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Deriving N=2 S-dualities from Scaling for Product Gauge Groups

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Abstract

S-dualities in scale invariant $N=2$ supersymmetric field theories with product gauge groups are derived by embedding those theories in asymptotically free theories with higher rank gauge groups. S-duality transformations on the couplings of the scale invariant theory follow from the geometry of the embedding of the scale invariant theory in the Coulomb branch of the asymptotically free theory.

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1 Introduction

S-duality is the quantum equivalence of classically inequivalent field theories. The paradigmatic example is the strong-weak coupling duality of $N = 4$ supersymmetric Yang-Mills under which theories with couplings τ and $-1/\tau$ are identified.

This letter discusses S-dualities for scale invariant $N = 2$ supersymmetric field theories with product gauge groups and matter in the fundamental representation [1, 2, 3, 4]. The main evidence for S-dualities in $N = 2$ theories has come from the spectrum of BPS saturated states [5] and from low-energy effective actions [6]. By relating S-duality transformations in scale invariant $N = 2$ gauge theories to global symmetries in asymptotically free theories, it was shown in [7] that S-dualities of gauge theories with a single gauge group factor are, in fact, *exact* equivalences of their quantum field theories. S-dualities of the theories with product gauge groups are expected to have a more complicated structure of identifications on the classical coupling space [1]. Nonetheless, we show that it is still possible to relate S-dualities of these theories to global symmetries of higher rank asymptotically free theories.

As for simple gauge groups, the basic idea is to regard the marginal couplings of the scale invariant theory as the lowest components of $N = 2$ vector multiplets—complex scalar "Higgs" fields—in an enlarged theory. Then the coupling space $\mathcal{M}$ of the scale invariant theory is realized as a submanifold of the Coulomb branch $\mathcal{C}$ of the enlarged theory. Any S-duality identifications of different points of $\mathcal{M}$ are interpreted as equivalences on $\mathcal{C}$. By choosing the enlarged theory appropriately, these equivalences on $\mathcal{C}$ can be made manifest as (spontaneously broken) global symmetries.

In the next section we review the derivation of the S-duality transformations for the gauge theories with simple gauge group [7]. The brane picture of the derivation is very illuminating and allows for the straightforward generalization to the theories with product gauge groups. In section 3 we consider scale invariant $\bigotimes_i SU(k_i)$ gauge theories and in section 4 we derive S-dualities of scale invariant $\bigotimes_i SO(k_{2i-1}) \otimes Sp(k_{2i})$, $\bigotimes_i SU(k_i) \otimes SO(n)$ and $\bigotimes_i SU(k_i) \otimes Sp(n)$ theories. We conclude with some comments in section 5.

2 Review of S-duality for simple gauge groups

The scale invariant $N = 2$ theories with a single SU, SO, or Sp gauge group and quarks in the fundamental (defining) representation all have low-energy effective theories that are invariant under identifications of τ under a discrete group isomorphic to[1] $\Gamma^0(2) \subset SL(2,\mathbf{Z})$ [8, 9]. $\Gamma^0(2)$ is the subset of $SL(2,\mathbf{Z})$ matrices with even upper off-diagonal entry, or equivalently, is the subgroup of $SL(2,\mathbf{Z})$ generated by $\widetilde{T}: \ \tau \to \tau+2$ and $S: \ \tau \to -1/\tau$, acting on the classical coupling space $\mathcal{M}_{\rm cl} = \{{\rm Im}\tau > 0\}$. (We have taken the gauge coupling to be $\tau = \frac{\vartheta}{\pi} + i\frac{8\pi}{g^2}$, differing by a factor of two from the usual definition.) $\Gamma^0(2)$ is characterized more abstractly as the group freely generated by two generators $\widetilde{T}$ and S subject to one relation $S^2 = 1$. Dividing $\mathcal{M}_{\rm cl}$ by the S-duality group gives the the quantum coupling space $\mathcal{M} = \mathcal{M}_{\rm cl}/\Gamma^0(2)$. Generally, the relations satisfied by the generators $\{\widetilde{T}, S\}$ encode the holonomies in $\mathcal{M}$ around the fixed points of the $\Gamma^0(2)$ action. For example, for the action given above, $\mathcal{M}$ has three such points: the weak coupling point $\tau = +i\infty$ fixed by $\widetilde{T}$, an "ultra-strong" coupling point $\tau = 1$ fixed by $\widetilde{T}S$, and a $\mathbf{Z}_2$ point $\tau = i$ fixed by S. In fact, this data plus the fact that the topology of $\mathcal{M}$ is that of a two sphere summarizes the physically meaningful information about the space of couplings. Its particular realization as a fundamental domain of $\Gamma^0(2)$ acting on the τ upper half plane is dependent on which coordinate τ we use; without an independent non-perturbative definition of τ, the only conditions we can physically impose are on its behavior at arbitrarily weak coupling. Thus the physical content of a statement of S-duality is nothing more than a characterization of the topology of the space of couplings and the holonomies around its singular points.

Consider the scale invariant $SU(r)$ theory with $2r$ fundamental quarks. The Coulomb branch of the theory is described by the curve [8]

$$y^2 = \prod_{a=1}^{r} (x-\phi_a)^2 + (h^2-1) \prod_{j=1}^{2r} (x-\mu_j - h\mu), \tag{1}$$

parameterized by a gauge coupling h, quark masses μ_j and μ, and Higgs vevs ϕ_a. h is a function of the coupling such that $h^2 \sim 1 + 64 e^{i\pi\tau}$ at weak coupling, μ_j (satisfying $\sum_j \mu_j = 0$) are the eigenvalues of the mass matrix transforming in the adjoint of the $SU(2r)$ flavor group, the singlet mass μ is charged under the $U(1)$ "baryon number" global symmetry, and the ϕ_a (satisfying $\sum_a \phi_a = 0$) are the eigenvalues of the adjoint Higgs field; only flavor and gauge invariant combinations of the μ_j and ϕ_a appear as coefficients in (1).

In the scale invariant theory (setting the masses to zero) in this parameterization of the curve, the coupling parameter space is the h plane. There are two weak coupling points $h = \pm 1$ and a point $h = \infty$ where the low energy effective theory on the Coulomb

[1]For the $SU(2)$ scale invariant theory the S-duality group is enlarged to $SL(2,\mathbf{Z})$ [6]; we return to this point in section 5.

branch is singular. (Though $h = 0$ is apparently also such a singular point, a more careful analysis of the low energy effective action as $h \to 0$ reveals no divergences.) The low energy effective action is invariant under the $\mathbf{Z}_2$ identification $h \to -h$ (with $\mu \to -\mu$). Identifying the h-plane under the action of this $\mathbf{Z}_2$ gives a quantum coupling space $\mathcal{M}$ with one weak coupling singularity ($h = 1$), a $\mathbf{Z}_2$ fixed point ($h = 0$), and a singular point ($h = \infty$). This matches the description given above of the fundamental domain of $\Gamma^0(2)$ and is the low energy evidence for this S-duality.

We now review the scaling argument of [7] which derives this S-duality by embedding the scale invariant theory in the asymptotically free $SU(r+1)$ theory with $2r$ quarks. We can flow to the scale invariant $SU(r)$ theory by Higgsing the $SU(r+1)$ theory so that $\phi_a = M,\ 1 \leq a \leq r$ and $\phi_{r+1} = -rM$ and assigning the singlet mass $\mu = M$ to keep the $2r$ quarks massless. These tunings are also valid in the quantum theory, since upon applying them to the asymptotically free $SU(r+1)$ curve

$$y^2 = \prod_{a=1}^{r+1} (x - \phi_a)^2 - \Lambda^2 (x - \mu)^{2r} \tag{2}$$

and shifting $x \to x + M$, the curve factorizes as

$$y^2 = x^{2r} \left[(x + (r+1)M)^2 - \Lambda^2 \right]. \tag{3}$$

We recognize the x^{2r} factor of (3) as the singularity corresponding to the scale invariant vacuum of the $SU(r)$ theory with $2r$ massless quarks. We identify the dimensionless parameter M/Λ which varies along the scale invariant singularity with (some holomorphic function of) the gauge coupling of the scale invariant theory. Denote by $\mathcal{G}$ the submanifold (parameterized by M) of scale invariant vacua of the Coulomb branch. From the degenerations of (3) $\mathcal{G}$ has a weak coupling point at $M = \infty$ and two ultra-strong coupling points (singularities) at $M = \pm\Lambda/(r+1)$. Furthermore there is a non-anomalous $\mathbf{Z}_2 \subset U(1)_R$ which acts on the Higgs fields as $\phi \to -\phi$ (and when appropriately combined with a global flavor rotation takes $\mu \to -\mu$), so that the M plane is identified under $M \to -M$,[2] giving a single ultra-strong coupling point and a $\mathbf{Z}_2$ orbifold point at $M = 0$, thus deriving the content of the conjectured S-duality for the SU series.

Although this argument used the low energy effective action of the asymptotically free theory, it provides more than just low energy evidence for the S-duality. By tuning

[2] Although one should not quotient the space of vacua by the action of a spontaneously broken global symmetry, because we are interpreting this submanifold of the Coulomb branch as the *coupling* space of the scale invariant theory, it is legitimate (indeed necessary) to do so.

vevs on the Coulomb branch $\mathcal{C}$ of the asymptotically free theory to approach the scale invariant fixed line $\mathcal{G} \subset \mathcal{C}$, we can deduce exact information about the scale invariant theories.[3] Since the manifold of fixed points $\mathcal{G}$ is related to the coupling space $\mathcal{M}$ of the scale invariant theory by a holomorphic map (by $N = 2$ supersymmetry), $\mathcal{G}$ must be a multiple cover of $\mathcal{M}$, and hence the topology of $\mathcal{G}$ will constrain the topology of $\mathcal{M}$, giving exact S-duality relations. These identifications are exact in the the scale invariant theory because by approaching $\mathcal{G}$ arbitrarily closely on the Coulomb branch we can make the effect of any irrelevant operators from the asymptotically free theory as small as we like. In essence, this argument assumes only that the scale invariant theory has a gap in its spectrum of dimensions of irrelevant operators.[4]

The explicit factorization of (3) into the conformal factor of the scale invariant $SU(r)$ theory and the factor responsible for additional singularities (which we interpreted as singularities on the space of couplings) is possible because of the special hyperelliptic representation of this curve. Solutions of $N = 2$ gauge theories with product gauge groups are encoded in curves which are not hyperelliptic. To generalize the preceding argument to product gauge groups it is useful to reformulate it in an M-theory/IIA brane language.

An *ad hoc* object from the field theory perspective—a Riemann surface whose complex structure encodes the low energy couplings on the Coulomb branch—is given a physical interpretation in the M-theory formulation of $N = 2$ gauge theories. Consider the type IIA configuration of intersecting branes depicted in Fig. 1. Two (NS) fivebranes extend in the directions $x^0, x^1, \ldots, x^5$ and are located at $x^7 = x^8 = x^9 = 0$ and at some specific values of x^6. $r + 1$ (D) fourbranes stretch between the fivebranes, extend over $x^0, \ldots, x^3$ and are located at $x^7 = x^8 = x^9 = 0$ and at a point in $v = x^4 + ix^5$. Note that these fourbranes are finite in the x^6 direction. This configuration represents the Coulomb branch vacua of an $SU(r + 1)$ gauge theory in the four dimensions $x^0, \ldots, x^3$, with the position of the fourbranes in v corresponding to the vevs of the adjoint scalar in the $N = 2$ vector multiplet. For the model in hand we choose $v = M$ for r of the fourbranes and $v = -rM$ for the remaining one. In Fig. 1 we have also shown $2r$ semi-infinite fourbranes which also extend over $x^0, \ldots, x^3$ and are located at $x^7 = x^8 = x^9 = 0$. r of these fourbranes end on the left fivebrane and the other r end on the right one and correspond to the $2r$ hypermultiplets (quarks) of the $SU(r + 1)$

[3]This same type reasoning is used in [10] to deduce exact equivalences between conformal field theories and supergravity theories.

[4]By the arguments of [11] this is plausibly also the condition for the scale invariant theory to be conformally invariant.

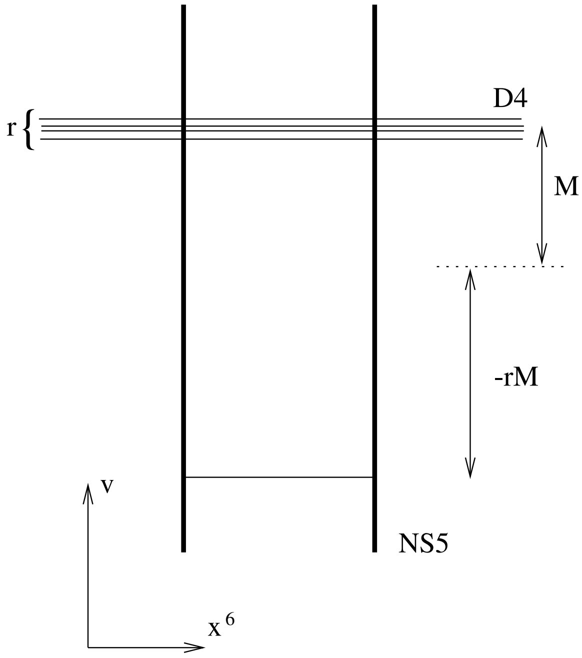

Figure 1: A configuration of fivebranes connected by parallel fourbranes realizing the embedding of the scale invariant $SU(r)$ gauge theory into an asymptotically free $SU(r+1)$ theory. The horizontal dotted line marks the position of the origin in the v-plane.

gauge theory. Their (fixed) v coordinates encode the masses of the hypermultiplets. With the choice of mass parameters as in (2), one gets the configuration of Fig. 1.

This brane configuration is singular where the ends of the fourbranes end on the fivebrane world volume. These singularities are resolved by lifting the construction to M theory [1] where the IIA fourbranes become M theory fivebranes wrapped around the eleventh dimension x^{10}. In fact, the whole configuration of branes in Fig. 1 appears as a single M theory fivebrane with world volume $\mathbf{R}^4 \times \Sigma$ where $\mathbf{R}^4$ is the four-dimensional low energy space-time and Σ is the Riemann surface describing the Coulomb branch of the $SU(r+1)$ gauge theory with $2r$ quarks. Let $t = e^{-s}$ with $s = (x^6 + ix^{10})/R$, R being the radius of the eleventh dimension. Then Σ is holomorphically embedded in $\mathbf{C}^2 = \{t, v\}$ as the curve [1]

$$(v - M)^r \; t^2 + \tilde{f} \; (v - M)^r (v + rM) \; t + (v - M)^r = 0. \tag{4}$$

From the Type IIA perspective (4) describes bending fivebranes as $t \to \infty$ or 0. Asymptotically, the separation of the fivebranes reads

$$s_2 - s_1 = \ln \frac{t_1}{t_2} \simeq 2 \ln \tilde{f} v \tag{5}$$

which describes the running of the β-function of the asymptotically free $SU(r+1)$ gauge theory with $2r$ quarks, with $1/\tilde{f}$ identified with the strong coupling scale Λ of the $SU(r+1)$ theory, and v interpreted as the energy scale at which we measure the coupling.

The scaling limit, implicit in (3), involves looking at the scale invariant $SU(r)$ theory at energy scales much smaller than either M or Λ. Only far below these scales can one sensibly talk about the scale invariance of the $SU(r)$ gauge theory which appears as an effective low energy description after integrating out modes charged under the decoupled $U(1)$ gauge factor. The decoupling of the $U(1)$ gauge factor from the brane point of view means that we consider a small region of the brane construction near $v = M$, which locally looks like a finite $SU(r)$ gauge theory. To describe the scale invariant geometry near $v = M$ we thus introduce a local coordinate $x = v - M$ and consider $x \ll \{\Lambda, M\}$ which is the geometrical realization of the scaling to the fixed point theories. From (4) we find the description of this region to be

$$x^r \left(t^2 + \frac{M(r+1)}{\Lambda}\, t + 1 \right) = 0, \tag{6}$$

which indeed describes the scale invariant $SU(r)$ theory (at the origin of its Coulomb branch) with coupling parameter

$$f = M(r+1)/\Lambda. \tag{7}$$

Thus, we have tuned to a one complex dimensional submanifold of the Coulomb branch $\{M\} \equiv \mathcal{G} \subset \mathcal{C}$ of the $SU(r+1)$ theory, which realizes some multiple cover $\widetilde{\mathcal{M}}$ of the coupling space of the scale invariant $SU(r)$ theory. At certain points in $\mathcal{G}$, specifically $M = \pm 2\Lambda/(r+1)$ and $M \to \infty$, (6) develops additional singularities that we interpret through (7) as singularities ("punctures") of $\widetilde{\mathcal{M}}$. $\widetilde{\mathcal{M}}$ is not quite the coupling manifold $\mathcal{M}$ of the $SU(r)$ theory since, as in the field theoretical arguments above, there is a $\mathbf{Z}_2$ identification $M \to -M$ coming from the unbroken subgroup of the $U(1)_R$ symmetry:

$$\mathcal{M} \simeq \widetilde{\mathcal{M}}/\mathbf{Z}_2. \tag{8}$$

This once again gives the expected S-duality of the $SU(r)$ theory.

S-dualities can be derived for the scale invariant SO and Sp theories along the same lines. One can also provide a similar interpretation of the derivation in brane language.

Before proceeding to the generalization of this construction to the product gauge group theories, we comment on the definition of S-duality *groups*. It is natural to define the S-duality group Γ by $\mathcal{M} = \mathcal{M}_{\rm cl}/\Gamma$. Using the fact that $\pi_1(\mathcal{M}_{\rm cl}) = 1$ one might be tempted to identify $\Gamma = \pi_1(\mathcal{M})$, the fundamental group of $\mathcal{M}$. However, as we have just seen, Γ does not act freely on $\mathcal{M}_{\rm cl}$, so $\pi_1(\mathcal{M})$ does not capture all of Γ. In particular, we must enlarge π_1 to include the holonomies around the orbifold singularities of $\mathcal{M}$ [12]. For example, we can consider fixed points of discrete identifications as being punctures, and identify a closed loop around a $\mathbf{Z}_2$ orbifold singularity $M = 0$ of $\mathcal{M}$ with the order two element of the S-duality group. Adding to this the other nontrivial element of the fundamental group of $\mathcal{M}$ represented by a closed loop around the $M^2 = 4\Lambda^2/(r+1)^2$ puncture, we recover the full S-duality group $\Gamma^0(2)$ of the scale invariant $SU(r)$ theory.

3 S-duality in scale invariant $\otimes$SU gauge theories

In this section we consider the "cylindrical" models of [1] and implement their non-perturbative duality group as global symmetries acting on the Coulomb branch of a higher rank theory.

Consider $N = 2$ supersymmetric gauge theory with gauge group

$$G = \bigotimes_{i=1}^{n} SU(k_i), \tag{9}$$

$n-1$ hypermultiplets in the bifundamental representation $(\mathbf{k_1}, \overline{\mathbf{k_2}}) \oplus (\mathbf{k_2}, \overline{\mathbf{k_3}}) \ldots \oplus (\mathbf{k_{n-1}}, \overline{\mathbf{k_n}})$ and d_i hypermultiplets in the fundamental representation of the $SU(k_i)$ gauge group factor. We denote this model as

$$\mathcal{F}_G = \mathbf{k_1}\ _{\mathbf{d_1}}| \ldots |\mathbf{k_i}\ _{\mathbf{d_i}}| \ldots |\mathbf{k_n}\ _{\mathbf{d_n}}. \tag{10}$$

Given a gauge group G there is a unique choice of d_i for which the beta functions all vanish, namely

$$d_i = 2k_i - k_{i-1} - k_{i+1} \tag{11}$$

(where we understand $k_0 = k_{n+1} = 0$). The Coulomb branch $\mathcal{C}$ is the manifold of vacua with zero hypermultiplet vevs and vector multiplet vevs in the complexified Cartan subalgebra of G. Locally $\mathcal{C} \simeq \mathbf{C}^r$, where $r = \text{rank } G$, with coordinates $\phi_j^{(i)}$, $j = 1, \ldots k_i$, $i = 1, \ldots n$. The origin of $\mathcal{C}$ ($\phi_j^{(i)} = 0$ for all $\{j, i\}$) describes the scale invariant vacuum, while elsewhere the scale invariance is spontaneously broken and the underlying theory generically flows to an infrared free $U(1)^r$ gauge theory. The couplings τ_{ab} of this low-energy effective $U(1)^r$ theory can be geometrically encoded as the complex structure (period matrix) of a genus-r Riemann surface Σ.

Following [1] for the given model Σ is

$$\begin{aligned} y^{n+1} &+ g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} \\ &+ \ldots + g_i(v)\prod_{s=1}^{i-1} J_s^{i-s}y^{n+1-i} + \ldots + \prod_{s=1}^{n} J_s^{n+1-s} = 0 \end{aligned} \tag{12}$$

with

$$\begin{aligned} J_i &= \prod_{j=1}^{d_i}(v - m_j^{(i)}) \\ g_i &= f_i \prod_{j=1}^{k_i}(v - \mu_i - \phi_j^{(i)}), \qquad \sum_{j=1}^{k_i-1} \phi_j^{(i)} = 0. \end{aligned} \tag{13}$$

Here $m_j^{(i)}$ are masses of the hypermultiplets in the fundamental representation of the jth gauge group factor, $\mu_{i+1} - \mu_i$ are the masses of the hypermultiplets in the bifundamental representations $(\mathbf{k_i}, \overline{\mathbf{k}}_{\mathbf{i+1}})$, and the f_i are some functions of the n gauge couplings τ_i of the different factors in G. The dependence of the coefficients of the polynomial (12) on the bares masses, vevs, and couplings can be determined by taking various weak coupling limits of the above curve (corresponding in the IIA picture to taking groups of NS fivebranes off to infinity), and matching to known lower rank results. This also determines the f_i as functions of the τ_i as (though we will not need this result in what follows)

$$f_i = (\sum_{j_1 > \cdots > j_i} s_{j_1} \cdots s_{j_i}) \left(\prod_{i=1}^{n+1} s_i\right)^{-i/(n+1)}, \tag{14}$$

where $s_i \equiv -\prod_{k=1}^{i-1} q_k$ (and $s_1 \equiv -1$) and $q_i = e^{i\pi\tau_i}$. Note that all these identifications (of the masses and vevs as well as the couplings) can be modified by adding terms which vanish at least as fast as a power law in the q_i at weak coupling ($q_i \to 0$). Such redefinitions just correspond to non-perturbative redefinitions of the parameters which we have no independent way of fixing. The dependence of the coefficients of polynomials describing the curves for non-scale invariant (asymptotically free) theories can be found from the above identifications by appropriately sending bare masses to infinity and couplings to zero. We used such parameter matchings to write equation (4) for the simple SU theory considered in the last section, and we will use such matchings several times more in the remainder of the paper. Some examples of these identifications had been derived previously in [13].

The classical space of couplings ${\mathcal{M}_{\rm cl}}^{(n)}$ is $\mathbf{C}^n$ corresponding to the n gauge couplings q_i. Just as in the case of a single gauge group the "quantum" coupling space $\mathcal{M}^{(n)}$ seen by the low energy effective theory—as encoded in the curve (12)—is quite different. It can be identified with the asymptotic positions of the M theory fivebrane on the cylinder parameterized by the complex coordinate $s = (x^6 + ix^{10})/R$. This identification can be expressed following [1] as

$$\mathcal{M}^{(n)} = \mathcal{M}_{0,n+3;2} \tag{15}$$

where $\mathcal{M}_{0,n+3;2}$ is the moduli space of a genus zero Riemann surface with $n+3$ marked points, two of which are distinguished and ordered while the other $n+1$ are indistinguishable. (In [1] the S-duality group Γ_n for these theories was identified with $\pi_1(\mathcal{M}_{0,n+3;2})$. This must be extended to include holonomies around the orbifold points as discussed in the last section.) In what follows we will present a scaling argument to derive this S-duality as an exact equivalence by deriving (15).

We start with an asymptotically free theory described in the brane language by

$$\mathcal{F}_{\widetilde{G}} = \mathbf{k_1+1}\ {}_{\mathbf{d_1}}|\ldots|\mathbf{k_i+i}\ {}_{\mathbf{d_i}}|\ldots|\mathbf{k_n+n}\ {}_{\mathbf{d_n}}. \tag{16}$$

We can flow to a scale invariant $\mathcal{F}_G$ model by tuning the Higgs vevs $\widetilde{\phi}_j^{(i)}$ of the $\mathcal{F}_{\widetilde{G}}$ model so as to break

$$\widetilde{G} = \bigotimes_{i=1}^{n} SU(k_i+i) \rightarrow \bigotimes_{i=1}^{n} U(1)^i\ \otimes\ G. \tag{17}$$

Classically, this is achieved by choosing the Higgs vevs of $\mathcal{F}_{\widetilde{G}}$ as

$$\begin{array}{lcl}
\phi_j^{(1)} = M_1,\ j=1,\ldots,k_1 & \text{and} & \phi_{k_1+1} = -k_1M_1 \\
\phi_j^{(2)} = M_1,\ j=1,\ldots,k_2 & \text{and} & \phi_{k_2+p} = -k_2M_1+\omega_2^pM_2,\ p=1,2 \\
\ldots & & \ldots \\
\phi_j^{(i)} = M_1,\ j=1,\ldots,k_i & \text{and} & \phi_{k_i+p} = -k_iM_1+\omega_i^pM_i,\ p=1,\ldots,i \\
\ldots & & \ldots \\
\phi_j^{(n)} = M_1,\ j=1,\ldots,k_n & \text{and} & \phi_{k_n+p} = -k_nM_1+\omega_n^pM_n,\ p=1,\ldots,n
\end{array} \tag{18}$$

where ω_p is the pth root of unity, $\omega_p = e^{2\pi i/p}$. In addition, to insure that all hypermultiplets of the $\mathcal{F}_G$ model are massless we choose zero bifundamental masses of $\mathcal{F}_{\widetilde{G}}$, $\widetilde{\mu}_i = 0$, $i=1,\ldots,n$, and tune all masses of the hypermultiplets in the fundamental representations of the $\widetilde{G}$ factors to $\widetilde{m}_j^{(i)} = M_1$, $j=1,\ldots,d_i$; $i=1,\ldots,n$.

These tunings are also valid quantumly at energy scales much lower than $\min_i M_i$. To see this, consider the curve $\widetilde{\Sigma}$ for the $\mathcal{F}_{\widetilde{G}}$ model

$$\begin{aligned}
y^{n+1} &+ \tilde{f}_1(v-M_1)^{k_1}(v+k_1M_1)y^n \\
&+ \tilde{f}_2(v-M_1)^{k_2}(v-M_1)^{d_1}y^{n-1}\prod_{p=1}^{2}(v+k_2M_1-\omega_2^pM_2)+\ldots
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \tilde{f}_i (v-M_1)^{k_i} (v-M_1)^{\sum_{s=1}^{i-1} d_s (i-s)} y^{n+1-i} \prod_{p=1}^{i} (v + k_i M_1 - \omega_i^p M_i) + \ldots \\
&+ \quad (v-M_1)^{\sum_{s=1}^{n} d_s (n+1-s)} = 0 \ .
\end{aligned} \tag{19}
$$

Shifting $v \to v + M_1$, in the decoupling limit, $v \ll \min_i M_i$, $\tilde{\Sigma}$ turns into

$$
\begin{aligned}
y^{n+1} \quad &+ \quad \tilde{f}_1 (k_1+1) M_1 v^{k_1} y^n + \ldots \\
&+ \quad \tilde{f}_i ((k_i+1)^i M_1^i - M_i^i) v^{k_i} \prod_{s=1}^{i-1} \left(v^{d_s}\right)^{i-s} y^{n+1-i} + \ldots \\
&+ \quad \prod_{s=1}^{n} \left(v^{d_s}\right)^{n+1-s} = 0,
\end{aligned} \tag{20}
$$

which we recognize as the curve for the scale invariant $\mathcal{F}_G$ model (12) with gauge coupling parameters

$$
f_i = \tilde{f}_i ((k_i+1)^i M_1^i - M_i^i). \tag{21}
$$

Note that the first $n-1$ gauge factors of the $\mathcal{F}_{\widetilde{G}}$ model have zero beta function, while the last one is asymptotically free. We therefore must assign a strong coupling scale Λ to this factor. By embedding the scale invariant model $\mathcal{F}_G$ into $\mathcal{F}_{\widetilde{G}}$ we effectively traded the n dimensionless couplings q_i of $\mathcal{F}_G$ for the n gauge invariant dimensionless parameters $x_i = (M_i/\Lambda)^i$ of the $\mathcal{F}_{\widetilde{G}}$ model. Let

$$
\widetilde{\mathcal{M}}^{(n)} = \{x_i, \ i = 1, \ldots, n\}. \tag{22}
$$

At special points in $\widetilde{\mathcal{M}}^{(n)}$, (20) develops addition singularities that should be thought of as the singularities in the quantum coupling space. This locus of singularities (punctures) in $\widetilde{\mathcal{M}}^{(n)}$ is easily extracted from (20) by redefining $y \to v^{k_1} y$ and using the scale invariance conditions (11) to get

$$
y^{n+1} + \tilde{f}_1 (k_1+1) x_1 y^n + \ldots + \tilde{f}_i ((k_i+1)^i x_1^i - x_i) y^{n+1-i} + \ldots + 1 = 0, \tag{23}
$$

and is given by the vanishing of the discriminant of (23). Singularities of (20) also arise when one of the roots of (23) goes either to zero or infinity, which signals the decoupling of the gauge group factors.

Global symmetries of the $\mathcal{F}_{\widetilde{G}}$ model induce identifications on the $\widetilde{\mathcal{M}}^{(n)}$ manifold. The classical $U(1)_R$ symmetry of $\mathcal{F}_{\widetilde{G}}$ is broken by the instantons of the last factor in $\widetilde{G}$ to a discrete subgroup $\mathbf{Z}_{n+1} = \{\omega_{n+1}^p, \ p = 0, \ldots, n\}$. The latter acts on the coordinates x_i as

$$
x_i \to \omega_{n+1}^{p \cdot i} \ x_i, \qquad p = 0, \ldots, n, \tag{24}
$$

under whose action we identify the quantum coupling space $\mathcal{M}^{(n)}$ with

$$\mathcal{M}^{(n)} \simeq \widetilde{\mathcal{M}}^{(n)}/\mathbf{Z}_{n+1}. \tag{25}$$

The punctures of the $\widetilde{\mathcal{M}}^{(n)}$ manifold comprise the vanishing locus of the discriminant of (23) along with points where one or more roots of (23) go to zero or infinity.

We can now show that $\mathcal{M}^{(n)} \simeq \mathcal{M}_{0,n+3,2}$. It is convenient to change coordinates $\{x_i\}$ on $\widetilde{\mathcal{M}}^{(n)}$ to $\{f_i\}$ through (21). This change is nonsingular and one-to-one. The $\mathbf{Z}_{n+1}$ symmetry of $\widetilde{\mathcal{M}}^{(n)}$ has the same action on f_i as on x_i. The polynomial determining the punctures then reads

$$y^{n+1} + f_1 y^n + \ldots + f_i y^{n+1-i} + \ldots + f_n y + 1 = 0. \tag{26}$$

Now, a configuration of $n+1$ unordered points on a genus zero Riemann surface with two marked points (zero and infinity)—a point in $\mathcal{M}_{0,n+3,2}$—can be thought of as the positions of the roots of the polynomial (26) which are encoded in the coefficients $\{f_i\}$. Furthermore, the $n+1$ sets of coefficients $\{\omega_{n+1}^p \cdot f_i\}$, $p = 0, \ldots, n$ are related by a rotation of the sphere (which preserves the special character of zero and infinity) and so should be identified. Dilatations of y are already fixed by the constant term in (26). The set $\{f_i\}$ maps to a single point in $\widetilde{\mathcal{M}}^{(n)}$, while the above identification is equivalent to the modding by $\mathbf{Z}_{n+1}$ in $\widetilde{\mathcal{M}}^{(n)}$. Thus,

$$\mathcal{M}^{(n)} = \widetilde{\mathcal{M}}^{(n)}/\mathbf{Z}_{n+1} \simeq \mathcal{M}_{0,n+3,2}, \tag{27}$$

as we sought to prove.

We can also show how decoupling of a single gauge factor in G produces $\mathcal{M}^{(n-1)}$ from $\mathcal{M}^{(n)}$. Decoupling the first gauge factor in the product chain is achieved by taking all $f_i \to \infty$, while keeping

$$\tilde{p}_i = \frac{f_{i+1}}{f_1^{1-i/n}} \tag{28}$$

finite. In this limit, $\widetilde{\mathcal{M}}^{(n)} \to \widetilde{\mathcal{P}}^{(n-1)}$ with the punctures of the $\widetilde{\mathcal{P}}^{(n-1)}$ manifold described by

$$y^n + p_1 y^{n-1} + \ldots + p_i y^{n-i} + \ldots + p_n y + 1 = 0, \tag{29}$$

which follows from (26) in the limit after rescaling $y \to y/f_1^{1/n}$. The $\mathbf{Z}_{n+1}$ action on $\widetilde{\mathcal{M}}^{(n)}$ induces a $\mathbf{Z}_n$ action on $\widetilde{\mathcal{P}}^{(n-1)}$

$$p_i \to \omega_{n+1}^{p\cdot i\cdot(1+1/n)}\, p_i = \omega_n^{p\cdot i} p_i. \tag{30}$$

Thus we conclude that indeed $\widetilde{\mathcal{P}}^{(n-1)} \simeq \widetilde{\mathcal{M}}^{(n-1)}$.

4 S-duality in other product gauge groups

$N=2$ gauge theories with products of SO and Sp groups were solved in [2, 3]. The derivation of S-dualities for these models essentially repeats the arguments of the previous section.

The model

$$\mathcal{F}_G = +\mathbf{2k_1}\ {}_{\mathbf{d_1}}| - \mathbf{2k_2}\ {}_{\mathbf{d_2}}|\ldots|(\mathbf{-1})^{\mathbf{i}}\mathbf{2k_i}\ {}_{\mathbf{d_i}}|\ldots|(\mathbf{-1})^{\mathbf{n}}\mathbf{2k_n}\ {}_{\mathbf{d_n}} \tag{31}$$

(where "+" signs denote SO gauge factors and "−" signs represent Sp gauge factors) is scale invariant if

$$d_i = 2k_i + 2(-1)^i - k_{i-1} - k_{i+1}, \qquad i=1,\ldots,n, \qquad k_0 \equiv k_{n+1} \equiv 0. \tag{32}$$

It can be obtained by Higgsing the

$$\mathcal{F}_{\widetilde{G}} = +\mathbf{2(k_1+1)}\ {}_{\mathbf{d_1}}| - \mathbf{2(k_2+2)}\ {}_{\mathbf{d_2}}|\ldots|(\mathbf{-1})^{\mathbf{i}}\mathbf{2(k_i+i)}\ {}_{\mathbf{d_i}}|\ldots|(\mathbf{-1})^{\mathbf{n}}\mathbf{2(k_n+n)}\ {}_{\mathbf{d_n}} \tag{33}$$

model with all hypermultiplets massless so that

$$\widetilde{G} \to \bigotimes_{i=1}^{n} U(1)^i \ \otimes\ G. \tag{34}$$

This is achieved by tuning

$$\begin{array}{l}
\widetilde{\phi}^{(i)}_j = 0, \qquad j=1,\ldots,k_i; \qquad i=1,\ldots,n \\
\widetilde{\phi}^{(i)}_{k_i+1} = M_1; \qquad i=1,\ldots,n \\
\ldots \\
\widetilde{\phi}^{(i)}_{k_i+r} = M_r; \qquad i=r,\ldots,n \\
\ldots \\
\widetilde{\phi}^{(i)}_{k_n+n} = M_n.
\end{array} \tag{35}$$

The curve $\widetilde{\Sigma}$ for the $\mathcal{F}_{\widetilde{G}}$ model then reads [2]

$$\begin{aligned}
v^2 y^{n+1} &+ \tilde{f}_1 v^{2k_1}(v^2-M_1^2)\ y^n \\
&+ \tilde{f}_2 v^{2k_2+2}(v^2-M_1^2)(v^2-M_2^2)v^{2d_1}\ y^{n-1} \\
&+ \ldots \\
&+ \tilde{f}_i v^{2k_i+1+(-1)^i} \prod_{r=1}^{i}(v^2-M_r^2)\ v^{2\sum_{s=1}^{i-1} d_s(i-s)}\ y^{n+1-i} \\
&+ \ldots \\
&+ v^{1+(-1)^{n+1}} v^{2\sum_{s=1}^{n} d_s(n+1-s)} = 0.
\end{aligned} \tag{36}$$

In the decoupling limit $v^2 \ll M_r^2$, $\tilde{\Sigma}$ reproduces the curve of the scale invariant $\mathcal{F}_G$ model with gauge couplings

$$f_i = (-1)^i \tilde{f}_i \prod_{r=1}^{i} M_r^2. \tag{37}$$

As for the product of SU gauge groups, $n-1$ factors of the $\mathcal{F}_{\widetilde{G}}$ model have zero beta function and the last one is asymptotically free. The instantons of the asymptotically free factor generate the strong coupling scale Λ which breaks the global $U(1)_R$ symmetry to a $\mathbf{Z}_{n+1}$ subgroup. The latter acts on the dimensionless gauge invariant coordinates M_i^2 as $M_i^2 \to \omega_{n+1}^p M_i^2$, and hence identifies the couplings under $f_i \to \omega_{n+1}^p\, f_i$, for $p = 0, \dots n$. We thus identify the the quantum coupling space $\mathcal{M}^{(n)}$ of the scale invariant $\mathcal{F}_G$ model with $\mathcal{M}^{(n)} \simeq \widetilde{\mathcal{M}}^{(n)}/\mathbf{Z}_{n+1}$ where $\widetilde{\mathcal{M}}^{(n)} = \{f_i\}$. Punctures on $\widetilde{\mathcal{M}}^{(n)}$ are identified with the points of vanishing discriminant of the polynomial (extracted from the $\mathcal{F}_G$ curve by the change of variables $y \to v^{2k_1-2}y$)

$$y^{n+1} + f_1 y^n + \ldots + f_i y^{n+1-i} + \ldots + 1 = 0 \tag{38}$$

along with values of f_i which where one or more roots of (38) go to zero or infinity. This is precisely the same description of $\mathcal{M}^{(n)}$ as we found in the last section, and so gives $\mathcal{M}^{(n)} \simeq \mathcal{M}_{0,n+3,2}$, which again is the answer expected on the basis of the low energy effective theory.

In fact, we find that all the scale invariant models studied in [2] have the coupling space $\mathcal{M}_{0,n+3,2}$, determined only by the number of gauge group factors. This is not surprising since the $N=2$ gauge theory with n factors in the gauge group was derived in [2] from a configuration with $n+1$ straight NS fivebranes which can be associated [1] with $n+1$ unordered points on a cylinder. The moduli space of these points have "punctures" (where the low energy effective description of the theory is singular) whenever two or more NS fivebranes coincide or move to infinity. This can be cast in the form of the vanishing of the discriminant of an $n+1$ order polynomial as in (38).

Unlike the previous example, the $\bigotimes_i SU(k_i) \otimes SO(N)$ and $\bigotimes_i SU(k_i) \otimes Sp(N)$ series of [4] have a different quantum coupling space (and therefore a different S-duality group). We will now show that in these cases the coupling space is $\mathcal{M}^{(n)} = \mathcal{Q}_{0,n,2}$, where $\mathcal{Q}_{0,n,2}$ is the moduli space of genus zero Riemann surfaces with two distinguished and marked points and n unordered pairs of points with the further condition that, in terms of any complex coordinate z on the sphere, the product of the coordinates within each pair is the same for all pairs.

We will write out the derivation only for the $\bigotimes_i SU(k_i) \otimes SO(k_n)$ model with even k_n, since the other cases can be analyzed in a similar way. We flow to the scale invariant model

$$\mathcal{F}_G = \mathbf{k_1}\ _{\mathbf{d_1}}|\ldots|\mathbf{k_i}\ _{\mathbf{d_i}}|\ldots|\mathbf{k_{n-1}}\ _{\mathbf{d_{n-1}}}|{+}\mathbf{k_n}\ _{\mathbf{d_n}} \tag{39}$$

with

$$d_i = 2k_i - k_{i-1} - k_{i+1}, \qquad i = 1,\ldots,n-1, \qquad k_0 \equiv 0, \text{ and } d_n = k_n - 2 - k_{n-1}, \tag{40}$$

by Higgsing the

$$\mathcal{F}_{\widetilde{G}} = \mathbf{k_1+2}\ _{\mathbf{d_1}}|\ldots|\mathbf{k_i+2i}\ _{\mathbf{d_i}}|\ldots|\mathbf{k_{n-1}+2n-2}\ _{\mathbf{d_{n-1}}}|{+}(\mathbf{k_n+2n})\ _{\mathbf{d_n}} \tag{41}$$

model with massless hypermultiplets, breaking the gauge group $\widetilde{G}$ as

$$\widetilde{G} \to \bigotimes_{i=1}^{n-1} U(1)^{2i} \ \otimes U(1)^n \ \otimes \ G. \tag{42}$$

This is achieved by tuning

$$\begin{array}{l}
\widetilde{\phi}^{(i)}_j = 0, \qquad j = 1,\ldots,k_i; \qquad i = 1,\ldots,n-1\\
\widetilde{\phi}^{(n)}_j = 0, \qquad j = 1,\ldots,k_n/2\\
\widetilde{\phi}^{(i)}_{k_i+1} = -\widetilde{\phi}^{(i)}_{k_i+2} = M_1; \qquad i = 1,\ldots,n-1\\
\widetilde{\phi}^{(n)}_{k_n/2+1} = M_1;\\
\ldots\\
\widetilde{\phi}^{(i)}_{k_i+2r-1} = -\widetilde{\phi}^{(i)}_{k_i+2r} = M_r; \qquad i = r,\ldots,n-1\\
\widetilde{\phi}^{(n)}_{k_n/2+r} = M_r;\\
\ldots\\
\widetilde{\phi}^{(n-1)}_{k_{n-1}+2n-1} = -\widetilde{\phi}^{(n-1)}_{k_{n-1}+2n} = M_n;\\
\widetilde{\phi}^{(n)}_{k_n/2+n} = M_n \ .
\end{array} \tag{43}$$

Introducing

$$\begin{array}{l}
g_i(v) = g_{2n-i}(-v) = \tilde{f}_i v^{k_i} \prod_{r=1}^{i} (v^2 - M_r^2), \ i = 1,\ldots,n\\
J_i(v) = J_{2n-i}(-v) = v^{d_i}, \ i = 1,\ldots,n-1\\
J_n(v) = (-1)^{d_n} v^4 \ v^{2d_n},
\end{array} \tag{44}$$

the curve $\widetilde{\Sigma}$ for the $\mathcal{F}_{\widetilde{G}}$ model becomes [4]

$$y^{2n} + g_1(v) y^{2n-1} + \ldots + g_i(v) \prod_{r=1}^{i-1} J_r^{i-r} y^{2n-i} + \ldots + \prod_{r=1}^{2n-1} J_r^{2n-r} = 0. \tag{45}$$

In the decoupling limit $v^2 \ll M_r^2$ it reduces to the scale invariant $\mathcal{F}_G$ model with gauge couplings

$$f_i = f_{2n-i} = (-1)^i \tilde{f}_i \prod_{r=1}^{i} M_r^2, \qquad i = 1, \ldots, n. \tag{46}$$

The SU factors of the $\widetilde{G}$ group have vanishing beta function and the SO factor is asymptotically free. The global $U(1)_R$ symmetry is broken to a $\mathbf{Z}_2$ discrete subgroup which acts on the gauge invariant parameters M_i^2 as $M_i^2 \to -M_i^2$, and hence identifies the couplings by $f_i \to -f_i$, $i = 1, \ldots, n$. The quantum coupling space $\mathcal{M}^{(n)}$ of the scale invariant $\mathcal{F}_G$ model is thus $\mathcal{M}^{(n)} \simeq \widetilde{\mathcal{M}}^{(n)}/\mathbf{Z}_2$ where $\widetilde{\mathcal{M}}^{(n)} = \{f_i\}$. Punctures on $\widetilde{\mathcal{M}}^{(n)}$ are at values of f_i for which either the discriminant of the polynomial (extracted from the curve of the $\mathcal{F}_G$ model by the change of variables $y \to v^{k_1} y$)

$$y^{2n} + f_1 y^{2n-1} + \ldots + f_{n-1} y^{n+1} + f_n y^n + f_{n-1} y^{n-1} + \ldots + f_1 y + 1 = 0 \tag{47}$$

vanishes or some or its roots go to zero or infinity. It is always possible to factorize a polynomial of the form (47) into one of the form

$$\prod_{i=1}^{n} (y^2 + z_i y + 1) = 0, \tag{48}$$

thus giving an isomorphism between $\widetilde{\mathcal{M}}^{(n)} = \{f_i\}$ and $\widetilde{\mathcal{Q}}^{(n)} = \{z_i \text{ modulo permutations}\}$. The $\mathbf{Z}_2$ identification $f_i \to -f_i$ becomes $z_i \to -z_i$. With this presentation it is clear that $\mathcal{M}^{(n)} = \widetilde{\mathcal{Q}}^{(n)}/\mathbf{Z}_2 \simeq \mathcal{Q}_{0,n,2}$.

5 Conclusion

In this letter we have derived the S-dualities of scale invariant $N = 2$ gauge theories with product gauge groups by embedding those theories in asymptotically free $N = 2$ theories with higher rank gauge groups and tuning to the scale invariant theories on the Coulomb branch. We related S-duality transformations on the couplings of the scale invariant theories to global symmetries acting on the Coulomb branch of the higher rank theories. Since these global symmetries are exact in the asymptotically free theories, this shows these S-dualities as exact equivalences of the scale invariant theories and not just as a property of their supersymmetric states.

The uniform way in which the complicated quantum coupling spaces of the $N = 2$ scale invariant theories were derived by this scaling procedure suggests that it should be effective more generally. In particular it would be interesting to extend this argument to study S-dualities in scale invariant $N = 1$ theories, some classes of which have been proposed and studied in [14, 15].

Another open question involves the physics of the "ultra-strong" coupling points in the coupling spaces of $N = 2$ theories. Unlike the $N = 4$ theories where the $SL(2, \mathbf{Z})$ S-duality identifies all the ultra-strong ($\mathrm{Im}\tau = 0$) points with the weak coupling limit of the theory, we have seen above that the S-dualities of the $N = 2$ scale invariant theories are generically smaller, and leave ultra-strong points (or manifolds) in their coupling spaces. One counterexample is the scale invariant $SU(2)$ theory with four fundamental quark hypermultiplets, studied in [6]. There it was found that the theory in fact has the whole $SL(2, \mathbf{Z})$ duality and no ultra-strong points. This could also be derived through scaling arguments [7] by comparing the embeddings of $SU(2)$ into $SU(3)$ and $Sp(4)$ asymptotically free theories. One should note that the scaling arguments presented in this paper do not claim to capture all possible S-dualities—there may be further identifications of the coupling space which are missed since our arguments only show that $\mathcal{M}^{(n)}$ is some multiple cover of the true coupling space.

This leaves open the possibility that there are further identifications of the coupling spaces of the $N = 2$ theories, perhaps relating the ultra-strong coupling points to some other weakly coupled physics. One place where we know such further identifications must exist are in theories with $SU(2)$ gauge group factors: for in the limit that the other factors decouple, the $SU(2)$ factors must have the full $SL(2, \mathbf{Z})$ duality of [6]. The new S-dualities in these theories will be explored in [16].

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References

[1] E. Witten, *Solutions of four-dimensional field theories via M theory*, hep-th/9703166, Nucl. Phys. **B500**(1997)3.

[2] K. Landsteiner, E. Lopez, and D.A. Lowe, *N=2 supersymmetric gauge theories, branes and orientifolds*, hep-th/9705199, Nucl. Phys. **B507**(1997)197.

[3] A. Brandhuber, J. Sonnenschein, S. Theisen, and S. Yankielowicz, *M theory and Seiberg-Witten curves: orthogonal and symplectic groups*, hep-th/9705232, Nucl. Phys. **B504**(1997)175.

[4] K. Landsteiner and E. Lopez, *New curves from branes*, hep-th/9708118.

[5] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. **B72**(1977)117; E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **B78**(1978)97; H. Osborn, *Topological charges for N=4 supersymmetric gauge theories and monopoles of spin 1*, Phys. Lett. **B83**(1979)321; A. Sen, *Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and SL(2,Z) invariance in string theory*, hep-th/9402032, Phys. Lett. **B329**(1994)217.

[6] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, hep-th/9408099, Nucl. Phys. **B431**(1994)484.

[7] P.C. Argyres, *S-duality and global symmetries in N=2 supersymmetric field theory*, hep-th/9706095, to appear in Adv. Theor. Math. Phys.

[8] P.C. Argyres, M.R. Plesser, and A.D. Shapere, *Coulomb phase of N=2 supersymmetric QCD*, hep-th/9505100, Phys. Rev. Lett. **75**(1995)1699.

[9] P.C. Argyres and A.D. Shapere, *The vacuum structure of N=2 super-QCD with classical gauge groups*, hep-th/9509175, Nucl. Phys. **B461**(1996)437.

[10] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, hep-th/9711200.

[11] J. Polchinski, *Scale and conformal invariance in quantum field theory*, Nucl. Phys. **B303**(1988)226.

[12] P.C. Argyres, *Duality in supersymmetric field theories*, hep-th/9705076, Nucl. Phys. B (Proc. Suppl.) **61A**(1998)149.

[13] J. Erlich, A. Naqvi, and L. Randall, *The coulomb branch of N=2 supersymmetric product group theories from branes*, hep-th/9801108.

[14] R. Leigh and M. Strassler, *Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory*, hep-th/9503121, Nucl. Phys. **B447**(1995)95.

[15] S. Kachru and E. Silverstein, *4d conformal field theories and strings on orbifolds*, hep-th/9802183; A. Lawrence, N. Nekrasov, and C. Vafa, *On conformal field theories in four dimensions*, hep-th/9803015; M. Bershadsky, Z. Kakushadze, and C. Vafa, *String expansion as large N expansion of gauge theories*, hep-th/9803076; Z. Kakushadze, *Gauge theories from orientifolds and large N limit*, hep-th/9803214.

[16] P.C. Argyres and A. Buchel, *New S-dualities in N=2 theories with SU(2)xSU(2) gauge groups*, to appear.