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OPTICAL STOCHASTIC COOLING AND REQUIREMENTS FOR A LASER AMPLIFIER

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Abstract

Implementation the quadrupole wiggler, optical amplifier and second wiggler as an analogues of a pick-up, an amplifier and a kicker in well known stochastic cooling method, drastically increases the bandwidth of the system. Some estimations made for amplifiers with Dye, Titanium Sapphire and CO_2.

1. Introduction

Stochastic cooling method was proposed about 25 years ago [1]. Basically the method deals with the fast feedback loop arranged in a storage ring and includes a pickup, an amplifier and a kicker. The cooling rates are defined by a bandwidth of the equipment listed above. The (anti)proton beams were the only subjects for application due to low rate of damping[1].

In [2] a drastic increase in the bandwidth was proposed by implementation the optical analogues of the pickup, the amplifier and the kicker into the feedback loop. A typical decrease in the damping times was estimated to be down to 10^4 of previous, coming to a millisecond level. Under this rates of cooling, the method, called Optical Stochastic Cooling (OSC), could be interesting not for the protons only, but for electrons also, as well as for m-mesons and multi charge ions. For electron and positron beams the method allows drastic reduction of final temperature with an optical amplifier of intermediate complexity.

The necessity to keep the bunch length unperturbed from pickup to the kicker within the level of the wavelength of the optical system is obvious. With the same accuracy the congruence of the light amplified and particle's beam must be arranged in the kicker as well. However, the transverse kick to the particles moved with relativistic speed co-directionally with the light, could be arranged through the energy change in the place of the particle's orbit with nonzero dispersion function only. This requires some functional relations between parameters of beam optics on the way between pickup and kicker. So, simultaneous cooling of transverse and longitudinal emittances yield some threshold value in the invariant emittance level as $g e_x^{th} \cong l g \cdot (D E / E)$, where $g e_x$--is the invariant radial emittance, l -- is a central wavelength of the optical amplifier (and all system), $g = E / mc^2$, $D E / E$-- is the initial relative energy spread in the beam. The above value is of the order $g e_x^{th} \cong 3 \div 5 \cdot 10^{-4}\, cm \cdot rad$ for $l \cong 1 mm$, what slightly exceeds the typical value for the damping ring for Next Generation of Linear Collider, $g e_x \cong 3 \cdot 10^{-4}\, cm \cdot rad$ [3]. In [4] some developments in OSC method made in attempt to eliminate dependence of initial emittance.

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[1] For electron-positron beams typical rate of damping due to synchrotron radiation is about 1-10 *msec*.

From the other hand, the investigations of all possible types of beam coolers as a damping ring, reside so called Kayak-paddle type racetrack[5] as a promising candidate for a machine with emittance of the order even $g\,e_x \cong 3\cdot 10^{-8} cm\cdot rad$. The reason for this kind of investigations was encouraged by desire to come to degeneration of the Fermi gas, what are the particles of electron or positron beam [5]. The emittance like just mentioned above, simply eliminates the troubles about threshold emittance at all.

Quadrupole wiggler as a kicker also rejects this problem.

Implementation of OSC method in Kayak-paddle type of cooler, allows further lowering the beam emittance down to 10^2 times. For Laser Acceleration concept this allows utilization of a $1m\,m$ laser for acceleration [6].

In this paper we reviewed the OSC method and made some practical estimations for different types of optical amplifiers.

2. Stochastic cooling

Cooling and lowering the beam emittance are synonymies for the beam technics. Normally the beam emittance e could be represented by a quadratic form

$$e = \Gamma\, x^2 + 2\cdot \mathrm{A}\, x\cdot x' + \mathrm{B}\, x'^2 = (x; x')\cdot \begin{pmatrix} \Gamma & \mathrm{A} \\ \mathrm{A} & \mathrm{B} \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad (1)$$

where $\mathrm{A}, \mathrm{B}, \Gamma$ are the parameters of this form. Basically (1) represents the ellipse. Maximal deflection of the beam envelope and its derivative could be represented as

$$x' \cong \sqrt{e\,\Gamma}\ \ ,\ x \cong \sqrt{e\,\mathrm{B}}\ ,\quad A = -\frac{1}{2}\frac{dB(s)}{ds},\quad \Gamma\,\mathrm{B} - \mathrm{A}^2 = 1, \qquad (2)$$

where s is a longitudinal coordinate.

For electrons, a synchrotron radiation treated as radiation of individual quants, carries out the particle's momenta in direction of instant motion[2]. This includes both longitudinal and transverse components. As usual, longitudinal component is restored by RF cavity. So after one turn the slope of trajectory is decreased. Characteristic frequency of radiation is $w_C \cong (3c/2r)g^3$ and corresponding wavelength $\lambda\!\!\!^{-} \cong c/w_C = (2/3)r/g^3$. For typical parameters, the number of the particles distributed longitudinally within the wavelength is much more than unity. Hence, the radiation within the bandwidth around w_C occurs only due to fluctuations of the number of the particles within this bandwidth selected[3]. So only $\sqrt{N}$ particles, where N is the total number of the particles in the beam, involved in radiation process[4], but all these particles radiate coherently. So, formally, intensity of radiation is proportional to $\left(\sqrt{N}\right)^2 = N$. The length of radiation formation is $l_F \cong r/g$, where r --is the bending radius. As at the distance l_F about $a = e^2/\hbar c \cong 1/137$ photons are radiated, the total number of photons radiated by each particle per turn will be $N_g \cong (a\cdot 2p\ r)/(r/g) \cong a\,g \cong g/137$.

For the protons under cooling , the synchrotron radiation is not significant. This is because the protons under cooling have energy below relativistic, while the bending radius R, which defined

[2] Within a small angle $\sim 1/g$.

[3] Or with other words, due to fluctuations in the longitudinal positions of these particles.

[4] As the bunch length is bigger, than $\sim c/w_C$

by a magnetic field, remains about the same as for electron machines.
So stochastic cooling (SC) method deals with finite number of particles in the bandwidth directly [1]. Transverse fluctuation of the center of gravity position of N particles each having transverse coordinate x_j is

$$< x > \cong \frac{\sum_{j=1}^{N_S} x_j}{N_S} \approx \frac{A}{\sqrt{N_S}} \ , \tag{3}$$

where A is an effective amplitude of transverse oscillation[5], N_S --is the number of the particles in the bandwidth. One can see that the lower number of the particles in the bandwidth is better. This value of average displacement generates the electrical signal in differential pick up[6], proportional to $< x >$. Further, after amplification, this signal applied to the dipole kicker at the place with maximal slope of particle's trajectory, so the betatron amplitude of oscillation changes. Mathematically this could be described by transformation

$$x_i^{new} \cong x_i^{old} - G\frac{\sum_{j=1}^{N_S} x_j^{old}}{N_s} \rightarrow x_i^{old} - G\frac{x_i^{old}}{N_S} - G\frac{\sum_{j\neq i}^{N_S} x_j^{old}}{N_S}, \tag{4}$$

where i marks selected particle, and where the part, associated with this selected particle was also detached from the sum. G is a normalized amplification in the feedback. One can see, that the damping associated with the signal from the particle itself is $\approx G\frac{x_i}{N_S} \cong G\frac{A}{N_S}$ and the rest part $\approx G\frac{\sum x_j}{N_S} \cong G\frac{A}{\sqrt{N_S}}$ represents the heating for this particle selected. Squaring one can obtain for dipole kicker [1]

$$\frac{d < x^2 >}{dn} + \frac{2G - G^2}{N_S} < x^2 >= 0 \, ,$$

where n marks the turns. Maximal cooling rate corresponds to G =1, this means, that the kicker must eliminate the amplitude, corresponding to the picked averaged displacement in the pick-up. One can see that the cooling rate is simply associated with the number of the particles in the bandwidth. This number is $N_S \sim \frac{N}{\Delta f}\frac{c}{l_B}$, where Δf -- is the bandwidth, l_B --is the bunch length[7], c-- is the speed of light. For typical proton cooling system with $\Delta f \approx 250\ MHz$, $N_S \approx 4 \cdot 10^6$.

2.1. Microwave pick up.

So as the optical amplifier becomes a natural analog of microwave amplifier, the analogues for a pick up and a kicker were found to be a quadrupole and a dipole wiggler correspondingly [2].

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[5] This corresponds to a model, where all particles have the same amplitude, but random betatron phases. An effective beam emittance is represented as ellipse. This still exact for the beams with constant density also.
[6] What includes the pick up electrodes itself plus differential amplifier or any other element, where signals from opposite electrodes are subtracted.
[7] For continues beam the orbit perimeter substitutes the bunch length.

For ordinary pick up the bandwidth can be estimated as $\sim c/D$, where D is a characteristic size of electrodes. Electromagnetic energy deposited in the electrodes could be estimated as $\sim q^2/D$, where q-- is an effective charge induced by the part of the beam inside the electrodes. Typical frequencies correspond to $\sim c/l_B \approx c/D$, so energy of the quanta at characteristic frequency is about $\hbar w \approx \hbar c/l_B \approx \hbar c/D$. For integrating pick up $q \cong eN_S$, so the number of the quants radiated is $N_g \cong (q^2/D)/(\hbar c/D) = (e^2/\hbar c)N_S^2 = a\ N_S^2$ what is proportional to the number of the particles in a power *two*. This means that all particles in the bandwidth radiate coherently. For differential pick up electrode $q \cong eN_S < x >_S /D$, where $< x >_S$ –is an averaged over bandwidth position of centroid. So in this case

$$N_g \cong (q^2/D)/(\hbar c/D) = \frac{e^2 N_S^2}{D^3} < x >_S^2 \frac{D}{\hbar c} = a\ N_S^2 \frac{A^2}{D^2 N_S} = a\ N_S \frac{A^2}{D^2}, \tag{5}$$

where we suggested, that $< x >_S \cong A/\sqrt{N_S}$. One can see, that for differential pick up electrode (see comment 6) the number of the quants is proportional to the number of the particles in the bandwidth in a power *one*.

More precisely, effective charge $q \cong eN_S < x >_S /D$ appears, when electrodes connected to so called *differential* amplifier. The same is valid if the signals from the opposite electrodes subtracted with help of passive element(s)[8]. In any case, one can say, that proportionality to the number of the particles in the bandwidth appears when the photons, radiated in each individual electrode, *destructively interfere* in the amplifier on in the passive element. One can also say that linear proportionality appears due to fluctuations of centroid, what in it's turn is proportional to $\sim A/\sqrt{N_S}$. In both cases, like in synchrotron radiation, the number of the photons, radiated by each particle in the bandwidth, defined by length of formation, is proportional to the fine structure constant, $a = e^2/\hbar c$.

To finalize the considerations of pick up, let us estimate the back reaction of the pick up to the particle. If the electrical field, generated by induced charge is $E_\perp$, then the transverse kick to the momenta of each particle will be

$$Dp \cong eE_\perp \frac{D}{c} \cong \frac{eU}{c} \cong \frac{e}{c}\frac{q}{D} = \frac{e}{c}\frac{eN_s}{D}\frac{< x >_S}{D} = \frac{e^2}{c}\frac{N_S}{D^2}\frac{A}{\sqrt{N_S}} = \frac{e^2}{c}\frac{A\sqrt{N_S}}{D^2}\ . \tag{6}$$

According to relation $\Delta p \Delta x \cong \hbar$, one can conclude, that uncertainty in location of each individual particle will be

$$\Delta x \cong \hbar/\Delta p = \frac{\hbar c}{e^2}\frac{D^2}{A\sqrt{N_S}} \cong 137\frac{D^2}{A\sqrt{N_S}}. \tag{7}$$

For the beam centroid uncertainty one need to average (7), so the beam uncertainty becomes

$$< \Delta x > \cong \frac{\hbar c}{e^2}\frac{D^2}{AN_S} \cong 137\frac{D^2}{AN_S} \tag{8}$$

For typical case with $D \cong 1cm$, $A \cong 0.1cm$, $N_S \cong 10^6$, $< \Delta x > \cong 1.37 \cdot 10^{-3} cm$. Formally for one electron, $N_S \cong 1$, resolution becomes bigger, than dimensions of the pick up.

[8] As a transformer with two oppositely connected primary coils, for example.

3. Optical Stochastic Cooling

So one can conclude, that the bandwidth is a crucial parameter for stochastic cooling. In [2] there was proposed to move the feedback loop bandwidth to the optical range. Typical optical amplifier on Dye or Ti Sapphire has a bandwidth on the level $6\cdot 10^{13} \div 10^{14}\,Hz$, what means the bandwidth increase up to $10^4 \div 10^5$ times. The rates of cooling, corresponding to this bandwidth could be interesting for electrons and positrons. The CO_2 amplifiers also demonstrate the ability to amplify a picosecond pulses [10]. Powerful CW CO_2 laser amplifier might be interesting for the proton cooling as well as for m - meson and multi-charged ions cooling.

3.1. Radiation from a quadrupole wiggler

Quadrupole wiggler is a series of quadrupoles with alternating sign of gradient installed along straight line with a period *2L*. Such arrangement increases the density of radiation in the bandwidth. Magnetic field in the aperture of the lens could be represented as $H_y(x) = g\cdot x$, $H_x(y) = g\cdot y$, where g – is a gradient. For a particle, which is going off the central trajectory, the field picture looks like a field in ordinary dipole wiggler with magnetic field proportional to the displacement. Betatron motion modulates the field with the frequency of transverse oscillations [7]. Electrical field in far zone can be represented as

$$\vec{E}(t) = -\frac{e}{cR}\frac{\vec{n}\,'((\vec{n}-\dot{\vec{b}})'\,\vec{n})}{(1-\vec{n}\vec{b})^3}\Bigg|_{t^{\ell}=t-R(t^{\ell})/c},$$

where $\vec{n}$ --is an unit vector in direction of observation, R—is the distance to the observer at the moment of radiation. In dipole approximation, $\vec{b}_{\perp} \le \frac{K}{g}$, where $K = \frac{egx2L}{2p\,mc^2} \le 0.7$, $\vec{b} = \vec{b}_{\parallel} + \vec{b}_{\perp} \cong \vec{k}b$, where $\vec{k}$ -- is an unit vector in longitudinal direction. The field becomes

$$\vec{E}(t) = -\frac{e}{cR}\cdot\frac{\dot{\vec{b}}_{\perp}}{1-b(\vec{n}\vec{k})}\Bigg|_{t'=t-R(t')/c} \cong -\frac{e}{cR}\cdot\frac{4g^2\,\dot{\vec{b}}_{\perp}}{(1+K^2+g^2J^2)^2}\Bigg|_{t'=t-R(t')/c}, \qquad (9)$$

where J -- is an angle between $\vec{k}$ and $\vec{n}$.

Transverse acceleration can be represented as

$$\left|c\dot{b}_{\perp i}\right| \cong \left|\frac{e\cdot\vec{k}\times\vec{H}_i}{mcg}\right| = \frac{e\cdot gx_i}{mcg}, \qquad (10)$$

so (9) can be rewritten as

$$\left|\vec{E}_i(t)\right| \cong -\frac{e}{mc^2g\,R}\cdot\frac{4g^2g}{(1+K^2+g^2J^2)^2}\cdot x_i\Bigg|_{t'=t-R(t')/c} = E_o.x_i(t'). \qquad (11)$$

In expression (11) we ignore the dependence $R(t')$ in denominator. Mostly important influence arises from $R(t')$ containing in x. So what is important, is that the electrical field is proportional to the particle's displacement in the lens. In the same manner one can consider the radiation in *sextupole and so on wiggler*. This may give an important information about *higher* moments in transverse beam distribution. To obtain the full radiation from the beam with finite transverse dimensions one needs to sum the fields (11) arisen from all particles. So the expression for the field becomes

$$\vec{E}_x(t) \cong -E_0 \cdot \sum_{i=1}^{N} x_i(t')\Big|_{t'=t-R(t')/c}, \quad \vec{E}_y(t) \cong -E_0 \cdot \sum_{i=1}^{N} y_i(t')\Big|_{t'=t-R(t')/c}.$$

The sum could be transformed into convolution (11) with normalized transverse distribution, what basically gives a *dipole* component of transverse distribution. Other types of wigglers (sextupole and so on) could be used for obtaining higher moments. Spectral angular density can be calculated in a usual way $\P^2 E / \P S \P w = c\left|\vec{E}(w)\right|^2$.

One can see from (9), that the time dependence of electric field strength $\vec{E}(t)$ copy the time dependence of the particle's acceleration in a time scale $1/(1-b(\vec{k}\vec{n}))$, what is basically a Doppler's factor, Fig.1.

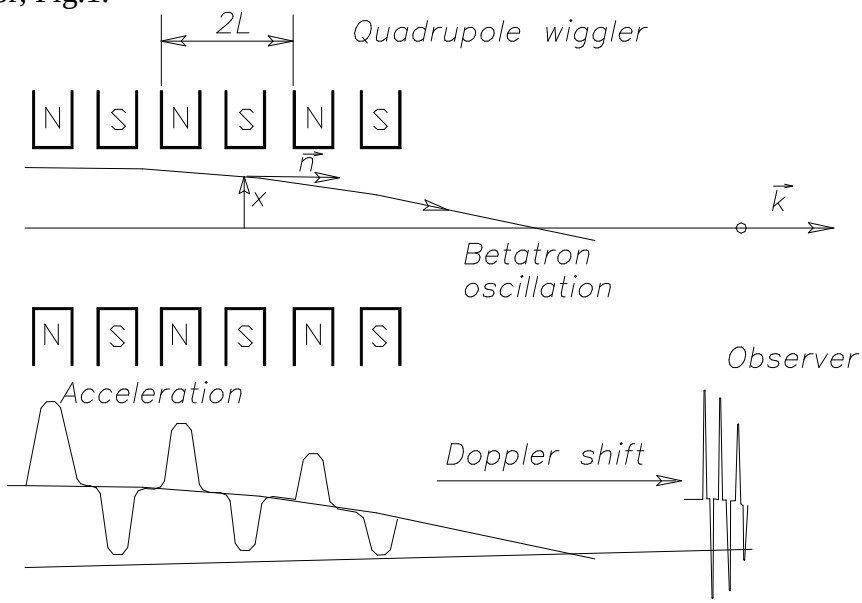

Fig. 1. Radiation from a quadrupole wiggler [7].

For transverse kick in a quadrupole wiggler, the time sequence in the picture must be reversed[9]. For the beam passing the quadrupole wiggler, for each particle with positive off axis position $+x$, there exists a particle with negative off axis position $-x$, so in a quadrupole field they have opposite acceleration, Fig.2, so the radiation in the forward direction interferes *destructively*.

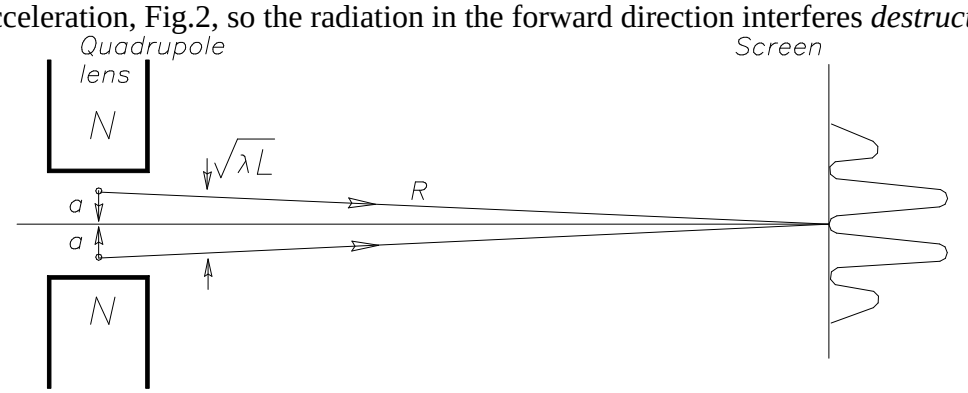

Fig. 2. Destructive interference of radiation in a quadrupole lens from two particles.

[9] So the radiation and the particle are moving to the left side in Fig.1. At the wiggler position the dispersion must be a nonzero one.

The size of coherence $\sqrt{l\ 2ML}$, where M – is a number of periods, is bigger than the transverse dimension of the beam

$$\sqrt{\lambda\ 2ML} \geq \sqrt{\varepsilon\ \mathrm{B}}\,, \tag{12}$$

where $l \cong L(1+K^2+g^2J^2)/g^2$ is the wavelength of the wiggler radiation, e, B -- emittance and envelope function - parameters of ellipse from (1), (2). So, radiation in *forward* direction occurs only due to fluctuation of the gravity center in a wiggler, similarly to the microwave pick up, described above. Using the electrical field representation (9), one can obtain the spectral density of photons as

$$\frac{dN_g}{d(w/w_{\max})} \cong 4p\,a\,\frac{K^2}{1+K^2}\cdot\frac{1}{2}\cdot\left[1-2\frac{w}{w_{\max}}+2\left(\frac{w}{w_{\max}}\right)^2\right], \tag{13}$$

where $w_{\max} = \dfrac{2p\,g^2c}{L\cdot(1+K^2)}$ is the maximal frequency in the spectrum corresponding to the first harmonic. In the spectrum region $w_{\max} \div w_{\max}\cdot(1-1/M)$, the number of photons becomes $N_g \cong 4a\,K^2/(1+K^2)$, with $K^2 \sim x^2$. So the number of photons radiated by the particle with amplitude A in the relative bandwidth $D\,f/f \cong 1/M$ becomes

$$N_g \cong a\left(\frac{A}{A_0}\right)^2 = a\,\frac{e}{e_0}\,, \tag{14}$$

where A_0 is a normalization amplitude, corresponding the beginning of the cooling, when K~0.7 and A – is an actual amplitude, e_0, e -- are the corresponding emittances.

If N_S is the number of the particles in the bandwidth, then total number of photons, radiated by these particles due to fluctuation of center of gravity position becomes $D\,N_g \cong a\,N_S e/e_0$. Total number of the photons, radiated by the beam with N particles, becomes

$$D\,N_g \cong a\,Ne/e_0\,. \tag{15}$$

Each of these photons has an energy

$$\hbar w \cong \frac{2p\,\hbar c g^2}{L\cdot(1+K^2+g^2J^2)}\,. \tag{16}$$

So the total energy carried by these photons becomes $W_{pick} \cong \hbar w\,D\,N_g$

3.2. Transverse kick over energy change

According to the idea of stochastic cooling the radiated electromagnetic wave must be applied to the particle in congruence with initial longitudinal distribution. This could be done if the photons and the beam are moving in the same direction. As the direct transverse kick is not possible under this condition, the transverse kick arranged through the *energy change* in the place, where dispersion of trajectory has some value $h(s)$. That means the amplitude of transverse betatron oscillations around new closed[10] orbit will change on $D\,x \cong h(s)\dfrac{D\,E}{E}$, where $\dfrac{D\,E}{E}$ -- is a relative energy change, see Fig.3. According to previous considerations,

[10] Corresponding to new energy.

$$D\,x \cong A/\sqrt{N_S} \cong h(s) D\,E/E\,. \tag{17}$$

If the dipole wiggler has the same period and undulatority factor[11], then relative energy change will be

$$\frac{D\,E}{E} \cong \frac{eE_\perp KLM}{Eg} = \frac{A}{h\sqrt{N_S}}\ , \tag{18}$$

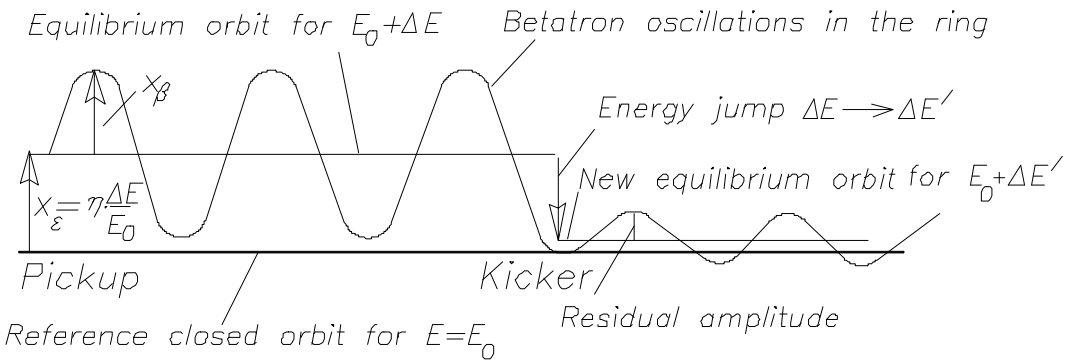

Fig.3. The transverse kick through energy change.

where $E_\perp$ -- is the electrical field strength in the electromagnetic wave after amplification. The last expression allows to calculate

$$E_\perp \cong \frac{AEg}{eKLM\,h\sqrt{N_S}} \tag{19}$$

3.3. Amplification.

So the total energy delivered by optical amplifier must be

$$W_{kick} \cong \frac{1}{4p} E_\perp^2 V\ , \tag{20}$$

where V – is an effective volume. One can estimate

$$V \cong p\,A_0^2 l_B \cong p\,2LMl\ \ l_B\,, \tag{21}$$

where transverse cross section chosen corresponding to the size of coherence (12). Combining (15), (16), (19), (20), (21) and estimating $h \cong A_0 \cdot (D\,E/E)^{-1}$ and $D\,f/f \cong 1/M$, one can obtain the amplification required from the optical amplifier as [2]

$$k = \sqrt{\frac{W_{kick}}{W_{pick}}} \cong \frac{g\ l_B}{r_0}\frac{D\,E}{E}\frac{1}{N}\frac{D\,f}{f}\ , \tag{22}$$

where $r_0 = e^2/mc^2$.

We would like to mention, that amplification, like described in (22) corresponds to G = 1 in (4). So the equation for a change of emittance will be

$$\frac{de}{d\,n} = -\frac{e}{N_S}\,, \tag{23}$$

[11] Typical magnetic field value in the wiggler is about 5-10G for the parameters under discussion. For quadrupole wiggler as a kicker, period must be the same. The fields in the quadrupole wiggler will be the same as in pick up wiggler for transformation matrix between pick up and kicker equals to -***I***.

where $N_S \sim \frac{N}{D\,f}\frac{c}{l_B} \cong MN\frac{l}{l_B} \cong MNL\frac{l_B}{g^2}$. This gives the cooling times $t \cong N_S T$ for a single cooling system. Cooling finishes when emittance reaches its final value like

$$e_{fin} \cong e_0 \frac{1}{a\,N_S}, \tag{24}$$

what corresponds to one photon in the coherence volume.

3.4. Quadrupole wiggler as a kicker.

In our considerations we used the dipole wiggler for energy change. For the same reason the second *quadrupole* wiggler could be used as well. In this case one needs to have the beam optical transformation from pickup to kicker with magnification module equal to one. In this case one can avoid the necessity to control the frequency shift arising from changing the amplitude, and, hence, K parameter in (16). All formulas for optical amplification still valid, because the principle of operating through energy change in a wiggling trajectory accompanied by the amplified radiation remains the same. Philosophically speaking, this is a fully reversible picture if the particle in the kicker is following a reversed trajectory. This could be arranged by transforming matrix equal to minus unity. One can see, that in a quadrupole kicker particle will receive a kick, proportional it's current transverse position. So the particles in the bandwidth having opposite amplitudes in a pickup (and in kicker) will obtain the kicks of opposite polarities, what is necessary. Indeed, in a dipole wiggler all particles in a sample will obtain the same kick. So the particles, for example, with opposite displacement will be overheated. As one can see from (4), that particle's decrement arising only from interaction with self field, but amplified only to the level, required to eliminate *average*[12] displacement in the pickup. Mathematically the action of quadrupole wiggler can be expressed as

$$x_i^{new} \cong x_i^{old} - G_Q \cdot x_i^{old} \cdot \frac{\sum_{j=1}^{N_S} x_j^{old}}{N_s} \rightarrow x_i^{old} - G_Q \cdot \frac{(x_i^{old})^2}{N_S} - G_Q \cdot x_i^{old} \cdot \frac{\sum_{j\neq i}^{N_S} x_j^{old}}{N_S},$$

Where G_Q is a normalized amplification. One can see from here, that the heating arising from the second term could be decreased significantly, as $G_Q \cdot x_i^{old} \cdot \frac{\sum_{j\neq i}^{N_S} x_j^{old}}{N_S} \rightarrow 0$.

3.5. The beam dynamics from the pickup to the kicker.

As one needs to cool both longitudinal and transverse emittance simultaneously, the system with dipole wiggler as a second kicker must satisfy some certain conditions.

Let us represent the transverse position of the particle as a sum of two eigenvectors called sine-like *S(s)* and cosine-like *C(s)* trajectories. Basically these vectors describe the trajectory with initial conditions like $x_0' = 0;\ x(s) = x_0 \cdot C(s)$ and $x_0 = 0;\ x(s) = x_0' \cdot S(s)$, where index 0 marks the longitudinal position of pick up. So transverse position for the particle becomes [11]

$$x(s) = x_0 \cdot C(s) + x_0' \cdot S(s) + D(s) \cdot (D\,E / E), \tag{25}$$

where dispersion *D(s)*, as one can see from (25), describes the trajectory for the particle with different value of energy and zero initial conditions for coordinate and it's derivative. We also omitted index i, numbering each individual particle. Basically

[12] For all sample, or for all particles within the bandwidth.

$$D(s) = -S(s)\cdot\int_{S0}^{S}\frac{C(t)}{r(t)}dt - C(s)\cdot\int_{S0}^{S}\frac{S(t)}{r(t)}dt \quad , \tag{26}$$

where $r(s)$ -- is a current bending radius. In accelerator's physics accepted to separate the coordinate arisen from betatron motion itself and the energy offset like $x = x_b + h\cdot(DE/E)$, where function $h(s)$ was used earlier, eq. (17). So (25) can be expanded

$$x(s) = x_{0b}\cdot C(s) + x'_{0b}\cdot S(s) + [h_0\cdot C(s) + h'_0\cdot S(s) + D(s)]\frac{DE}{E} \quad . \tag{27}$$

One can see from (26), (27), that in absence of magnets, $r\to\infty$, dispersion is absent also. Expression (27) can be also rewritten as

$$x^{kicker} = x_b^{kick} + [h^{kick} + D^{kick}]\frac{DE}{E}, \tag{28}$$

while the same for the pickup $$x^{pick} = x_b^{pick} + h^{pick}\frac{DE}{E} \tag{29}$$

Simultaneous cooling requires, that $h^{kick} + D^{kick} = -h^{pick}$, so $D^{kick} \cong -2\cdot h^{pick}$ and at least one of the integrals in (26) must be not equal to zero. Meanwhile the longitudinal motion of the particle with nonzero initial conditions and the energy offset could be described by

$$D\,l = -\int_{S0}^{S}\frac{x}{r}dt = -x_0\int_{S0}^{S}\frac{C}{r}dt - x'_0\int_{S0}^{S}\frac{S}{r}dt - \frac{D\,E}{E}\int_{S0}^{S}\frac{D}{r}dt \quad . \tag{30}$$

So as for example $\int\frac{S}{r}dt \cong 2h_0$, then the lengthening becomes

$$D\,l \cong -x'_0 2h_0 = -[x'_{0b} + h'_0\cdot(DE/E)]\cdot 2h_0 . \tag{31}$$

For a practical reason it is easier to have the dispersion function at the kicker positive and the same as in the pick up. In this case one needs to have $x^{pick} = -x^{kick}$ [13]. One can see from (27), that in this case $h^{kick} = -h^{pick}$, because of same C and S --functions involved in transformation for x and h. In this case one need to keep $D^{kick} \cong 2\cdot h^{pick}$. The absolute value for the lengthening remains the same. As we suggested, that $\sqrt{eB} \approx h\frac{DE}{E}$, so the lengthening can be expressed as

$$D\,l \cong -x'_0 2h_0 \cong -\left[\sqrt{\frac{e}{B}} + h'_0\cdot\frac{DE}{E}\right]\cdot 2\cdot\frac{\sqrt{eB}}{(DE/E)} = -2\left[e\cdot(DE/E)^{-1} + h'_0\cdot\sqrt{eB}\right]. \tag{32}$$

As the lengthening must remain below the wavelength of radiation, $|D\,l| \le l$, (32) yields the threshold emittance value $e_{th} \cong l\cdot\frac{DE}{E}$ if the derivative of h-function in the pick up wiggler is chosen close to zero $h'_0 \cong 0$. The last expression rewritten for *invariant* emittance becomes

$$g\,e_{th} \cong l\cdot g\,\frac{DE}{E} \cong \frac{l}{l_B}\cdot g\,l_B\frac{DE}{E}, \tag{33}$$

where $g\,l_B\cdot(DE/E)$ -- is so called *invariant longitudinal emittance*. For $l \approx 1mm = 10^{-4}cm$, $DE/E \cong 10^{-3}$, $g \cong 10^3$ (500 *MeV*), numerical value for the threshold emittance becomes

[13] This corresponds to the betatron transformation matrix equal to -***I***, or *C(at kicker)*= -1.

$g\,e_{th} \cong 10^{-4} cm \cdot rad$. The damping rings for linear colliders have typical values of emittances $g\,e_x \cong 3 \cdot 10^{-4} cm \cdot rad$, $g\,e_{yx} \cong 3 \cdot 10^{-6} cm \cdot rad$ [3]. For Kayak-Paddle type ring [5] equilibrium emittances can reach $g\,e_x \cong 1 \cdot 10^{-7} cm \cdot rad$, $g\,e_{yx} \cong 1 \cdot 10^{-9} cm \cdot rad$, so there is no apparent limitations for this type of ring at all.

One can also arrange the cooling system so that it is able to cool only the particles with lower derivatives of trajectory, that satisfy the condition (31). For example one can screen the radiation from some parts of the beam to avoid heating the core by peripheral parts of emittance. One positive property of quadrupole wiggler as a pick up, is that the central parts of the beam, what have higher values of slope x_0' are moving *closer* to the center of the lens, see (1), and, hence, radiate less. One can see also, that the particles with higher slope x_0' in the pick up (and having maximal lengthening), will be at the center in the quadrupole wiggler used as a kicker if transformation matrix $\boldsymbol{-I}$ (or for cosine trajectory $C(s \to kick) = -1$). That will reduce the heating from interaction with amplified radiation.

So general conclusion is that the threshold emittance is not a problem for OSC in many interesting cases.

The method described in [4] uses two dipole wigglers for manipulation with the beam emittance. As the dipole wiggler *does not* give any information about beam emittance, this information acquired from (30). So the lengthening of the trajectory, proportional to the beam emittance (coordinate or derivative) is used to put the selected particle onto proper phase of amplified radiation. This changes the energy of the particle in accordance with its initial emittance. The difficulty of this method is obvious: cooled beam radiates the same amount of radiation as initial one. So one needs to manipulate with amplification to keep cooling process steady. Indeed, in quadrupole wiggler cooled particles have lower amplitudes and, hence, move in lower field (zero at the axis) and radiate less.

4. Optical amplifier

Formulas (19), (20), (22) give an idea about amplification required and the power contained in the laser flash. Two examples considered below use these formulas.

Example 1.

For $N \cong 10^{10}$, $l_B \cong 15cm$, M = 5, $(D\,E/E) \cong 10^{-3}$, $g \cong 10^3$ (500 *MeV*), $l \cong 1m\ m$, optical amplifier must be able to have amplification about 300, peak power about 5 *kW*, average power about 25 *W* with repetition rate *f* of the order of 10 *MHz*. Number of the particles in the bandwidth $N_S \cong 3 \cdot 10^5$ defines the number of the turns and the damping time $t_C \cong N_S / f \cong 30ms$. Emittance reduction $e_f / e_0 \cong 1/a\ N_S \cong 10^{-3}$.

Example 2.

For $N \cong 10^8$, $l_B \cong 5cm$, M = 5, $(D\,E/E) \cong 10^{-4}$, $g \cong 10^3$ (500 *MeV*), $l \cong 1m\ m$, optical amplifier must be able to have amplification about 100, peak power about 225 *W*, average power about 0.075 *W* with repetition rate of 2 *MHz*[14]. Number of the particles in the bandwidth $N_S \cong 2 \cdot 10^3$ defines the number of the turns and the damping time $t_C \cong N_S / f \cong 1ms$. Emittance reduction $e_f / e_0 \cong 1/a\ N_S \cong 7 \cdot 10^{-2}$.

[14] Low frequency of revolution is result of the presence of long straight sections in a Kayak-Paddle type cooler.

The parameters described above look far or less realistic from the energetics. Optical amplifier needs to be done with lowest phase distortion and have minimal time delay.

4.1. Dye amplifier

For Dye amplifier with Rh6J, the operating wavelength remains within $l \approx 340 \div 540\,nm$. Life time of the states excited is $t_L \cong 5ns$, absorption cross section $s_{01} \cong 2 \cdot 10^{-16} \div 4 \cdot 10^{-16}\,cm^2$, density of Dye molecules $n_0 \approx 10^{17}\,cm^{-3}$. The last numbers give for absorption length value $l_{ab} \cong \frac{1}{n_0 s_{01}} \cong 0.05 \div 0.1\,cm$. So the pumping area could be estimated as $S_{pump} \cong l \cdot l_{ab} \cong 5 \cdot 10^{-6}\,cm^2$.

Saturation power density could be calculated as $P_{sat} \cong \frac{\hbar w\, n_0 l_{ab}}{t_L} \le 100\,kW/cm^2$ and the power of radiation required to pump the dye becomes $I = P_{sat} \cdot S_{pump} \cong 0.5\,W$/stage only. The pumping time required $t_{pump} \cong \frac{\hbar w\, S_{pump} n_0 l_{ab}}{I} \cong 0.15\,ns$. For pumping the Nitrogen laser with $l = 337nm$ could be used here as well as Xe laser with $l = 308nm$ or $l = 248nm$. Amplification can be calculated as $k \cong \exp[s_{10}(l) \cdot n_1 \cdot l] \approx 7$ for the length $l \cong l_{ab}$ and $n_1 \approx n_0$. So *three* stages can give $7^3 = 343$ in amplification. One can increase the length of amplification with transverse pumping, instead of longitudinal, supposed above. For this purpose a cylindrical lens can be used. This arrangement can reduce power density to $10\,kW/cm^2$, and come to amplification per stage obtained about $k \cong 2 \cdot 10^4$ [9].

4.2. Ti:Al$_2$O$_3$ amplifier

For Titanium Sapphire $t_L \cong 3.5\,ms$, $s_{01}(490nm) \cong 10^{-19}\,cm^2$, $s_{10}(790nm) \cong 3 \cdot 10^{-19}\,cm^2$, $n_0 \approx 10^{20}\,cm^{-3}$. Absorption length value becomes $l_{ab} \cong 1/(n_0 s_{01}) \cong 0.1\,cm$, the pumping area $S_{pump} \cong l \cdot l_{ab} \cong 7 \cdot 10^{-6}\,cm^2$. Saturation power density becomes $P_{sat} \cong \frac{\hbar w\, n_0 l_{ab}}{t_L} \le 1\,MW/cm^2$ and the power of radiation required to pump the *Ti:Al$_2$O$_3$* becomes $I = P_{sat} \cdot S_{pump} \cong 7\,W$. Amplification can be calculated as $k \cong \exp[s_{10}(l) \cdot n_1 \cdot l] \approx \exp(s_{10}/s_{01}) \approx 20$ for the length $l \cong l_{ab}$. So *two* stages can give a resulting amplification up to $20^2 = 400$. For pumping the Argonne laser can be used here.

4.3. CO$_2$ amplifier

CO$_2$ amplifier becomes interesting for cooling the protons and m -mesons due to it's huge power (CW up to 200 *kW*) available. In [10] there is described the concept of (sub)picosecond amplifier with *CO*$_2$. To achieve the necessary bandwidth, there was proposed to use a high pressure (10-15 *atm*) gas mixture, containing oxygen isotopes in proper combination. A picosecond pulse has a bandwidth of the order $D f \cong 1/t \cong 10^{12}\,Hz$, what is good enough for many applications for proton, meson and heavy ions machines. For m -mesons, for example, according to (22), amplification required is r_0 / r_m times bigger, what is about 100 times. Mostly adequate regime for this amplifier is CW operation. In this case the problems with repetition rate could be diminished. One can imaging that gas volume used for few rays. Appropriate scale of proton and heavy ion machines makes installation with *CO*$_2$ amplifier adequate.

5. Installation in a damping ring

An example of installation of OSC system for electron/positron cooler is represented in the Fig. 4 below. This is so called Kayak-paddle cooler [5]. Straight section has wigglers and RF cavities installed in series. Equilibrium value of emittance in this system defined by the wiggler field and period. Energy of the cooler is around 500 *MeV* - to keep quantum excitations at the low level. Narrowing of straight section made for compacted design and in attempt to keep the cost of the ring lower. The end of the straight section arranged with quadrupole wigglers what is FODO structure. So dual bore lenses could be used here. The lenses of such type described in [12]. For the parameters we used above, the period of the lenses must be $2L \cong 2g^{2} l / (1+K^{2})$ what is about 2 *m*. More hard radiation from the dipole wigglers installed in straight section has different wavelength and does not interfere with optical amplifier operating at lower wavelength around $1\,mm$. Optical amplifier installed at stabilized platform. For the fine phase adjustments one can use the dual prism system.

Initial phase adjustments made by movement the table (trombone). Optical telescopes at both ways used for proper conjunction the radiation from the wiggler and amplifier.

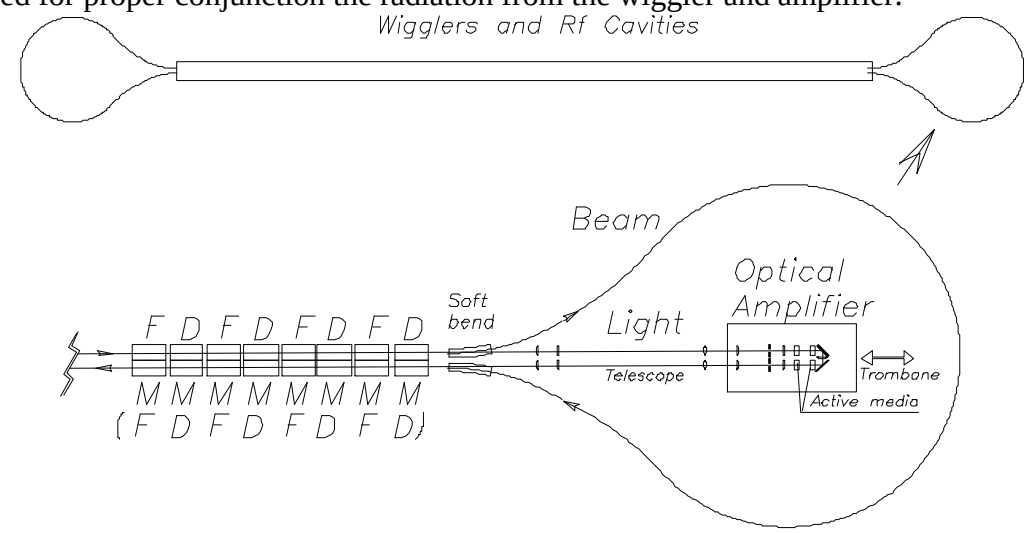

Fig. 4. Kayak-paddle cooler with OSC equipment.

The big path difference between the light and beam is possible in this cooler, what makes it possible to neutralize delay in materials of amplifier and windows easily. Active media three-four stages with Dye could be placed on the table symmetrically. The necessary optics for optical pumping also installed on the same table.

6. Conclusion

The OSC method makes it possible to cool the beams at lower energy, where quantum excitation of emittance suppressed $\propto g^{5}$ for fixed radius. Relatively low beam population required for a laser acceleration, $\propto 10^{6}$ makes problem with intra- beam scattering avoidable. The method could be interesting for proton, m - meson and heavy ion cooling as well.

In conclusion author thanks M. Zolotorev and A. Zholents for numerous discussions on the subject described.

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Summary

Optical amplifier is important component of the optical stochastic cooling method. In this method the quadrupole wiggler, optical amplifier and second wiggler used as analogues of a pick-up, an amplifier and a kicker in well-known stochastic cooling method. Optical amplifier must be able to give amplification about 1000 a peak power about 1 *kW*, average power about 10*W*. Repetition rate must be of the order of few *MHz*. Some estimations made for Dye, Titanium Sapphire and CO_2 amplifiers.