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D-particles on $T^4/\mathbf{Z}_n$ orbifolds and their resolutions

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ABSTRACT

We formulate the effective field theory of a D-particle on orbifolds of T^4 by a cyclic group as a gauge theory in a V-bundle over the dual orbifold. We argue that this theory admits Fayet-Iliopoulos terms analogous to those present in the case of noncompact orbifolds. In the $n = 2$ case, we present some evidence that turning on such terms resolves the orbifold singularities and may lead to a $K3$ surface realized as a blow up of the fixed points of the cyclic group action.

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Introduction

The investigation of the effective short-distance geometric description of space-time is one of the most fascinating ways of exploring the fundamental properties of string theory. The first order perturbative analysis at weak string coupling ('stringy geometry') reduces to questions about the moduli space of (a certain class of) conformal field theories. This analysis reveals [1] a rich phase structure of the conformal field theory moduli space, unifying a large class of what were once thought to be unrelated models and allowing for topology changing transitions among them.

The corresponding nonperturbative analysis is still in its infancy, but it has been known for some time [2, 3, 4, 5] that qualitatively new phenomena, involving rather drastic departures from the geometry of the classical moduli space should occur in this regime. Lacking a general nonperturbative formulation of string theory compactifications, the best one can do at this stage is to approach the problem by using nonperturbative objects as probes of the relevant moduli space, much as one uses conformal field theory to probe the pertubative approximation. In this light, a first understanding of how the perturbative picture of [1] is corrected by nonperturbative effects can be gained by considering 'brany-geometry', i.e. the moduli space of various D-branes present in a string-theoretic compactification. To carry out such an investigation one needs a description of the low energy dynamics of D-branes which is reasonably easy to manipulate. In the case of flat space-time, such an effective low energy description is provided [6] by the reduction of the D=10, N=1 supersymmetric Yang Mills theory to the D-brane worldvolume. Unfortunately, the appropriate description in the case of a nontrivial spacetime geometry is far from being understood properly, despite various attempts [7]. Although some tantalizing conjectures have been made by exploiting rather abstract features expected of such a description [8, 9, 10], the rigorous physical formulation of the problem is lacking.

One of the processes of 'brany-geometry' that one would like to investigate is the resolution of orbifold singularities by D-branes. The case of flat *noncompact* cyclic orbifolds of the type $C^d/\mathbb{Z}_n$ was considered in [11, 12, 13, 14, 15, 16, 17]. There it was shown that the basic physical mechanism for the orbifold resolution was the posssibility to turn on certain vacuum expectation values in the twisted sectors of the orbifold, thus inducing nontrivial Fayet-Iliopoulos terms in the effective supersymmetric theory of the D-branes.

The present work is an attempt to understand the analogous process in the case of *compact* orbifolds. As we will see, the resolution process is considerably more subtle. The reason for this is that the compactness of the original orbifold leads us to a problem of moduli spaces of a certain class of gauge connections because the effective theory is inherently a gauge *field* theory. The mathematical difficulty is comparable with the one encountered when generalizing from linear algebra to functional analysis. Namely, we will discover that both the geometric and the analytic aspects of our problem are nontrivial, unlike the case of [11, 13].

To avoid unnecessary technicalities, we will focus on the case of one D-particle on abelian orbifolds of T^4. This admits an effective low-energy description via a supersymmetric theory along the lines of [18]. In section 1, we review the resolution process of noncompact orbifolds from a point of view which is relevant for this paper and we explain the origin of the difficulty in the compact case. In section 2, we formulate the relevant low-energy theory starting from the abstract set-up of [19] and we show that it leads to a problem of equivariant gauge connections on a V-bundle [1] over the orbifold. In section 3, we give a clear account of how the singular moduli space arises in the case of $T^4/\mathbb{Z}_2$, which is the focus of the rest of the paper. In section 4, proceeding along the lines of[11, 12], we argue that the orbifold gauge theory admits a certain class of localized Fayet-Iliopoulous terms, leading to a moduli space of singular gauge connections. In section 5 we give some evidence that this provides a resolution of the original singular moduli space. Finally, section 6 presents our conclusions and speculations. During the preparation of this paper, we became aware of [21], which has some overlap

[1]The concept of a V -bundle turns out to be the proper mathematical framework for formulating our orbifold theory. Although this paper can be read without any understanding of this concept, we will occasionally state some results in this language, since it allows us to give a precise meaning to the so-called 'singular bundles' sometimes mentioned in the matrix-theory literature. A V-bundle E, say, over $T^4/\mathbb{Z}_n$, is simply a vector bundle F over T^4 together with linear identifications ϕ_α among its fibers, giving an action of $\mathbb{Z}_n$ by bundle automorphisms which cover the action of $\mathbb{Z}_n$ on T^4. Thus, for any element α of $\mathbb{Z}_n$, and for any point $x \in T^4$, one has a linear isomorphism $\phi_{\alpha,x}$ between the fiber F_x of F over x and the fiber F_y over the image $y = \alpha \cdot x$ of x. This specifies the way in which the fibers of F are to be identified under the action of the orbifold group. One imposes the compatibility condition $\phi_{\beta,\alpha\cdot x} \circ \phi_{\alpha,x} = \phi_{\beta+\alpha,x}$. In general, the action of $\mathbb{Z}_n$ on T^4 has a number of fixed points. If x_0 is such a point, then the maps ϕ_{α,x_0} give a representation of $\mathbb{Z}_n$ in the fiber F_{x_0}, called the *isotropy representation* at x_0. A V-bundle is characterized by the topological invariants of the underlying vector bundle and by its isotropy representations at the fixed points of the orbifold action. The simplest example of a V-bundle –which will occur below– is the usual twisted product $T^4 \times_{\mathbb{Z}_n} R$ (also called a product V-bundle), where $R = (\mathbb{C}^r, \rho)$ is a representation space for $\mathbb{Z}_n$. This has trivial underlying bundle $F = T^4 \times \mathbb{C}^r$ and V-structure given by the constant identifications $\phi_\alpha(x) = \rho(\alpha)$, $\forall x \in T^4$. It has isotropy representations equal to ρ at each of the fixed points. Given a V-bundle E, one can consider connections on the underlying vector bundle F, which are compatible with the V-bundle structure (i.e. with the maps $\phi_{\alpha,x}$). Such objects are called *invariant, equivariant* or $V-$connections on E, and are nothing else than usual connections A on the underlying bundle subject to the condition that their pull-backs ${}^\alpha A$ by the orbifold action are gauge-equivalent with A:

$$
{}^\alpha A = \sigma_\alpha (d + A) \sigma_\alpha^{-1} \quad ,
$$

with $\sigma_\alpha = \phi_{\alpha,(-\alpha)\cdot x}$. The gauge transformations σ_α implementing this equivalence encode the V-bundle structure. If the underlying bundle F is equipped with a hermitian metric such that $\phi_{\alpha,x}$ are unitary for all α, x, then E is called a hermitian V-bundle. In this case, those unitary automorphisms U of F which commute with ϕ_α are called *V-gauge transformations* of E. They satisfy the projection condition:

$$
U((-\alpha) \cdot x) = \sigma_\alpha(x) U(x) \sigma_\alpha(x)^{-1}
$$

and act on connections A in the usual manner. Then the projection condition satisfied by U assures that the gauge transform of a V-connection is still a V-connection. For more information about V-bundles and invariant connections the reader is reffered to [20].

with our work.

1 General considerations

1.1 The quotienting procedure

The general method for obtaining an effective field theory of D-particles on a quotient space of the form $K = \mathbb{R}^4/G$, with G a discrete group, is to consider a system of D-particles on the covering space, together with their images, and to project the resulting field theory onto its G-invariant part. The basic example of this procedure is the case of one D-particle over an orbifold of the type $\mathbb{C}^2/\mathbb{Z}_n$, which is obtained [11] by letting $G = \mathbb{Z}_n$ act on $\mathbb{C}^2$ via its fundamental representation ρ_f and on the Chan-Paton factors via its regular representation ρ_{reg}. In this case, one has a system of n images distributed over $\mathbb{C}^2$ in a $\mathbb{Z}_n$-invariant way. More precisely, there are two types of variables entering the resulting supersymmetric quantum mechanics: $X^\mu(x)(\mu = 5..10)$ and $X^a(x)(a = 1..4)$, corresponding to the position of the D-particles in the 'flat' directions, respectively in the orbifold directions. These carry $n \times n$ Chan-Paton matrix indices. General considerations [22] instruct one to keep only the $\mathbb{Z}_n$ -invariant part of X^μ, X^a, where $\mathbb{Z}_n$ acts on X^μ, X^a as:

$$\begin{aligned} X^\mu &\to \rho_{reg}(\alpha) X^\mu \rho_{reg}(\alpha)^{-1} \\ X^a &\to \rho_f(\alpha)\rho_{reg}(\alpha) X^a \rho_{reg}(\alpha)^{-1} \quad . \end{aligned}$$

The extra-factor $\rho_f(\alpha)$ in the action on X^a reflects the nontrivial transformation law of the coordinates parallel to the orbifold directions. The compact part of moduli space of vacua coincides [11] with the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ itself. More general actions on the Chan-Paton factors are possible and were discussed in some detail in [11]. Some of these do not have a standard D-particle interpretation, corresponding instead to bound states containing fractional numbers of D-particles and/or D2-branes [23]. In general, it turns out [11, 12] that the moduli space of the resulting quantum mechanics is given by a singular hyperkahler quotient of matrix data.

A second application of the quotienting procedure was proposed in [18] for the case $G = \mathbb{Z}^4$. This leads to compactifications of D-particle systems on a four-torus $T = \mathbb{R}^4/\mathbb{Z}^4$. Once again, the theory of r D-particles on T is obtained by considering their system of images on $\mathbb{R}^4$ under the $\mathbb{Z}^4$-action. This time, however, the number of images is infinite, which leads to an effective description given by a supersymmetric $U(r)$ gauge field theory defined on the dual torus T'. The images are distributed according to the action of $\mathbb{Z}^4$ on $\mathbb{R}^4$, but there is again considerable freedom in the choice of the action on the Chan-Paton factors. This Chan-Paton representation is reflected in the topology of the bundle over T' in which the effective field theory is defined. In fact, Taylor's original approach takes the Chan-Paton representation to be the regular representation of $\mathbb{Z}^4$, which leads to a trivial bundle over T', but more general representations can be considered, and they lead to nontrivial bundles. Such compactifications can be interpreted as systems of D-particles over T which also contain D2 and D4-branes. In general, the moduli space of supersymmetric vacua is given by the moduli space of instantons in the bundle. In the case of the trivial bundle, this is just the moduli space of flat connections. Again the simplest case is that of one D-particle, which leads to a $U(1)$ gauge theory and to the moduli space of flat line bundles over T', which coincides — as expected — with the original torus T.

Note that the situation in the compact case is more involved because of the presence of an infinity of images. While in the noncompact case there exists a quantum-mechanical description of the moduli space, in the compact case only a field-theoretic description is available. The gauge symmetry is infinite dimensional for the latter.

Viewed more abstractly, both the construction of [11] and that of [18] are procedures for quotienting the effective theory on the covering space by a discrete group–finite, in the first case, and infinite in the second. In general, then, one can follow the same pattern for describing D-particles over $\mathbb{R}^4/G$ for any discrete group G. Our focus in this paper will be on the case of the semidirect product of $\mathbb{Z}^4$ and $\mathbb{Z}_n$, with appropriate integer n, which leads to a theory of D-particles on $T^4/\mathbb{Z}_n$. In fact, we will be primarily concerned with the case of one D-particle only. This can be described in a way which is very similar to that of [18], by considering an image of the D-particle for each element of G, as we will explain in detail in the next section. However, it is by now well understood that another and particularly useful way of formulating this sort of problem is via the more abstract approach of [19]. This starts with a supersymmetric 'quantum mechanics' of variables $X^\mu, \mathcal{X}^a$ valued in a Hilbert space $\mathcal{H}$ and imposes projection conditions with respect to a unitary representation U of G in $\mathcal{H}$. The freedom in the choice of U corresponds essentially to the freedom in the choice of the action of G on the Chan-Paton factors. Then the case of finite-dimensional $\mathcal{H}$ allows us to describe quotients $\mathbb{R}^4/G$ for a finite group G, while the case of an infinite (but discrete) G can be described by taking $\mathcal{H}$ to be a (separable) infinite-dimensional Hilbert space[2]. This approach has major advantages in terms of generality and clarity as well as in terms of computational power. As we will show explicitly in the next section, its relation to the more intutive approach of [18] is similar to the relation between the abstract operator formalism of quantum mechanics and its formulation in a particular representation: Taylor's description appears simply by chosing a particular pair of bases of the Hilbert space $\mathcal{H}$: a basis leading to the D-particle representation and a basis leading to the D4-brane representation. From this point of view, the Fourier transform of [18] is simply the change of basis between these, while the compactification procedure of [19] is the overarching 'operator formulation'.

[2]Moreover, if one replaces U by a *projective* representation of G, then one obtains noncommutative-geometric compactifications as in [19]. In this paper, however, we will only consider proper representations.

1.2 The resolution mechanism in the noncompact case

In the case of $\mathbb{C}^2/\Gamma$, it was shown in [11] that the effective quantum mechanics describing the D-particle system admits Fayet-Iliopoulous terms. More precisely, one can turn on such a term for each central generator of the effective gauge group. (The gauge group is generated by the unitary gauge transformations that commute with the orbifold projection, while the effective gauge group is obtained by modding out a $U(1)$ factor which acts trivially on X^μ, X^a.) The moduli space of the resulting theory gives a resolution of the orbifold, and, in the Matrix theory limit at least, this can be interpreted as a resolution of the space itself. In fact, the quantum-mechanical description gives the resolved moduli space as a hyperkahler quotient, in a form which is *identical* to the hyperkahler quotient description given in [24] to the minimal resolution of the orbifold singularity. The resolved space X is a smooth hyperkahler manifold which is asymptotically locally euclidean (i.e. it is an ALE space).

For compact orbifolds, it is not immediately clear how such a procedure would be applied. Indeed, what we will need is some analogue of the resolution mechanism of [11], given that fundmental degrees of freedom are not just finite-dimensional matrices but equivariant and intrinsically nonabelian connections. By analogy with the process of [11], we could hope that, starting with the infinite lattice of D-particles on R^4, one finds a *natural* mechanism for producing a deformation of orbifold gauge field theory — *still defined over* $T'/\mathbb{Z}_n$ — whose vacuum moduli space gives a resolution of the moduli space of the original system.

2 Formulation of $T^4/\mathbf{Z_n}$ D-particle orbifolds

2.1 The abstract set-up

To obtain a compactification preserving half of the supersymmetry of the type IIA theory, we start with a two-dimensional hermitian vector space V and a maximal rank lattice $\Lambda \subset V$. We let $<,>$ denote the hermitian scalar product on V and $(,) := Re <,>$ the associated euclidean product on the underlying real vector space $V_R \approx \mathbb{R}^4$. If $\Lambda' := \{x \in V | (x,t) \in \mathbb{Z},\ \forall t \in \Lambda\}$ is the dual lattice, we have dual 4-tori $T = V/\Lambda$ and $T' = V/\Lambda'$. We consider a faithful representation $\gamma : \mathbb{Z}_n \to SU(V)$ of $\mathbb{Z}_n$ by *special* unitary transformations of V. To have a well-defined quotient, we must assume that Λ is invariant with respect to this action:

$$\gamma(\alpha)(\Lambda) = \Lambda \quad ,$$

for all $\alpha \in \mathbb{Z}_n$ [3].

[3] Note that V admits a unique quaternionic structure compatible both with its original complex structure and its hermitian metric. With respect to this quaternionic structure, $SU(V)$ is identified with the group of unitary symplectic automorphisms of V and γ acts by elements of $USp(V)$. Therefore, T and T' can be viewed as flat quaternionic manifolds in a natural way. The reason for requiring γ to act by special unitary

Consider the group G of real affine transformations of V generated by translations with $t \in \Lambda$ and by the linear transformations $\gamma(\alpha)(\alpha \in \mathbb{Z}_n)$. This is given by actions $v \to \gamma(\alpha)v + t$ on the vectors v of V. As an abstract group, G is a semidirect product of Λ and $\mathbb{Z}_n$, having Λ as a normal subgroup. Its composition law is given by:

$$(t', \alpha') \cdot (t, \alpha) = (\gamma(\alpha')t + t', \alpha + \alpha')$$

and the inversion by $(t, \alpha)^{-1} = (-\gamma(-\alpha)t, -\alpha)$. Here $\alpha \in \mathbb{Z}_n$ and $t \in \Lambda$.

To formulate the orbifold problem along the lines of [19], we consider a Hilbert space $\mathcal{H}$, a unitary representation U of G in $\mathcal{H}$ and selfadjoint operators $X^\mu, \mathcal{X}^a$ in $\mathcal{H}$, subject to the 'projection conditions':

$$U(t, \alpha)^{-1} X^\mu U(t, \alpha) = X^\mu \tag{1}$$

$$U(t, \alpha)^{-1} X U(t, \alpha) = \gamma(\alpha) X + 2\pi t \; Id_{\mathcal{H}} \quad . \tag{2}$$

where $X = \mathcal{X}^a \otimes e_a$ is a V-valued operator, with $e_1..e_4$ a real basis of V. In (2), $U(t, \alpha)$ acts $\mathcal{X}^a$, while $\gamma(\alpha)$ acts on the basis elements e_a. Physically, $X^\mu, \mathcal{X}^a$ describe 'infinite by infinite' Chan-Paton matrices, while U gives the action of the group G on the Chan-Paton factors. Therefore, the abstract projection conditions above are nothing else than the adaptation of the usual orbifold projection conditions on open strings [22] to the case of the infinite group G.

2.2 The D-particle basis

To describe one D-particle on $T/\mathbb{Z}_n$, we choose U to be the regular representation of G. This is characterized up to unitary equivalence by the requirement that there exist a vector $h \in \mathcal{H}$ such that the set $\{|t, \alpha> := U(t, \alpha)h | t \in \Lambda, \alpha \in \mathbb{Z}_n\}$ is an orthonormal basis of $\mathcal{H}$. In this case:

$$U(t', \alpha')|t, \alpha> = |\gamma(\alpha')t + t', \alpha + \alpha'> = |(t', \alpha') \cdot (t, \alpha)> \quad ,$$

so that, defining $X^\mu_{s,\alpha;t,\beta} := < s, \alpha | X^\mu | t, \beta > \in \mathbb{C}$ and $X_{s,\alpha;t,\beta} := < s, \alpha | X | t, \beta > \in V$, the projection conditions become:

$$X^\mu_{(t',\alpha')\cdot(s,\alpha);(t',\alpha')\cdot(t,\beta)} = X^\mu_{s,\alpha;t,\beta}$$
$$X_{(t',\alpha')\cdot(s,\alpha);(t',\alpha')\cdot(t,\beta)} = \gamma(\alpha') \cdot X_{s,\alpha;t,\beta} + 2\pi t' \delta_{\alpha,\beta} \delta_{s,t} \quad .$$

transformations is that we wish to preserve this quaternionic structure, which assures us that orbifolding by $\mathbb{Z}_n$ breaks exactly half of supersymmetry. Alternatively, we wish to have a Calabi-Yau orbifold, so that γ must preserve triviality of the canonical line bundle, and this requires $\det\gamma(\alpha) = 1$. Examples of lattices invariant under such $\mathbb{Z}_n$ actions are easily constructed. The simplest case is the action of $\mathbb{Z}_2$ by negation on V, which preserves any lattice Λ. Another is the case of the 'fundamental' representation of $\mathbb{Z}_4$, defined by $\gamma_f(\alpha) = \mathrm{diag}(i^\alpha, i^{-\alpha})$; lattices invariant under this action can be obtained by taking $\Lambda := (\mathbb{Z} + i\mathbb{Z})\epsilon_1 + (\mathbb{Z} + i\mathbb{Z})\epsilon_2$, with (ϵ_1, ϵ_2) the complex basis of V which diagonalizes γ_f. Although the quaternionic structure will be important in section 4, we will present our discussion in the simpler language of hermitian vector spaces. This amounts to singling out a preferred complex structure.

As the states of the above basis are labelled by the group elements (t, α), we can think of our Hilbert space as the space of square-summable sequences $\phi : \Lambda \to \mathbb{C}$, $\sum_{t,\alpha} |\phi(t, \alpha)|^2 < \infty$. Then the group G acts by $[U(t', \alpha')\phi](t, \alpha) = \phi((t', \alpha')^{-1} \cdot (t, \alpha))$, which is the defining property of the regular representation. The abstract variables X^μ, X become infinite matrices $X^\mu_{s,\alpha;t,\beta}$, $X_{s,\alpha;t,\beta}$ with entries in $\mathbb{C}$, respectively V. Their physical interpretation is as Chan-Paton matrices of open strings connecting pairs of D-particles (s, α) and (t, β). We can interpret the D-particle indexed by (t, α) as the image by the element $(t, \alpha) \in G$ of the D-particle $(0, 0)$. This is the direct generalization of the representation used in [18] for the case of one D-particle on a torus, and we call it the 'D-particle' representation. A two-dimensional model of the situation is drawn below (for the case $n = 2$).

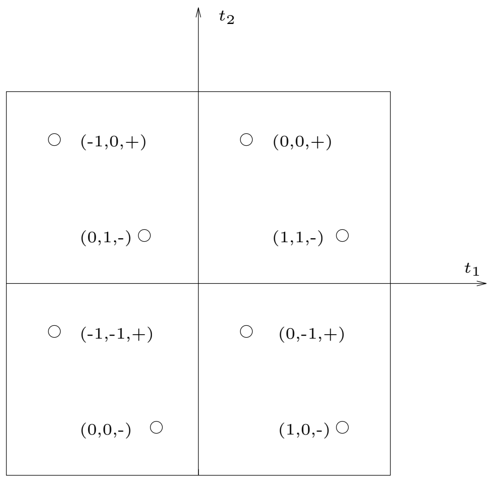

Figure 1: A two-dimensional model of one D-particle on the covering space together with some of its images. For clarity we denoted the elements $\hat{0}, \hat{1}$ of $\mathbb{Z}_2$ by $+ = \hat{0}$ and $- = \hat{1}$. The first two coordinates give the lattice translation vector $t = (t_1, t_2)$. All images are obtained by acting with various (t_1, t_2, α) on $(0, 0, +)$.

2.3 The 'intermediate' basis

The action of the $\mathbb{Z}_n$ subgroup of G can be diagonalized by performing a discrete Fourier transform. For this, let $\xi := e^{\frac{2\pi i}{n}}$. Since:

$$\sum_{\alpha \in \mathbb{Z}_n} \xi^{\alpha\beta} = n\delta_{\beta,0}$$

(where $\delta_{\beta,0}$ is taken in $\mathbb{Z}_n$, i.e. $\delta_{n,0} = \delta_{0,0} = 1$), we can define a new orthonormal basis of $\mathcal{H}$ by:

$$|t,\alpha) := \frac{1}{\sqrt{n}} \sum_{\beta\in\mathbb{Z}_n} \xi^{\alpha\beta}|t,\beta> \quad . \tag{3}$$

In this basis, the action of G becomes:

$$U(t',\alpha')|t,\alpha) = \xi^{-\alpha\alpha'}|\gamma(\alpha')t + t',\alpha) \quad .$$

We call this the 'intermediate' representation. The discrete Fourier transform from the D-particle representation to the intermediate representation is similar to the transform effected in [11, 13] in order to diagonalize the $\mathbb{Z}_n$ action on the Chan-Paton factors. This representation has an interpretation similar to that of the D-particle basis and is further discussed in the appendix.

2.4 The D4-brane basis

The T-dual description of our system can be obtained by a further change of basis of $\mathcal{H}$[4] :

$$|x,\alpha > := \sum_{t\in\Lambda} e^{2\pi i(x,t)}|t,\alpha), \tag{4}$$

[4]We consider the Fourier transform from the space of sequences enumerated by Λ to the space of distributions on the dual torus $T' = V/\Lambda'$. For each $t \in \Lambda$, we have a function $e^{2\pi i(x,t)}$ on V. This is Λ'-periodic and induces a function on T', which we still denote by $e^{2\pi i(x,t)}$. To define the Fourier transform we consider the natural measure $d\mu_{T'}(x) := [dx]$ on T', induced from the Lesbesgue measure on the covering space and normalized such that the elementary cells of Λ' have volume 1. The Fourier coefficients of a function f on T' are then given by:

$$f_t := \int_{T'} [dx] f(x) e^{2\pi i(x,t)} \ (t\in\Lambda) \quad ,$$

with the inversion formula:

$$f(x) = \sum_{t\in\Lambda} f_t e^{-2\pi i(x,t)} \quad .$$

This extends to distributions on T', giving the completeness and orthogonality relations :

$$\sum_{t\in\Lambda} e^{2\pi i(x-x',t)} = \delta_{T'}(x-x') \quad ; \qquad \int_{T'} [dx] e^{2\pi i(x,t-t')} = \delta_{t,t'}$$

Here the difference $x - x'$ is understood as an operation in the group $(T',+)$, while $\delta_{T'}(x)$ is the Dirac distribution on T', normalized with respect to the above measure:

$$\int_{T'} [dx]\delta_{T'}(x) = 1 \quad .$$

for $x \in T'$ and $\alpha \in \mathbb{Z}_n$. We have the orthogonality and completeness conditions:

$$< x, \alpha | y, \beta >= \delta_{T'}(x-y)\delta_{\alpha,\beta}$$
$$\int_{T'} [dx] \sum_{\alpha \in \mathbb{Z}_n} |x, \alpha >< x, \alpha| = Id_{\mathcal{H}} \quad .$$

In this 'D4-brane' representation, $\mathcal{H}$ is the Hilbert space $L^2(T') \otimes \mathbb{C}^n$, and the action of G becomes:

$$U(t', \alpha')|x, \alpha >= \xi^{-\alpha\alpha'} e^{-2\pi i(\gamma(\alpha')x, t')} |\gamma(\alpha')x, \alpha >$$

(here we used unitarity of $\gamma(\alpha')$ and the fact that $(,) = Re <,>$). This shows that $U(t, 0)$ has (generalized) eigenvalues $e^{-2\pi i(x,t)}$ with generalized eigenspaces $I_x := \mathrm{Span}((|x\alpha >)_{\alpha \in \mathbb{Z}_n})$. Since condition (1) requires X^μ to preserve these subspaces, we must have:

$$X^\mu |x, \alpha >= X^\mu_{\beta,\alpha}(x)|x, \beta > \quad .$$

Hence X^μ become multiplication operators by the hermitian matrices $X^\mu(x) := (X^\mu_{\alpha,\beta}(x))_{\alpha,\beta \in \mathbb{Z}_n}$. Then taking $t = 0$ in (1) gives:

$$X^\mu_{\alpha,\beta}(\gamma(\alpha')x) = \xi^{-(\alpha-\beta)\alpha'} X^\mu_{\alpha,\beta}(x) \quad .$$

2.5 The D4-brane orbifold gauge theory

Orbifold connections

To solve (2), define the operator P by $P|t, \alpha) = 2\pi t|t, \alpha)$ and let $X := P + A$. Since P satisfies $U(t', \alpha')^{-1} P U(t', \alpha') = \gamma(\alpha')P + 2\pi t'$, we can rewrite (2) as:

$$U(t', \alpha')^{-1} A U(t', \alpha') = \gamma(\alpha')A \tag{5}$$

Taking $\alpha' = 0$ shows that A must have the form:

$$A|x, \alpha >= A_{\beta,\alpha}(x)|x, \beta > \quad ,$$

which gives a matrix multiplication operator $A(x) = (A_{\alpha,\beta}(x))_{\alpha,\beta \in \mathbb{Z}_n}$ (with entries in V). Then (5) becomes:

$$A_{\alpha,\beta}(\gamma(\alpha')x) = \xi^{-(\alpha-\beta)\alpha'} \gamma(\alpha') A_{\alpha,\beta}(x) \tag{6}$$

Since in the D4-brane representation we have $P = +i\nabla_x$, with ∇ the gradient associated to the metric induced by $(,)$ on T', we obtain:

$$X = i[\nabla_x - iA(x)] \quad .$$

To formulate this more geometrically, pick any real basis $e_a (a = 1..4)$ of V and expand:

$$A(x) = \sum_{a=1..4} \mathcal{A}^a(x) e_a$$
$$X = \sum_{a=1..4} \mathcal{X}^a e_a \quad .$$

Defining:

$$\mathcal{A}_a(x) := g_{ab} \mathcal{A}^b(x)$$

(with $g_{ab} = (e_a, e_b)$) and lowering indices in (6) we obtain:

$$\mathcal{A}_a(\gamma(\alpha')x) = \rho_{reg}(-\alpha')\gamma(-\alpha')_{ba}\mathcal{A}_b(x)\rho_{reg}(\alpha') \quad ,$$

where we again used unitarity of $\gamma(\alpha)$, the fact that $(,) = Re <,>$ and we let ρ_{reg} denote the regular matrix representation of $\mathbb{Z}_n$: $\rho_{reg}(\alpha) = \mathrm{diag}(1, \xi^\alpha, \xi^{2\alpha} ... \xi^{(n-1)\alpha})$. Replacing α' by $-\alpha$, we can rewrite this as:

$$\gamma(-\alpha)^t_{ab}\mathcal{A}_b(\gamma(-\alpha)x) = \rho_{reg}(\alpha)\mathcal{A}_a(x)\rho_{reg}(-\alpha) \quad .$$

Defining the matrix-valued one-form $A = \sum_{a=1..4} \mathcal{A}_a(x) dx^a$ [5], the last relation becomes:

$$\gamma(-\alpha)^* A = \rho_{reg}(\alpha) A \rho_{reg}(\alpha)^{-1} \quad , \tag{7}$$

where $\gamma(-\alpha)^*$ denotes the pull-back. A can be viewed as a connection in a rank n hermitian bundle I over T', whose fiber over x is given by the vector space $I_x = \mathrm{Span}(|x\alpha>)_{\alpha \in \mathbb{Z}_n}$. The elements $|x\alpha>$ are then identified with sections $s_\alpha(x)$ of I. This gives a global orthonormal frame $(s_\alpha)_{\alpha \in \mathbb{Z}_n}$ of I, which shows that I is the trivial rank n bundle. Then A is a connection matrix in this frame, with associated covariant derivative operator:

$$-i\mathcal{X}_a = -ig_{ab}\mathcal{X}^b = \partial_a - i\mathcal{A}_a(x) \quad .$$

The conclusion is that $-i\mathcal{X}_a$ becomes the covariant derivative of a $U(n)$ gauge connection on a trivial bundle over T', subject to the projection condition (7). This requires the connection one-form to be invariant under the action of the orbifold group, up to the constant gauge transformation $\rho_{reg}(\alpha)$. This type of condition is natural when defining connections on an orbifold: the projection condition specifies the way in which the connection 'twists' around the fixed point. The gauge transformations associated to $\alpha \in \mathbb{Z}_n$ induce identifications between the fibers of the underlying bundle, which make it into a V-bundle. The identifications describing the V-bundle structure can be used to induce a natural action of the orbifold group on the space of sections. Then (7) is equivalent to the condition that the covariant differentiation operator be equivariant with respect to this action on sections, which can be interpreted as a compatibility

[5]From now on, A will denote this one-form and not the object $A(x) \in V$ considered previously. We hope that this does not produce any confusion.

condition between the connection and the V-structure. Thus, the mathematical meaning of (7) is that A is an equivariant connection in the V-bundle associated to I and to the identifications induced by $\rho_{reg}(\alpha)$. Since I is the trivial rank n bundle over T' and the gauge transformations $\rho_{reg}(\alpha)$ defining the V-structure are constant over T', we have a 'product' V-bundle structure $E = T' \times_{\mathbb{Z}_n} (\mathbb{C}^n, \rho_{reg})$ associated to the regular representation of $\mathbb{Z}_n$.

Orbifold gauge transformations

Let us discuss the equivalence relation needed to build the moduli space. Clearly we should identify two representations (X^i, X, U) and (X'^i, X', U') in $\mathcal{H}$ if there exists a *unitary* operator U in $\mathcal{H}$ such that U commutes with all $U(t,\alpha)$ and intertwines X^μ, X and X'^μ, X'. The group of such operators U is the 'projected symmetry group'. In the D4-brane basis, the condition that U commutes with all $U(t,\alpha)$ constrains U to be a local $U(n)$-valued multiplication operator $U(x)$, satisfying the projection:

$$U(\gamma(-\alpha)(x)) = \rho_{reg}(\alpha) U(x) \rho_{reg}(\alpha)^{-1} \quad . \tag{8}$$

The action $X \rightarrow UXU^{-1}$ of U on X is reflected by the usual gauge transformation law

$$A \rightarrow UAU^{-1} - idUU^{-1}$$

while the action $X^\mu \rightarrow UX^\mu U^{-1}$ on X^μ becomes:

$$X^\mu(x) \rightarrow U(x) X^\mu(x) U(x)^{-1} \quad .$$

Thus, we can view $X^\mu(x)$ as scalar fields on T' in the adjoint representation of $U(n)$.

Mathematically, (8) means that U defines a 'V-gauge transformation' of E, i.e. a unitary bundle automorphism which commutes with the V-structure. Thus, we should identify two V-connections A, A' if they are V-gauge equivalent in the V-bundle E.

The compact part of the moduli space

To construct the physically interesting moduli space we must also impose the vacuum conditions stemming from the standard super-Yang-Mills action. Considering a real basis $e_1..e_4$ of V as before and defining $g_{ab} := (e_a, e_b)$ $(a, b = 1..4)$, we can write the bosonic part of the action as [6]:

$$\begin{aligned} S = & \frac{1}{2\rho g} \int dt \; \{Tr(\dot{X}^\mu \dot{X}^\mu) + g_{ab} Tr \; (\dot{\mathcal{X}}^a \dot{\mathcal{X}}^b)\}. \\ & + \frac{1}{4\rho g} \int dt \; \{Tr \; ([X^\mu, X^\nu][X^\mu, X^\nu]) + 2 g_{ab} Tr \; ([X^\mu, \mathcal{X}^a][X^\mu, \mathcal{X}^b])\} \\ & + \frac{1}{4\rho g} \int dt \; \{g_{ac} g_{bd} Tr \; ([\mathcal{X}^a, \mathcal{X}^b], [\mathcal{X}^c, \mathcal{X}^d])\} \quad . \end{aligned}$$

[6] We will allways use Tr to denote the 'total' trace in the Hilbert space $\mathcal{H}$ and tr to denote the matrix trace in $u(2)$.

Here ρ is a normalization factor, which is needed to eliminate the 'infinite measure' of the $\mathbb{Z}^4$ symmetry group of the projected theory. Formally, ρ is given by $\rho = \sum_{t\in\Lambda} 1 = \delta_{T'}(0)$. In this paper, we will treat the issue of such infinities quite formally, avoiding a serious discussion of regularization procedures.

The vacuum constraints are :

$$[X^\mu, X^\nu] = 0 \tag{9}$$

$$[X^\mu, \mathcal{X}^a] = 0 \tag{10}$$

$$[\mathcal{X}^a, \mathcal{X}^b] = 0 \quad . \tag{11}$$

In the D4-brane basis, these become:

$$[X^\mu(x), X^\nu(x)] = 0 \tag{12}$$

$$\mathcal{D}_a^{(adj)} X^\mu = \partial_a X^\mu(x) - i[\mathcal{A}_a(x), X^\mu(x)] = 0 \tag{13}$$

$$\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a - i[\mathcal{A}_a, \mathcal{A}_b] = 0 \quad , \tag{14}$$

where $\mathcal{D}^{(adj)}$ is the covariant derivative associated to A in the adjoint representation.

The adjoint scalars $X^\mu(x)$ are sections of a trivial Lie agebra bundle (of type $u(n)$) over T'; their modes describe the oscillations of the D4-brane in the uncompactified directions. The full system of constraints is a coupled nonlinear system, somewhat similar to the Yang-Mills-Higgs equations. One can obtain a simplification by assuming that $X^\mu(x)$ are slowly varying over T'. Under this adiabatic approximation the total moduli space becomes a fibration over the space of constant matrices $X^\mu \in u(n)$ satisfying (9), whose fibre is the moduli space of solutions to (13) and (14). Then the D-particle has a well-defined position in the uncompactified space directions (specified by the common eigenvalues of X^μ), so that the brany-geometry of the uncompactified part of the space reduces to its classical geometry. This is the limit we are interested in in this paper, since we want to probe the brany-geometry of the compact part of the space only. To further simplify the problem we will take $X^\mu = 0$, $\forall \mu$. Then *the desired moduli space is given by the moduli space of flat V-connections in the V-bundle E, modulo V-gauge transformations.*

It is expected that this moduli space coincides with the original orbifold $T/\mathbb{Z}_n$. With the above formulation, we wish to present a clear proof of this claim. We will proceed by considering an equivariant form of the argument which relates flat connections to their holonomy. Further, to simplify the presentation, we will only consider the case $n = 2$, which is the focus of the rest of this paper, but similar arguments can be applied for other cases. We will also restrict to $SU(2)$ connections, which amounts to imposing the tracelessness condition on $\mathcal{A}_a$ at every point of T'. (General $U(n)$ connections can be studied by similar methods.)

3 The singular $SU(2)$ moduli space for the case $n = 2$

When $n = 2$ we let $\mathbb{Z}_2$ act on V by negation $x \to -x$, $\forall x \in V$. Any lattice in V is invariant under this action, which descends to the usual negation of the complex torus $T' = V/\Lambda'$. It is well-known [25, 26] that the negation of T' has 16 fixed points. These form a subgroup of $(T', +)$ which is isomorphic [7] with $(\mathbb{Z}_2)^4$.

The regular representation of $\mathbb{Z}_2$ is generated by the Pauli matrix $\rho_{reg}(\hat{1}) = \sigma_3$. The projection condition on the connection becomes:

$$\sigma_3 \mathcal{A}_a(x) \sigma_3 = -\mathcal{A}_a(-x) \quad , \tag{15}$$

while the projection condition on gauge transformations is:

$$\sigma_3 U(x) \sigma_3 = U(-x) \quad .$$

These restrict $\mathcal{A}_a$, U to the forms:

$$\begin{aligned} \mathcal{A}_a(x) &= \begin{pmatrix} o_1(x) & e_1(x) \\ e_2(x) & o_2(x) \end{pmatrix} \quad , \\ U(x) &= \begin{pmatrix} e_1(x) & o_1(x) \\ o_2(x) & e_2(x) \end{pmatrix} \quad , \end{aligned} \tag{16}$$

where e_k, o_k denote even, respectively odd complex-valued functions defined over T'. It is important to realize that, in the above relations, $-x$ represents the negation on T' (the opposite of the element x in the abelian group $(T', +)$). Thus, a point $x \in T'$ is a solution of the equation $x = -x \Leftrightarrow 2x = 0$ if and only if it is a fixed point of the $\mathbb{Z}_2$ action. We will denote the fixed points by $x_0 ... x_{15}$.

3.1 Characterization of the moduli space

Choose a fixed global trivialization of the underlying bundle I of E over T'. Then any $su(2)$ connection A on I is represented by a globally defined $su(2)$-valued 1-form over T'. Any such 1-form is induced by a periodic 1-form $A(v)$ defined on the covering space V. Here periodicity means :

$$\tau_t^*(A) = A, \ \forall t \in \Lambda'$$

[7] The fixed points x_l are the projections modulo Λ' of some points m_l of the covering space V. The fixed-point condition $x_l = -x_l \Leftrightarrow 2x_l = 0$ (in $(\mathrm{T}', +)$) is equivalent to the constraint $m_l \in \frac{1}{2}\Lambda'$. Thus the fixed points are in bijection with the elements of the quotient group $(\frac{1}{2}\Lambda')/\Lambda' \approx (\mathbb{Z}_2)^4$.

where $\tau_t(v) := v + t$ ($v \in V, t \in \Lambda'$) is the translation by t on V. If one choses an arbitrary real basis $e_1..e_4$ of V, in which A has components $\mathcal{A}_a$, then periodicity reads:

$$\mathcal{A}_a(v+t) = \mathcal{A}_a(v),\ \forall t \in \Lambda' \quad .$$

Reduction to parallel transport operators

Clearly $A(x)$ is flat on T' if and only if $A(v)$ is flat on V. In this case, the parallel transport of $A(v)$ gives a well-defined $SU(2)$ matrix-valued function $P(v)$ on V (depending on our trivialization of I). We choose $0 \in V$ (with projection $x_0 = O \in T'$) as a base point for defining $P(v)$. Then $P(v)$ satisfies:

$$\begin{aligned} dP(v) &= iA(v)P(v) \\ P(0) &= 1 \quad . \end{aligned}$$

Together with Λ'-periodicity of $A(v)$, this equation immediately leads to the property:

$$P(v+t) = P(v)P(t),\ \forall v \in V,\ \forall t \in \Lambda' \tag{17}$$

(that is, $P(v)$ is Λ'-'quasiperiodic'). It is easy to see that the $\mathbb{Z}_2$ projection condition on A implies:

$$P(-v) = \sigma_3 P(v) \sigma_3 \quad . \tag{18}$$

The gauge transformations $U(x)$ of $A(x)$ are induced by Λ'-periodic $SU(2)$-valued functions $U(v)$ on V. They act on $P(v)$ by :

$$P(v) \to U(v)P(v)U(0)^{-1} \quad . \tag{19}$$

Moreover, the projection condition on $U(x)$ is equivalent to :

$$U(-v) = \sigma_3 U(v) \sigma_3 \quad . \tag{20}$$

Conversely, if one is given a map $P : V \to SU(2)$ satisfying (17) and (18), then $A(v) := -i(dP(v))P(v)^{-1}$ is a Λ'-periodic 1-form on V which induces a flat $SU(2)$ connection $A(x)$ on T' satisfying the $\mathbb{Z}_2$ projection condition (15). Moreover, if $U(v)$ is a periodic $su(2)$-valued function on V which satisfies (20), then the transformation (19) of P is equivalent to the gauge transformation of $A(x)$ by the associated projected gauge group element $U(x)$. Hence the desired moduli space is given by:

$$\mathcal{M}_{flat} = \frac{\{P : V \to SU(2) | P \text{ satisfies (17) and (18)}\}}{\{P(v) \to U(v)P(v)U(0)^{-1} | U \text{ is } \Lambda' \text{−periodic and satisfies (20)}\}} \quad . \tag{21}$$

Reduction to the holonomy representation

To express the moduli space in terms of finite-dimenional matrices, we consider the restriction $S := P|_{\Lambda'}$ of P to the lattice Λ'. The quasiperiodicity property (17) of P implies that $S : \Lambda' \to SU(2)$ is a unitary representation of the discrete abelian group $(\Lambda', +)$ (the *holonomy representation*):

$$S(t + t') = S(t)S(t'),\ \forall t, t' \in \Lambda' \quad . \tag{22}$$

The projection condition (18) induces a constraint on $S(t)$:

$$S(t)^{-1} = \sigma_3 S(t) \sigma_3 \quad . \tag{23}$$

On the other hand, the gauge transformations (19) (with a Λ'-periodic $U(v)$) induce actions on the holonomy:

$$S(t) \to W S(t) W^{-1} \quad , \tag{24}$$

where $W := U(0) \in SU(2)$. The projection condition (20) implies:

$$W = \sigma_3 W \sigma_3 \quad . \tag{25}$$

Thus, we have a well-defined map from $\mathcal{M}_{flat}$ as given in (21) to the quotient of all representations $S : \Lambda' \to SU(2)$, satisfying (23), divided by conjugations (24), with W satisfying (25).

If two maps $P(v), P'(v)$, satisfying (17) and (18) are such that $S'(t) = WS(t)W^{-1}$, with W satisfying (25), then it is easy to see that the map $U(v) := P'(v)WP(v)^{-1}$ is Λ'-periodic and satisfies (20). Then $U(0) = W$ and we have $P'(v) = U(v)P(v)U(0)^{-1}$, so that P and P' are gauge-equivalent. This shows that the above map is one to one.

On the other hand, given any representation $S : \Lambda' \to SU(2)$ which satisfies (23), one can immediately construct a parallel transport operator $P(v)$, satisfying (17,18) and such that $P|_{\Lambda'} = S$ [8]. It follows that the above map is onto.

In conclusion, our moduli space is, equivalently, given by:

$$\mathcal{M}_{flat} = \frac{\{S : \Lambda' \to SU(2) | S \text{ satisfies (22) and (23)}\}}{\{S(t) \to WS(t)W^{-1} | W \in SU(2) \text{ satisfies (25)}\}} \quad . \tag{26}$$

Computation of the moduli space

To make this completely explicit, let $\tau_1..\tau_4$ be an integral basis of the lattice Λ' and $S_a := S(\tau_a)$. Note that τ_a correspond to a basis of cycles on the torus, while S_a give the holonomies of A around those cycles. The projection condition (23) requires

[8]This can be done as follows. Pick an integral basis $\tau_a (a = 1..4)$ of Λ' and let $S_a := S(\tau_a)$. Then $S_a^{-1} = \sigma_3 S_a \sigma_3$ and $[S_a, S_b] = 0$. Write $S_a = e^{is_a}$, with s_a mutually commuting hermitian matrices satisfying $\sigma_3 s_a \sigma_3 = -s_a$. Then one can take $P(v) = e^{iv^a s_a}$, for all $v = v^a \tau_a \in V$. In particular, this gives a constant representative $\mathcal{A}_a = s_a$ of the V-gauge equivalence class of connections having holonomies $(S_1, ..., S_4)$ around the cycles of T' associated to $\tau_1..\tau_4$, as announced in the note on page 16.

$S_a^{-1} = \sigma_3 S_a \sigma_3$, while the condition (22) that S_a generate a representation of the abelian group $(\Lambda', +)$ is equivalent to the requirement that they commute:

$$[S_a, S_b] = 0 \ \forall a, b = 1..4 \quad . \tag{27}$$

Therefore, the desired moduli space is given very explicitly as follows:

$$\mathcal{M}_{flat} = \frac{\{(S_1, S_2, S_3, S_4)|S_a \in SU(2)\ ,\ [S_a, S_b] = 0,\ \forall a, b = 1..4 \text{ and } \sigma_3 S_a \sigma_3 = S_a^{-1}\}}{\{(S_a)_{a=1..4} \to (W S_a W^{-1})_{a=1..4}|W \in SU(2) \text{ and } \sigma_3 W \sigma_3 = W\}} \quad .$$

A simple matrix computation now shows that $\mathcal{M}_{flat}$ coincides (topologically) with the original orbifold $K = T/\mathbb{Z}_2$. Indeed, the general $SU(2)$ solution of the projection constraint (23) is:

$$S_a = \begin{pmatrix} u_a & v_a \\ -v_a^* & u_a \end{pmatrix}$$

with $u_a \in \mathbb{R}, v_a \in \mathbb{C}$ satisfying $u_a^2 + |v_a|^2 = 1$. Writing $v_a = e^{i\phi_a} s_a$ (with $s_a \in \mathbb{R}$ positive or negative and $\phi_a \in [0, \pi)$), the commutation relations (27) are equivalent with:

$$e^{2i(\phi_a - \phi_b)} = 1 \quad ,$$

which shows that ϕ_a are all equal. Denoting their common value by $\phi \in [0, \pi)$, the general solution for S_a is:

$$S_a := \begin{pmatrix} u_a & e^{i\phi} s_a \\ -e^{-i\phi} s_a & u_a \end{pmatrix} \quad ,$$

where $u_a, s_a \in \mathbb{R}$ satisfy $u_a^2 + s_a^2 = 1$ (they parametrize a circle S^1).

The general $SU(2)$ solution of (25) is:

$$W(\eta) = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix} \quad ,$$

with $\eta \in (-\pi, \pi]$.

Then W acts on $(S_1..S_4)$ via (24) by shifting ϕ and possibly inverting the sign of s_a:

$$\begin{aligned} \phi &\to \phi' \\ s_a &\to (-1)^r s_a \quad , \end{aligned}$$

where $r \in \mathbb{Z}$ and $\phi' \in [0, \pi)$ are defined by $\phi + 2\eta = \phi' + r\pi$ (u_a are unchanged under this transformation). If some s_a is nonzero, then we can perform the transform by $W(-\frac{1}{2}\phi)$ to go to a 'gauge' in which $\phi = 0$. Then $W(\frac{1}{2}\pi) = W(-\frac{1}{2}\pi) = -1_2$ gives a residual identification $s_a \to -s_a$. If all s_a are zero, then $u_a = \pm 1$ and we have a fixed point under the action of W. Thus, we obtain a quotient of the form $(S^1)^4/\mathbb{Z}_2$, where

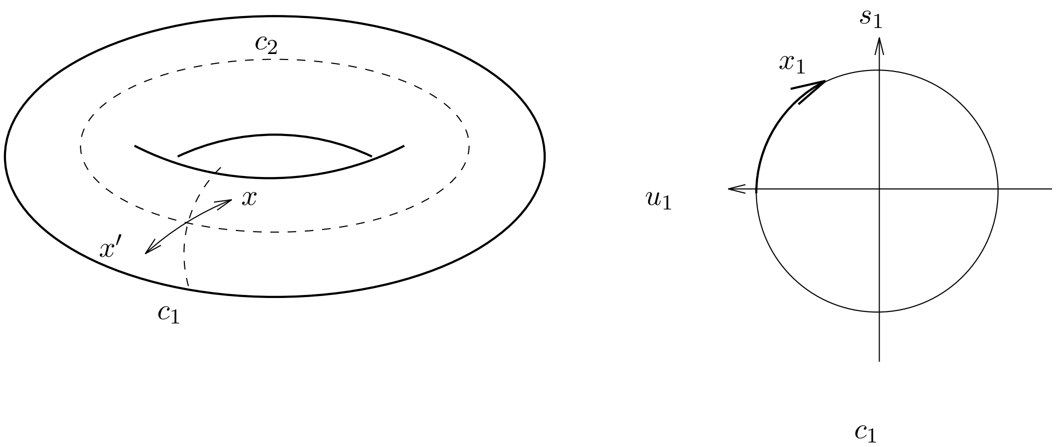

Figure 2: Two circles c_1, c_2 on a torus associated to a basis τ_1, τ_2 of the lattice Λ (in the two-dimensional case). A point $x = (x_1\tau_1 + x_2\tau_2)(\text{mod } \Lambda')$ is shown together with its $\mathbb{Z}_2$ image $x' = ((-x_1)\tau_1 + (-x_2)\tau_2)(\text{mod } \Lambda')$. The right part of the figure shows our coordinates s_1, u_1 for the first circle. The reflection $s_a \to -s_a$ corresponds to $x_a \to -x_a$ i.e. to $x \to x'$.

the a-th S^1 is parametrized by (u_a, s_a) and $\mathbb{Z}_2$ acts by $s_a \to -s_a$. This is topologically the same as $T/\mathbb{Z}_2$, as one can see from the following two-dimensional model.

For further use, note that the singular points of the moduli space (given by $s_a = 0$) correspond to central holonomies, i.e. to holonomy operators $S_a = \pm 1$. There are $2^4 = 16$ such possibilities, as expected. If $S_a = \epsilon_a 1_2$, with $\epsilon_a \in \{-1, 1\}$ is a central holonomy, then a a constant representative of the associated V-gauge equivalence class is given by $\mathcal{A}_a = \frac{1}{2}(1-\epsilon_a)\pi\sigma_2$, with $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ the second Pauli matrix.

4 The moment map and Fayet-Iliopoulos terms

We obtained a supersymmetric field theories with 8 supercharges and $SU(2)_R$ R-symmetry, and may expect a triplet of Fayet-Iliopulos terms for each central generator of the (projected) gauge group. Naively, there appears to be no such central generator since the theory is intrinsically nonabelian even in case of single D-particle on $T^4/\mathbb{Z}_2$. However, since the abstract formulation of our system is formally nothing but a supersymmetric quantum mechanics, although infinite-dimensional, one may try to find a central generator at that level. In this section, we follow this idea for the case of $T^4/\mathbb{Z}_2$ orbifolds. As we will discover, the particular nature of the $\mathbb{Z}_2$ projection dictates that the center of the projected gauge group is nontrivial but is generated by *localized* gauge transformations in the D4-brane basis. Thus, one is lead to consider Fayet-Ilioupoulos terms which – in the D4-brane basis – are localized at the fixed points of the $\mathbb{Z}_2$ action on T' (In the D-particle picture, such terms are completely delocalized on the lattice Λ. See the appendix for detailed description of the Fayet-Iliopoulos terms in the D-particle picture).

4.1 Formal considerations

The formal aspects of the situation are most clearly discussed at the level of the representation-independent formulation. For this purpose, we treat $\mathcal{X}^a$ as formal algebraic quantities.

The moment map

The projected symmetry group $\mathcal{G}$ acts by unitary symplectic automorphisms of the space of operators X, which has a natural quaternionic structure, induced from that of the vector space V (since X are vector-valued with vector indices corresponding to V). Therefore, we have an associated quaternionic moment map $\vec{m}$, taking X into the Lie algebra Lie($\mathcal{G}$) of $\mathcal{G}$.

To formulate this precisely, we choose $(e_a)_{a=1..4}$ to be the real basis associated to a complex orthonormal basis $\epsilon_k (k = 1, 2)$ of V:

$$e_i = \epsilon_i$$
$$e_{i+2} = i\epsilon_i$$

$(i = 1, 2)$. Then $(e_a)_{a=1..4}$ is automatically orthonormal. The expression of the moment map is most transparent in the associated complex coordinates:

$$X^k = \mathcal{X}^k + i\mathcal{X}^{k+2}$$
$$X^{\bar{k}} = \mathcal{X}^k - i\mathcal{X}^{k+2}$$

$(k = 1, 2)$, where the complex and real parts of the moment map are:

$$m_c = [X^1, X^2]$$
$$m_r = \frac{i}{2}([X^1, X^{\bar{1}}] + [X^2, X^{\bar{2}}]) \quad .$$

Writing $m_c = m_1 + im_2$ and $m_r = -im_3$, with m_k hermitian matrices gives the usual real components $(m_1, m_2, m_3) = \vec{m}$.

Lowering indices with $g_{ij} = g_{\bar{i},\bar{j}} = 0, g_{i,\bar{j}} = g_{\bar{i},j} = \frac{1}{2}\delta_{ij}$, we have $X_i = \frac{1}{2}X^{\bar{i}}$, $X_{\bar{i}} = \frac{1}{2}X^i$, so that:

$$X_k := \frac{1}{2}(\mathcal{X}_k - i\mathcal{X}_{k+2})$$
$$X_{\bar{k}} := \frac{1}{2}(\mathcal{X}_k + i\mathcal{X}_{k+2})$$

$(k = 1, 2)$. The hermicity conditions are equivalent with $X_k^+ = X_{\bar{k}}$. In terms of the covariant quantities $X_k, X_{\bar{k}}$, the real components of the moment map are $m_1 = 2\mu_1, m_2 = 2\mu_2, m_3 = 2\mu_3$ [9], with:

$$\begin{aligned} \mu_1 &= [X_2, X_1] + [X_{\bar{1}}, X_{\bar{2}}] \\ \mu_2 &= i([X_2, X_1] - [X_{\bar{1}}, X_{\bar{2}}]) \\ \mu_3 &= [X_1, X_{\bar{1}}] + [X_2, X_{\bar{2}}] \quad . \end{aligned}$$

Defining $Q_{ab} := [\mathcal{X}_a, \mathcal{X}_b]$ $(a, b = 1..4)$, we can rewrite this as:

$$\begin{aligned} \mu_1 &= iQ_{14}^{(a)} \\ \mu_2 &= -iQ_{12}^{(a)} \\ \mu_3 &= iQ_{13}^{(a)} \quad , \end{aligned} \tag{28}$$

where $Q_{ab}^{(a)} := \frac{1}{2}(Q_{ab} - \tilde{Q}_{ab})$ is the antiself-dual part of the antisymmetric tensor Q_{ab}. Here $\tilde{Q}_{ab} = \frac{1}{2}\epsilon_{abcd}Q_{cd}$, with $\epsilon_{1234} = +1$.

The superpotential and Fayet-Iliopoulos terms

In the basis (28) the scalar potential becomes:

$$W = \frac{1}{\rho} Tr \sum_{a,b=1..4} [\mathcal{X}_a, \mathcal{X}_b][\mathcal{X}_a, \mathcal{X}_b] \quad , \tag{29}$$

up to a prefactor which we ignore. Let $\frac{1}{\sqrt{2}}\lambda_m$ be self-adjoint elements which form a (real) orthonormal basis of the Lie algebra of the gauge group $\mathcal{G}$ (and thus a complex orthonormal basis of its complexification), with respect to the scalar product $< A, B >= Tr(A^+ B)$ [10]. Then a direct computation as in [12] gives the identity:

$$W = -\frac{8}{\rho} Tr(\vec{\mu}^2) = -\frac{4}{\rho} \sum_m [Tr(\lambda_m \vec{\mu})]^2 \quad .$$

Now let $\frac{1}{\sqrt{2}}\lambda_s$ be an orthonormal basis of the center $\mathrm{Lie}_0(\mathcal{G})$ of the Lie algebra of $\mathcal{G}$, and λ_n such that $\frac{1}{\sqrt{2}}\lambda_n, \frac{1}{\sqrt{2}}\lambda_s$ form an orthonormal basis of $\mathrm{Lie}(\mathcal{G})$. Then one can introduce a triplet of auxiliary fields $\vec{D}_s$ for each s. After integrating out the auxiliary fields, one obtains the modified scalar potential:

$$W' = -\frac{4}{\rho}(\sum_n [Tr(\lambda_n \vec{\mu})]^2 + \sum_s [Tr(\lambda_s \vec{\mu}) - \vec{\xi}_s]^2) \tag{30}$$

[9] We use the quantities μ_1, μ_2, μ_3 in order to avoid some cumbersome factors later.

[10] Below we will allow m to take on continuous values as well. In this case the sums over m should be interpreted as integrals and $\delta_{m,n}$ as meaning $\delta(m - n)$

where $\vec{\xi}_s$ are the associated triplets of Fayet-Iliopoulos parameters. Then the vacuum constraint $W' = 0$ requires:

$$Tr(\lambda_n \vec{\mu}) = 0$$
$$Tr(\lambda_s \vec{\mu}) = \vec{\xi}_s$$

i.e.:

$$\vec{\mu}(X) = \frac{1}{2} \sum_s \vec{\xi}_s \lambda_s \quad .$$

4.2 The D4-brane representation

The moment map

After implementing the translation projection, $\mathcal{X}_a$ become covariant derivatives:

$$\mathcal{X}_a = i\mathcal{D}_a = i(\partial_a - i\mathcal{A}_a),$$

while the commutators are:

$$Q_{ab} = [\mathcal{X}_a, \mathcal{X}_b] = +i\mathcal{F}_{ab}$$

with:

$$\mathcal{F}_{ab} = +i[\mathcal{D}_a, \mathcal{D}_b] = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a - i[\mathcal{A}_a, \mathcal{A}_b] \quad .$$

Therefore, $\vec{\mu}$ becomes the multiplication operator by the function $\vec{\mu}(x)$ whose real components are (cf 28):

$$\mu_1 = -\mathcal{F}^{(a)}_{14}$$
$$\mu_2 = \mathcal{F}^{(a)}_{12}$$
$$\mu_3 = -\mathcal{F}^{(a)}_{13} \quad .$$

Here $\mathcal{F}^{(a)}$ is the antiself-dual part of $\mathcal{F}$.

The central part of the gauge group in the case of $T^4/\mathbb{Z}_2$

Let us determine the central part $\mathrm{Lie}_0(\mathcal{G})$ of the Lie algebra $\mathrm{Lie}(\mathcal{G})$. It is immediate that there are two types of central elements of the Lie algebra, namely:

$$\lambda(x) = f(x)1_2 \tag{31}$$

with $f(-x) = f(x)$, and:

$$\lambda_l(x) = \sigma_3 \delta_{T'}(x - x_l) \tag{32}$$

where x_l are the 16 fixed points of the $\mathbb{Z}_2$ action. (We normalize λ_l such that $tr \int_T \lambda_l(x)\lambda_{l'}(x) = 2\delta_{T'}(x_l - x_l'))$.

To justify this, let $\phi(x)$ be an element of $\mathrm{Lie}_0(\mathcal{G})$ (this is an $u(2)$ matrix-valued function on T'). Since ϕ belongs to $\mathrm{Lie}(\mathcal{G})$, it must satisfy the infinitesimal form of the projection condition on the gauge tranformations:

$$\sigma_3\phi(x)\sigma_3 = \phi(-x) \quad .$$

On the other hand, being a central element of $\mathrm{Lie}(\mathcal{G})$, it must commute with all (projected) gauge transformations:

$$U(x)\phi(x)U(x)^{-1} = \phi(x) \quad . \tag{33}$$

Applying the second condition for the constant $SU(2)$ gauge transformations $U(x) = e^{i\alpha\sigma_3}$, $\alpha \in \mathbb{R}$ (which clearly satisfy the projection condition) and combining with the first relation shows that $\phi(-x) = \phi(x)$. Thus ϕ must have the form:

$$\phi(x) = \left(\begin{array}{cc} e_1(x) & o_1(x) \\ o_2(x) & e_2(x) \end{array} \right) \quad ,$$

with e_k, o_k even, respectively odd functions on T'. Now consider a general $SU(2)$ gauge transformation satisfying the projection conditions:

$$U(x) = \left(\begin{array}{cc} u(x) & v(x) \\ v^*(x) & u^*(x) \end{array} \right) \quad ,$$

with $u_1(-x) = u(x), v(-x) = -v(x)$ and $|u(x)|^2 + |v(x)|^2 = 1$. Then (33) reduces to the condition $e_1(x)v(x) = v(x)e_2(x)$. If x is not a fixed point of the $\mathbb{Z}_2$ action, then $x \neq -x$ and it is always possible to construct a smooth *odd* function v such that $|v| \leq 1$ and e takes a nonzero value at x. Then we must have $e_1(x) = e_2(x) := f(x)$ and $U(x) = f(x)1_2$. However, if x is a fixed point, then the evenness condition requires $v(-x) = -v(x) = v(x)$ (since $x = -x$), hence v must be zero at x. It follows that ϕ is constrained to be proportional to the identity matrix at all points except for the fixed points. At the fixed points $x_l (l = 0..15)$, $\phi(x_l)$ can be any real diagonal matrix, i.e. any real linear combination of 1_2 and σ_3. If one requires ϕ to be smooth over T', this latter freedom is, of course, irrelevant. However, if one allows for singular $\phi(x)$, then one obtains the central generators (31,32).

The Fayet-Iliopoulos constraints

For what follows we will set the Fayet-Iliopoulous parameters associated to (31) to zero, i.e. we impose the tracelessness constraint:

$$tr(\vec{\mu}(x)) = \vec{0} \quad .$$

This is automatically satisfied for $su(2)$ connections, which is the case we are considering. Then turning on Fayet-Iliopoulous parameters $\vec{\xi}_l$ associated to the 16 central generators (32) leads to the vacuum constraints:

$$\vec{\mu}(x) = \frac{1}{2} \sum_{l=0..15} \vec{\xi}_l \sigma_3 \delta_{T'}(x - x_l) \quad , \tag{34}$$

i.e.:

$$\begin{aligned} \mathcal{F}^{(a)}_{14} &= -\mu_1 = -\frac{1}{2} \sum_l \xi^1_l \sigma_3 \delta_{T'}(x - x_l) \\ \mathcal{F}^{(a)}_{12} &= \mu_2 = \frac{1}{2} \sum_l \xi^2_l \sigma_3 \delta_{T'}(x - x_l) \\ \mathcal{F}^{(a)}_{13} &= -\mu_3 = -\frac{1}{2} \sum_l \xi^3_l \sigma_3 \delta_{T'}(x - x_l) \quad . \end{aligned} \tag{35}$$

The action of the symmetry group, the description of its center, and the form of the Fayet-Iliopoulos terms in the D-particle representation are discussed in the Appendix.

5 The moduli space in the presence of Fayet-Iliopoulous terms

5.1 Naive considerations

The equations (35) completely determine $F^{(a)}$. Since we chose to work with the regular representation on the Chan-Paton factors, the gauge bundle in the D4-brane representation is trivial (a fact that ensures, for example, that no additional topological terms arise when interpreting the Hilbert space manipulations of subsection 4.1 in the gauge theory realization of subsection 4.2). In particular, this means that the second Chern class of the gauge bundle vanishes, giving us a relation between $F^{(s)}$ and $F^{(a)}$:

$$\int_T dx \ [tr(\mathcal{F}^{(s)}_{ab}\mathcal{F}^{(s)}_{ab}) - tr(\mathcal{F}^{(a)}_{ab}\mathcal{F}^{(a)}_{ab})] = 0 \quad .$$

With our form of the Fayet-Iliopoulous constraints, the quantity $\int_T dx \ tr(\mathcal{F}^{(a)}_{ab}\mathcal{F}^{(a)}_{ab})$ is divergent, so the same should be true of $\int_T dx \ tr(\mathcal{F}^{(s)}_{ab}\mathcal{F}^{(s)}_{ab})$. It seems reasonable to expect that the mechanism for obtaining such an infinity is common for $F^{(s)}$ and $F^{(a)}$. Therefore, we propose that the self-dual part also has to be of the form for any vacuum configuration;

$$\mathcal{F}^{(s)}_{14} = \frac{1}{2} \sum_l \eta^1_l \sigma_3 \delta_{T'}(x - x_l)$$

$$\mathcal{F}^{(s)}_{12} = \frac{1}{2}\sum_l \eta^2_l \sigma_3 \delta_{T'}(x - x_l) \tag{36}$$
$$\mathcal{F}^{(s)}_{13} = \frac{1}{2}\sum_l \eta^3_l \sigma_3 \delta_{T'}(x - x_l) \ .$$

If this is indeed the case, then cancellation of $tr(\mathcal{F}^{(s)}_{ab}\mathcal{F}^{(s)}_{ab})$ and $tr(\mathcal{F}^{(a)}_{ab}\mathcal{F}^{(a)}_{ab})$ in c_2 occurs locally and requires that $\vec{\eta}_l$ satisfy:

$$\vec{\eta}_l^2 = \vec{\xi}_l^2$$

for all $l = 0..15$. That is, each $\vec{\eta}_l$ is constrained to lie on a 2-sphere of radius $|\vec{\xi}_l|$. This gives 16 2-spheres, each of which parametrizes local behaviour of F at each of 16 fixed points, given the FI parameters $\vec{\xi}_I$.

To understand the situation better, note that, since the punctured torus $T'_p := T' - \{x_0...x_{15}\}$ has $\pi_1 = \mathbb{Z}^4$, a connection obeying (35) has a well-defined parallel transport operator $P(v)$ as in section 3, which determines the connection uniquely. In particular, we have an associated holonomy representation $S := P|_{\Lambda'}$. If a (complex) gauge in which (36) holds exists, then, as in section 3, S determines A up to gauge transformations which are smooth over the *punctured* torus T'_p. However, such a gauge transformation $U(x)$ may become singular at the puncture points $x_0..x_{15}$, so that A is not determined by S up to a gauge transformation which is smooth over *whole of* T'. Therefore, if – as is natural – we build the moduli space by using gauge transformations which are smooth over the entire T', then the map ψ from gauge equivalence classes of connections (which are the points of the moduli space $\mathcal{M}_\xi$) to equivalence classes of holonomies modulo transformations of type (24) may not be one to one. In a gauge in which (35,36) hold, fixing the equivalence class $[S]$ of the holonomy modulo transformations (24) determines A only up to a gauge transformation $U(x)$ which may be singular at $x_0...x_{15}$. However, in order to fix A up to a gauge transformation which is smooth over the entire T', one also needs to specify the asymptotic behaviour of $\mathcal{A}_a(x)$ as x approaches any of the fixed points. This asymptotic behaviour is encoded by the parameters $\vec{\eta}_0..\vec{\eta}_{15}$.

At this point, it seems reasonable that the moduli space $\mathcal{M}_\xi$ can be identified with a subset of the direct product $(S^2)^{16} \times (T/\mathbb{Z}_2)$, where the first factor correponds to $\vec{\eta}_0..\vec{\eta}_{15}$, while the second factor corresponds to projected monodromies. Note, however, that $\mathcal{M}_\xi$ coincides with $(S^2)^{16} \times (T/\mathbb{Z}_2)$ only if the parameters $\vec{\eta}_0..\vec{\eta}_{15}$ and S can be varied independently. To show that $\mathcal{M}_\xi$ is at least toplogically equivalent to a K3 surface, it would suffice to show that the preimage $\psi^{-1}([S])$ contains exactly one point unless $[S]$ corresponds to a fixed point of $T/\mathbb{Z}_2$ (i.e. unless S is a central holonomy), while $\psi^{-1}([S])$ would be a 2-sphere for each central $[S]$ To establish this, one may try to deduce a contraint between $\vec{\eta}_0..\vec{\eta}_{15}$ and $[S]$, which would fix the values of $\vec{\eta}_l$ for all noncentral $[S]$, while allowing for a nontrivial set of solutions $\vec{\eta}_0..\vec{\eta}_{15}$ (presumably parametrized by the points of a two-sphere) for a central $[S]$.

Even granted that (35) can be imposed (which is not clear by any means), implementing this idea is a rather difficult task, essentially due to the fact that the equations (35) are not mathematically well-defined as they stand. Indeed, we obtained these equations by rather cavalier formal manipulations, but, in order to give them a clear meaning, one needs to specify a regularization of the commutators $[\mathcal{A}_a, \mathcal{A}_b]$ entering $\mathcal{F}_{ab}$. The reason is that, if one wants to satisfy (35) in distributions, then $A(x)$ must scale at least as $\frac{1}{|x-x_l|^2}$ near the fixed points. In this case, the commutators $[\mathcal{A}_a, \mathcal{A}_b]$ scale at least as $\frac{1}{|x-x_l|^4}$, which means that they cannot be locally integrable functions around the fixed points (and thus do not define a distribution in a canonical way). Therefore, it is not immediately clear what the meaning of the object $[\mathcal{A}_a, \mathcal{A}_b]$ is as a distribution. To make strict sense out of (35), one needs to give an appropriate definition ('regularization' in the sense of distribution theory) of this commutator, compatible with the formal computations that we carried out in order to arrive at (35)[11].

Alternatively, one can think about this in the D-particle representation as follows [12]. As discussed in the appendix, in order to solve the Fayet-Iliopoulous constraints (35), one must include field configurations $A(t)$ which do not decrease fast at infinity. Specifying the precise class of such configurations is equivalent to specifying the class of connections $\mathcal{A}_a$ in which we look for a solution of (35). Depending on the precise class of $A(t)$ chosen, one will obtain, via the Fourier transform, a different interpretation of $\mathcal{F}_{ab}$ as a distribution. We see that the above line of thinking leads to rather nontrivial technical problems. Therefore, in the next subsection we will give an alternate – but still not rigorous – argument.

5.2 K3 surface

One may hope that $\mathcal{M}_\xi$ would give a smooth and compact manifold which continuously reduces to the singular moduli space $T/\mathbb{Z}_2$ in the limit $\xi_l = 0, \forall l = 0 \dots 15$. Such a smooth and compact moduli space must be a K3 surface by a general argument. The equations (35) present $\mathcal{M}_\xi$ as an (infinite form of a) hyperkahler quotient, where the space being quotiented is the affine space $\mathcal{Q}$ of all (suitable) connections (the latter space has a natural hyperkahler structure, induced from the hyperkahler structure of V). Assuming that a suitable analogue of the argument of [27] could be made in our case, $\mathcal{M}_\xi$ inherits a hyperkahler structure from $\mathcal{Q}$, in a manner controlled by the FI parameters $\vec{\xi}_0 \dots \vec{\xi}_{15}$. Given that $\mathcal{M}_\xi$ continuously reduces to $T/\mathbb{Z}_2$ for $\xi_l \to 0$, it must be a a four-manifold. Since a smooth, compact and connected hyperkahler four-manifold can only be a complex two-dimensional torus or a K3 surface, and since $\mathcal{M}$ reduces to $T^4/\mathbb{Z}_2$ in the limit of zero Fayet-Iliopoulos terms, the resolved moduli space

[11] This can be viewed essentially as the problem of giving an adequate meaning to a certain type of distribution product.

[12] The solution of the projection conditions as well as the counterparts of our Fayet-Iliopoulos terms in the D-particle respresentation are discussed in detail in the appendix.

can only be a K3.

5.3 Parameter count

As a further check, let us see whether we have the correct number of moduli. The blow-up of $T/\mathbb{Z}_2$ at the 16 fixed points gives a Kummer surface X, carrying the famous 16 lines associated to the exceptional divisors [28]. These are rational curves Δ_l ($l = 0..15$) of self-intersection -2 and give elements of the Picard lattice. When talking about the blow-up, we are considering everything in the algebraic-geometric approach and therefore we fix a choice of complex structure by taking a definite embedding in some projective space. For a fixed complex structure, the Kahler form ω is characterized by its periods Ω_l along the 16 two-cycles $[\Delta_l] \in H_2(X, \mathbb{R})$(the areas of the 2-spheres Δ_l). This gives 16 real parameters characterizing the Kahler structure. To specify a *hyperkahler* structure on X, we have to give the periods of three Kahler forms $\omega^{(k)}(k = 1..3)$ along $[\Delta_l]$. This gives a total of 48 real parameters, which is the same as the number of real parameters $\xi_l^{(k)}$ ($l = 0..15, k = 1..3$) controlling the moduli space $\mathcal{M}_\xi$.

5.4 Sheaf interpretation

The singular connections discussed above may appear to be less strange if one considers the problem from the point of view of holomorphic sheaves. For this, let us start by reformulating the moduli space of flat connections in the holomorphic language. The Hitchin-Kobayashi correspondence (see, for example, [29]) tells us that giving a flat connection on the (differentiable) hermitian bundle I is equivalent to giving a semistable holomorphic structure on I. This makes I into a holomorphic vector bundle over T'. The tracelessness condition on the connection is equivalent to the requirement that the determinant line bundle $detI = \Lambda^2 I$ be holomorphically trivial.

Given any flat connection on I, one can find a gauge in which its connection matrix is constant. Since such a self-adjoint matrix is diagonalizable by a constant unitary gauge transformation, it follows that flat connections on T' are reducible. Therefore, the associated holomorphic bundle I decomposes as a direct sum of flat holomorphic line bundles:

$$I = L \oplus L^{-1} \quad ,$$

with $c_1(L) = 0$. Thus, the moduli space of flat $SU(2)$ connections on T' coincides with the moduli space of holomorphic line bundles L on T', which is isomorphic with the dual torus T. This is the holomorphic version of the the argument giving the moduli space of flat connections on T'. In the case of $T/\mathbb{Z}_2$, one has to consider the obvious equivariant adaptation of the above.

It is a general fact that moduli spaces of self-dual connections over a compact manifold admit compactifications via moduli spaces of sheaves. The typical examples are Gieseker's compactification (via moduli spaces of Gieseker stable torsion-free sheaves) and the generally smaller compactification proposed by Maruyama[30], which is essentially obtained by restricting the class of sheaves allowed. Since torsion-free sheaves are essentially a singularization of vector bundles, one may expect that there exists a degenerate form of the Hitchin-Kobayashi correspondence, relating torsion-free sheaves to certain singular gauge connections. Indeed, such a correpondence was proposed in [31] for the case of reflexive sheaves (which are a particular class of torsion-free shaves). Moreover, it was noted in [30] that the sheaves achieving the Maruyama compactification are related to singular limits of instantons.

It is tempting, therefore, to suppose that the moduli space of connections obeying (35) could be related to an appropriate class of holomorphic sheaves over $T'/\mathbb{Z}_2$. To make this slightly more explicit, consider setting ξ_l^c to zero. If an appropriate version of the usual relation between Kahler and Geometric Invariant Theory quotients can be established in our case, then $\mathcal{M}_{(\xi_r,0)}$ could be identified with a moduli space of connections subject only to the (homogeneous) complex moment map equations, modulo *complexified* gauge transformations. Provided that a suitable sheaf interpretation of our connections can be found, this would coincide with the moduli space of an appropriate class of equivariant sheaves, similar to what happens in the case of moduli spaces of instantons over the noncompact orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ [32]. Since sheaf moduli spaces can be studied by the powerful methods of algebraic geometry, this may provide a better approach to the problem at hand. In this context, it may be noted that moduli spaces of sheaves often provide compact resolutions of moduli spaces of holomorphic bundles, and that there are situations (as that discussed in [32]) when certain moduli spaces of equivariant sheaves realize the resolution of orbifolds. We hope to pursue this approach further elsewhere.

6 Conclusions

We gave a clear formulation of the effective field theory of one D-particle over $T^4/\mathbb{Z}_n$ orbifolds and an explicit derivation of the associated singular moduli space, as a moduli space of equivariant flat connnections in a product V-bundle over the orbifold. We showed that a straightforward modification of the procedure of [11, 12] leads to localized Fayet-Iliopoulous terms and to a theory whose vacua are described by an 'infinite' version of a hyperkahler quotient. We also presented some evidence that the resulting moduli space may provide a resolution of the compact orbifold.

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A The form of the Fayet-Iliopoulos terms in the intermediate representation

In the 'intermediate' representation (3), the projection constraints on $X_{\alpha,\beta}(s,t) := (s,\alpha|X|t,\beta) \in V$ become:

$$\rho_{reg}(\alpha')X(\gamma(\alpha')s+t',\gamma(\alpha')t+t') = \gamma(\alpha')X + 2\pi t' 1_n \delta_{s,t} \quad , \tag{37}$$

where we combined the vectors $X_{\alpha,\beta}(s,t)$ into the n by n matrix $X(s,t) = (X_{\alpha,\beta}(s,t))_{\alpha,\beta=1..n}$. Defining $A(s) := X(s,0)$ as in [18], the projection conditions by $\mathbb{Z}^4$ translations (obtained by taking $\alpha' = 0$ in (37)) require:

$$X(s,t) = A(s-t) + 2\pi t 1_n \delta_{s,t} \quad , \tag{38}$$

while the $\mathbb{Z}_n$ projection (corresponding to $t' = 0$) constrains $A(t)$ to satisfy:

$$\rho_{reg}(\alpha)A(t)\rho_{reg}(-\alpha) = \gamma(\alpha)A(\gamma(-\alpha)t) \quad .$$

Equation (38) shows that $A_{\alpha,\beta}(s-t) = (s\alpha|A|t\beta)$, where A is the Hilbert space operator used in section 2.

Defining $U(s,t) = (U_{\alpha,\beta}(s,t))_{\alpha,\beta=1..n}$, with $U_{\alpha,\beta}(s,t) = (s,\alpha|U|t,\beta)$, the projection condition on the gauge group is:

$$\rho_{reg}(\alpha')U(\gamma(\alpha')s+t',\gamma(\alpha')t+t')\rho_{reg}(-\alpha') = U(s,t) \quad .$$

For $\alpha' = 0$, this gives $U(s,t) = U(s-t)$, where $U(t) := U(t,0)$, while for $t' = 0$ it becomes:

$$\rho_{reg}(\alpha)U(t)\rho_{reg}(-\alpha) = U(\gamma(-\alpha)t) \quad .$$

In general, an operator R satifies the projection condition by translations:

$$U(t,0)^{-1}RU(t,0) = R \tag{39}$$

if and only if its 'intermediate' representation $R(s,t) = ((s,\alpha|R|t,\beta))_{\alpha,\beta=1..n}$ has the form:

$$R(s,t) = R(s-t,0) := R(s-t) \quad .$$

If $R,\ S$ are two such operators, then their operator product $C = RS$ also satisfies (39) and its intermediate representation $C(s) = C(s,0)$ is given by:

$$C = R * S \ ,$$

where $R * S$ is the convolution product of matrix valued sequences defined over Λ:

$$(R * S)(t) = \sum_{u,v \in \Lambda,\ u+v=t} R(u)S(v) \quad .$$

In particular, unitarity of a gauge transformation U is equivalent to the constraint:

$$U * U^+(t) = 1_2,\ \forall t \in \Lambda \quad ,$$

(where $U^+(t) = U(-t)^+$), while its action on A is reflected by the transformation:

$$A(t) \to (U * A * U^{-1})(t) + 2\pi \sum_{u+v=t} U(u)vU^+(v)$$

of the matrix-valued sequence $A(t)$. Since the identifications used to build the moduli space are given by such nonlocal transformations, identifying the center of the projected gauge group in the D-particle picture is difficult. This is why we presented most of our discussion in the D4-brane description, where the relevant arguments are more transparent. Having identified the center of the gauge group $\mathcal{G}$ in that picture, one can wonder what it corresponds to in the more intuitive language of D-particles. One would also like to have an understanding in this language of the reason for the existence of the nontrivial central elements (32).

To answer these questions, let us first construct the D-particle analogues of our central generators. If R is any operator obeying (39), then the change of basis (4) between the intermediate and the D4-brane representations corresponds to the Fourier transform:

$$\begin{aligned} R(t) &= \int_{T'} [dx] R(x) e^{2\pi i(t,x)} \\ R(x) &= \sum_{t \in \Lambda} R(t) e^{-2\pi i(t,x)} \quad . \end{aligned}$$

(In the D4-brane representation, the projection condition by translations is equivalent to the requirement that R be diagonal in the dual torus variables, i.e. $< x, \alpha|R|y, \beta >= R_{\alpha,\beta}(x)\delta_{T'}(x-y)$. Then $R(x)$ in the above relation is defined by $R(x) = (R_{\alpha,\beta}(x))_{\alpha,\beta=1..n}$.)

Now consider the case $n = 2$. Then the $\mathbb{Z}_2$ projection on the gauge group is (in the intermediate representation):

$$\sigma_3 U(t) \sigma_3 = U(-t) \quad .$$

In the D4-brane representation, an element of the Lie algebra of $\mathcal{G}$ is a matrix-valued function $\lambda(x)$ on T', associated to an operator λ in $\mathcal{H}$ which obeys (39). Hence in the intermediate representation it corresponds to a matrix-valued sequence:

$$\lambda(t) = \int_{T'} [dx] \lambda(x) e^{2\pi i(t,x)} \quad .$$

Therefore, the diagonal generators (31) give sequences of the form:

$$\lambda(t) = \hat{f}(t)1_2 \quad , \tag{40}$$

with $\hat{f}(t) = \int_{T'}[dx]f(x)e^{2\pi i(t,x)}$ the Fourier transform of the scalar function f, while the other 16 generators correspond to:

$$\lambda_l(t) = \sigma_3 e^{2\pi i(t,x_l)} \quad . \tag{41}$$

To understand directly why the latter generators are central in the projected gauge group, note that any fixed point $x_l \in T'$ is induced from a point $m_l \in \frac{1}{2}\Lambda'$ on the covering space (m_l is determined up to an element of Λ'). Therefore, we have $(t, x_l) \equiv (t, m_l)(\bmod \mathbb{Z}) \in \frac{1}{2}\mathbb{Z}$, $\forall t \in \Lambda$, so that $e^{2\pi i(t,x_l)} = e^{-2\pi i(t,x_l)}$. Moreover, we have $(2t, x_l) \equiv (2t, m_l)(\bmod \mathbb{Z}) \in \mathbb{Z}$, $\forall t \in \Lambda$, which implies $e^{2\pi i(v,x_l)} = e^{2\pi i(v-2t,x_l)}$, $\forall v \in \Lambda$. Using these two facts, we can see directly that λ_l commutes with any element U of the projected gauge group:

$$(U * \lambda_l)(t) = \sum_{u+v=t} U(u)\sigma_3 e^{2\pi i(v,x_l)} =$$
$$\sum_{u+v=t} \sigma_3 U(-u)e^{2\pi i(v,x_l)} = \sum_{u+v=-t} \sigma_3 e^{2\pi i(v,x_l)}U(u) =$$
$$\sum_{u+v=t} \sigma_3 e^{2\pi i(v,x_l)}U(u) = (\lambda_l * U)(t) \quad .$$

In the second line we made the change of variables $u \to -u, v \to -v$ and used the first property discussed above, while in the third line we made the change of variables $v \to v - 2t$ and used the second property.

If $l = 0$, then $x_0 = 0$ and the sequence λ_0 is constant:

$$\lambda_0(t) = \sigma_3, \ \forall t \quad .$$

If $l = 1..15$, then m_l must belong to $\frac{1}{2}\Lambda' - \Lambda'$ and the sequence λ_l is never constant. It consists of alternating matrices $+\sigma_3, -\sigma_3$:

$$\lambda_l = (-1)^{2(t,x_l)}\sigma_3 \quad .$$

To make this more explicit, choose dual integral bases $(\tau_1..\tau_4)$, $(\tau'_1..\tau'_4)$ of Λ, Λ':

$$(\tau_a, \tau'_b) = \delta_{ab}, \ \forall a, b = 1..4 \quad .$$

Pick representatives $m_r := \frac{1}{2}(r_1\tau'_1 + ... + r_4\tau'_4)$ of the fixed points, with $r = (r_1...r_4)$, $r_a \in \{0, 1\} \approx \mathbb{Z}_2$, $\forall a = 1..4$. This corresponds to indexing the fixed points by $r \in (\mathbb{Z}_2)^4$. Then:

$$\lambda_r(t) = \sigma_3(-1)^{r_1t^1+...r_4t^4} \quad ,$$

for all $t = t^1\tau_1 + .. + t^4\tau_4 \in \Lambda$. A two-dimensional model of the situation is given below.

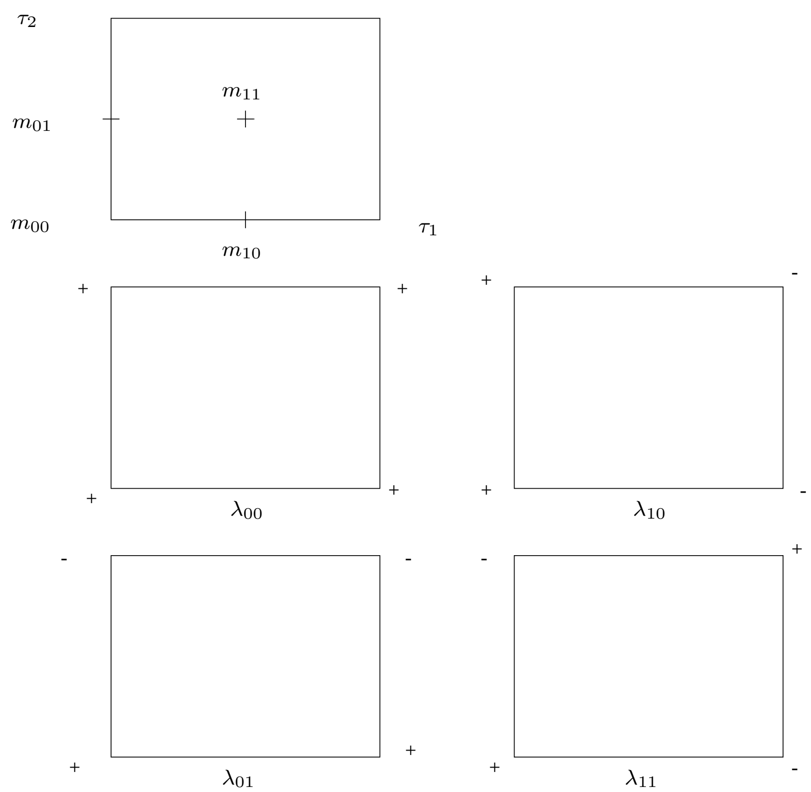

Figure 3: The sign prefactors of the first components of λ_r.

As expected, the Fourier transform maps the localized central elements (32) in the D4-brane picture into completely delocalized elements (41). The singularities in the gauge connection needed to obey the Fayet-Iliopoulos constraints (34) in the D4-brane picture are reflected in the necessity to include delocalized configurations $A(t)$, i.e. configurations for which $A(t)$ does not decrease fast enough at infinity. The fact that the Fayet-Iliopoulos terms are delocalized in the D-particle picture means that they are essentially a 'large volume' effect. The ultimate reason for this is, of course, the fact that we projected the system by the integral translation group $\mathbb{Z}^4$. Once this projection is implemented (at least in the regular representation, as we did in this paper), the only localized central generators still allowed in the D-particle picture are the diagonal elements (40).

Finally, let us write down the Fayet-Iliopoulos constraints in the intermediate representation. The abstract operators $Q_{ab} = [\mathcal{X}_a, \mathcal{X}_b]$ of section 4 obey (39) and are given by:

$$Q_{ab}(t) := Q_{ab}(t,0) = [\mathcal{A}_a, \mathcal{A}_b]_*(t) + 2\pi(t_a\mathcal{A}_b - t_b\mathcal{A}_a) \quad ,$$

where $[\mathcal{A}_a, \mathcal{A}_b]_* := \mathcal{A}_a * \mathcal{A}_b - \mathcal{A}_b * \mathcal{A}_a$. Hence $\vec{\mu}$ also obeys (39) and gives a sequence $\vec{\mu}(t) := \vec{\mu}(t,0)$, with:

$$\begin{aligned} \mu_1(t) &= iQ^{(a)}_{14}(t) \\ \mu_2(t) &= -iQ^{(a)}_{12}(t) \\ \mu_3(t) &= iQ^{(a)}_{13}(t) \quad . \end{aligned}$$

Then the moment map constraints are given by:

$$\vec{\mu}(t) = \frac{1}{2}\sum_{r\in(\mathbb{Z}_2)^4} \vec{\xi}_r \lambda_r(t) \quad .$$

This is a sistem of nonlocal, nonlinear equations for $\mathcal{A}_a(t)$.

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