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Integrability in Classical and Quantum Field Theory and the Bukhvostov-Lipatov Model[1]

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Abstract. A brief overview of the notions of classical and quantum integrability in $1+1$ space-time dimensions is given with an emphasis on the existence of infinite number of conserved quantities in $(1+1)D$ field theories. Some methods for building such conserved quantities are outlined and are then applied to study the integrability of the Bukhvostov-Lipatov model, a 2-field generalization of the sine-Gordon/massive Thirring model. The integrability properties of both the classical/quantum and the bosonic/fermionic versions of the model are discussed. It is shown that the *classical fermionic* model possesses a conserved current of Lorentz spin 3. It is then established that the conservation law is spoiled at the quantum level – a fact that indicates that the *quantum* Bukhvostov-Lipatov model is not integrable.

I INTEGRABILITY IN $1+1$ SPACE-TIME DIMENSIONS

A Classical Integrability

Let the time evolution of a physical system be described by some (generally non-linear) partial differential equations. The system is said to be *classically integrable* if there exist *exact integrals* to its *classical equations of evolution*.

The pioneering work in the field of classically integrable non-linear systems was a 1967 paper by Gardner, Greene, Kruskal, and Miura [1]. The authors were able to introduce a non-linear change of variables that makes the Korteweg-de Vries equation linear and exactly solvable. In 1968 Lax generalized their work by

[1]) Talk given at MRST XX, McGill University, Montréal, May 13–15, 1998.

developing a general procedure [2] for associating to a system of *non-linear* PDEs a *linear operator*, whose *eigenvalues* are *integrals* of the non-linear system.

In the following decades exact solutions to a number of non-linear classical systems have been found and the importance of the field has grown for several reasons:

The evolution of some very important *physical systems* is governed by non-linear equations that admit exact solutions. Their range of *applicability* includes fluid dynamics, non-linear optics, and condensed matter theory.

Most known classically integrable systems exhibit the presence of "particle-like" solutions to the non-linear equations of motion – solitary waves or *solitons*. Their properties present interest to many areas of physics, both applicable, such as optic fiber design, and purely theoretical, such as string theory.

The solutions of some integrable systems can be used as 0^{th}-order approximations in perturbation theory to study non-integrable systems. Such refined perturbative techniques often are more powerful than conventional perturbation theory.

Some examples of integrable non-linear partial differential equations (PDEs) are:

– Korteweg-de Vries equation (KdV): $\phi_t + 6\phi\phi_x + \phi_{xxx} = 0$, [1, 2].

– Non-linear Schödinger equation: $i\phi_t = -\phi_{xx} + |\phi|^2\phi$, [1, 2].

– sine-Gordon equation (sG): $\phi_{tt} - \phi_{xx} = \sin\phi$, [3].

A comprehensive introduction to the subject of classical integrability and the theory of solitons can be found in [4].

B Quantum Integrability in $1 + 1$ Space-Time Dimensions

Because of the specific (*strongly restrictive*) features of particle kinematics in $(1 + 1)$-dimensional space-time, there exist scattering theories which allow *precise computation* of all elements of the *exact* particle S-matrix. This is possible because the following 2 (related) properties, each of which can be taken as a *definition of quantum integrability* in $(1 + 1)D$:

Definition A. A QFT in 2 space-time dimensions is *integrable* if there exists an *infinite set* of independent, mutually commuting *integrals of motion*

$$[Q_r, H] = 0\ ,\quad [Q_r, Q_{r'}] = 0\ ,\quad r = 1, 2, 3, \ldots\ . \tag{I.1}$$

Definition B. A QFT in 2 space-time dimensions is *integrable* if the N-paticle S-matrix can be *factorized* into a product of $\frac{N(N-1)}{2}$ 2-particle S-matrices, as if the process of N-particle scattering were reduced to a *sequence of pair collisions*; thus, allowing all the matrix elements of S to be determined exactly.

It can be shown that **Property B** follows from **Property A**. Indeed, in $1+1$ dimensions the *only symmetry transformations* on the space-time are the *translations* and the *Lorentz boosts* That's why, in the limits $t \to \pm\infty$, any conserved tensorial quantity of rank r reduces to a polynomial in the asymptotic particle 2-momenta. In a diagonal basis this polynomial could be written as

$$\hat{Q}_r = \sum_{a \in IN} \hat{p}_a^r = \sum_{b \in OUT} \hat{p}_b^r \ . \tag{I.2}$$

The last result is important because it has 2 consequences:

1. **No particle production (the S-matrix is *elastic*).**
   Let $|p_1, p_2, \ldots, p_a, \ldots, p_{N_a}\rangle_{IN}$ and $\left|p'_1, p'_2, \ldots, p'_b, \ldots, p'_{N_b}\right\rangle_{OUT}$ be asymptotic particle states at $t \to \pm\infty$. If we act on these states with the conserved charge operator Q_r, we obtain, according to (I.2), that

   $$\sum_{a=1}^{N_a} p_a^r = \sum_{b=1}^{N_b} p_b^r \ . \tag{I.3}$$

   Since there are infinite number of such conserved charges in the theory, the particle 2-momenta sholud satisfy *infinite number of conservation laws* like (I.3) with $r = 1, 2, 3, \ldots$. This is only possible if the *set of the ingoing 2-momenta is equal to the set of outgoing 2-momenta*:

   $$\{p_1, p_2, \ldots, p_a, \ldots, p_{N_a}\}_{IN} = \left\{p'_1, p'_2, \ldots, p'_b, \ldots, p'_{N_b}\right\}_{OUT} , \quad N_a = N_b = N \ .$$

   This implies, in particular, that the total particle number and the number of particles of given mass, $m^2 = p_\mu p^\mu$, is conserved. Therefore,

   $$S_{fi} = {}_{OUT}\langle p'_1, p'_2, \ldots, p'_b, \ldots, p'_N | p_1, p_2, \ldots, p_a, \ldots, p_N\rangle_{IN} \ \text{ is } \textit{elastic}.$$

2. **Multiparticle collisions reduce to a sequence of pair collisions (the S-matrix is *factorizable*).**
   Since there exists an infinite set of integrals of motion in our theory, it is natural to assume that there is a *particular symmetry transformation* that reduces at $t \to \pm\infty$ to *independent translations* of the individual particles. By performing such translations on can get *arbitrarily large space-time separations* of all the regions where pair collisions occur. Therefore, the pair collisioins can be treated as independent. The existence of infinite number of symmetries further implies that the time order of the collisions is irrelevant and, therefore, different permutations of the 2-particle S-matrices in the factorized product will give rise to the same multiparticle S-matrix (*Yang-Baxter equations*).

The above 2 properties, together with the Yang-Baxter equations and the requirements for *unitarity*, *analyticity*, and *crossing symmetry*, for many $(1+1)D$ QFTs provide enough constraints to compute the S-matrix *exactly*.

A more extensive review of the properties of the S-matrix in $(1+1)D$ can be found in [5].

II METHODS FOR BUILDING INTEGRALS OF MOTION OF HIGHER TENSORIAL RANK

A Classical Integrals of Motion

In a remarkable paper [2] Lax formulated a powerful technique for integrating exactly systems of non-linear evolution equations. Let

$$u_t = K(u) \tag{II.1}$$

be some non-linear PDE and let $u(x,t)$ be a solution to (II.1). Lax showed that if we can associate some *selfadjoint* linear differential operator $\hat{L}_u$ with the function u, such that $\hat{L}_u$ is *unitary invariant* under the evolution of u according to (II.1), then the *eigenvalues of* $\hat{L}_u$ *are integrals of motion* of (II.1).

If we find the operator $\hat{L}_u$ for a certain PDE, it generates an *infinite series of integrable PDEs*. The higher order PDEs in the series can be used to build integrals of motion of higher rank and, thus, to construct and *infinite series of conservation laws* of the classical system. Using this and a variety of other methods, infinite sets of conserved quantities of increasing spin have been constructed for the classical MT, Korteweg-de Vries (KdV), and sG models [1–3, 6, 7].

B Quantum Integrals of Motion

A powerful tool for building conserved quantities for $2D$ *quantum* models is the technique of *perturbed conformal field theory* [8]. By treating a $2D$ QFT as a perturbed CFT, one can make use of conformal invariance to find an infinite number of conserved quantities in the conformal theory. It is then possible to study which (if any) of these conservation laws survive the perturbation. Zamolodchikov's paper [8] provides an easy way of computing the conserved current densities explicitly.

In CFT any conserved density T_{s+1} is a holomorphic function and $\partial_{\bar{z}} T_{s+1} = 0$. In the perturbed QFT that is no longer true and we can compute $\partial_{\bar{z}} T_{s+1}$, using Zamolodchikov's formula [8, eq.(3.14)]:

$$\partial_{\bar{z}} T_{s+1} = \lambda \oint_z \frac{d\zeta}{2\pi i} : \Phi(\zeta, \bar{z}) :: T_{s+1}(z) : \tag{II.2}$$

If the RHS of (II.2) can be expressed as a total ∂_z-derivative of some local operator Θ_{s-1}, there exists a spin s conservation law surviving in the perturbed QFT:

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}, \qquad \partial_z \overline{T}_{s+1} = \partial_{\bar{z}} \overline{\Theta}_{s-1} \ , \tag{II.3}$$

T_{s+1} and Θ_{s-1} being the quantum conserved densities.

III THE BUKHVOSTOV-LIPATOV MODEL

The bosonic Bukhvostov-Lipatov model (BL) is a 2-field generalization of the sine-Gordon model (sG)[2]. It was first introduced by Bukhvostov and Lipatov [10] in a study of the $O(3)$-nonlinear σ-model and has drawn recent attention in works by Fateev [11] and Lesage, Saleur, Simonetti [12]. The model is defined by the action

$$S = \frac{1}{4\pi} \int dtdx \left[\frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \lambda : \cos \hat{\beta}_1 \phi_1 :: \cos \hat{\beta}_2 \phi_2 : \right] , \tag{III.1}$$

where $\hat{\beta}_i \equiv \frac{\beta_i}{\sqrt{4\pi}}$, $i = 1, 2$. Using the and Coleman's bosonization prescription [13], Bukhvostov and Lipatov mapped[3] the quantum bosonic theory (III.1) to a dual theory of two Dirac fermions[4]:

$$\begin{aligned} S_{\text{full}} = \frac{1}{4\pi} \int dtdx \Big\{ & \frac{i}{2} \Psi^{\mathcal{D}} \gamma^\mu \partial_\mu \Psi - \frac{i}{2} \partial_\mu \Psi^{\mathcal{D}} \gamma^\mu \Psi + \frac{i}{2} X^{\mathcal{D}} \gamma^\mu \partial_\mu X - \frac{i}{2} \partial_\mu X^{\mathcal{D}} \gamma^\mu X \\ & - m \left(\Psi^{\mathcal{D}} \Psi + X^{\mathcal{D}} X \right) - g \left(\Psi^{\mathcal{D}} \gamma^\mu \Psi \right) \left(X^{\mathcal{D}} \gamma^\mu X \right) - g' \left[\left(\Psi^{\mathcal{D}} \gamma^\mu \Psi \right)^2 + \left(X^{\mathcal{D}} \gamma_\mu X \right)^2 \right] \Big\} . \end{aligned} \tag{III.2}$$

where

$$g = \pi^2 \left(\frac{1}{\hat{\beta}_1^2} - \frac{1}{\hat{\beta}_2^2} \right) , \qquad g' = \frac{\pi^2}{2} \left(\frac{1}{\hat{\beta}_1^2} + \frac{1}{\hat{\beta}_2^2} - 4 \right) . \tag{III.3}$$

The above model is obviously a 2-field generalization of the massive Thirring model (MTM). It involves 2 types of 4-fermion interactions: the interaction of the MT model with coupling g' and a new interaction term with coupling g.

In [10] Bukhvostov and Lipatov studied the integrability of the model (III.2), using Bethe ansatz [14]. They constructed the S-matrix for the *pseudoparticles* and showed that it satisfies Yang-Baxter equations in 2 cases:

$$1.\ \ g = 0 \ , \qquad\qquad g' : \text{unrestricted} \ , \tag{III.4}$$

$$2.\ \ g' = 0 \ , \qquad\qquad g : \text{unrestricted} \ . \tag{III.5}$$

In Case 1 (III.2) reduces to two decoupled copies of the MT model, known to be integrable both classically and quantum mechanically [6,15]. In Case 2 one obtains a new model, to which we will refer as *"the fermionic BL model"*:

$$S_{BL} = S_{\text{full}} \big|_{g'=0} \ . \tag{III.6}$$

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2) Generalizations of the sine-Gordon model have been studied as early as 1981 [9].

3) After a rotation of the boson fields: $\phi_1 = \frac{1}{2\hat{\beta}_1} (\vartheta_1 + \vartheta_2)$ and $\phi_2 = \frac{1}{2\hat{\beta}_2} (\vartheta_1 - \vartheta_2)$.

4) Here $\Psi^{\mathcal{D}}$ denotes the *Dirac conjugate* of Ψ, $\Psi^{\mathcal{D}} \equiv \Psi^\dagger \gamma^0$. We use this unconventional notation rather than the usual "$\overline{\Psi}$" to avoid confusion with the *antichiral spinor component*, $\overline{\psi}$.

Based on their result for the pseudoparticle S-matrix, Bukhvostov and Lipatov conjectured that the quantum field theory (III.6) is integrable. The actual *physical* states, however, have *not* been constucted in their paper. To the best of our knowledge, computing the physical S-matrix and, thus carrying out Bethe ansatz for the model consistently to the end, still remains an open problem.

The **classical bosonic model** (III.1) was studied by Ameduri and Efthimiou. Using *Painlevé analysis*, they proved [16] that the model is *not* classically integrable. In the following sections we discuss in some details the integrability of the **classical fermionic model** (III.6) and the **quantum integrability** of the models (III.1) and (III.2), dual to each other in the quantum case. We focus, in particular, on constructing conserved quantities of higher tensorial rank in these theories.

A Classical Fermionic Model

In Eucledian space-time, parametrized by the light cone coordinates $z = \frac{1}{2}(\tau + ix)$, $\bar{z} = \frac{1}{2}(\tau - ix)$, and in terms of spinor components[5], the action (III.6) becomes

$$S_E = \frac{1}{4\pi}\int d\tau dx \left[\frac{1}{2}\left(\psi_+\partial_{\bar{z}}\psi_- + \psi_-\partial_{\bar{z}}\psi_+ + \overline{\psi}_+\partial_z\overline{\psi}_- + \overline{\psi}_+\partial_z\overline{\psi}_-\right) + \right.$$
$$\left. - im\left(\overline{\psi}_+\psi_- + \overline{\psi}_-\psi_+\right) - g\left(\psi_+\psi_-\overline{\chi}_+\overline{\chi}_- + \overline{\psi}_+\overline{\psi}_-\chi_+\chi_-\right) + (\psi \leftrightarrow \chi)\right] , \quad \text{(III.7)}$$

which gives rise to the *classical* equations of motion

$$\partial_{\bar{z}}\psi_\pm(z,\bar{z}) = -im\overline{\psi}_\pm \pm 2g\psi_\pm\overline{\chi}_+\overline{\chi}_- , \qquad \partial_z\overline{\psi}_\pm(z,\bar{z}) = im\psi_\pm \pm 2g\overline{\psi}_\pm\chi_+\chi_- \quad \text{(III.8)}$$

and the equations obtained with the substitutiuon $\psi \leftrightarrow \chi$.

Because of the space and time translational invariance, the theory possesses a trivial conserved charge of spin 1 — the energy-momentum tensor. Finding conserved quantitites of higher Lorentz spin is equivalent to finding non-trivial integrals of motion and is considered to be a strong indication for the classical integrability of the theory. We have been able to show [17] that the fermionic BL model (III.7) has a classically conserved charge of spin 3 in the bulk:

$$Q_3 = \int_{-\infty}^{+\infty} dx\,(T_4 - \Theta_2) \quad \text{and} \quad \overline{Q}_3 = \int_{-\infty}^{+\infty} dx\,\left(\overline{T}_4 - \overline{\Theta}_2\right) , \qquad \text{(III.9)}$$

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[5] We use the following conventions:

$$\Psi \to \begin{pmatrix}\psi_- \\ \overline{\psi}_-\end{pmatrix},\ \Psi^\dagger \to \left(\psi_+\ \overline{\psi}_+\right);\ \ \Psi^{\mathcal{D}} \equiv \Psi^\dagger\gamma^0;\ \ \gamma_0 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} ,\ \ \gamma_1 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

and, similarly, for X. The components of the metric tensor are $[g_{\mu\nu}] = [g^{\mu\nu}] = \mathrm{diag}\,[1, -1]$.

where the densities T_4 and Θ_2 are given by:

$$T_4 = \psi_+\partial_z^3\psi_- + 6g\partial_z\psi_+\partial_z\psi_-\chi_+\chi_- + \ (+\leftrightarrow -) + \ (\psi \leftrightarrow \chi) \ , \tag{III.10}$$

$$\begin{aligned}\Theta_2 &= im\overline{\psi}_+\partial_z^2\psi_- + 2mg\overline{\psi}_+\partial_z\psi_-\chi_+\chi_- + 4mg\overline{\psi}_+\psi_-\chi_+\partial_z\chi_- + \\ &+ 4mg\overline{\psi}_+\psi_-\partial_z\chi_+\chi_- - 4im^2g\psi_+\psi_-\chi_+\chi_- + \ (+\leftrightarrow -) + \ (\psi \leftrightarrow \chi) \ .\end{aligned} \tag{III.11}$$

(similarly, for $\overline{T}_4$ and $\overline{\Theta}_2$) and satisfy the continuity equation (II.3) with $s = 3$. The spin 3 charge (III.9) is conserved also in the theory on the half-line for specific types of boundary actions [17]. We intend to generalize our result to conserved quantities of arbitrary spin by using methods similar to the ones described in Refs. [1–3, 6, 7].

We have also found that the *more general* 2-fermion model (III.2) does *not* possess a classically conserved spin 3 current for arbitrary values of the couplings g and g'. Classical conservations laws of spin 3 exist only in the special cases $g = 0$ (2×MTM) and $g' = 0$ (BL model), i. e. the model (III.2) is classically integrable precisely in the cases (III.4) and (III.5), suggested by Bukhvostov and Lipatov.

B Quantum Integrability

In this section we will study the modifications to the classical conservation law due to quantum corrections, using *perturbed conformal field theory*[6]. Boson-fermion duality in 2 space-time dimensions allows us to write our expression for the classically conserved fermionic current (III.10) in bosonic form and then to study its quantum conservation in the model (III.1), treating the term $\lambda : \cos\hat{\beta}_1\phi_1 \cos\hat{\beta}_2\phi_2 :$ as a single relevant perturbation to the conformal theory of massless free bosons.

Using the substitutions $\psi_\pm =:e^{\pm i\vartheta_1}:$, $\chi_\pm =:e^{\pm i\vartheta_2}:$ and the vertex operator product expansions from CFT [18], we can map each term of T_4 to its dual operator in the bosonic theory. Since quantum corrections will, in principle, modify the coefficients of the classical T_4, we prefer to leave them as arbitrary functions of g and fix them later while computing the conserved current via perturbed CFT. Finally, using the relations in Footnote 3, we can write T_4 in terms of the boson fields ϕ_1 and ϕ_2 as

$$\begin{aligned}T_4 = &\sum_{m+n=2} a_{mn} : \left(\partial_z^2\phi_1\right)^m \left(\partial_z^2\phi_2\right)^n : + \sum_{k+l=4} b_{kl} : (\partial_z\phi_1)^k (\partial_z\phi_2)^l : \\ &+ c_1 : (\partial_z\phi_1)^2 \partial_z^2\phi_2 : + c_2 : \partial_z^2\phi_1 (\partial_z\phi_2)^2 : ; \qquad {\scriptstyle m,n=0,1,2;\ k,l=0,1,2,3,4}\ .\end{aligned} \tag{III.12}$$

This is, in fact, the *most general ansatz* for T_4 for the double cosine model. It includes, up to addition of total ∂_z-derivatives, *all* operators of mass dimension 4 with arbitrary coefficients, functions of $(\hat{\beta}_1, \hat{\beta}_2)$ or, via (III.3), of g. As it turns out, the requirement that T_4 be conserved in the perturbed CFT is very restrictive and gives enough information to compute the exact form of these functions.

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6) See Section II.B and Ref. [8].

In starting this calculation, our goal was to find *all* the conditions on the couplings $\hat{\beta}_1$ and $\hat{\beta}_2$ for which the spin 3 charge is conserved. We expected that the 'BL manifold' (III.5) would be one of the integrable cases and then, by fermionizing back, we would be able to obtain an exact quantum expression for the fermionic T_4. At the end, we found that the spin 3 current is conserved only in 3 cases:

$$\hat{\beta}_1^2 - \hat{\beta}_2^2 = 0 \longrightarrow \quad g = 0 \ , \tag{III.13}$$

$$\hat{\beta}_1^2 + \hat{\beta}_2^2 = 1 \longrightarrow \quad \left(\frac{2g'}{\pi^2} + 2\right)^2 - \left(\frac{g}{\pi^2}\right)^2 = 4 \ , \tag{III.14}$$

$$\hat{\beta}_1^2 + \hat{\beta}_2^2 = 2 \longrightarrow \quad \left(\frac{2g'}{\pi^2} + 3\right)^2 - \left(\frac{g}{\pi^2}\right)^2 = 1 \ . \tag{III.15}$$

The first manifold is trivial: when $\hat{\beta}_1^2 = \hat{\beta}_2^2$ the double cosine model decouples into 2 sine-Gordon models, integrable both classically and quantum mechanically. The second two[7] quantum integrable[8] manifolds have been previously identified by Fateev [11] and by Lesage, Saleur, and Simonetti [12]. On the BL manifold (III.5) the charge (III.9) is *not quantum conserved*, except in the trivial case when the manifolds (III.13), (III.5), and (III.14) intersect each other (free fermion point).

The above result becomes even more clear when we look at the fermionic parameter space. As we showed at the end of Section III.B, the general model has a *classically* conserved charge of spin 3 only if either $g = 0$ or $g' = 0$. Therefore, the *classical integrable manifolds* are simply the axes of the (g, g')-plane:

We see that the manifold $g = 0$ is present both in the classical and in the quantum case. This merely reflects the fact that massive Thirring model is integrable both classically and quantum mechanically. In other words, if we start with the classical MT model, turning on quantum corrections will *not* take the classical conservation law away from the vertical axis. The coefficients of T_4 will be modified but they will still remain functions of g' only and will be independent of g.

In contrast, if we start with the classical fermionic BL model, with T_4 depending on g only (III.10), quantum corrections will not only modify the coefficients of the conserved current but will also introduce new terms, involving g'-dependence. Looking at Figure 1, we see that the g-axis, on which the model is integrable classically, is 'deformed' into the hyperbola (III.14) in the quantum case. The only point on the classical manifold where integrability is preserved by quantum mechanics is the origin, which, of course, is simply the the *free point* $(g = g' = 0)$ of the fermionic theory.

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[7] We concentrate here on the first non-trivial manifold, (III.14), since this is the one with physical significance (tunneling in quantum wires) [12].

[8] Another model that is *not integrable classically*, but is *quantum integrable* on certain sub-manifolds of the parameter space has been previously discussed by Shankar [9].

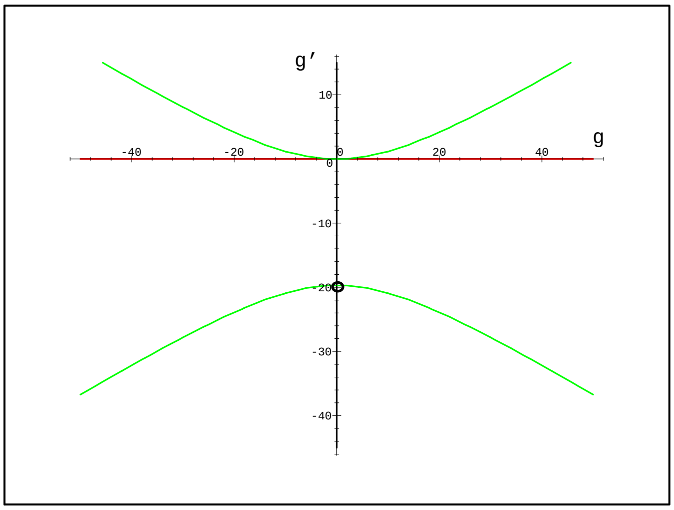

FIGURE 1. Fermionic parameter space.

IV CONCLUSION

We have shown that the fermionic Bukhvostov-Lipatov model, given by the action (III.6), admits a nontrivial classical integral of motion of spin 3. This conservation law holds quantum mechanically only at the free fermion point $g = 0$ and is spoiled by quantum corrections for generic values of the coupling g. The more general fermionic model (III.2) admits a quantum conservation law of spin 3 for the specific relation (III.14) between the couplings g and g'. The study of the spin 3 conservation laws, therefore, suggests that the integrable manifold (g : free, $g' = 0$) proposed by Bukhvostov and Lipatov does not survive in the quantum field theory.

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