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Transcendental numbers and the topology of three-loop bubbles

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We present a proof that all transcendental numbers that are needed for the calculation of the master integrals for three-loop vacuum Feynman diagrams can be obtained by calculating diagrams with an even simpler topology, the topology of spectacles.

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Feynman diagrams belong to the basic objects needed in the study of present phenomenological elementary particle physics. They provide a simple and convenient language for the bookkeeping of perturbative corrections involving many-fold integrals. In order to arrive at physical predictions the many-fold integrals have to be evaluated explicitly. If one is only aiming at practical applications a numerical evaluation of the integrals may be sufficient. However, their analytical evaluation is theoretically more appealing. This line of research has been vigorously pursued during the last few decades. Major breakthroughs were marked by the introduction of the integration-by-parts technique [1] and the algebraic approach [2] which lead to an intensive use of computers for the necessary symbolic calculations. The new techniques allow one to systematically classify the structure of integrands and to undertake the rather massive computations necessary for multi-loop calculations which may include tens of thousands of Feynman diagrams (see e.g. refs. [3,4]). Still the analytical evaluation of the basic prototype integrals remains an art and even has given rise to a new field of research, namely the study of the transcendental structure of the results as well as their connection to topology and number theory (see e.g. refs. [5–7]).

In general, Feynman integrals are not well defined and require regularization. The dimensional regularization scheme is the regularization scheme favoured by most of the practitioners. In contrast to the Pauli-Villars regularization scheme and other subtraction techniques, the dimensional regularization method is the most natural one in the sense that it preserves various features of the diagrams. For example, massless diagrams (i.e. with vanishing bare mass) remain massless under dimensional regularization.

The transcendental structure of massless multi-loop integrals is rather well understood. It is expressible mainly in terms of Riemann's ζ-function, $\zeta(L)$ where L depends on the topology of the diagram and the number of loops. In contrast to this, the massive case is more complicated and has a richer transcendental structure. At present this field is being very actively studied and unites the community of phenomenological physicists and pure mathematicians. Very recently some novel and striking results concerning three-loop vacuum bubbles have been discovered in this field [8].

The integration-by-parts technique within dimensional regularization [1] reduces the calculation of a general three-loop vacuum diagram to several master configurations. This reduction involves only algebraic manipulations and is universal for any given space-time dimension D (see e.g. ref. [2]). A general strategy for reducing all three-loop vacuum diagrams to a finite set of fixed master integrals through recurrence relations was described in ref. [9]. The analytical structure of the remaining unknown master integrals with tetrahedron topology has been identified with the help of ultraprecise numerical methods [8]. These new achievements allowed one to obtain the complete analytical expression for the three-loop ρ-parameter of the Standard Model which was known before only numerically [8,10,11]. The results can be written in terms of a finite set of transcendental numbers called primitives. Which of these primitives enter the final result for a particular diagram depends on how the masses are distributed along the lines of a specific diagram.

The main objects of the calculation in [8] were the finite parts F_i of the tetrahedron diagrams (we adopt the notation of ref. [8]). The diagrams were considered in four-dimensional space-time while the overall ultraviolet divergence appears as a simple pole in $\varepsilon=(4-D)/2$ within dimensional regularization. By extensive use of ultra-high precision numerical calculations it was found that only two new transcendental numbers $U_{3,1}$, $V_{3,1}$ and the square of Clausen's dilogarithm $\mathrm{Cl}_2(\theta)=\mathrm{Im}(\mathrm{Li}_2(e^{i\theta}))$ (for discrete values of its argument) enter the final results. The presence of the square of Clausen's dilogarithm $\mathrm{Cl}_2(\theta)^2$ (being the square of Clausen's dilogarithm appearing already at the two-loop level) had been conjectured on the basis of the assumption that the primitives form an algebra [8]. The quantity $U_{3,1}$ is related to the master integral B_4 found earlier and is expressible through the polylogarithm value $\mathrm{Li}_4(1/2)$ [2]. The quantity $V_{3,1}$ which emerges in the analysis of vacuum diagrams with tetrahedron topology appears to be entirely new [8].

In the present note we discuss some new results obtained in the field of three-loop vacuum bubble diagrams. We show how to identify the above mentioned primitives in the simpler spectacle topology or in the even simpler water melon topology (see e.g. ref. [12]). Our calculations are done within two-dimensional space-time where the ultraviolet divergences are less severe and the integrals and final results are simpler. By keeping masses finite we have good infrared behaviour. Our previous results tell us that the transcendental structure for the water melon topology in two dimensions is the same as in the case of four-dimensional space-time [12,13]. There is strong evidence that this is also true for other topologies and for any even space-time dimensionality.

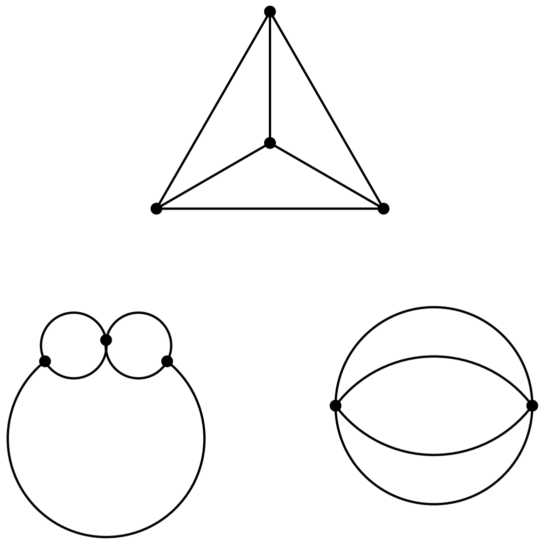

FIG. 1. The three three-loop vacuum bubble topologies that are the subject of this note: The tetrahedron topology (top), the spectacle topology (bottom left), and the water melon topology (bottom right)

Diagrams with tetrahedron topology (Fig. 1) have been analyzed in [8] using arbitrary combination of massless and massive lines albeit with a single mass scale m. Analytical results for all possible mass configurations containing only few transcendental numbers have been obtained with the help of ultra-high precision (thousands decimal points) numerical calculations [8]. The key observation presented in this note is that all necessary transcendental numbers that appear in the tetrahedron case can already be found in the simpler spectacle and water melon topology shown in Fig. 1. These topologies are sufficiently simple to allow one to perform all necessary integrations analytically. We present analytical results for the three-loop spectacle and water melon diagrams, i.e. their transcendentality structure, without ever using any numerical tools.

The main building block for the treatment of three-loop vacuum bubbles is the one-loop two-line massive correlator $\Pi(p^2)$ in $D=2-2\varepsilon$ dimensional (Euclidean) space-time,

$$\begin{aligned}\Pi(p^2) &= \int\frac{d^Dk}{((p-k)^2+m^2)(k^2+m^2)} \\ &= \frac{2^{3+2\varepsilon}\pi^{1-\varepsilon}\Gamma(1+\varepsilon)}{(p^2+4m^2)^{1+\varepsilon}}\ {}_2F_1\left(1+\varepsilon,\frac{1}{2};\frac{3}{2};\frac{p^2}{p^2+4m^2}\right)\end{aligned} \tag{1}$$

with ${}_2F_1(a,b;c;z)$ being the hypergeometric function. An alternative representation of the correlator is obtained through the dispersion relation

$$\Pi(p^2)=\int_{4m^2}^{\infty}\frac{\rho(s)ds}{s+p^2} \tag{2}$$

with

$$\rho(s)=\frac{(s-4m^2)^{\varepsilon}}{2\pi\sqrt{s(s-4m^2)}}\ \frac{\pi^{1/2-\varepsilon}}{2^{4\varepsilon}\Gamma(1/2+\varepsilon)}. \tag{3}$$

In order to reproduce the transcendental structure of the finite parts of the tetrahedron in four dimensions we need a first order ε expansion of water melons and spectacles near two-dimensional space-time. Note that these diagrams are well-defined and ultraviolet finite in two dimensions and, formally, require no regularization. However, the sought-for transcendental structure appears only in higher orders of the ε expansion while the leading order is simple and contains only the standard transcendental numbers such as $\zeta(3)$ or $\ln(2)$. Therefore we write

$$\Pi(p^2)=\Pi_0(p^2)+\varepsilon\Pi_1(p^2)+O(\varepsilon^2) \tag{4}$$

and keep only the first order in ε which happens to be sufficient for our goal of finding all the transcendental numbers appearing in the tetrahedron case. Using either the explicit formula in Eq. (1) or the dispersive representation Eq. (2) with the spectral density given by Eq. (3) and expanded to the first order in ε, we find

$$\Pi_0(4m^2\sinh^2(\eta/2))=\frac{\eta}{4\pi m^2\sinh\eta} \tag{5}$$

where the variable η has been introduced for convenience, $\sqrt{p^2}=2m\sinh(\eta/2)$. In the first order of the ε expansion we have

$$\Pi_1(4m^2\sinh^2(\eta/2))=\frac{f(e^{-\eta})}{4\pi m^2\sinh\eta} \tag{6}$$

with

$$\begin{aligned}f(t)&=2\mathrm{Li}_2(-t)+2\ln t\ln(1+t)-\frac{1}{2}\ln^2(t)+\zeta(2)\\ &=2\int_0^t\frac{\ln u}{1+u}du-\frac{1}{2}\ln^2 t+\zeta(2).\end{aligned} \tag{7}$$

The integral for the water melon diagram is given by

$$W = 2\pi m^2 \int \Pi(p^2)^2 d^2p = W_0 + \varepsilon W_1 + O(\varepsilon^2). \quad (8)$$

Note that here and later we use a two-dimensional integration measure. This prescription differs from the standard dimensional regularization but suffices for our purposes and makes the final expressions simpler. The use of a D-dimensional integration measure would not change the functional structure of the integrands and would simply generate some additional terms that can be analyzed within the same technique. Upon expanding the integral in Eq. (8) in powers of ε we obtain the leading term

$$W_0 = 2\pi m^2 \int \Pi_0(p^2)^2 d^2p = \frac{7}{8}\zeta(3) \quad (9)$$

and the first order term

$$W_1 = 4\pi m^2 \int \Pi_0(p^2)\Pi_1(p^2) d^2p = \int_0^1 \frac{2\ln t}{1-t^2} f(t)dt. \quad (10)$$

The spectacle diagram is given by the integral

$$S = 2\pi m^4 \int \frac{\Pi(p^2)^2}{p^2+M^2} d^2p = S_0 + \varepsilon S_1 + O(\varepsilon^2), \quad (11)$$

where the single "frame" propagator has a mass M which differs from the other mass parameter m in the "rim" propagators. The expression for the leading order term S_0 is simple (see ref. [12]) while the first order term S_1 (which is of interest for us here) reads

$$\begin{aligned} S_1 &= \int_0^1 \frac{2t\, f(t) \ln t\, dt}{(1-t^2)(t^2 - 2t\cos\theta + 1)} \\ &= \int_0^1 \frac{2t\, f(t) \ln t\, dt}{(1-t^2)(\lambda_0 - t)(\bar\lambda_0 - t)} \end{aligned} \quad (12)$$

where $\lambda_0 = e^{i\theta}$, $\cos\theta = 1 - M^2/2m^2$.

By partial fractioning the rational expressions in the integrands of Eqs. (10) and (12) we find that the most complicated integral in both cases has the form

$$\int_0^1 \frac{\ln t}{\bar\lambda - t} f(t)dt = 2I(\lambda) + 3\mathrm{Li}_4(\lambda) - \zeta(2)\mathrm{Li}_2(\lambda) \quad (13)$$

where $I(\lambda)$ is a generic nonreducible term which cannot be expressed with the help of the transcendentality structure that occured earlier on. One has

$$I(\lambda) = \int_0^1 dt \frac{\ln t}{\bar\lambda - t} \int_0^t du \frac{\ln u}{1+u} \quad (14)$$

while the integral of the last two terms in Eq. (7) is explicitly expressed through polylogarithms Li_2 and Li_4. For the relevant values of λ this generic integral in Eq. (14) contains all the primitives $U_{3,1}$, $V_{3,1}$ and also Clausen's polylogarithms Cl_2 and Cl_4.

The value $\lambda = 1$ occurs in both water melon and spectacle cases. For this value we obtain

$$I(1) = \frac{17\pi^4}{1440} + 2U_{3,1} \quad (15)$$

with the explicit expression for the primitive $U_{3,1}$ [8]

$$U_{3,1} = \frac{1}{2}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2(2) - \frac{1}{12}\ln^4(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

where $\zeta(4) = \pi^4/90$ and $\zeta(2) = \pi^2/6$.

The part present in the spectacle diagram in Eqs. (11) and (12) depends on the mass ratio. For $M = m$ we have $\theta = \pi/3$, so $\lambda_0 = e^{i\pi/3}$ is one of the sixth order roots of unity. This observation discloses the special role of the sixth order roots of unity which had been observed before in ref. [8]. For $\lambda = e^{i\pi/3}$ we obtain

$$\begin{aligned} I(e^{i\pi/3}) &= \frac{197\pi^4}{38880} - \frac{1}{3}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + 2V_{3,1} + \frac{5i\pi^3}{162}\ln 3 \\ &+ \frac{13}{108} i\pi^2 \mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{35i}{18}\mathrm{Cl}_4\left(\frac{\pi}{3}\right). \end{aligned} \quad (16)$$

The primitive $V_{3,1}$ is given by

$$V_{3,1} = \sum_{m>n>0} (-1)^m \cos\left(\frac{2\pi n}{3}\right) \frac{1}{m^3 n}. \quad (17)$$

In the case $M = \sqrt{3}m$ we end up with $\lambda_0 = e^{2i\pi/3}$, another sixth order root of unity. For this value of λ we obtain

$$\begin{aligned} I(e^{2i\pi/3}) &= -\frac{79\pi^4}{12960} + \frac{1}{3}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) \\ &+ \frac{7i\pi^2}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{11i}{6}\mathrm{Cl}_4\left(\frac{\pi}{3}\right). \end{aligned} \quad (18)$$

This expression is simpler and does not contain the new primitive $V_{3,1}$.

The case $M = 2m$ is a degenerate one, $\lambda_0 = e^{i\pi} = -1$ and the expression for I reduces to ζ-functions only,

$$I(-1) = -\frac{\pi^4}{288} = -\frac{5}{16}\zeta(4). \quad (19)$$

Finally, the complete expression for the spectacle (or its most interesting part S_1) can be easily found by collecting the above results. For the standard arrangement of masses $M = m$ we have

$$S_1 = -\frac{251\pi^4}{58320} + 4U_{3,1} - \frac{16}{3}V_{3,1} + \frac{8}{9}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right). \quad (20)$$

We have thus fulfilled our promise and have discovered all magic numbers $U_{3,1}$, $V_{3,1}$ and $\mathrm{Cl}_2^2(\pi/3)$ which one encounters in the evaluation of tetrahedron vacuum diagrams using different combinations of massless and massive lines with a single mass scale m. We have found the magic numbers in the simpler three-loop water melon and spectacle topologies. The reason for this accomplishment can be read off from our analytical expressions: all integrals (or the functional structures of integrands) which appeared in the calculation of the finite parts of the tetrahedron diagrams are contained in our basic object $I(\lambda)$. We see the origin of the important role played by the sixth order root of unity: for a single mass m this number appears as a root of the denominator in the spectacle diagram. Within the analysis presented in ref. [8] the special role of this sixth order root of unity had no rational explanation. Having discovered this, we extended the analysis to arbitrary values of λ in Eqs. (12), (13) and (14) by introducing a second mass parameter M in the spectacle diagram given by Eq. (11). Concerning possible future extensions of our approach we emphasize that the appearance of the relevant transcendental structure at the level of the simpler topologies is quite an essential simplifying feature if one wants to proceed to even higher-loop calculations. For example, the simplicity of the water melon topology makes them computable with any number of loops [13]. In this sense the present calculations can be considered to be a first step towards the evaluation of four-loop vacuum bubbles. Our result finally leads us to a conjecture about the calculability of the three-loop master integrals. If they are reducible to spectacle and water melon diagrams at the analytical level, this observation may lead to a way beyond the time-consuming integration-by-parts technique for the evaluation of three-loop diagrams.

To conclude, by analyzing massive three-loop vacuum bubbles belonging to the spectacle topology class within dimensional regularization for two-dimensional space-time, we discovered and identified analytically all transcendental numbers which were previously found by numerical methods for the three-loop tetrahedron topology.

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