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The Nonperturbative Gauge Coupling of $N=2$ Supersymmetric Theories

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Abstract

The topology of the quantum coupling space and the low energy effective action on the Coulomb branch of scale invariant $N=2$ $SU(n)$ gauge theories pick out a preferred nonperturbative definition of the gauge coupling τ. The S-duality group acts on this τ as a subgroup of $SL(2,\mathbf{R})$ generated by $T:\ \tau\to\tau+2$ and $S:\ \tau\to -1/[\sin^2(\pi/n)\,\tau]$.

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1 Introduction and discussion

The quantum coupling space of scale invariant $N = 2$ supersymmetric gauge theories is a subset of the classical one obtained by discrete identifications under the action of the S-duality group. These S-duality identifications imply an exact quantum equivalence between classically inequivalent theories. However, while the topology of the quantum coupling space has an invariant meaning, its parametrization is unambiguous only at weak coupling. Different parametrizations can give rise to different S-duality group actions on the gauge couplings.

In one nonperturbative definition of the coupling, the S-duality group was found to be $\Gamma^0(2) \subset SL(2, \mathbf{Z})$ for the scale invariant $SU(n)$ $N = 2$ SQCD theories [1]. This S-duality group is generated by $T: \ \tau \to \tau + 2$ and $S: \ \tau \to -1/\tau$ subject to the single constraint $S^2 = 1$. (We take the gauge coupling to be $\tau = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}$, differing by a factor of two from the usual definition.) A different nonperturbative parameterization of the $SU(3)$ gauge coupling proposed in [2] gives an S-duality group with the same T generator, but a different $\tilde{S}$ generator, $\tilde{S}: \ \tau \to -4/(3\tau)$. Although there exists a unique conformal map between the fundamental domains of gauge couplings in these two parameterizations, the corresponding S-duality groups are not isomorphic as abstract groups.

In this letter we argue that the topology of the quantum coupling space together with the low energy effective action on the Coulomb branch pick out a preferred nonperturbative definition of the gauge coupling. We explicitly compute the S-duality group associated with this definition of the coupling. We find that for the finite $N = 2$ SQCD with $SU(n)$ gauge group the S-duality group $\mathcal{G}_n$ is the subgroup of $SL(2, \mathbf{R})$ generated by

$$\begin{aligned} T: &\qquad \tau \to \tau + 2, \\ S_n: &\qquad \tau \to \frac{-1}{\sin^2(\pi/n)\ \tau}, \end{aligned} \tag{1}$$

acting on the classical coupling space $\{\mathrm{Im}\tau > 0\}$. For the $SU(2)$ theory $\mathcal{G}_2 \simeq \Gamma^0(2)$, though this duality group is naturally enlarged to $SL(2, \mathbf{Z})$ [3, 4] as we discuss below. For $SU(3)$, $\mathcal{G}_3$ coincides with the S-duality group proposed from a different perspective in [2]. The $\mathcal{G}_n$ type subgroups of $SL(2, \mathbf{R})$ were studied by Hecke in [5] in the context of Dirichlet series.

Before turning to the nonperturbative definition of the gauge coupling in the next section, we wish to point out some properties of the result (1). The S-duality group

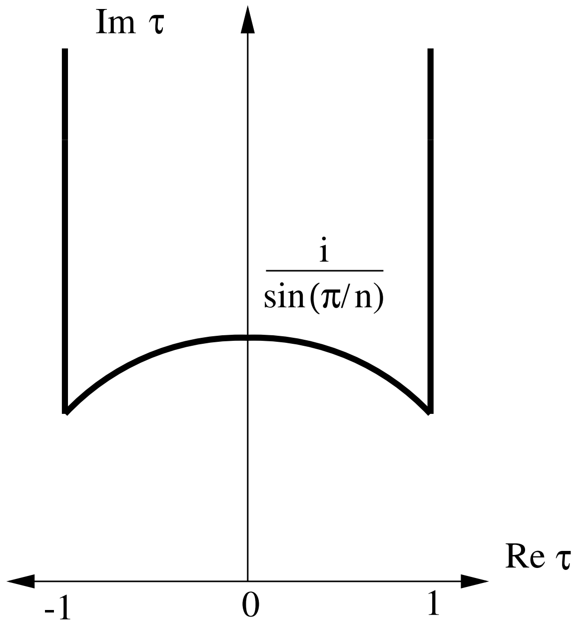

Figure 1: Fundamental domain in the τ plane of $\mathcal{G}_n$. Weak coupling is at $\tau = +i\infty$, the $\mathbf{Z}_2$ singularity is at $\tau = +i/\sin\frac{\pi}{n}$, and the conifold singularity is at $\tau = \pm 1 + i\cot\frac{\pi}{n}$.

$\mathcal{G}_n$ is not freely generated by T and S_n, but is subject to two constraints:

$$S_n^2 = 1 \tag{2}$$

and

$$\begin{array}{rclr} \left(S_n\, T^{-1}\right)^{2n} & = & 1, & n = 2k+1 \\ \left(S_n\, T^{-1}\right)^{n} & = & 1, & n = 4k \\ \left(S_n\, T^{-1}\right)^{n/2} & = & 1, & n = 4k+2. \end{array} \tag{3}$$

The fundamental domain of $\mathcal{G}_n$ is shown in Fig. 1. It is defined by $-1 \leq \mathrm{Re}\tau \leq 1$ and $|\tau| \geq 1/\sin(\pi/n)$ with edges identified.

This fundamental domain has three special points: a weak coupling singularity at $\tau = +i\infty$, a $\mathbf{Z}_2$ orbifold point at $\tau = +i/\sin(\pi/n)$, and a conifold singularity with opening angle $\pi(1-\frac{2}{n})$ at $\tau = \pm 1 + i\cot(\pi/n)$. For the special cases $n = 3, 4, 6, \infty$ the conifold points are actually $\mathbf{Z}_6$, $\mathbf{Z}_4$, $\mathbf{Z}_3$, and $\mathbf{Z}_2$ orbifold points. It is an open question whether there is an alternative characterization of the physics of these special strongly coupled theories. In any case, we find that dividing the classical coupling space by the action of the $\mathcal{G}_n$ S-duality group removes all ultrastrong ($\mathrm{Im}\tau = 0$) coupling singularities with zero opening angle (except in the SU(2) case). This suggests that further identification of strongly coupled $SU(n)$ theories with some other weakly coupled theories are unlikely.

Note that this preferred parameterization of τ is more natural from the large n point of view. In 't Hooft's large n limit [6], $g^2_{\rm eff} = g^2\, n$ is kept constant as $n \to \infty$. Note that for large n, the strong coupling points in the fundamental domain of $\mathcal{G}_n$ are at $g^2_{eff} \sim 1$, and that the strong-weak coupling duality S_n (1) acts as $\tau_{\rm eff} \to -1/\tau_{\rm eff}$ where $\tau_{\rm eff} = \pi\tau/n$ is the 't Hooft coupling.

It would be interesting to extend this construction to theories with other simple and semi-simple gauge groups. In particular, extension to the elliptic models of [7] may permit a comparison with the $SL(2,\mathbf{Z})$ S-duality group of $N = 4$ gauge theories.

2 Topological definition of the coupling

The quantum coupling space of scale invariant $N = 2$ SQCD has isolated singularities at special couplings where the *whole* Coulomb branch is singular. Just as traversing paths around the singularities on the Coulomb branch generate elements of the low energy electric-magnetic (EM) duality group (reflected in monodromies of the BPS spectrum), we argue that monodromies of the BPS spectrum around the singularities of the quantum coupling space, $\mathcal{F}$, generate the S-duality group.

Consider the $N = 2$ supersymmetric gauge theory with $SU(n)$ gauge group and $2n$ hypermultiplets in the fundamental representation. At a generic point on the Coulomb branch the gauge group is broken to $U(1)^{n-1}$ whose effective couplings τ_{ij} form a section of an $Sp(2n-2,\mathbf{Z})$ bundle on the Coulomb branch reflecting the EM duality identifications of the low energy effective description. The matrix of the effective couplings was identified in [1] with the complex structure (the period matrix) of the genus $n-1$ hyperelliptic curve Σ_n

$$y^2 = P^2(x) - f\ Q(x) \tag{4}$$

where $P(x) = x^n - \sum_{\ell=2}^{n} u_\ell\, x^{n-\ell}$ and $Q(x) = \prod_{j=1}^{2n}(x - m_j)$. The moduli u_ℓ parametrize the Coulomb branch, m_j are hypermultiplet mass parameters, and f is a function of the gauge coupling τ. At weak coupling $\tau \to +i\infty$, m_j coincide with physical masses of the hypermultiplets and $f \sim e^{i\pi\tau}$. At finite τ, mass parameters m_j in principle can depend both on physical masses and τ while f, being a flavor singlet, is expected to depend only on τ.

The complex structure of Σ_n degenerates whenever the discriminant of (4) vanishes. At fixed coupling f and mass parameters m_j, these singularities of the low-energy effective action are resolved by including in the effective description states that become massless there. The charge vectors of BPS states which are massive in the vicinity of a singularity undergo $Sp(2n-2, \mathbf{Z})$ EM duality group monodromies upon traversing closed paths in the Coulomb branch around the singularity.

Equivalently, we can think about the same singularities as singularities in the coupling parameter space, which we denote $\mathcal{F}$, at a fixed vacuum u_ℓ with given mass parameters m_j. We call such singularities "specific singularities" since their locations depend on the specific values of vacuum moduli and mass parameters. Specific singularities are not the only singularities in $\mathcal{F}$. The complex structure of (4) degenerates also at the "special singularities" $f = f_s \equiv \{0, 1, \infty\}$. These values of the coupling parameter are special in that the whole Coulomb branch becomes singular whenever $f = f_s$. Thus we can think of the quantum coupling space $\mathcal{F}$ as a three punctured sphere. More generally, in $N = 2$ scale invariant theories with simple gauge group, the gauge coupling τ is a section of a holomorphic line bundle over a three punctured sphere whose structure group is identified with the S-duality group. One of the punctures corresponds to weak coupling and the other two to special strongly coupled theories.

As an illustration of the difference between specific and special singularities, consider the scale invariant $SU(3)$ theory. The Coulomb branch of this theory is described by a curve Σ_3:

$$y^2 = \left(x^3 - u_2 x - u_3\right)^2 - f x^6. \tag{5}$$

The complex structure of Σ_3 degenerates when the discriminant of the right hand side of (5) vanishes,

$$u_3^{10}(f-1)f^3\left(f - \left[1 - \frac{4u_2^3}{27u_3^2}\right]^2\right) = 0, \tag{6}$$

and for $f \to \infty$ with u_ℓ kept finite. (In the latter case by an appropriate rescaling of x and y, Σ_3 reduces to the singular curve $y^2 = x^6$.) Clearly, $f = \{0, 1, \infty\}$ are always singularities of the low-energy effective action irrespective of the choice of the vacuum moduli u_ℓ. These are the special singularities of the gauge parameter space $\mathcal{F}$. The fourth singularity in $\mathcal{F}$ is at $f = [1 - (4u_2^3/27u_3^2)]^2$. This "specific" singularity differs from the previous three in that its position in $\mathcal{F}$ depends on the choice of the Coulomb branch vacuum. While there are always three special singularities at fixed positions in $\mathcal{F}$ for any rank of the gauge group, the number of specific singularities is rank dependent. For example, the scale invariant $SU(2)$ theory does not have specific singularities at all. Note that the $SU(3)$ vacuum with $u_2 = 0$ and $u_3 \neq 0$ is special in that the specific singularity coincides with the $f_s = 1$ special singularity.

Like the monodromies of the BPS spectrum around the singularities on the Coulomb branch, their monodromies around f_s on the coupling space $\mathcal{F}$ encode information about the S-duality group. But while S-duality transformations on the gauge coupling τ must be the same for any choice of vacuum moduli and hypermultiplet mass parameters, the monodromies in $\mathcal{F}$ around f_s actually depend on this choice. This presents a puzzle: How can the S-duality group information which is invariant under changes in the vacuum moduli be extracted from these monodromies? The key to solving this puzzle is to realize that, in principle, noncontractable loops in $\mathcal{F}$ can be generators of both the S-duality and the low energy EM duality groups. In fact, nontrivial loops around the specific singularities in $\mathcal{F}$ have nothing to do with the S-duality group: any such loop can be deformed in the combined Coulomb branch and quantum coupling space to a loop around a singularity on the Coulomb branch at a fixed value of the gauge coupling parameter f.

So an S-duality group (and hence the coupling τ it acts on) can be defined as the subgroup of the $Sp(2n-2,\mathbf{Z})$ EM duality group generated by the monodromies along some choice of basis cycles around the special singularities f_s in $\mathcal{F}$. How those cycles are chosen to go around the specific singularities in $\mathcal{F}$ affects the resulting S-duality group.

We now note that there is a *unique* choice of $SU(n)$ vacuum on the Coulomb branch, namely $u_\ell = 0$, $\ell = 2,\ldots,n-1$, and $u_n \equiv u \neq 0$ with mass parameters $m_j = 0$, for which the above ambiguity in the definition of the S-duality group disappears.[1] This is simply because for this choice of vacuum *all* the singularities in $\mathcal{F}$ are "special singularities". (The importance of this Coulomb branch submanifold was stressed in [8, 9] where S-duality identifications were analyzed from an algebraic point of view.) It is this unique and natural choice which gives us our nonperturbative definition of the coupling τ.

An explicit formula relating τ to f is obtained as follows. In our special choice of vacuum, there is an unbroken global $\mathbf{Z}_n$ discrete subgroup of the anomaly-free $U(1)_R$ which significantly simplifies the computation of the periods. In fact, all the periods can be computed exactly (see the Appendix for details) and we find

$$a_i(f,u) \;=\; \frac{\omega_n^i - 1}{\omega_n - 1}\, a(f,u),$$

[1]Note that we have actually specified a one complex dimensional submanifold of the Coulomb branch; however, all these vacua are physically equivalent since they only depend on the single dimensionful vev u which spontaneously breaks the scale invariance of the theory.

$$b_i(f,u) \;=\; (\omega_n^{i-1}-\omega_n^i)\; b(f,u) - \omega_n^i\; a(f,u), \tag{7}$$

where $\omega_n = e^{2\pi i/n}$,

$$\begin{aligned} a(f,u) &= \frac{-2i\; u^{1/n}}{n} \frac{\mathcal{B}(\frac{1}{2},\frac{1}{2})}{(1+\sqrt{f})^{1/n}}\; {}_2F_1\Big(\frac{1}{2},\frac{1}{n},1,\frac{2\sqrt{f}}{1+\sqrt{f}}\Big), \\ b(f,u) &= \frac{2\; u^{1/n}}{n} \frac{\mathcal{B}(\frac{1}{2},\frac{1}{n})}{(1+\sqrt{f})^{1/n}}\; {}_2F_1\Big(\frac{1}{2},\frac{1}{n},\frac{1}{2}+\frac{1}{n},\frac{1-\sqrt{f}}{1+\sqrt{f}}\Big), \end{aligned} \tag{8}$$

$\mathcal{B}$ is a standard beta-function, and ${}_2F_1$ is a hypergeometric function. We thus have the exact relation

$$b_i = \frac{1}{2}\;\tau\; C_{ij}\; a_j \tag{9}$$

where C_{ij} is a Cartan matrix of $SU(n)$ (depending on the precise choice of basis of "electric" and "magnetic" periods a_i and b_i), and

$$\tau \equiv 2i\; \frac{\mathcal{B}(\frac{1}{2},\frac{1}{n})}{\mathcal{B}(\frac{1}{2},\frac{1}{2})} \frac{{}_2F_1\Big(\frac{1}{2},\frac{1}{n},\frac{1}{2}+\frac{1}{n},\frac{1-\sqrt{f}}{1+\sqrt{f}}\Big)}{{}_2F_1\Big(\frac{1}{2},\frac{1}{n},1,\frac{2\sqrt{f}}{1+\sqrt{f}}\Big)} - \frac{2\;\omega_n}{1-\omega_n}. \tag{10}$$

Since the relation (9) is preserved by the monodromies in $\mathcal{F}$, this picks out (10) as the natural nonperturbative definition of the coupling.

Indeed, it is easy to check that at weak coupling, $f \to 0$, $\tau \simeq -(i/\pi)\ln f$, which is the perturbative answer. The monodromy around $f = 0$ generates $\tau \to \tau + 2$ while the monodromy around $f = \infty$ generates $\tau \to -1/(\tau\;\sin^2\frac{\pi}{n})$, together generating the S-duality group $\mathcal{G}_n$ of (1). It is also easy to see from (10) that the special singularities $f = \infty, 1$ are mapped to the $\tau = +i/\sin(\pi/n)$ $\mathbf{Z}_2$ point and the $\tau = \pm 1 + i\cot(\pi/n)$ conifold point, respectively.

An interesting feature of the $\mathcal{G}_n$ S-duality identifications is that they remove all the ultrastrong ($\mathrm{Im}\tau = 0$) points in the classical coupling space except for the $SU(2)$ theory. Here $\mathcal{G}_2 \simeq \Gamma^0(2)$, which is only a subgroup of the full conjectured $SL(2,\mathbf{Z})$ duality group [3]. The existence of an ultrastrong singularity $\tau = 1$ in the fundamental domain of $\mathcal{G}_2$ is a clue that the full S-duality group bigger. Indeed, using the isomorphism between $SU(2)$ and Sp(2) gauge groups it was shown in [4] that there are extra identifications over the gauge parameter space $\mathcal{F}$ of the $SU(2)$ theory

$$\frac{2f(-1-\sqrt{1-f}+f)}{(-1-\sqrt{1-f}+3f/2)^2} \;\longleftrightarrow\; f \;\longleftrightarrow\; \frac{2f(-1+\sqrt{1-f}+f)}{(-1+\sqrt{1-f}+3f/2)^2} \tag{11}$$

which imply that the three punctured sphere is actually a triple cover the true quantum coupling space. The identifications (11) translate through (10) into

$$\tau + 1 \longleftrightarrow \tau \longleftrightarrow \frac{2-\tau}{1-\tau}, \tag{12}$$

enlarging $\Gamma^0(2)$ to the full $SL(2,\mathbf{Z})$ S-duality of the scale invariant $SU(2)$ theory.

Acknowledgments

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Appendix

We compute the periods of the $SU(n)$ $N=2$ gauge theory with $2n$ massless quarks in the fundamental representation of the gauge group in the $\mathbf{Z}_n$ invariant vacuum.

The $\mathbf{Z}_n$ invariant vacuum, $\mathcal{C}_n$, is characterized by the Coulomb branch moduli $u_\ell = 0$, $\ell = 2, \ldots, n-1$ and $u_n \equiv u \neq 0$. The curve $\Sigma_{\mathcal{C}_n}$ describing the low-energy effective action at $\mathcal{C}_n$ is

$$y^2 = (x^n - u)^2 - f\ x^{2n}. \tag{13}$$

It is nonsingular unless f is at a special singularity in $\mathcal{F}$.

For a nonsingular gauge parameter f the low-energy gauge group at $\mathcal{C}_n$ is $U(1)^{n-1}$. We choose these $U(1)$ factors to be aligned with the simple roots of the $SU(n)$ group which ensures that at weak coupling ($f \to 0$) the complex structure of (13) is proportional to a Cartan matrix of $SU(n)$. Up to an overall irrelevant normalization, electric a_i and magnetic b_j cycles are represented by the periods of the holomorphic one-form $\lambda = (u/y)dx$ over a symplectic homology basis of $\Sigma_{\mathcal{C}_n}$:

$$a_i = \oint_{\alpha_i} \lambda, \qquad b_j = \oint_{\beta_j} \lambda. \tag{14}$$

The homology basis may be constructed as follows. Start with the non-symplectic basis $\{\tilde{\alpha}_i, \tilde{\beta}_j\}$ shown in Fig. 2. The basis

$$\alpha_i = \sum_{k=1}^{i} \tilde{\alpha}_k, \qquad i = 1, \ldots, n-1$$

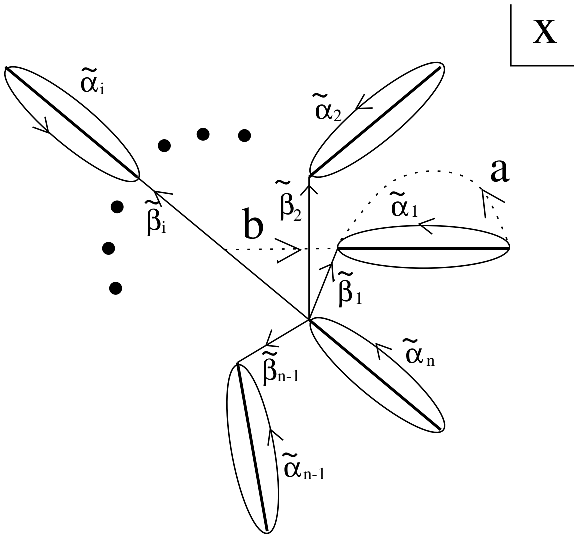

Figure 2: Representation of a sheet in two-sheeted covering of $\Sigma_{\mathcal{C}_n}$ with (non-symplectic) basis $\{\tilde\alpha_i, \tilde\beta_j\}$. The dark lines represent the cuts on the x-plane. Only half of the magnetic cycles $\tilde\beta_j$ are visible. The dotted lines represent the a and b integrals.

$$\beta_j = \tilde\beta_j - \tilde\beta_{j+1} - \tilde\alpha_{j+1}, \qquad j = 1, \ldots, n-1 \tag{15}$$

then has canonical intersections $\alpha_i \cdot \beta_j = \delta_{ij}$, $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and the required alignment of the low-energy $U(1)$ gauge factors. (In (15) $\tilde\beta_n$ should be set to zero.) The periods of the non-symplectic basis,

$$\tilde a_i = \oint_{\tilde\alpha_i} \lambda, \qquad \tilde b_j = \oint_{\tilde\beta_j} \lambda, \tag{16}$$

are simply related to the a and b integrals of Fig. 2 by the $\mathbf{Z}_n$ symmetry of the branched x-plane:

$$\begin{aligned} \tilde a_i &= 2\ \omega_n^{i-1}\ a, \\ \tilde b_i &= 2\ (\omega_n^{i-1} - \omega_n^{n-1})\ b, \end{aligned} \tag{17}$$

where

$$\begin{aligned} a = \int_{x_-}^{x_+} \frac{u\,dx}{\sqrt{(x^n-u)^2 - f\ x^{2n}}} &= -i\frac{u^{1/n}}{n}\ \frac{\mathcal{B}(\frac{1}{2},\frac{1}{2})}{(1+\sqrt{f})^{1/n}}\ {}_2F_1\Big(\frac{1}{2},\frac{1}{n},1,\frac{2\sqrt{f}}{1+\sqrt{f}}\Big), \\ b = \int_{0}^{x_+} \frac{u\,dx}{\sqrt{(x^n-u)^2 - f\ x^{2n}}} &= \frac{u^{1/n}}{n}\ \frac{\mathcal{B}(\frac{1}{2},\frac{1}{n})}{(1+\sqrt{f})^{1/n}}\ {}_2F_1\Big(\frac{1}{2},\frac{1}{n},\frac{1}{2}+\frac{1}{n},\frac{1-\sqrt{f}}{1+\sqrt{f}}\Big), \end{aligned} \tag{18}$$

where $\omega_n = e^{2\pi i/n}$ and $x_\pm = u^{1/n}/(1\pm\sqrt{f})^{1/n}$. Combining (15), (17) and (18) we obtain (7).

References

[1] P.C. Argyres, M.R. Plesser, and A.D. Shapere, *Coulomb phase of N=2 supersymmetric QCD*, hep-th/9505100, Phys. Rev. Lett. **75**(1995)1699.

[2] J. A. Minahan and D. Nemeschansky *Hyperelliptic curves for Supersymmetric Yang-Mills*, hep-th/9507032, Nucl. Phys. **B464**(1996)3.

[3] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, hep-th/9408099, Nucl. Phys. **B431**(1994)484.

[4] P.C. Argyres, *S-duality and global symmetries in N=2 supersymmetric field theory*, hep-th/9706095, Adv. Theor. Math. Phys. **2**(1998)61

[5] E. Hecke, *Lectures on Dirichlet-series, modular functions and quadratic forms*, Vandenhoeck & Ruprecht(1983).

[6] G. 't Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. **72**(1974)461.

[7] E. Witten, *Solutions of four-dimensional field theories via M theory*, hep-th/9703166, Nucl. Phys. **B500**(1997)3.

[8] O. Aharony and S. Yankielowicz, *Exact Electric-Magnetic Duality in N=2 Supersymmetric QCD Theories*, hep-th/9601011, Nucl. Phys. **B473**(1996)93.

[9] J. A. Minahan and D. Nemeschansky, $N=2$ *Super Yang-Mills and Subgroups of* $SL(2,Z)$, hep-th/9601059, Nucl. Phys. **B468**(1996)72.