(0,-1/\sqrt{2};0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{28},\mathbf{1})(0,0;+1/\sqrt{2},0)$ $(\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{28}})(0,0;0,-1/\sqrt{2})$ $(\mathbf{8},\overline{\mathbf{8}};\mathbf{1},\mathbf{1})(+1/\sqrt{8},-1/\sqrt{8};0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{8},\overline{\mathbf{8}})(0,0;+1/\sqrt{8},-1/\sqrt{8})$ $(\mathbf{8},\mathbf{1};\overline{\mathbf{8}},\mathbf{1})(+1/\sqrt{8},0;-1/\sqrt{8},0)$ $(\mathbf{1},\mathbf{8};\mathbf{1},\overline{\mathbf{8}})(0,+1/\sqrt{8};0,-1/\sqrt{8})$	16	4
$\mathbb{Z}_4$	2	$[SU(4)\otimes SU(4)$ $\otimes U(1)^2]^2$	$(\mathbf{6},\mathbf{1};\mathbf{1},\mathbf{1})(+1,0;0,0)$ $(\mathbf{1},\overline{\mathbf{6}};\mathbf{1},\mathbf{1})(0,-1;0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1})(0,0;+1,0)$ $(\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{6}})(0,0;0,-1)$ $(\mathbf{4},\overline{\mathbf{4}};\mathbf{1},\mathbf{1})(+1/2,-1/2;0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{4},\overline{\mathbf{4}})(0,0;+1/2,-1/2)$ $2\times(\mathbf{4},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})(+1/2,0;-1/2,0)$ $2\times(\mathbf{1},\mathbf{4};\mathbf{1},\overline{\mathbf{4}})(0,+1/2;0,-1/2)$	14	6
$\mathbb{Z}_4$	4	$[SU(2)\otimes SU(2)$ $\otimes U(1)^2]^2$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})(+\sqrt{2},0;0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})(0,-\sqrt{2};0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})(0,0;+\sqrt{2},0)$ $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})(0,0;0,-\sqrt{2})$ $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})(+1/\sqrt{2},-1/\sqrt{2};0,0)$ $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})(0,0;+1/\sqrt{2},-1/\sqrt{2})$ $4\times(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})(+1/\sqrt{2},0;-1/\sqrt{2},0)$ $4\times(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})(0,+1/\sqrt{2};0,-1/\sqrt{2})$	13	7

TABLE I. The massless spectrum of the six dimensional Type IIB orientifolds on $T^4/\mathbb{Z}_N$ for $N=2,4$, and various values of b (the rank of B_{ij}). The semi-colon in the column "Charged Hypermultiplets" separates 99 and 55 representations.

REFERENCES

[1] M. Green and J. Schwarz, Phys. Lett. **B149** (1984) 117.

[2] M. Dine, N. Seiberg and E. Witten, Nucl. Phys. **B289** (1987) 589;
J. Atick, L. Dixon and A. Sen, Nucl. Phys. **B292** (1987) 109;
M. Dine, I. Ichinoise and N. Seiberg, Nucl. Phys. **B293** (1987) 253.

[3] For a partial list, see, *e.g.*,
M. Berkooz and R.G. Leigh, Nucl. Phys. **B483** (1997) 187;
C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Ya.S. Stanev, Phys. Lett. **B385** (1996) 96;
Z. Kakushadze, Nucl. Phys. **B512** (1998) 221;
Z. Kakushadze and G. Shiu, Phys. Rev. **D56** (1997) 3686; Nucl. Phys. **B520** (1998) 75;
G. Zwart, Nucl. Phys. **B526** (1998) 378;
Z. Kakushadze, G. Shiu and S.-H.H. Tye, Nucl. Phys. **B533** (1998) 25;
R. Blumenhagen and A. Wisskirchen, Phys. Lett. **B438** (1998) 52;
Z. Kakushadze, Phys. Lett. **B434** (1998) 269; Nucl. Phys. **B535** (1998) 311; Phys. Rev. **D58** (1998) 101901;
Z. Kakushadze and S.-H.H. Tye, Phys. Rev. **D58** (1998) 126001.

[4] E. Gimon and J. Polchinski, Phys. Rev. **D54** (1996) 1667.

[5] E. Gimon and C.V. Johnson, Nucl. Phys. **B479** (1996) 285;
A. Dabholkar, J. Park, Nucl. Phys. **B472** (1996) 207; Nucl. Phys. **B477** (1996) 701.

[6] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724.

[7] J. Lykken, Phys. Rev. **D54** (1996) 3693.

[8] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429** (1998) 263, Phys. Rev. **D59** (1999) 086004;
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436** (1998) 257.

[9] G. Shiu and S.-H.H. Tye, Phys. Rev. **D58** (1998) 106007.

[10] Z. Kakushadze and S.-H.H. Tye, Nucl. Phys. **B548** (1999) 180.

[11] G. Aldazabal, A. Font, L. E. Ibanez and G. Violero, Nucl. Phys. **B536** (1998) 29.

[12] L. E. Ibanez, R. Rabadan, A. M. Uranga, Nucl. Phys. **B542** (1999) 112; hep-th/9905098
G. Aldazabal, D. Badagnani, L.E. Ibanez and A.M. Uranga, JHEP **9906** (1999) 031;
E. Poppitz, Nucl. Phys. **B542** (1999) 31;
M. Cvetic, L. Everett, P. Langacker and J. Wang, hep-th/9903051;
Z. Lalak, S. Lavignac and H.P. Nilles, hep-th/9903160.

[13] M. Berkooz, R.G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and E. Witten, Nucl. Phys. **B475** (1996) 115.

[14] C.A. Scrucca and M. Serone, hep-th/9907112.

[15] M. Bianchi, Pradisi and A. Sagnotti, Nucl. Phys. **B376** (1992) 365;
M. Bianchi, Nucl. Phys. **B528** (1998) 73.

[16] E. Witten, JHEP **9802** (1998) 006.

[17] Z. Kakushadze, G. Shiu, and S.-H. H. Tye, Phys. Rev. **D58** (1998) 086001.

[18] R. Blumenhagen and L. Gorlich, Nucl. Phys. **B551** (1999) 601.

[19] S. Chaudhuri, G. Hockney and J. Lykken, Phys. Rev. Lett. **75** (1995) 2264;
S. Chaudhuri and J. Polchinski, Phys. Rev. **D52** (1995) 7168.

[20] A. Sen and S. Sethi, Nucl. Phys. **B499** (1997) 45.

[21] A. Sen, hep-th/99709159 and reference therein.
[22] M. Bershadsky, T. Pantev and V. Sadov, hep-th/9805056.
[23] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory: volume 2*, Cambridge University Press (1987).
[24] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234** (1983) 269.
[25] M.B. Green, J.H. Schwarz and P.C. West, Nucl. Phys. B254 (1985) 327.
[26] J. Erler, J. Math. Phys. 35 (1994) 1819.
[27] J.H. Schwarz, Phys. Lett. **B371** (1996) 223.
[28] D. Mumford, *The Tata Lectures on Theta I*, Birkhäuser, (1983).
[29] Y. Cai and J. Polchinski, Nucl. Phys. **B296** (1988) 91;
C.G. Callan, C. Lovelace, C.R. Nappi, S.A. Yost Nucl. Phys. **B293** (1987) 83.
[30] L.E. Ibáñez, J. Mas, H.P. Nilles and F. Quevedo, Nucl. Phys. **B301** (1988) 157;
A. Font, L.E. Ibáñez, F. Quevedo and A. Sierra, Nucl. Phys. **B331** (1990) 421.
[31] A. Buchel, G. Shiu and S.-H.H. Tye, in preparation.
[32] M. Douglas and C. Hull, JHEP **9802** (1998) 008.