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Dyons in $N = 4$ Supersymmetric Theories and Three-Pronged Strings

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ABSTRACT

We construct and explore BPS states that preserve 1/4 of supersymmetry in $N = 4$ Yang-Mills theories. Such states are also realized as three-pronged strings ending on D3-branes. We correct the electric part of the BPS equation and relate its solutions to the unbroken abelian gauge group generators. Generic 1/4-BPS solitons are not spherically symmetric, but consist of two or more dyonic components held apart by a delicate balance between static electromagnetic force and scalar Higgs force. The instability previously found in three-pronged string configurations is due to excessive repulsion by one of these static forces. We also present an alternate construction of these 1/4-BPS states from quantum excitations around a magnetic monopole, and build up the supermultiplet for arbitrary (quantized) electric charge. The degeneracy and the highest spin of the supermultiplet increase linearly with a relative electric charge. We conclude with comments.

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1 Introduction

Among supersymmetric theories that are known to admit a strong-weak coupling duality, $N = 4$ $D = 4$ supersymmetric Yang-Mills field theories are perhaps the easiest and most straightforward to study. In its Coulomb phase, the solitonic spectra are scrutinized in great detail, where a manifest strong-weak coupling duality was observed among the charged BPS particles that break exactly half of supersymmetry. This includes the usual BPS magnetic monopoles and standard dyonic excitations thereof whose electric charges are proportional to the magnetic charge. These BPS monopoles and dyons break half of $N = 4$ supersymmetry, and duality predicts that they are all in the $N = 4$ vector multiplet with the maximum spin 1, a short multiplet of degeneracy $2^4 = 16$.

There are, however, other kinds of supersymmetric states which break 3/4 of supersymmetry. Such states would come in an intermediate multiplet which contains spin 3/2 or higher. It is only very recently that their properties are explored. Most notable is a work by O. Bergman [1] who constructed such dyons as three-pronged strings that end on three parallel D3-branes. Here, we recapitulate this construction.

Recall that $N = 4$ $D = 4$ $U(n) = SU(n) \times U(1)$ Yang-Mills theory is a world-volume theory of n parallel D3-branes [2]. The Coulomb phase of the $U(n) \to U(1)^n$ theory is parameterized by six adjoint Higgs expectations, whose $6n$ eigenvalues encode the positions of the n D3-branes in the internal part R^6 of the spacetime $R^6 \times R^{3+1}$. One special feature of the D3-brane is that it is self-dual under the $SL(2, Z)$ U-duality of the type IIB string theory. As far as the low energy world-volume physics goes, a practical consequence of this is that any (q, g)-string may end on the D3-brane. Here q and g are the charges with respect to the two antisymmetric tensor fields $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ that lives, respectively, in the NS-NS sector and in the Ramond-Ramond sector of the type IIB theory. With respect to the unbroken $U(1)$ associated with the D3-brane where a (q, g) string end, then, such an end-point appears as a particle of q electric and g magnetic charges. The familiar BPS (q, g) dyons of $SU(n)$ theory corresponds to a straight (q, g) string segment that connects a pair of D3-branes.

A novelty comes from the fact that three-pronged strings are also in the spectrum of string theory/M theory. They can be used to connect a set of three D3-branes. The three segments that meet at a single junction must have different (q, g)'s to preserve some supersymmetry [3, 4], so the resulting BPS state has its electric charge not proportional to its magnetic charge. Typically, it will break 3/4 of the $N = 4$ supersymmetry.[3] We will use the phrase "1/4-BPS state" to distinguish

[3] Three-pronged strings can also generate BPS states in $N = 2$ theories [5]. In such cases, they actually break only

from the usual BPS states that break only half of supersymmetry. For instance, suppose that we have $SU(n)$ broken down to $U(1)^{n-1}$. Pick a pair of roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = 1$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = -1/2$. A state of magnetic charge $m\boldsymbol{\alpha} + m\boldsymbol{\beta}$ and of electric charge $n\boldsymbol{\alpha}$ would then be 1/4-BPS.

Now the question is how these 1/4-BPS states are realized on the field theory side. One might be tempted to look for a spherically symmetric soliton. In fact, very recently, a special class of 1/4-BPS states in $SU(3)$ theory was found in a spherically symmetric ansatz [6]. In terms of roots, these BPS configurations carry magnetic charge of $2\boldsymbol{\alpha} + 2\boldsymbol{\beta}$. However, as will become clear in later sections, the existence of these solutions is quite accidental and fails to illuminate how the general 1/4-BPS dyons are constructed in the field theory language. One severe problem is that if their electric charge is, say, of the form $q\boldsymbol{\alpha}$, the real number[4] q is determined uniquely by the Higgs vev's. (In the spherically symmetric case of the total magnetic charge, $\boldsymbol{\alpha} + \boldsymbol{\beta}$, for instance, q has to vanish for all vev's.) Because of this, at generic point of vacuum moduli space, BPS configurations of properly quantized electric charge ($q =$ integer) cannot be realized as a spherically symmetric classical soliton.

In general, we expect the BPS configurations to be of an elongated shape. Roughly speaking, it will consist of a pair of dyonic cores which are bound but separated by some distance R. This is due to a delicate balance between static electromagnetic force and scalar Higgs force. (See section 3.) Once we realize this, it is almost obvious that the amount of electric charge has to depend on the separation R as well as Higgs vev's. What one misses by insisting the spherical symmetry is this extra parameter R. With this picture in mind, it is now clear that a BPS configuration of given electric and magnetic charges will have some definite length R that parameterizes the deviation from the spherical symmetry.

This begs for another question: what happens in the limit of $R \to \infty$? Since it is electromagnetic and Higgs interaction that separates the two dyonic cores, a change in R implies a change in electric charge. At $R \to \infty$, the electric charge of the 1/4-BPS state reaches a limiting value. In all cases we consider, the charge will actually reach its maximum possible value. Trying to put an even larger electric charge will result in an instability and cause the two cores to fly away from each other. The upper bound on the electric charge can be also translated into a lower bound on a linear combination of Higgs vev's with any given electric charge, in which form the instability was found

half of supersymmetry.

[4]Recall the electric charge is not quantized in classical dyon solutions, unlike the magnetic charge which is quantized topologically.

in the three-pronged string configuration in Ref. [1].

The paper is organized as follows. In section 2, we derive the BPS bound of the energy functional and write down the complete set of equations that 1/4-BPS dyons must satisfy. This corrects and generalizes those in Ref. [7]. The magnetic part of the equations are unaffected by the electric part. Given any purely magnetic BPS solutions, the electric part is determined by solving a single *four-dimensional* covariant Laplace equation of an adjoint scalar. The existence of its solutions is tied to the existence of $U(1)$ gauge zero-modes of the purely magnetic soliton, which completes the existence proof of all the expected 1/4-BPS dyonic states corresponding to three-pronged strings. In section 3, we take the specific example of $SU(3)$ broken to $U(1)^2$. The 1/4-BPS dyonic configuration of magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is constructed, from which we extract the relationship between Higgs vev's, electric charges, and the separation length R. Important but technical details involve ADHMN construction, which we put in the appendices. We digress in section 4, and compare the field theory results to those from D-brane/three-pronged string picture. The instability bound is compared with that from the string construction, and a perfect fit is found.

In section 5, we present an alternate construction of the 1/4-BPS dyons via exciting compactly supported eigenmodes around spherically symmetric monopoles of magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$. The correct supermultiplet structure of 1/4-BPS states are shown to be reproduced, after a careful consideration of low energy eigenmodes. The approximation, however, ignores some backreaction of the bosonic background to excitation of these eigenmodes, which puts a stringent criteria on the validity of the construction. Because of this, in particular, it is impossible to see the instability in this second picture. In section 6, we use this construction to build up the supermultiplet structure of dyons of arbitrary quantized electric charge. Finally in section 7, we conclude with comments on unresolved issues.

2 BPS Energy Bound and Equations

Since the electric part of the BPS equations we found is different from what is commonly known [7], we will rederive the BPS energy bound and equations from scratch. Also there are several interesting new comments to be made about the BPS field configurations. We start by considering the bosonic Lagrangian of the $N=4$ supersymmetric Yang-Mills theories. With the gauge group $SU(n)$ with hermitian generators T^a in the n dimensional representation with the normalization $\mathrm{tr}\, T^aT^b = \delta^{ab}/2$, we introduce the gauge field $A_\mu = A_\mu^a T^a$ and six Higgs fields $\phi_I = \phi_I^a T^a$, $I = 1, ..., 6$. The bosonic Lagrangian density is

$$\mathcal{L} = \mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_I D^\mu\phi_I - \frac{1}{2}\sum_{I,J=1}^{6}(-ie[\phi_I,\phi_J])^2\right\}, \tag{2.1}$$

where $D_\mu\phi_I = \partial_\mu\phi_I - ie[A_\mu,\phi_I]$.

2.1 BPS bound

The energy density is

$$\begin{aligned}
\mathcal{H} &= \mathrm{tr}\left\{(E_i)^2 + (B_i)^2 + (D_0\phi_I)^2 + (D_i\phi_I)^2 + \sum_{I<J}(-ie[\phi_I,\phi_J])^2\right\} \\
&= \mathrm{tr}\left\{(a_I E_i + b_I B_i - D_i\phi_I)^2 + (D_0\phi_I)^2 + \sum_{I<J}(-ie[\phi_I,\phi_J])^2\right\} \\
&+ 2\,\mathrm{tr}\left\{E_i D_i a\cdot\phi + B_i D_i b\cdot\phi\right\},
\end{aligned} \tag{2.2}$$

where a_I, b_I are two arbitrary six-dimensional unit vectors orthogonal to each other, $a\cdot\phi \equiv a_I\phi_I$ and $b\cdot\phi \equiv b_I\phi_I$. The cross terms can be rewritten as,

$$\mathrm{tr}\, B_i D_i b\cdot\phi = \partial_i\left(\mathrm{tr}\, b\cdot\phi B_i\right), \tag{2.3}$$

$$\mathrm{tr}\, E_i D_i a\cdot\phi = \partial_i\left(\mathrm{tr}\, a\cdot\phi E_i\right) - ie\,\mathrm{tr}\,\left(D_0\phi_I[a\cdot\phi,\phi_I]\right), \tag{2.4}$$

where we used the Bianchi identity $D_iB_i = 0$ and the Gauss law,

$$D_iE_i - ie[\phi_I, D_0\phi_I] = 0. \tag{2.5}$$

Denote collectively by ζ_I, the components of ϕ_I which are orthogonal to both a_I and b_I. We split the energy density from the scalar fields into two parts;

$$(D_0 a\cdot\phi)^2 + (D_0 b\cdot\phi)^2 + (-ie[a\cdot\phi, b\cdot\phi])^2, \tag{2.6}$$

and

$$(D_0\zeta_I)^2 + (-ie[a\cdot\phi,\zeta_I])^2 + (-ie[b\cdot\phi,\zeta_J])^2 + \sum_{I<J}(-ie[\zeta_I,\zeta_J])^2, \tag{2.7}$$

then complete the squares in the energy density as,

$$\begin{aligned}
\mathcal{H} &= \mathrm{tr}\left\{(E_i - D_i a\cdot\phi)^2 + (B_i - D_i b\cdot\phi)^2 + (D_0 a\cdot\phi)^2 + (D_0 b\cdot\phi - ie[a\cdot\phi, b\cdot\phi])^2\right\} \\
&+ \mathrm{tr}\left\{(D_0\zeta_I - ie[a\cdot\phi,\zeta_I])^2 + (D_i\zeta_I)^2 + (-ie[b\cdot\phi,\zeta_J])^2 + \sum_{I<J}(-ie[\zeta_I,\zeta_J])^2\right\} \\
&+ 2\partial_i\,\mathrm{tr}\,\left\{a\cdot\phi E_i + b\cdot\phi B_i\right\}.
\end{aligned} \tag{2.8}$$

Every term except those in the last line is nonnegative, so the total energy is bounded by the contribution from the latter;

$$\mathcal{E} = \int d^3x \, \mathcal{H} \; \geq \; \mathrm{Max} \, (a_I Q_I^E + b_I Q_I^M), \tag{2.9}$$

with

$$Q_I^E \;=\; 2 \int d^3x \, \partial_i \, (\mathrm{tr} \, \phi_I E_i), \tag{2.10}$$

$$Q_I^M \;=\; 2 \int d^3x \, \partial_i \, (\mathrm{tr} \, \phi_I B_i). \tag{2.11}$$

One most stringent bound must be found by varying a_I and b_I and achieving the maximum. The quantities Q_I^E and Q_I^M can be evaluated by converting to boundary integrals, and clearly depends on the asymptotics only.

The expression $a_I Q_I^E + b_I Q_I^M$ is maximized only if the two unit vectors lie on the plane spanned by Q_I^M and Q_I^E. Assuming this, let α be the angle between Q_I^M and Q_I^E, and θ the one between b_I and Q_I^M. The extrema occur if and only if

$$\pm a_I Q_I^M = b_I Q_I^E, \tag{2.12}$$

which can be translated to an equivalent condition;

$$\tan\theta = \frac{\pm Q^E \cos\alpha}{Q^M \pm Q^E \sin\alpha}. \tag{2.13}$$

Q^M and Q^E are the magnitude of vectors Q_I^M and Q_I^E. The two positive extrema are the two central terms of $N = 4$ supersymmetry algebra,,

$$Z_\pm = \sqrt{(Q^M)^2 + (Q^E)^2 \pm 2 Q^M Q^E \sin\alpha}. \tag{2.14}$$

The true BPS bound for $N = 4$ theory is then,

$$\mathcal{E} \geq \mathrm{Max} \, (Z_+, Z_-). \tag{2.15}$$

2.2 BPS Equations in Generic $N = 4$ Vacua

The BPS bound is saturated when every bulk term in the energy density vanishes, from which we obtain total eight sets of equations. The first part is the most familiar;

$$B_i = D_i b \cdot \phi. \tag{2.16}$$

This is the usual BPS equation that admits magnetic monopole solutions. Note that this magnetic equation can be solved independently, regardless of remaining equations. The other BPS equations influence only the choice of the unit vector b_I. This fact is of crucial important when we construct the BPS solution later.

The second, electric part is made of several equations

$$E_i = D_i a \cdot \phi, \tag{2.17}$$

$$D_0 a \cdot \phi = 0, \tag{2.18}$$

$$D_0 b \cdot \phi = -ie[b \cdot \phi, a \cdot \phi]. \tag{2.19}$$

Using the latter two, we reduce the Gauss law (2.5) to

$$D_i E_i = e^2\,[b \cdot \phi, [b \cdot \phi, a \cdot \phi]] + e^2[\zeta_I, [\zeta_I, a \cdot \phi]]. \tag{2.20}$$

Combining this with Eq. (2.17) into a single second order linear differential equation, we find that

$$D_i D_i\, a \cdot \phi = e^2\,[b \cdot \phi, [b \cdot \phi, a \cdot \phi]] + e^2[\zeta_I, [\zeta_I, a \cdot \phi]]. \tag{2.21}$$

which is a linear equation for $a \cdot \phi$ once ζ_I's are given.

So far we have not required that the spatial gauge field A_i be time-independent. If we choose such a gauge, one sees easily that Eq. (2.17) is solved by

$$A_0 = -a \cdot \phi. \tag{2.22}$$

In this gauge, $D_0\zeta_I - ie\,[a \cdot \phi, \zeta_I] = \partial_0 \zeta_I = 0$, which requires ζ_I be time-independent. Other ζ_I equations require them to be covariantly constant ($D_i\zeta_I = 0$), commute with $b\cdot\phi$, and also commute among themselves: In the unitary gauge where $b\cdot\phi$ is diagonal, the ζ's are all diagonal, constant, and uniform, and also commute with the A_i's. The latter condition implies that each ζ_I is proportional to the identity in each irreducible block(s) spanned by nontrivial part of the configurations A_i and $b \cdot \phi$.[5] Imagine that one think of the magnetic solution to Eq. (2.16) as embedded along a subgroup of the original gauge group; the expectation value ζ_I's must be invariant under such a subgroup.

Now Eq. (2.21) is a zero-eigenvalue problem of a nonnegative operator acting on $a \cdot \phi$ linearly. Under the boundary condition that $a \cdot \phi(\infty)$ should commute with the asymptotics of $b \cdot \phi$ and

[5] If we were considering more general configurations with many three-pronged strings connected to form a string web, this would translate to the requirement that the BPS string web is planar in the internal space R^6.

ζ_I, its solutions have nontrival behavior only in the said irreducible block(s). Thus ζ_I should also commute with $a \cdot \phi$. With such expectation value ζ_I's, Eq. (2.21) reduces to,

$$D_i D_i \, a \cdot \phi = e^2 \, [b \cdot \phi, [b \cdot \phi, a \cdot \phi]]. \tag{2.23}$$

This is a four-dimensional covariant Laplacian for an adjoint scalar field, provided that we identify $D_4 \equiv -ieb \cdot \phi$. A more restricted version of this equation, where one assumes $[b \cdot \phi, a \cdot \phi] = 0$ as well, has appeared and been used in existing literatures [7, 6]. Thus, we find two sets of relevant BPS equations, given by Eq. (2.16) and (2.23), that must be solved to produce classical 1/4-BPS configurations. (See Appendix E for a discussion about the energy density of BPS configurations.)

2.3 Dyons and the Scalar BPS Equation

The general configuration will have both magnetic and electric charges. Along, say, $-z$ axis, the asymptotic behavior of the Higgs fields will be

$$b \cdot \phi \simeq b \cdot \phi(\infty) - \frac{\mathbf{g} \cdot \mathbf{H}}{4\pi r}, \tag{2.24}$$

$$a \cdot \phi \simeq a \cdot \phi(\infty) - \frac{\mathbf{q} \cdot \mathbf{H}}{4\pi r}. \tag{2.25}$$

The $n-1$ dimensional vectors $\mathbf{g}$ and $\mathbf{q}$ are the magnetic and the electric charge, respectively, while $\mathbf{H}$ generates the Cartan subalgbra of $SU(n)$.

We need to solve the first order equation (2.16) and the second order equation (2.23). The first order equation is the well-understood BPS equation for monopoles [8]. Let the vacuum expectation values of the Higgs be such that

$$b \cdot \phi(\infty) = \mathbf{h} \cdot \mathbf{H} = \mathrm{diag}(h_1, h_2, .., h_n), \tag{2.26}$$

where $\sum_a h_a = 0$ and $h_1 < h_2 < ... < h_n$.[6] The magnetic charge of any BPS configuration should satisfy the topological quantization,

$$\mathbf{g} \cdot \mathbf{H} = \sum_{r=1}^{n-1} \frac{4\pi}{e} \, l_r \boldsymbol{\beta}_r \cdot \mathbf{H} = \frac{2\pi}{e} \, \mathrm{diag}(-l_1, l_1 - l_2, l_2 - l_3, ..., l_{n-1}) \tag{2.27}$$

with nonnegative integers l_r. One interprets such configurations as being made of $n-1$ species of fundamental monopoles, where l_r is the number of the r-th fundamental monopole associated with the simple root $\boldsymbol{\beta}_r$. The conditions on the diagonal ζ_I's can be translated quite easily now.

[6]These quantities h_i can be thought of as projected coordinate values of the n D3-brane positions along the b_I direction. Thus, the gauge symmetry could be still broken even when some of h_i's coincide.

Generically, ζ_I must have vanishing inner products with all $\boldsymbol{\beta}_r$ whenever $l_r \neq 0$. Only exception is when a consecutative chain of $\boldsymbol{\beta}_r$ is such that $l_s = \cdots = l_{s+t}$ and the corresponding monopoles are "coincident." In that case, ζ_I must have a vanishing inner product with $\sum_{r=s}^{r=s+t} \boldsymbol{\beta}_r$ but not necessarily with individual $\boldsymbol{\beta}_r, \ldots, \boldsymbol{\beta}_{r+s}$.

The second-order BPS equation (2.23) is to be solved in the background of purely magnetic solutions to $B_i = D_i\,(b\cdot\phi)$. While we will come back to actual solutions for specific examples next section, it is important to note that the existence of the solution is already well established. In fact, we know the exact number of linearly independent solutions. This is because any gauge zero mode of a BPS monopole solution is automatically a solution to Eq. (2.23).

Recall that the conventional way of finding zero-modes of BPS monopoles is to perturb $B_i = D_i\Phi$ and impose the background gauge $D_i\delta A_i = ie\,[\Phi, \delta\Phi]$ [8]. For a gauge zero-mode, say, generated by a gauge function Λ, the linearized BPS equations are always satisfied since both B_i and $D_i\Phi$ are gauge-covariant. Only the gauge-fixing condition is nontrivial,

$$D_i\delta A_i = ie\,[\Phi, \delta\Phi] \qquad \Rightarrow \qquad D_iD_i\Lambda = e^2[\Phi, [\Phi, \Lambda]]. \tag{2.28}$$

Inserting the solution to $B_i = D_i(b\cdot\phi)$ as the background field, and replacing Λ by $a\cdot\phi$, we realize that this is identical to Eq. (2.23). The number of solutions to this covariant Laplace equation, must equal the number of unbroken $U(1)$ generators that act nontrivially on the monopole solution. There must be at least one and at most $n-1$.

Where is the electric charge located? When magnetic monopoles described by the first BPS equation (2.16) are well separated from each other, the field configuration outside the core region would be purely abelian and so cannot carry any electric charge. Each fundamental monopole may carry only its own type of electric charge, that is, $\boldsymbol{\beta}_r$ monopoles can carry only $\boldsymbol{\beta}_r$ electric charge for any simple roots $\boldsymbol{\beta}_r$. One could say that generic 1/4-BPS configurations are made of classically bound (two or more) 1/2-BPS dyons.

One might think that there is something odd about what we are doing here. After all, what we mean by $b\cdot\phi$ and $a\cdot\phi$ do depend on what kind of electric and magnetic charges we have, yet we seem to have fixed b_I even before turning on the electric charge. But what matters at the end of the day is that we get a set of field configurations that solve all BPS equations simultaneously for some b_I and a_I. The BPS bound is a mini-max problem where one tries to obtain a most stringent lower bound for all reasonably smooth configurations. The simple fact that a configuration saturates a lower bound, implies that the bound it saturates is actually the maximum possible of all lower bounds. In section 3, we shall see how this is realized in a concrete way.

3 1/4-BPS Soliton in the $SU(3)$ Theory

As an example, let us consider the $SU(3)$ gauge group. Following the strategy outlined in the previous section, we start with a purely magnetic BPS configuration of a pair of distinct monopoles. The configuration must solve only the magnetic part of BPS equations, and the scalar BPS equation will be solved in that background.

If we let $b\cdot\phi(\infty)$ be equal to diag (h_1, h_2, h_3) with $h_1 < h_2 < h_3$ and $h_1 + h_2 + h_3 = 0$, the two fundamental monopoles would have magnetic charges,[7]

$$4\pi\,\boldsymbol{\alpha}\cdot\mathbf{H} = 2\pi\ \mathrm{diag}(-1,+1,0), \tag{3.1}$$

$$4\pi\,\boldsymbol{\beta}\cdot\mathbf{H} = 2\pi\ \mathrm{diag}(0,-1,+1). \tag{3.2}$$

We will label these monopoles by their charge vector in root space; $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Throughout the rest of the paper, we will consider 1/4-BPS configurations with magnetic charge of $\boldsymbol{\alpha}+\boldsymbol{\beta}$. Accordingly, the asymptotic behavior of $b\cdot\phi$ would be

$$b\cdot\phi \simeq \mathrm{diag}(h_1,h_2,h_3) - \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{H}}{r}. \tag{3.3}$$

From the work of E. Weinberg [8], we learn that the separation between the two monopole cores is an arbitrary parameter, which we denote by R. R uniquely determines A_i and $b\cdot\phi$ up to overall position, spatial orientation, and internal gauge angles. The explicit form of the field configuration can be obtained in principle from the ADHMN formalism [9, 10]. The latter is summarized in Appendices A and B. Recently, E. Weinberg and one of the authors (P.Y.) have found the explicit A_i and $b\cdot\phi$ configuration for these two monopoles by exploring the Nahm's formalism [11].

Now the difficult part is to solve the covariant Laplace equation;

$$D_i^2\Lambda = [b\cdot\phi,[b\cdot\phi,\Lambda]]. \tag{3.4}$$

Once this is done, we simply take $a\cdot\phi$ to be a linear combination of all possible solutions Λ. We know, from the arguments of previous section, there exist two linearly independent solutions. We already know one such solution, since $D_i^2(b\cdot\phi) = D_iB_i = 0$ and $b\cdot\phi$ obviously commutes with itself. How do we find the other solution? There have been several works on the finding the solution of the covariant Laplacian of the adjoint Higgs field around the instanton background [12]. This

[7] Unless noted otherwise, we will suppress the electric coupling constant e from now on.

can be generalized to the magnetic monopole background, which can be obtained as a limit of an instanton on $R^3 \times S^1$ with nontrivial Wilson loop [13, 14, 15]. Appendix B and C provide the detailed discussion of the solution for the covariant four-dimensional Laplacian. Especially, a single instanton in the $SU(3)$ case are made of three monopoles, two of which correspond to two simple roots and one of which does to one minimal negative root. This additional monopole solution depends on the x_4 coordinate of S^1 and here we take the limit where this additional monopole is taken to spatial infinity.

We will refer all detailed computation of the $SU(3)$ case to the Appendix D. In this section, we will simply borrow the result and use it for the study of (unquantized) 1/4-BPS configurations. Combine the Higgs expectation values to $\mu_2 = h_3 - h_2$ and $\mu_1 = h_2 - h_1$. For $SU(3)$ case, there are two independent solutions to the covariant Laplace equations, since there are two unbroken $U(1)$'s acting on the pair of monopole solutions. We will only need their asymptotic forms, which can be read off from Eq. (D8).

As mentioned above, the first is proportional to the Higgs field $b \cdot \phi$ itself, whose asymptotics are

$$\Lambda_T \simeq \mathrm{diag}\left(h_1 + \frac{1}{2r}, h_2, h_3 - \frac{1}{2r}\right), \tag{3.5}$$

while the second is a bit more involved

$$\Lambda_R \simeq \mathrm{diag}\left(\mu_2 + \frac{p_1}{2r}, -(\mu_1+\mu_2) + \frac{p_2 - p_1}{2r}, \mu_1 - \frac{p_2}{2r}\right). \tag{3.6}$$

The real numbers p_1 and p_2 are defined to be

$$\begin{aligned} p_1 &= \frac{\mu_1 - \mu_2 - 2(\mu_1 + 2\mu_2)\mu_2 R}{\mu_1 + \mu_2 + 2\mu_1\mu_2 R}, \\ p_2 &= \frac{\mu_1 - \mu_2 + 2(2\mu_1 + \mu_2)\mu_1 R}{\mu_1 + \mu_2 + 2\mu_1\mu_2 R}. \end{aligned} \tag{3.7}$$

R is again the separation between the two monopoles, as naturally occurs in the standard form of monopole moduli space metric or in the Nahm data.

The scalar field $a \cdot \phi$ and thus A_0 would be in general a linear combination of Λ_T and Λ_R. Denote the respective coefficients by ξ and η;

$$\begin{aligned} a \cdot \phi(\infty) &= \xi\, \mathrm{diag}(h_1, h_2, h_3) + \eta\, \mathrm{diag}(\mu_2, -\mu_2 - \mu_1, \mu_1) \\ &= \xi \mathbf{h} \cdot \mathbf{H} + 2\eta\,(\mu_1 \boldsymbol{\beta} \cdot \mathbf{H} - \mu_2\, \boldsymbol{\alpha} \cdot \mathbf{H}). \end{aligned} \tag{3.8}$$

The resulting electric charge is such that

$$\mathbf{q} = q_\alpha \boldsymbol{\alpha} + q_\beta \boldsymbol{\beta}, \tag{3.9}$$

where

$$
\begin{aligned}
q_\alpha &= 4\pi(\xi + \eta p_1), \\
q_\beta &= 4\pi(\xi + \eta p_2).
\end{aligned}
\tag{3.10}
$$

For any nonzero separation R, the electric charge is misaligned against the magnetic charge unless $\eta = 0$. For $R = 0$, however, electric charge is proportional to $\boldsymbol{\alpha} + \boldsymbol{\beta}$. For any R, it is easy to double-check that the BPS configuration indeed saturate the most stringent BPS bound. All one need to ensure is that the angle θ between Q_I^M and b_I is unchanged as the electric charge is turned on, which is in turn guaranteed as Eq. (2.12) holds. This is always true for the solution we obtained.

The resulting 1/4-BPS configuration is then composed of a pair of distinct monopole separated by a distance R, and on top of which the time-like gauge potential $A_0 = -a \cdot \phi$ is turned on to carry the additional electric charge whose relative value is completely determined by R. The $\boldsymbol{\alpha}$ monopole would carry q_α electric charge and the $\boldsymbol{\beta}$ monopole would carry q_β electric charge. The relative electric charge $(q_\beta - q_\alpha)/2$ is the part of the electric charge orthogonal to the magnetic charge and is given by

$$
\Delta q = 8\pi\eta \, \frac{(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)R}{\mu_1 + \mu_2 + 2\mu_1\mu_2 R}, \tag{3.11}
$$

and responsible for the electromagnetic repulsion which must be balanced against the Higgs attraction. Note that Δq is a monotonic function of R. In particular, $R = 0$ implies that $\Delta q = 0$ as well. When the two constituent monopoles form a single spherically symmetric configuration, they can be 1/2-BPS but not 1/4-BPS.

As Δq increases, R increases, and at some critical charge, the separation diverges, $R \to \infty$. This of course signals that the BPS configuration no longer exists as a single particle state. Two solitonic cores are separated by arbitrarily large distance once Δq reaches its maximum possible value,

$$
\Delta q_{\rm cr} = 4\pi\eta \, \frac{(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}{\mu_1\mu_2}, \tag{3.12}
$$

at which point the instability sets in. While we carried out the analysis with arbitrary electric charges, it is simply a matter of putting particular values of R if one wishes to extend the result to properly quantized dyons.

Before closing this section, we would like to clarify how a spherically symmetric 1/4-BPS dyon is possible for higher magnetic charges. As we just saw, the only spherically symmetric solution with magnetic charge corresponding to a root, say $\boldsymbol{\alpha} + \boldsymbol{\beta}$, is the ones that break half of supersymmetry. They cannot possess any relative electric charge. However, when the magnetic charge is a double, say $2\boldsymbol{\alpha} + 2\boldsymbol{\beta}$, the analog of this 1/2-BPS, purely magnetic state is not spherically symmetric. The situation is analogous to having a pair of identical $SU(2)$ monopoles as close to each other as possible, if we consider the $SU(2)$ as embedded inside $SU(3)$ along $\boldsymbol{\alpha} + \boldsymbol{\beta}$. We know from early works on $SU(2)$ monopoles that this configuration is cylindrically symmetric, and of toroidal shape [16]. As we turn on relative electric charge and thereby reduce the state to 1/4-BPS, all four constituents, two $\boldsymbol{\alpha}$'s and two $\boldsymbol{\beta}$'s, begin to move away from one another, and eventually become independent. It is then conceivably that, at some specific electric charge, the all four soliton cores are separated just right so that they actually form a spherically symmetric shape. The one solution found in Ref. [6], is an example of this phenomenon.

4 Three-Pronged String and Instability

Let us compare the above result against the string picture. For the purpose of this section, we will pretend that string tension is not quantized, since in the end the physics of instability can be understood classically. Let us consider the specific configuration with the q fundamental strings and g D-strings so that, in the field theoretic context, this is translated to a magnetic charge $g(\boldsymbol{\alpha} + \boldsymbol{\beta})$ and the electric charge $q_\alpha \boldsymbol{\alpha}$. Take $\xi = -p_2\eta$ so that $q_\beta = 0$ of Eq. (3.10), then the dyonic solution of previous section acquires an electric charge along $\boldsymbol{\alpha}$ only,

$$\mathbf{q} = 4\pi\eta(p_1 - p_2)\boldsymbol{\alpha}. \tag{4.1}$$

Let $q \equiv q_\alpha = 4\pi\eta(p_1 - p_2)$.

Let X^I_{21} be the six-dimensional displacement between the first and the second D3-branes, and similarly X^I_{32} be the one between the second and the third D3-branes. The projection along b_I is determined by the Higgs vev $b \cdot \phi(\infty)$;

$$b_I X^I_{21} = h_2 - h_1, \qquad b_I X^I_{32} = h_3 - h_2, \tag{4.2}$$

and similarly $a \cdot \phi(\infty)$ of Eq. (3.8) determines the projection along a_I. The vectors Q^E_I and Q^M_I are then,

$$Q^M_I = g X^I_{31} = g\,(X^I_{32} + X^I_{21}), \tag{4.3}$$

$$Q^E_I = q X^I_{21}, \tag{4.4}$$

where

$$X^I_{21} = (h_2 - h_1)\, b_I - \eta(2\mu_2 + \mu_1 + p_2\mu_1)\, a_I, \tag{4.5}$$

$$X^I_{32} = (h_3 - h_2)\, b_I + \eta(2\mu_1 + \mu_2 - p_2\mu_2)\, a_I. \tag{4.6}$$

A simple generalization of Bergman's calculation shows again that the energy of the string configuration coincides with the field theoretic one if we identify the string tension of (q,g) string to be $\sqrt{q^2+g^2}$ in the field theory unit. If we quantize the system, q becomes the number of the fundamental strings. The same consideration tells us that the angle ω between the $(0,g)$ string and the (q,g) string as they meet at the junction is solely determined by their tension, and thus by $g=4\pi$ and q,

$$\cos(\pi-\omega)=\frac{g}{\sqrt{g^2+q^2}}. \tag{4.7}$$

This angle ω is depicted in Fig. 1.

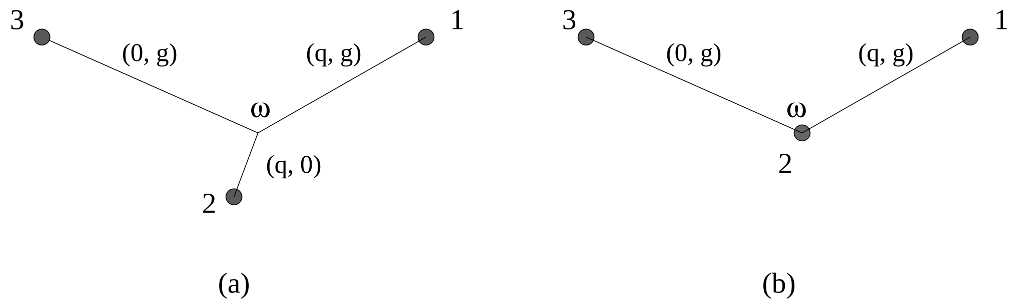

FIG. 1. Configurations of Three-Pronged Strings when it is (a) stable or (a) at the threshold of instability. We labeled the D3-branes by numeral 1,2,3 in accordance with the choice of basis in section 3.

The three-pronged string becomes marginally (un)stable whenever any one of the string has zero length. This happens either due to change of Higgs vev's or due to change in electric charge/coupling. In Fig. 1, we described the case where the Higgs vev's change. When the fundamental string become arbitrarily short so that the second D3-brane coincides with the junction at the center, the string configuration is made only of $(0,g)$ and (q,g) strings. The Higgs force is still attractive but not strong enough compare with the repulsive force from the presence of the relative electric charge; the system is no longer classically bound. In this limit, the angle between X^I_{21} and X^I_{32} must become $\pi-\omega$. Indeed it is not hard to show that

$$\frac{X_{21}\cdot X_{32}}{|X_{21}||X_{32}|}\leq\cos(\pi-\omega), \tag{4.8}$$

where the equality holds precisely when Higgs vev's and electric charge are such that $R\to\infty$. Thus we find the same instability in both string and field theory pictures.

There are other kind of instability, for instance, when (q,g) string becomes arbitrarily short. Clearly there is no static electromagnetic force between the electric and magnetic charges. In this case, the cause of instability in field theoretical term, turned out be due to the repulsion from the Higgs interaction. This is the limit where $\mu_1 = h_2 - h_1 = 0$ in the field theory, and where X^I_{12} and $X^I_{13} = X^I_{12} + X^I_{23}$ becomes mutually orthogonal in the string picture.

5 1/4-BPS Dyons from Quantum Excitations

In principle, the supermultiplet structure of the 1/4-BPS states should be recovered from low energy quantum mechanics of the above solitonic solution. However, in this paper, we will take a shortcut, and ask the question of degeneracy by presenting an alternate construction of these dyonic states. For simplicity, we will confine the present discussion to the case of $SU(3)$.

We start with the spherically symmetric magnetic monopole solution obtained by an $SU(2)$ embedding along the root $\boldsymbol{\alpha} + \boldsymbol{\beta}$ with the single nonuniform Higgs $b \cdot \phi$. If $a \cdot \phi$ vanished, the monopole would have 8 bosonic and 8 fermionic zero modes. In a generic vacuum where $\langle a \cdot \phi \rangle \neq 0$, however, half of these 16 zero modes are lifted and acquire finite energy. Of the remaining 4 bosonic zero modes, three corresponds to translations and one is generated by global $U(1)$ transformations. There are also 4 fermionic zero modes, quantization of which imparts a $N = 4$ vector multiplet structure, thus the degeneracy 2^4, to the soliton.

A minimal 1/4-BPS states should have a degeneracy factor of 2^6 and highest spin 3/2. To see how such structures arise, we need to pay close attention to those modes lifted by $\langle a \cdot \phi \rangle \neq 0$. Fermionic modes are easiest to follow. Introduce a basis for Dirac matrices where γ^0 is diagonal and γ^5 is off-diagonal,

$$\gamma^0 = -i \otimes \sigma^3, \tag{5.1}$$

$$\gamma^k = \sigma^k \otimes \sigma^2, \tag{5.2}$$

$$\gamma^5 = 1 \otimes \sigma^1, \tag{5.3}$$

with 2 by 2 Pauli matrices σ^i's. Using $SO(6)$ R-symmetry, one can bring the Dirac equation to the following form,

$$\gamma^0 \left[i\gamma^k D_k + \gamma^5 \, b \cdot \phi \pm i \, a \cdot \phi \right] \Psi_\pm = \epsilon \, \Psi_\pm, \tag{5.4}$$

written in the time-independent form with the energy eigenvalue ϵ. Here we used a static gauge with the purely magnetic background solution. $N = 4$ theory has two (adjoint) Dirac fermions, which together lift to a Dirac spinor in 6-dimensions. The two are of opposite six-dimensional chiralities, and the subscript $\pm$ refers to this fact.

Decomposing the Dirac spinors as $\Psi = (\chi, \psi)^T$ in term of two-component spinors, and defining an operator $\mathcal{D} \equiv i\sigma^k D_k + i\, b\cdot\phi$, the Dirac equations is rewritten as,

$$\mathcal{D}\psi_\pm \pm [a\cdot\phi, \chi_\pm] = \epsilon\chi_\pm, \tag{5.5}$$

$$\mathcal{D}^\dagger\chi_\pm \mp [b\cdot\phi, \psi_\pm] = \epsilon\psi_\pm. \tag{5.6}$$

Recall that, given a BPS background monopole configuration that satisfies $B_k = D_k(b\cdot\phi)$, the operator $\mathcal{D}$ has zero modes while $\mathcal{D}^\dagger$ does not. When $a\cdot\phi = 0$, each Dirac fermion contributes 4 zero modes ($E = 0$); they solve $\mathcal{D}\psi = 0$ and $\chi = 0$. The 4 solutions to $\mathcal{D}\psi = 0$ can be labeled by the representation under the embedded $SU(2)$. The adjoint representation of the gauge group $SU(3)$ is decomposed into a triplet, a pair of doublet, and a singlet with respect to the $SU(2)$ embedded along $\boldsymbol{\alpha}+\boldsymbol{\beta}$. The singlet is associated with the generator $\boldsymbol{\alpha}\cdot\mathbf{H} - \boldsymbol{\beta}\cdot\mathbf{H}$, while the two doublets are associated with the pairs $(E_\alpha, E_{-\beta})$ and $(E_\beta, E_{-\alpha})$. The triplet would contribute two zero modes, and each doublet would contribute one, which account for all four solutions to $\mathcal{D}\psi = 0$.

By construction of Eq. (5.4), the uniform field $a\cdot\phi$ is orthogonal to the total magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$;

$$a\cdot\phi = v\,(\boldsymbol{\alpha}\cdot\mathbf{H} - \boldsymbol{\beta}\cdot\mathbf{H}), \tag{5.7}$$

which has a nontrivial commutator only with isospin doublets, and even then acts on each as an multiplication by a number. With $a\cdot\phi \neq 0$, therefore, those modes from the isospin triplets commutes with $b\cdot\phi$ and survives as zero modes. As mentioned above, quantization of these leads to a vector multiplet structure of degeneracy $2^4 = 16$.

The other four from isospinor doublets can no longer be zero modes, however, and are promoted to finite energy eigenmodes of the form [17],

$$\Psi_\pm = e^{-i\epsilon t}\begin{pmatrix} 0 \\ \psi \end{pmatrix}. \tag{5.8}$$

The isospin doublet, 2-component spinor ψ is exactly of the same mode that solves $\mathcal{D}\psi = 0$, and thus are normalizable. They are compactly supported around the monopole core. The energy eigenvalue ϵ equals $\pm 3v/2$ for the first doublet and $\mp 3v/2$ for the second doublet. This is because

$$[a\cdot\phi, E_\alpha] = \frac{3v}{2}E_\alpha, \qquad [a\cdot\phi, E_{-\beta}] = \frac{3v}{2}E_{-\beta}, \tag{5.9}$$

and similarly for E_β and $E_{-\alpha}$ with a negative sign. Filling the Dirac sea up to $\epsilon = 0$, creation (or annihilation) of one of these eigenmodes will result in a quantum excitation that costs a positive energy $|\epsilon| = |3v/2|$.

To check against the BPS mass formula, we need the behavior of electric field at large distances when one of these modes is turned on. From various considerations, it is well known that these modes from gauge doublets carry no angular momentum. This can be surmised from the angular momentum formula, $\mathbf{J} = \mathbf{L} + \mathbf{s} + \mathbf{t}$, where the $SU(2)$ gauge generators $\mathbf{t}$ are added to orbital and spin angular momenta. The solution to $\mathcal{D}\psi = 0$ with an $SU(2)$ doublet ψ is unique and spherically symmetric ($\mathbf{L}^2 = 0$), hence must be of the form,

$$\psi_\pm \quad \propto \quad \frac{1}{\sqrt{2}}|E_\alpha, s_z = -1/2\rangle - \frac{1}{\sqrt{2}}|E_{-\beta}, s_z = +1/2\rangle, \tag{5.10}$$

from the first doublet, and

$$\psi_\pm \quad \propto \quad \frac{1}{\sqrt{2}}|E_\beta, s_z = -1/2\rangle - \frac{1}{\sqrt{2}}|E_{-\alpha}, s_z = +1/2\rangle, \tag{5.11}$$

from the second. The isospin and the spin are correlated in such a way that $\mathbf{J}^2 = (\mathbf{s} + \mathbf{t})^2 = 0$. From this, we learn that the mode by itself carries an electric charge of $\pm(\boldsymbol{\alpha} - \boldsymbol{\beta})/2$, or the relative charge is $\Delta q = \mp 1/2$.

However, there is a well known subtlety associated with turning on such a mode from a gauge doublet. Because it acquires a phase of -1 upon a gauge rotation corresponding to the center of $SU(2)$, its excitation must be accompanied by a half-integer momentum along internal phase angle of the background monopole. This leads to additional electric charges of the form $(m/2)(\boldsymbol{\alpha} + \boldsymbol{\beta})$ for any odd integer m. The minimal states are those with $m = \pm 1$. Combining this with the fermionic contribution, we find the electric charges are $\pm\boldsymbol{\alpha}$ or $\mp\boldsymbol{\beta}$. With two Dirac spinors $\Psi_\pm$ then, quantization leads to eight minimal dyonic excitations, which split into four pairs of identical electric charges, $\boldsymbol{\alpha}$, $-\boldsymbol{\beta}$, $\boldsymbol{\beta}$, $-\boldsymbol{\alpha}$. Excitation energy due to the half-integer momentum $m/2 = \pm 1/2$ is of second order in electric charge, and will not affect the leading approximation.

Does the leading excitation energy $|\epsilon| = |3v/2|$ agree with the general BPS mass formula? In the limit of small electric coupling,[8] the central charges may be expanded as

$$Z_\pm = \sqrt{(Q^M)^2 + (Q^E)^2 \pm 2Q^M Q^E \sin\alpha} \simeq Q^M \pm Q^E \sin\alpha + \cdots . \tag{5.12}$$

The actual BPS bound is $\mathrm{Max}(Z_+, Z_-)$, so the first order correction due to the electric charge is

$$|Q^E \sin\alpha| \simeq |\mathrm{tr}\,((a\cdot\phi)(\boldsymbol{\alpha}\cdot\mathbf{H}))\,| = |\mathrm{tr}\,((a\cdot\phi)(\boldsymbol{\beta}\cdot\mathbf{H}))\,| = \left|\frac{3v}{2}\right|, \tag{5.13}$$

[8] We remind readers that Q^E has a factor of e while Q^M has a factor of $1/e$. We suppressed e from notations in section 3 and thereafter.

which coincides with $|\epsilon| = |3v/2|$, as it should if the dyonic state is indeed 1/4-BPS. The bosonic counterpart of this eigenmode analysis should proceed similarly, except that the corresponding eigenmodes will come in a pair of spin doublets rather than four spin singlets. The final result is, then, for each electric charge, $\boldsymbol{\alpha}$, $-\boldsymbol{\beta}$, $\boldsymbol{\beta}$, $-\boldsymbol{\alpha}$, there are $2+2=4$ dyonic excitations due to the gauge-doublet eigenmodes: the net degeneracy of the resulting dyon is $4\times 2^4 = 2^6$ for each electric charge, where we take into account the extra degeneracy of 2^4 due to the four fermionic zero modes from $SU(2)$ triplets. The spin content of each dyon multiplet is that of two $N=4$ vector multiplets (from fermionic eigenmodes) plus a tensor product of a spin doublet and one $N=4$ vector multiplet (from bosonic eigenmodes). This is precisely the 1/4-BPS multiplet of highest spin 3/2. The four types of 1/4-BPS dyons correspond to the four different string configurations depicted in Fig. 2.

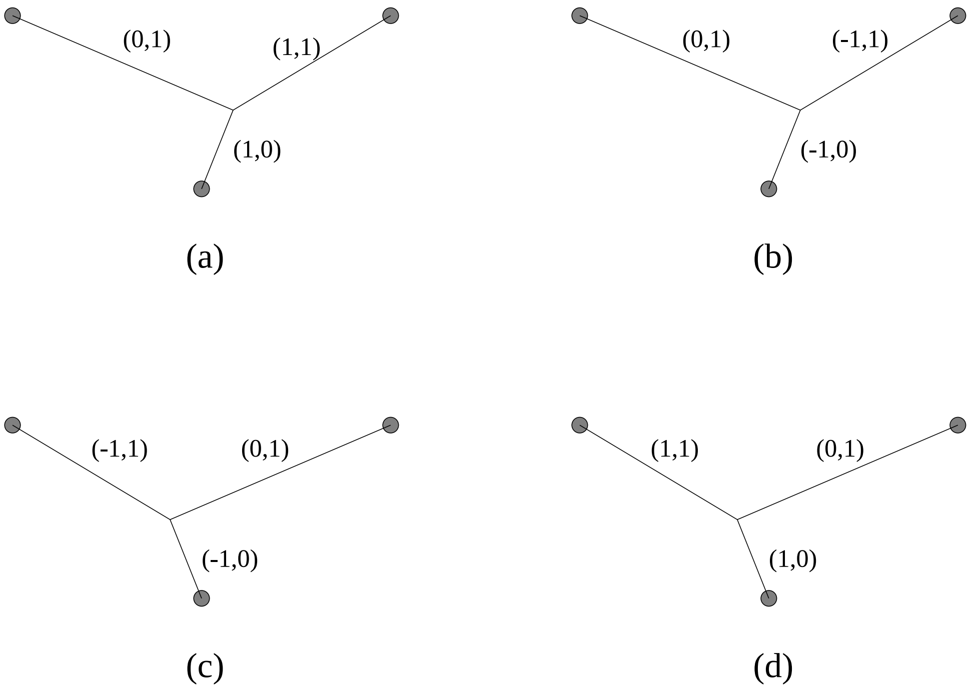

FIG. 2. Four different minimal dyonic states of magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$. Electric charges are respectively (a) $\boldsymbol{\alpha}$, (b) $-\boldsymbol{\alpha}$, (c) $\boldsymbol{\beta}$, and (d) $-\boldsymbol{\beta}$.

Some discussion is due on the validity of the approximation. Note that the expansion of the BPS mass formula proceeded with the assumption,

$$Q^M \gg Q^E \sin\alpha \gg \frac{(Q^E\cos\alpha)^2}{Q^M}, \tag{5.14}$$

which is obtained by expanding the BPS bound. It is clear from the subleading contributions to dyon energies that these criteria are necessary for a successful match between the BPS mass and the energy found from eigenmode analysis. The first condition simply says that the excitation energy should be much smaller that the mass of the bare soliton itself, and is to be expected. What does the second condition do?

The present approximation takes into account only part of the backreactions. It does address the change in long-range electric field in response to the excitation, but ignored its counterpart in magnetic soliton structure. This is of course why we seem to obtain spherically symmetric configuration, even though we clearly demonstrated that this should happen rarely in exact dyonic states. The consequence is that our choice of b_I is independent of the electric charge being turned on, such that b_I is in fact parallel to Q_I^M. To obtain the correct BPS bound, in reality, the angle θ between b_I and Q_I^M must be given by

$$\tan\theta = \frac{\pm Q^E\cos\alpha}{Q^M \pm Q^E\sin\alpha} \simeq \pm\frac{Q^E\cos\alpha}{Q_M}, \tag{5.15}$$

where we used the first condition $Q^M \gg Q^E\sin\alpha$. The BPS bound,

$$b_I Q_I^M + a_I Q_I^E, \tag{5.16}$$

then contains an error of order

$$\delta\theta^2 Q^M \pm \delta\theta Q^E\cos\alpha \sim \frac{(Q^E\cos\alpha)^2}{Q^M}, \tag{5.17}$$

where $\delta\theta' \equiv \theta - \theta' = \theta$, due to the incorrect angle $\theta' = 0$. Since we ignore the magnetic backreaction to the quantum excitation, we must require this error be negligible against the first order estimate, which explains the second condition. It also explains why we do not find the phenomenon of instability in the present set-up. Bergman's criteria tells us that it occurs when $(Q^E\cos\alpha)^2$ is comparable to $Q^M Q^E \sin\alpha$, where magnetic backreaction to the quantum excitations are of a first order effect, instead of being a second order effect. Instability cannot be probed without taking into account the reaction of magnetic soliton to the quantum excitation. In this sense, the two constructions we gave are complimentary to each other; the first gave us the understanding of the dynamics while the second is better suited for state counting.

6 Degeneracy and Supermultiplet Structure of Dyons

In the previous section, we saw how the supermultiplet of degeneracy 2^6 arises in case of minimally charged BPS states. The method we developed is applicable for 1/4-BPS states with higher electric charges, and we will summarize the general supermultiplet structure. Let us parameterize the quantized electric charge by writing

$$\mathbf{q} = q_\alpha \boldsymbol{\alpha} + q_\beta \boldsymbol{\beta} = \frac{k}{2}(\boldsymbol{\beta} - \boldsymbol{\alpha}) + \frac{m}{2}(\boldsymbol{\alpha} + \boldsymbol{\beta}). \tag{6.1}$$

with integers k and m. Consistent quantization requires that m is odd(even) whenever k is odd(even). The relative charge of the system is given by $\Delta q = (q_\beta - q_\alpha)/2 = k/2$. The integer k corresponds to the number of excited eigenmodes while $m/2$ is the momentum along a internal $U(1)$ angle of the magnetic solitons. The case of no relative electric charge $\Delta q = 0$ corresponds to the usual BPS dyon that breaks half of supersymmetry, which come in an $N = 4$ vector multiplet. The case of $\Delta q = \pm 1/2$ was addressed in the previous section. The supermultiplet structure found there can be summarized in terms of the eigenvalues under one of the angular momentum operators, J_3,

J_3	3/2	1	1/2	0	-1/2	-1	-3/2
Degeneracy	1	6	15	20	15	6	1

The total degeneracy is 2^6, which, for 1/4-BPS state, is the smallest while being also consistent with supersymmetry. Call this multiplet G_0. This multiplet can be seen as a tensor product between $N = 4$ vector multiplet with a $N = 1$ chiral multiplet.

Higher charged states with $|\Delta q| \geq 1$ is obtained by exciting appropriate eigenmodes $k = 2|\Delta q|$ times. Given a fixed electric charge, there are always two bosonic and two fermionic eigenmodes at disposal. There are $k + 1$ states where no fermionic modes are excited, $2k$ states where one fermionic modes are excited, and $k - 1$ states where both fermionic modes are excited. Combining the degeneracy from four fermion zero modes of the center of mass motion, we then find the total degeneracy of $4k \times 2^4 = 4(2|\Delta q|) \times 2^4 = (2|\Delta q|) \times 2^6$. For detailed spin content, we only need to recall that 2^4 has the vector structure and that bosonic excitations carry extra spin of $\pm 1/2$. The result is the sum of $2|\Delta q|$ tables identical to the above, except that J_3 eigenvalues are shifted,

$J_3 - S$	3/2	1	1/2	0	-1/2	-1	-3/2
Degeneracy	1	6	15	20	15	6	1

with S ranging from $-|\Delta q| + 1/2$ to $|\Delta q| - 1/2$ in step of 1. The resulting supermultiplet has a tensor product structure $G_0 \otimes [|\Delta q| - 1/2]$ where we denoted by $[|\Delta q| - 1/2]$ the spin $|\Delta q| - 1/2$ representation of the angular momentum. The highest spin of such a multiplet is $|\Delta q| + 1$. From construction, it is easy to see that $|\Delta q|$ of this arises from bosonic excitations. The only fermionic contribution comes from the four fermionic zero modes, which tops out at 1.

This bosonic spin has a rather interesting explanation in the context of classical dyonic configurations of section 3. Consider the limit of large Higgs vev's. In this limit, the solution degenerates to a pair of point-like dyons of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ type, each carrying electric charges q_α and q_β. The conserved angular momentum is known to contain an anomalous contribution in this situation,

$$\mathbf{J} = \mathbf{L} + \frac{g\Delta q}{4\pi}\hat{\mathbf{R}} \tag{6.2}$$

proportional to the relative electric charge $\Delta q = (q_\beta - q_\alpha)/2$. The unit vector $\hat{\mathbf{R}}$ points from $\boldsymbol{\alpha}$ dyon to $\boldsymbol{\beta}$ dyon. With the unit magnetic charges, $g = 4\pi$, the anomalous angular momentum is exactly $|\Delta q|$, as expected. (We fully expect that a classical field theoretic calculation of the anomalous angular momentum for the 1/4-BPS configurations will reproduce the answer (6.2) obtained in the point-like dyon limit. See Appendix E for a simple expression for the angular momentum.)

7 Conclusion

In this paper we explored 1/4-BPS states in $N = 4$ supersymmetric theories which correspond to three-pronged strings ending on D3-branes in Type IIB string theory. 1/4-BPS configurations typically consist of two (or more) dyonic cores, which are positioned so that static electromagnetic force is perfectly balanced against scalar Higgs force. The marginal instability previously found in string picture is shown to arise from the excessive repulsion from either electromagnetic or Higgs interaction. An alternate construction using the finite energy excitations around purely magnetic soliton also revealed supermultiplet structure of 1/4-BPS states with arbitrary relative electric charge. The degneracy and the highest spin the supermultiplets grow linearly with the relative charge. In the minimal cases, the multiplet has the degeneracy of 2^6 with the highest spin 3/2.

In principle, the question of degeneracy and supermultiplet structures can also be addressed by considering low energy quantum mechanics of the classical 1/4-BPS solution we found. This would necessarily involve zero-mode analysis of these nonspherical solitons, which we did not attempt.

Our constructions can be generalized to the case of multi-pronged string configurations in larger gauge groups. In the small coupling limit, the same eigenmode analysis should produce the dyonic states of higher magnetic and electric charges. Also classically, one can distribute many monopole in the background, and solve for possible electric configurations. We expect to find multi-dyon configurations hung together by the delicate balance of static forces. We should be able to exploit the ADHMN formalism as in this work to explore these field configurations. One interesting case is when the gauge symmetry is partially restored as in Ref. [18]. For solutions whose net magnetic charge is abelian, the configuration typically consists of massive magnetic cores surrounded by nonabelian magnetic clouds. It would be interesting to see if any new physics arises by considering 1/4-BPS version of such nonabelian configurations.

While we considered only $N = 4$ theories so far, it is clear that the methods developed here can be applied to $N = 2$ theories with minimal modifications. $N = 2$ supersymmetry algebra possesses half the supersymmetry generators and also only one central charge, so we naturally expect the spectrum be qualitatively different. This is quite apparent from the point of view adopted in section 5, since reducing supersymmetry involves removing one of the two adjoint Dirac spinors. In fact, there appears to be no guarantee that the present constructions produce proper 1/2-BPS states. It may in general depend on the particular electromagnetic charges, Higgs vev's, and other details of the theory. We are currently exploring some of the issues.

As this work was being completed, two related papers [19, 20] have appeared.

Acknowledgments

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Appendix A: The ADHM Formalism

The ADHM formalism [9] for k instantons of the SU(n) gauge theory starts with a $(n+2k)\times 2k$ matrix

$$\Delta = \begin{pmatrix} \lambda_{n\times 2k} \\ \mu_{2k\times 2k} \end{pmatrix} + \begin{pmatrix} 0 \\ I_{2k\times 2k} \end{pmatrix} x, \tag{A1}$$

where $x = x_\alpha e_\alpha$ and $e_\alpha = (i\sigma_j, 1)$ [9]. Finding the $(n+2k)\times 2k$ matrix v such that

$$\Delta^\dagger v = 0, \quad v^\dagger v = I_{n\times n}, \tag{A2}$$

we can construct the anti-hermitian gauge field

$$A_\alpha = v^\dagger \partial_\alpha v. \tag{A3}$$

The condition for the field strength to be self-dual is that

$$(\Delta^\dagger \Delta)_{2k\times 2k} = f^{-1}_{k\times k} I_{2\times 2}. \tag{A4}$$

This implies that $\mu = \mu_\alpha e_\alpha$ with hermitian matrices $(\mu_\alpha)_{k\times k}$ and that

$$i\eta^i_{\alpha\beta}[\mu_\alpha, \mu_\beta] + tr_2(\sigma^i \lambda^\dagger \lambda) = 0, \tag{A5}$$

where $e^\dagger_\alpha e_\beta = \delta_{\alpha\beta} + i\eta^i_{\alpha\beta}\sigma^i$ with anti-self-dual ‘t Hooft tensor $\eta^i_{\alpha\beta}$. The inverse $k\times k$ matrix f satisfies equation

$$\left\{(\mu_\alpha + x_\alpha)^2 + \frac{1}{2}\mathrm{tr}_2 \lambda^\dagger \lambda\right\} f = I_{k\times k}. \tag{A6}$$

We can choose the v such that

$$v_{(n+2k)\times n} = \begin{pmatrix} I_{n\times n} \\ u_{2k\times n} \end{pmatrix} N^{-\frac{1}{2}}, \tag{A7}$$

where $N = 1 + u^\dagger u$ is an $n\times n$ hermitian matrix [15]. The ADHM equation becomes

$$(\mu^\dagger + x^\dagger)u + \lambda^\dagger = 0. \tag{A8}$$

The gauge field becomes

$$A_\alpha = N^{-\frac{1}{2}}(u^\dagger \partial_\alpha u)N^{-\frac{1}{2}} + N^{-\frac{1}{2}}\partial_\alpha N^{-\frac{1}{2}}. \tag{A9}$$

The self-dual field strength is then given by

$$F_{\alpha\beta} = 2iN^{-\frac{1}{2}} u^\dagger f \bar{\eta}_{\alpha\beta} u N^{-\frac{1}{2}}, \tag{A10}$$

where $e_\alpha e^\dagger_\beta = \delta_{\alpha\beta} + i\bar{\eta}_{\alpha\beta}$ where $\bar{\eta}_{\alpha\beta}$ is the self-dual ’t Hooft tensor.

The construction has redundancy,

$$\lambda \to \lambda U, \quad \mu \to U^\dagger \mu U, \quad u \to U^\dagger u, \tag{A11}$$

where U belongs to $U(k)$. The number of parameters of μ_α and λ are

$$\mu_\alpha : 4k^2, \qquad \lambda : 4nk. \tag{A12}$$

The number of the conditions (A5) are $3k^2$ and the number of $U(k)$ elements is k^2. Thus the net number of independent variables for a k instanton in $SU(n)$ is

$$4k^2 + 4nk - 3k^3 - k^3 = 4nk. \tag{A13}$$

Appendix B: The Nahm Formalism of Calorons

We consider instanton solutions on $R^3 \times S^1$ with nontrivial Wilson loop, which can be regarded as the infinite infinite number of instantons which is quasi-periodic along x_4 axis [14, 11, 15]. We analyze these calorons by extending the method in Ref. [15] to the case of $SU(n)$ gauge group, along the way, by connecting to the Nahm's formalism /citenahm. We choose the unit interval of the x_4 to be $[0, \beta]$ and imagine the number of instantons in a given interval is k. The ADHM matrices becomes

$$\Delta(x) = \begin{pmatrix} \lambda_l \\ \mu_{ll'} \end{pmatrix} + \begin{pmatrix} 0 \\ x\delta_{ll'} \end{pmatrix}, \tag{B1}$$

where l, l' are integers. Here $\mu_{ll'}$ for each ll' is a $2k \times 2k$ matrix and λ_l for each l is a $2k \times n$ matrix.

We consider the gauge field to be quasi-periodic so that

$$A_\alpha(\mathbf{x}, x_4 + \beta) = e^{i\beta \mathbf{h}\cdot\mathbf{H}} A_\alpha(\mathbf{x}, x_4) e^{-i\beta \mathbf{h}\cdot\mathbf{H}}. \tag{B2}$$

This is equivalent to considering the periodic field configurations with the asymptotic value at spatial infinity to be

$$< A_\alpha >= i\mathbf{h} \cdot \mathbf{H}\delta_{\alpha 4}. \tag{B3}$$

Note that $\mathbf{h} \cdot \mathbf{H} = \sum_{a=1}^n h_a P_a$ such that $\sum_a h_a = 0$ with P_a being the projection operator to the a component of any n-dimensional vector. We can choose the gauge so that

$$h_1 < h_2 < ... < h_n < h_1 + \frac{2\pi}{\beta}. \tag{B4}$$

The condition (B2) can be satisfied if

$$u_l(\mathbf{x}, x_4+\beta) = u_{l-1}(\mathbf{x}, x_4) e^{-i\beta \mathbf{h}\cdot\mathbf{H}}, \tag{B5}$$

which in turn can be satisfied if

$$\lambda_l^\dagger = \lambda_{l-1}^\dagger e^{-i\beta \mathbf{h}\cdot\mathbf{H}}, \tag{B6}$$

$$\mu_{ll'} = \mu_{(l-1)(l'-1)} - \beta e_4 \delta_{ll'}. \tag{B7}$$

These relations lead to

$$\lambda_l^\dagger = \lambda_0^\dagger e^{-i\beta l \mathbf{h}\cdot\mathbf{H}}, \tag{B8}$$

$$\mu_{ll'}^\alpha = T_{ll'}^\alpha - l\beta\delta_{\alpha 4}, \delta_{ll'}, \tag{B9}$$

such that $T_{ll'}^\alpha = T_{(l-1)(l'-1)}^\alpha$. Note that $(\Delta^\dagger\Delta)_{ll'}(x_4+\beta) = (\Delta^\dagger\Delta)_{(l-1)(l'-1)}(x_4)$ and so $f_{ll'}(x_4+\beta) = f_{(l-1)(l'-1)}(x_4)$.

We introduce the Fourier transformation of these matrices:

$$\lambda^\dagger(t) = \sum_l e^{i\beta t l}\lambda_l^\dagger, \tag{B10}$$

$$T_\alpha(t) = \sum_l e^{i\beta t l} T_{l0}^\alpha, \tag{B11}$$

$$u(t) = \sqrt{\frac{\beta}{2\pi}} \sum_l e^{i\beta t l} u_l, \tag{B12}$$

$$f(t,t') = \frac{\beta}{2\pi} \sum_{ll'} e^{i\beta t l} f_{ll'} e^{-i\beta t' l'}. \tag{B13}$$

Note that $T_\alpha(t)$ is hermitian $k\times k$ matrix and periodic under $t \to t + 2\pi/\beta$, $\lambda^\dagger(t)$ is $n\times 2k$ and periodic, and $u(t)$ is $n\times 2k$ and periodic. The function $f(t,t')$ is periodic under shift of t, t' with $2\pi/\beta$.

Furthermore, from Eqs. (B8) and (B10), we get

$$\lambda^\dagger(t) = \frac{2\pi}{\beta}\lambda_0^\dagger \sum_a \delta(t-h_a) P_a. \tag{B14}$$

From the property that $u(t, x_4+\beta) = u(t, x_4) e^{i\beta(t-\mathbf{h}\cdot\mathbf{H})}$, we can introduce

$$u_*(t; \mathbf{x}, x_4) = u(t; \mathbf{x}, x_4) e^{-i x_4 (t-\mathbf{h}\cdot\mathbf{H})}, \tag{B15}$$

such that $u_*(t+2\pi/\beta) = u_*(t) e^{i2\pi x_4/\beta}$ and $u_*(x_4+\beta) = u_*(x_4)$.

In the Fouriered functions, the consistent condition (A5) becomes the Nahm equation for a caloron [13, 14],

$$\partial_t T_i - i[T_4, T_i] = \frac{i}{2}\epsilon_{ijk}[T_j, T_k] + \frac{1}{2}\mathrm{tr}_{\,2}\sigma_i w^\dagger \sum_a \delta(t-h_a)P_a w, \tag{B16}$$

where $w = \sqrt{2\pi/\beta}\lambda_0$. The ADHMN equation (A8) for $u(t)$ becomes

$$\Big[e_4^\dagger(i\partial_t + T_4 + x_4) + e_i^\dagger(T_i + x_i)\Big]u(t) + w^\dagger \sum_a \delta(t-h_a)P_a = 0. \tag{B17}$$

In terms of the quasi-periodic $u_*(t)$, the above equation becomes

$$\Big[i\partial_t + T_4 - i\sigma_i(T_i + x_i)\Big]u_*(t) + w^\dagger \sum_a \delta(t-h_a)P_a = 0. \tag{B18}$$

This is the standard Nahm equation for magnetic monopoles [10].

In this process the normalization factor $N^{-\frac{1}{2}}$ becomes

$$N^{-\frac{1}{2}} = e^{i\mathbf{h}\cdot\mathbf{H}x_4} N_*^{-\frac{1}{2}} e^{-i\mathbf{h}\cdot\mathbf{H}x_4}, \tag{B19}$$

where $N_* = 1+\int_0^{2\pi/\beta} dt u_*^\dagger u_*$ is single-valued under $x_4 \to x_4+\beta$. After singular gauge transformation $e^{i\mathbf{h}\cdot\mathbf{H}x_4}$, the gauge field becomes single-valued and is given by

$$\begin{aligned} A_{*4} &= N_*^{-\frac{1}{2}} i\sum_a h_a P_a N_*^{-\frac{1}{2}} + N_*^{-\frac{1}{2}}\int_0^{2\pi/\beta} dt\, itu_*^\dagger(t)u_*(t)N_*^{-\frac{1}{2}}, \\ A_{*i} &= N_*^{-\frac{1}{2}}\partial_i N_*^{-\frac{1}{2}} + N_*^{-\frac{1}{2}}\int_0^{2\pi/\beta} dt u_*^\dagger(t)\partial_i(u_*(t)N_*^{-\frac{1}{2}}), \end{aligned} \tag{B20}$$

which is the standard form of the Nahm construction for the self-dual magnetic monopoles [10].

We redefine the Green function $f_*(t,t;x_4) = e^{-ix_4 t}f(t,t';x_4)e^{ix_4 t'}$, which is single-valued in x_4 but multi-valued in t. It satisfies

$$(i\partial_t + T_4)^2 f_* + (T_i + x_i)^2 f_* + \frac{1}{2}W(t)f_* = \delta(t-t'), \tag{B21}$$

where

$$W(t) = \mathrm{tr}\, w^\dagger \Sigma_a \delta(t-h_a)P_a w. \tag{B22}$$

The single-valued self-dual field strength becomes

$$F_{*\alpha\beta} = N_*^{-\frac{1}{2}}\left\{\int dtdt' u_*^\dagger(t)f_*(t,t')\bar{\eta}_{\alpha\beta}u_*(t')\right\}N_*^{-\frac{1}{2}}. \tag{B23}$$

Appendix C: The Adjoint Scalar Field

The general method to find the solution of the covariant Laplacian for a scalar field in the adjoint representation has been developed in the instanton background [12]. We start with a general form

$$\Phi(x) = v^\dagger Q v, \tag{C1}$$

where Q is an hermitian $(n+2k)\times(n+2k)$ matrix. We assume that Q is independent of x and takes the ansatz

$$Q = \begin{pmatrix} q_{n\times n} & 0 \\ 0 & p_{k\times k} I_{2\times 2} \end{pmatrix}. \tag{C2}$$

Using the fact that the projection operator $P = vv^\dagger = I - \Delta f \Delta^\dagger$, one can show that

$$D_\alpha^2 \Phi = 4N^{-\frac{1}{2}} u^\dagger f \Big[\mathrm{tr}_2 \Big(\lambda^\dagger q \lambda - \frac{1}{2}\{\lambda^\dagger \lambda, p\} \Big) - [\mu_\alpha, [\mu_\alpha, p]] \Big] f u N^{-\frac{1}{2}}, \tag{C3}$$

where tr_2 is a trace over 2-dimensional part of matrices. With two hermitian $k\times k$ matrices,

$$W = \mathrm{tr}_2 \lambda^\dagger \lambda, \quad \Lambda = \mathrm{tr}_2 \lambda^\dagger q \lambda, \tag{C4}$$

the condition for the scalar field to satisfy the covariant Laplace equation $D_\alpha^2\Phi = 0$ becomes a condition on the matrix p,

$$-[\mu_\alpha, [\mu_\alpha, p]] - \frac{1}{2}\{W, p\} + \Lambda = 0. \tag{C5}$$

Note that the above equation determines p for a given infinitesimal generator q of $SU(n)$. Especially when $q = I_{n\times n}$, we can see $p = I_{k\times k}$ solves the above equation.

For similar scalar fields in any caloron background, we extend the method described in Appendix B. We generalize Eq. (C2) to an infinite dimensional matrix, and then the analogy of Eq. (C1) would be

$$\Phi = N^{-\frac{1}{2}} q N^{-\frac{1}{2}} + N^{-\frac{1}{2}} u_l^\dagger p_{ll'} u_{l'} N^{-\frac{1}{2}}. \tag{C6}$$

Similar to the gauge field, the adjoint Higgs scalar field should satisfy the quasi-periodic condition $\Phi(\mathbf{x}, x_4+\beta) = e^{i\beta \mathbf{h}\cdot\mathbf{H}} \Phi(\mathbf{x}, x_4) e^{-i\beta\mathbf{h}\cdot\mathbf{H}}$. Thus the above ansatz is consistent with Eq. (B19) only if

$$[\mathbf{h}\cdot\mathbf{H}, q] = 0. \tag{C7}$$

This equation implies that there are only $n-1$ independent q's when the gauge symmetry is maximally broken or all h_a are different.

To consider the similar solution around magnetic monopoles, we again Fourier transform p matrix,

$$p(t) = \sum_l e^{i\beta t l} p_{l0}. \tag{C8}$$

Then, we can re-express Eq. (C5) as an ordinary differential equation for $k \times k$ hermitian matrix $p(t)$,

$$[\partial_t - iT_4, [\partial_t - iT_4, p(t)]] - [T_i(t), [T_i(t), p(t)]] - \frac{1}{2}\{W(t), p(t)\} + \Lambda(t) = 0, \tag{C9}$$

where $W(t) = \mathrm{tr}\,_2 w^\dagger \sum_a \delta(t - h_a) P_a w$ and $\Lambda(t) = \mathrm{tr}\,_2 w^\dagger \sum_a \delta(t - h_a) P_a q w$. For such a solution $p(t)$, after a gauge transformation by $e^{-ix_4 \mathbf{h}\cdot\mathbf{H}}$, the single-valued solution of adjoint scalar Laplace equation is given by

$$\Phi_* = N_*^{-\frac{1}{2}} q N_*^{-\frac{1}{2}} + N_*^{-\frac{1}{2}} \int_0^{2\pi/\beta} dt\, u_*^\dagger(t) p(t) u_*(t) N_*^{-\frac{1}{2}}. \tag{C10}$$

Appendix D: The SU(3) Case

We first consider the Nahm data for three monopoles which makes a single instanton on $R^3 \times S^1$, or a caloron [13, 14, 15]. As shown in Appendix B, the Nahm equation is defined over three auxiliary time interval, $[t_1, t_2], [t_2, t_3], [t_3, t_1 + \frac{2\pi}{\beta}]$, where β is the circumference of S^1. The Nahm equation is almost trivial and the Nahm data gives the position vectors of magnetic monopoles as follows:

$$\begin{aligned}
\mathbf{T}_1 &= -\mathbf{x}_\alpha = (0, 0, R), \;\; t \in (t_1, t_2), \\
\mathbf{T}_2 &= -\mathbf{x}_\beta = (0, 0, 0), \;\; t \in (t_2, t_3), \\
\mathbf{T}_3 &= -\mathbf{x}_3 = (0, 0, -K), \;\; t \in (t_3, t_1 + \frac{2\pi}{\beta}),
\end{aligned} \tag{D1}$$

where $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ are the positions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ monopoles, and $\mathbf{x}_3$ is the position of the third monopole. For convenience, we put the third monopole at the z axis and later on take it to infinity by pushing $K \to \infty$. The distance between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ monopoles are R. The jumping condition (B16) satisfied by this Nahm data as follows:

$$\begin{aligned}
w_1^\dagger &= \begin{pmatrix} \sqrt{2(K+R)} \\ 0 \end{pmatrix}, \\
w_2^\dagger &= \begin{pmatrix} 0 \\ \sqrt{2R} \end{pmatrix}, \\
w_3^\dagger &= \begin{pmatrix} 0 \\ \sqrt{2K} \end{pmatrix}.
\end{aligned} \tag{D2}$$

Then one can find the $A_i, b\phi$ field configurations by the ADHMN method, as explored in detail in Refs,[11, 14]

For given solutions of the corresponding ADHMN equation, there exist a general method to find the solution of the covariant four-dimensional Laplacian satisfied by the adjoint Higgs field, as summarized in Appendix C. For a single caloron as in our case, we need to find a continuous and periodic function $p(t)$ on $[t_1, t_1 + \frac{2\pi}{\beta}]$, for a given $q \in SU(3)$ which commutes with the asymptotic Higgs value $\mathbf{h} \cdot \mathbf{H}$. The differential equation (C9) for the periodic $p(t)$ in our context is given by

$$\begin{aligned}
&\partial_t^2 p(t) - 2(K+D)(p(t) - q_1)\delta(t - h_1) \\
&\quad -2D(p(t) - q_2)\delta(t - h_2) - 2K(p(t) - q_3)\delta(t - h_3) = 0,
\end{aligned} \tag{D3}$$

where $q = \mathrm{diag}(q_1, q_2, q_3)$ and $q_1 + q_2 + q_3 = 0$. This equation is very simple to solve, especially in the limit where $K \to \infty$.

There are two independent q matrices:

$$\begin{aligned}
q_T &= \mathrm{diag}(h_1, h_2, h_3), \\
q_R &= \mathrm{diag}(\mu_2, -\mu_2 - \mu_1, \mu_1),
\end{aligned} \tag{D4}$$

where $\mu_2 = h_3 - h_2$ and $\mu_1 = h_2 - h_1$, so that $\mathrm{tr}\, q_T\, q_R = 0$. For each q, there exists a corresponding $p(t)$. Especially in the relevant interval $t \in [h_1, h_3]$, for q_T,

$$p_T = t. \tag{D5}$$

For q_R,

$$p_R(t) = \begin{pmatrix} p_1(t - h_2) + c, & t \in [h_1, h_2] \\ p_2(t - h_2) + c, & t \in [h_2, h_3] \end{pmatrix}, \tag{D6}$$

where

$$\begin{aligned}
c &= h_2 + \frac{1}{2R}(p_2 - p_1), \\
p_1 &= \frac{\mu_1 - \mu_2 - 2(\mu_1 + 2\mu_2)\mu_2 R}{\mu_1 + \mu_2 + 2\mu_1\mu_2 R}, \\
p_2 &= \frac{\mu_1 - \mu_2 + 2(2\mu_1 + \mu_2)\mu_1 R}{\mu_1 + \mu_2 + 2\mu_1\mu_2 R}.
\end{aligned} \tag{D7}$$

The p_T can be regarded case where $p_1 = p_2 = 1$.

Following the ADHMN method of the $b\phi$ field closely, as explored in Ref. [11], we can solve easily the ADHMN equations (B18) for a given Nahm data (D1) and (D2). Especially one can see easily that the solutions ADHMN equation for the interval $[t_3, t_1 + \frac{2\pi}{\beta}]$ goes to zero like $1/\sqrt{K}$, similar to the $SU(2)$ case in Ref. [14]. Thus, there will be no nontrivial contribution from the interval $[t_3, t_1 + \frac{2\pi}{\beta}]$. Then, we can now construct the solution of the second BPS equation (2.23) by using Eq. (C10) of Appendix C. From Eq. (C10) and the solution of the ADHMN equation in Ref. [11], we can easily construct the 3×3 adjoint Higgs field which satisfies the second BPS equation (2.23). The solution is

$$\Lambda(x) = \begin{pmatrix} \phi_{(1)} & \phi_{(3)} \\ \phi^\dagger_{(3)} & \phi_{(2)} \end{pmatrix}, \tag{D8}$$

where

$$\begin{aligned}
\phi_{(1)} &= N^{-\frac{1}{2}}(p_1 K_L + p_2 K_R) N^{-\frac{1}{2}} + cI_{2\times 2}, \\
\phi_{(2)} &= 2RL^2\ (0,1)(p_1 N_L^{-1} K_L N_L^{-1} + p_2 N_R^{-1} K_R N_R^{-1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c - \frac{p_2 - p_1}{2R} S^\dagger S, \\
\phi_{(3)} &= N^{-\frac{1}{2}} \left(-p_1 K_L N_L^{-1} + p_2 K_R N_R^{-1} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{2R} L,
\end{aligned} \tag{D9}$$

where $\mathbf{y}_1 = \mathbf{x} - \mathbf{x}_1$, $\mathbf{y}_2 = \mathbf{x} - \mathbf{x}_2$, and

$$\begin{aligned}
N_L &= \frac{1}{|\mathbf{y}_1|} \sinh(\mu_1 y_1) e^{-\mu_1 \mathbf{y}_1\cdot\sigma}, \\
N_R &= \frac{1}{|\mathbf{y}_2|} \sinh(\mu_2 y_2) e^{\mu_2 \mathbf{y}_2\cdot\sigma}, \\
N &= N_L + N_R, \\
K_L &= \frac{1}{2y_1} \hat{\mathbf{y}}_1 \cdot \sigma [\mu_1 e^{-2\mu_1 \mathbf{y}_1\cdot\sigma} - N_L], \\
K_R &= \frac{1}{2y_2} \hat{\mathbf{y}}_2 \cdot \sigma [\mu_2 e^{2\mu_2 \mathbf{y}_2\cdot\sigma} - N_R], \\
L &= \frac{1}{\sqrt{(y_1 \coth \mu_1 y_1 + y_2 \coth \mu_2 y_2)^2 - R^2}}, \\
S^\dagger S &= \frac{y_1 \coth \mu_1 y_1 + y_2 \coth \mu_2 y_2 - R}{y_1 \coth \mu_1 y_1 + y_2 \coth \mu_2 y_2 + R}.
\end{aligned} \tag{D10}$$

When $p_1 = p_2 = 1$, we have the solution corresponding to the p_T, which is of course the original Higgs field, $b\phi$, itself.

Here only useful part of this explicit solution is its asymptotic form in the limit where $|\mathbf{x}| >> R, \mu_1^{-1}, \mu_2^{-1}$. As in Ref [11], we can find the asymptotic form of this solution easily. In the unitary gauge, its asymptotic limit of Eq. (D8) for q_T and q_R of Eq. (D4) become Eqs. (3.5) and (3.6) in Section 3.

Appendix E: Energy Density and Angular Momentum

Here we want to point out that energy density and total angular momentum become considerably simpler for the self-dual configurations. Using the self-dual equations, one can also simplify the energy density to be

$$\mathcal{H}(\mathbf{x}) \quad = \quad \mathrm{tr} \left\{ E_i^2 + B_i^2 + (D_0 b\cdot\phi)^2 + (D_i b\cdot\phi)^2 + (-ie[a\cdot\phi, b\cdot\phi])^2 \right\}$$

$$= \partial_i^2 \mathrm{tr}\,[(a\cdot\phi)^2 + (b\cdot\phi)^2] \tag{E1}$$

where we used the result that $D_0\zeta_I = ie[a\cdot\phi, \zeta_I] = 0$.

The most general BPS solutions carry both electric and magnetic charges and will have nonzero angular momentum in general. The angular momentum of a BPS configuration is

$$\begin{aligned} J^i &= -2\int d^3x\ \epsilon_{ijk}\, x^j\, \mathrm{tr}\left\{\epsilon_{klm}E_l B_m + D_0\phi_I D_k\phi_I\right\} \\ &= -2\int d^3x\ (x^j\partial_i - \delta_i^j x^l\partial_l)\,\mathrm{tr}\,(a\cdot\phi D_j\, b\cdot\phi) \end{aligned} \tag{E2}$$

The angular momentum is a vector quantity and so should depend on the internal structure of the BPS configuration. While we do not pursue in the paper, we expect that both energy density and angular momentum can be simplified further.

References

[1] O. Bergman, *Three-pronged strings and 1/4 BPS states in N=4 Super-Yang-Mills Theory*, hep-th/9712211

[2] E. Witten, Nucl. Phys. **B460**, 335 (1996), hep-th/9510135.

[3] O. Aharony, J. Sonnenschein and S. Yankielowicz, Nucl. Phys. **B474**, 309 (1996), hep-th/9603009; J.H. Schwarz, *Lectures on Superstring and M-theory dualities*, hep-th/9607201.

[4] K. Dasgupta and S. Mukhi, *BPS dynamics of triple (p,q) string junction*, hep-th/9711094;
A. Sen, *String network*, hep-th/9711130;
S.J. Rey and J.T. Yee, *BPS dynamics of triple (p, q) string junction*, hep-th/9711202;
M. Krogh and S. Lee, *String network from M-theory*, hep-th/9712050;
Y. Matsuo and K. Okuyama, *BPS condition of string junction from M-theory*, hep-th/9712070.

[5] O. Bergman and A. Fayyazuddin, *String junctions and BPS states in Seiberg-Witten theory*, hep-th/9802033; A. Mikahilov, N. Nekrasov and S. Sethi, *Geometric realizations of BPS states in N=2 theories*, hep-th/9803142.

[6] K. Hasimoto, H. Hata and N. Sasakura, *3-string junction and BPS saturated solutions in SU(3) supersymmetric Yang-Mills theory*, hep-th/9803127.

[7] C. Fraser and T. J. Hollowood, Phys. Lett. **B402**, 106 (1997), hep-th/9704011.

[8] E.J. Weinberg, Nucl. Phys. **B167**, 500 (1980).

[9] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Mannin, Phys. Lett. **185B**, 185 (1978); N.H. Christ, E.J. Weiberg and N.K. Stanton, Phys. Rev. D **18**, 2013 (1978); E. Corrigan, D. Fairlie, P. Goddard, and S. Templeton, Nucl. Phys. **B140**, 31 (1978); E. Corrigan, P. Goddard, and S. Templeton, *ibid*, **B151**, 93 (1979).

[10] W. Nahm, Phys. Lett. **90B**, 413 (1980); in *Monopoles in quantum field theory*, edited by N. S. Craigie et al. (World Scientific, Singapore, 1982); in *Structural Elements in Particle Physics and Statistical Mechanics*, edited by J. Honerkamp et al. (Plenum, New York, 1983).

[11] E.J. Weinberg and P. Yi, *Explicit multimonopole solutions in $SU(N)$ gauge theory*, hep-th/9803164.

[12] N. Dorey, V.V. Khoze and M.P. Mattis, Phys. Rev. D **54**, 1921 (1996), hep-th/9603136; H. Osborn, Ann. Phys. (N.Y.) **135**, 373 (1981).

[13] W. Nahm, *Selfdual monopoles and calorons*, in *Group Theoretical methods in physics,* edited by D. Denardo et al. (Springer-Verlag, Berlin, 1984); H. Garland and M. Murray, *Kac-Moody monopoles and periodic instantons,* Commun. Math. Phys. **120**, 335 (1988).

[14] K. Lee and P. Yi, Phys. Rev. D **56** (1997) 3711, hep-th/9702107; K. Lee, *Instantons and magnetic monopoles on $R^3 \times S^1$ with arbitrary simple gauge groups*, hep-th/9802012; K. Lee and C. Lu, *SU(2) calorons and magnetic monopoles*, hep-th/9802108.

[15] T. C. Kraan and P. van Baal, *Exact T-duality between caloron and Taub-NUT spaces*, hep-th/9802049.

[16] R. S. Ward, Commun. Math. Phys. **79**, 317 (1981).

[17] M. Henningson, Nucl. Phys. **B461**, 101 (1996), hep-th/9510138.

[18] K. Lee, E. J. Weinberg, and P. Yi, Phys. Rev. D **54**, 6351 (1996).

[19] T. Kawano and K. Okuyama, *String network and 1/4 BPS states in N=4 SU(n) supersymmetric Yang-Mills theory*, hep-th/9804139.

[20] O. Bergman and B. Kol, *String webs and 1/4 BPS monopoles*, hep-th/9804160.