Perturbed Conformal Field Theory: A Tool for Investigating Integrable Models

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Abstract. Zamolodchikov's formalism of perturbed conformal field theory is reviewed, and its applications to the analysis of new families of integrable models are discussed. We examine the integrability properties of a class of generalized sine-Gordon models.

INTRODUCTION

Until the relatevily recent developments in the study of supersymmetric gauge theories [1], which have made it possible to obtain exact information about four-dimensional quantum field theories (QFTs)[1], the study of QFTs in two dimensions was the only way to gain an exact understanding of their structure. For many of these models it has been possible to compute their full S-matrix, thereby determining completely the spectrum of the theory. Progress in the computation of the exact correlation functions has also been made[2].

In recent years many two-dimensional integrable models have found an important application in the description of condensed matter systems (see [4] and references therein.)

The field of integrable models also displays a wealth of sofisticated mathematical structures—*e.g.* infinite dimensional Lie algebras and quantum groups.

For all these reasons the open problem of a classification of two-dimensional integrable models is very important, both for its theoretical and practical aspects. In this talk I wish to review an important approach to the problem, due to A.B. Zamolodchikov [5], where massive models are regarded as perturbations of a conformal field theory (CFT). I will also very briefly mention current applications of the method.

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1) For a review, see [2].
2) For a review, see [3].

CONFORMAL FIELD THEORY

A physical system at a second order phase transition can be described in terms of a conformally invariant quantum field theory, which can be regarded as a fixed point of the renormalization group transformations (for a review and a rich set of references see for instance [6].)

In dimension higher than two the group of conformal transformations is finite dimensional, so that the presence of conformal symmetry is not enough for a complete characterization of a QFT. In two dimensions the conformal group is infinite dimensional (any analytic function generates a local conformal transformation); this fact powerfully constrains the structure of a two-dimensional CFT, to allow for a complete classification based on the representation theory of the conformal algebra.

In this section we recall some basic notions of CFT in two dimensions, mainly with the purpose of establishing our notation[3]. We will work with complex coordinates

$$z = x + iy, \quad \overline{z} = x - iy. \tag{1}$$

As a consequence of translation and scale invariance, the complex components of the energy-momentum tensor

$$T \equiv T_{zz}, \quad \overline{T} \equiv T_{\overline{z}\overline{z}}, \quad \Theta \equiv 4T_{z\overline{z}}, \tag{2}$$

will satisfy the following relations:

$$\Theta = 0, \quad \partial_{\overline{z}} T = 0, \quad \partial_z \overline{T} = 0. \tag{3}$$

Eq. (3) implies that T is an analytic function of the coordinates, while $\overline{T}$ is antianalytic. One can therefore define a Laurent mode expansion for the energy-momentum tensor:

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}. \tag{4}$$

From the operator product expansion (OPE) of T with itself

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \tag{5}$$

one can easily show that the Laurent modes L_n in (4) satisfy the following algebra, known as the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n-1)\delta_{n+m,0}. \tag{6}$$

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3) For a comprehensive set of reprints see [7].

The constant c in (5) and (6) is known as the central charge, and can be considered the defining parameter of the CFT. Relations similar to (4)–(6) hold for the antianalytic component $\overline{T}$.

A field $\Phi(z,\overline{z})$ that under the conformal transformation $z \to f(z)$, $\overline{z} \to \overline{f}(\overline{z})$ changes according to the rule

$$\Phi(z,\overline{z}) \to \left(\frac{df}{dz}\right)^{\Delta} \left(\frac{d\overline{f}}{d\overline{z}}\right)^{\overline{\Delta}} \Phi\left(f(z), \overline{f}(\overline{z})\right) \tag{7}$$

is called a primary field of conformal weight $(\Delta, \overline{\Delta})$; the spin of the field and its scaling dimension are therefore $(\Delta - \overline{\Delta})$ and $(\Delta + \overline{\Delta})$.

The OPE of a primary field Φ of weight $(\Delta, \overline{\Delta})$ with the energy-momentum tensor is

$$T(z)\Phi(w) = \frac{\Delta}{(z-w)^2}\Phi(w) + \frac{\partial_w \Phi}{z-w}. \tag{8}$$

As already mentioned, it is possible to classify the operator content of a conformal field theory according to the irreducible representations of the Virasoro algebra, with the primary fields playing the role of the highest weight vectors. An important class of conformal fields theories is formed by the so-called minimal models, for which the operator algebra (the set of all OPEs) closes on a finite number of primary fields. The central charge of a given unitary minimal model is given by

$$c = 1 - \frac{6}{p(p+1)}, \quad p = 1, 2, \ldots\ . \tag{9}$$

INTEGRALS OF MOTION IN CFT

One can easily prove the existence of an infinite number of conserved currents in a given CFT. The action of the Virasoro modes L_n on the space of local fields can be naturally defined:

$$\left(L_n\Phi\right)(z,\overline{z}) = \oint_z d\zeta (\zeta - z)^{n+1} T(\zeta)\Phi(z,\overline{z}) \tag{10}$$

If we act with L_n on the identity operator $\mathbf{I}$ we generate an infinite tower of holomorphic fields: using (10) one finds

$$(L_{-2}\mathbf{I}) = T(z), \tag{11}$$

$$\vdots \tag{12}$$

$$(L_{-n}\mathbf{I}) = \frac{1}{(n-2)!}\partial_z^{n-2} T(z), \tag{13}$$

$$(L_{-2}L_{-2}\mathbf{I}) = : T^2(z) : . \tag{14}$$

We see that all the currents so defined are holomorphic, and therefore satisfy a conservation equation of the type

$$\partial_{\overline{z}}\left(L_{-m}\ldots L_{-n}\mathbf{I}\right)(z)=0. \tag{15}$$

We conclude that in a CFT there exists an infinite number of conserved currents. In the rest of the talk we will generically label them by T_s, where s refers to the spin of the current.

We are now ready to address the central question: If a perturbation that spoils the conformal invariance is added to the theory, will these conservation laws remain valid? The question is of crucial importance; as discussed in the talk by B. Gerganov at this conference, the existence of an infinite number of conservation laws is enough to establish the integrability of the model.

PERTURBATIONS OF A CFT

Let us consider a theory defined by the action

$$S_{\text{pert}}=S_{\text{CFT}}+\lambda_r\int\Phi_r(x,y)d^2x, \tag{16}$$

where S_{CFT} is the (formal) action of a conformal field theory, and Φ_r is a relevant field of conformal dimension (Δ_r,Δ_r); the dimension of λ_r therefore is $(1-\Delta_r,1-\Delta_r)$.

Let us consider one of the conserved currents T_s of spin s of the given CFT. In the perturbed theory T_s will no longer be holomorphic; rather it will satisfy an equation of the type

$$\partial_{\overline{z}}T_s(z,\overline{z})=\lambda_r R^{(1)}_{s-1}+\ldots+\lambda_r^n R^{(n)}_{s-1}+\ldots\ . \tag{17}$$

For dimensional consistency the dimension of each of the operators $R^{(n)}_{s-1}$ must be $(s-n\epsilon,1-n\epsilon)$, where $\epsilon\equiv 1-\Delta_r$. Assuming that the dimension of the fields in the perturbed theory (16) remains identical to the dimension of the corresponding fields in the CFT[4], we can immediately conclude that, since there are no fields of negative dimension, the series (17) is actually finite. Moreover, fore some specific choice of the perturbing operator Φ_r, only the first term can arise:

$$\partial_{\overline{z}}T_s=\lambda_r R^{(1)}_{s-1}. \tag{18}$$

The operator $R^{(1)}_{s-1}$ appearing in (18) can be calculated from the OPE in the conformal theory between T_s and the perturbation Φ_r; indeed one has [5]

$$\partial_{\overline{z}}T_s=\lambda_r\oint_z\frac{d\zeta}{2\pi i}\Phi_r(\zeta,\overline{z})T_s(z). \tag{19}$$

One can prove, for the sake of illustration, the following facts:

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4) For a discussion of this subtle point, see the original paper [5].

1. The energy-momentum tensor is conserved for any choice of the perturbation. One has

$$\partial_{\bar{z}} T = \partial_z \Theta, \tag{20}$$

where $\Theta = \lambda_r(\Delta_r - 1)\Phi_r$.

2. For the current T_4, one in general has

$$\partial_{\bar{z}} T_4 = \lambda_r \left(\Delta_r - 1\right) \left(2L_{-2}L_{-1} + \frac{\Delta_r - 3}{6} L_{-1}^3\right) \Phi_r. \tag{21}$$

Therefore, as we see from (21), in general the conservation law is spoiled, since the r.h.s. cannot be written as a total ∂_z derivative.

For some choice of the perturbing operator though, it is possible to write the r.h.s. of (18) as a ∂_z derivative:

$$\partial_{\bar{z}} T_s = \partial_z \Theta_{s-2}, \tag{22}$$

which implies that the conservation law, albeit modified, survives the addition of a perturbation, so that the resulting massive field theory remains integrable.

Prototypical cases are the minimal models S_p perturbed by any one of the operators[5] $\Phi_{1,3}$, $\Phi_{1,2}$, $\Phi_{2,1}$:

$$S_p^{(1,3)} = S_p + \lambda \int \Phi_{1,3} d^2x, \tag{23}$$
$$S_p^{(1,2)} = S_p + \lambda \int \Phi_{1,2} d^2x, \tag{24}$$
$$S_p^{(2,1)} = S_p + \lambda \int \Phi_{2,1} d^2x, \tag{25}$$

for which an infinite number of conserved currents exist, with spins given by

$$s = 2n - 1, \quad n = 1, 2, \ldots \quad \text{for} \quad S_p^{(1,3)}, \tag{26}$$

and

$$s = 1, 6n \pm 1, \quad n = 1, 2, \ldots \quad \text{for} \quad S_p^{(1,2)},\ S_p^{(2,1)}. \tag{27}$$

[5] The indices refer to the standard labeling of the Kac table (see [6]).

FURTHER APPLICATIONS

The importance of the formalism of perturbed CFT lies on the fact that the simple formula (19) can be applied to test for the presence of conserved currents in more general cases. Recently it has allowed for the construction of conservation laws (both local and non-local) in multi-field generalizations of the sine-Gordon model [4], [8]:

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2} (\partial_\mu \phi_i)^2 + \lambda \cos(\beta_1 \phi_1) \cos(\beta_2 \phi_2), \tag{28}$$

and

$$\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2} (\partial_\mu \phi_i)^2 + \lambda \left[\cos(\alpha_1 \phi_1 + \alpha_2 \phi_2) e^{\beta \phi_3} + \cos(\alpha_1 \phi_1 - \alpha_2 \phi_2) e^{-\beta \phi_3} \right], \tag{29}$$

provided that the couplings are constrained as follows:

$$\beta_1^2 + \beta_2^2 = 4\pi, \quad \text{or} \quad 8\pi, \tag{30}$$

$$\alpha_1^2 + \alpha_2^2 - \beta^2 = 4\pi, \tag{31}$$

for the models (28), (29) respectively.

An extensive investigation of further generalizations of this class of models is the goal of our current investigations [9].

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