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F-theory and Linear Sigma Models

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ABSTRACT

We present an explicit method for translating between the linear sigma model and the spectral cover description of $SU(r)$ stable bundles over an elliptically fibered Calabi-Yau manifold. We use this to investigate the 4-dimensional duality between $(0, 2)$ heterotic and F-theory compactifications. We indirectly find that much interesting heterotic information must be contained in the 'spectral bundle' and in its dual description as a gauge theory on multiple F-theory 7-branes. A by-product of these efforts is a method for analyzing semistability and the splitting type of vector bundles over an elliptic curve given as the sheaf cohomology of a monad.

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1 Introduction

Heterotic string theory on Calabi-Yau 3-folds has long constituted a phenomenologically interesting, yet technically difficult, class of string compactifications. Such compactifications naturally lead to chiral matter in the low energy effective action as well as $N = 1$ supersymmetry in spacetime, which underscores their phenomenological potential. The general such compactification has $(0, 2)$ world sheet supersymmetry. The lack of left moving supersymmetry severely hampers a direct perturbative study of such models and has been the subject of much subtle and inconclusive analysis in the past.

The beautiful construction [1] of $(0, 2)$ heterotic models as infrared renormalization fixed points of certain $(0, 2)$ linear sigma models was initially proposed as a controlled method for building $(0, 2)$ nonlinear sigma models with a fair chance of yielding perturbatively well defined theories. Due to its simple and direct connection with the formalism of toric geometry, it has enabled a rather explicit study of this accessible class of heterotic $(0, 2)$ compactifications.

The recent nonperturbative string revolution offers the potential of much new insight into both the formal and phenomenological aspects of these models. In particular, F-theory gives an alternate description of (some of) these models in essentially geometrical terms, thereby circumventing many subtle features of the heterotic formulation. Specifically, the well known topological and geometrical properties of Calabi-Yau 4-folds (which are elliptically and $K3$ fibered) are thought to encompass the data of an elliptically fibered Calabi-Yau 3-fold together with a stable, holomorphic $E_8 \times E_8$ vector bundle. The heterotic — F-theory duality implies that a nonperturbative formulation of $(0, 2)$ heterotic compactifications should cure the perturbative problems which many such theories apparently exhibit. In particular, F-theory gives an alternate description of these models in essentially geometrical terms. If the duality of F-theory and heterotic compactifications is taken at face value, one reaches the conclusion that essentially any [1] stable $E_8 \times E_8$ vector bundle over an elliptically fibered Calabi-Yau manifold should lead to a well-defined heterotic model once one properly includes the full quantum effects. In this context, F-theory gives us an unexpected way of exploring the full moduli space of $(0, 2)$ heterotic compactifications, including its nonperturbative components. In fact, in the spirit of most duality conjectures, this duality is thought to extend to the full nonperturbative structures on each side, including various extended brane configurations that can arise. We see that the potential for progress in understanding chiral $N = 1$ compactifications is considerable, if the map between their F-theory and heterotic descriptions can be made precise.

A big step in this direction was taken in the beautiful paper of Friedman, Morgan and Witten [3], where an explicit conjecture for the map from a certain part of the heterotic bundle data to the dual F-theory data was proposed. At the same time, three powerful methods for constructing stable heterotic bundles were explained and developed. In this paper we will concentrate on the best understood of these – the spectral cover description.

[1] An exception to this rule is given by heterotic compactifications whose F-theory dual may generate a superpotential [2]

As explained in [3], [4], this equivalent description of the heterotic bundle proves to be extremely useful in analyzing $(0,2)$ compactifications.

From an abstract perspective, giving an $SU(r)$ heterotic bundle V of rank r in a certain component of the moduli space of stable bundles over an *elliptically fibered* threefold is equivalent to giving a pair (Σ, L) where Σ is an r-fold cover of the base of the fibration and L is a line bundle over Σ. In heterotic compactifications on a Calabi-Yau 3-fold, Σ will be a complex surface, called the spectral cover associated to V due to its origins in the theory of algebraically integrable systems. Beyond giving an equivalent description of a large class of stable bundles (thus giving an effective way to implement the stability condition), this device has proven very useful in constructions of moduli spaces of stable vector bundles over elliptic fibrations [5]. Moreover, as explained in [3, 4], this description of the heterotic bundle allows one to make rather precise duality statements between certain heterotic bundle moduli and corresponding moduli in F-theory.

The purpose of this paper is to connect the explicit description of the only well-understood perturbative heterotic $(0,2)$ compactifications — those realized via $(0,2)$ linear sigma models — to the comparatively abstract spectral cover description of [3, 4]. When combined with the work of [3], this will give us a framework for understanding certain aspects of the duality in a computationally accessible class of models. Going through this analysis we will discover some rather surprising facts, indicating that there are still some gaps in our understanding of the proposed duality.

In section 2 we briefly review the general form of F-theory—heterotic duality, as well as the spectral cover construction. In section 3 we review the linear sigma model construction of heterotic compactifications. In section 4 we present an explicit method for recovering the spectral cover of an $SU(r)$ bundle built via a $(0,2)$ linear sigma model. In order to avoid unnecessary mathematical complications, only the 'generic'[2] case is studied. A complete mathematical discussion of the problem (including the 'nongeneric' case) will be given elsewhere [6]. In section 5 we investigate a number of examples. We discover that the spectral cover of the simplest *purely perturbative* models that we are able to construct has a degenerate form. In section 6 we briefly discuss the region of the moduli space to which we think our models belong, leaving a more detailed investigation of these issues to be reported elsewhere. Finally, in section 7 we offer our conclusions and speculations.

While no detailed understanding of the more subtle aspects of the theory of spectral covers are needed for reading this paper, we do assume a general familiarity with both the spectral cover construction and with the toric description of $(0,2)$ linear sigma models. An appendix summarizing certain basic results on stable vector bundles over an elliptic curve has been included for the convenience of the non-expert reader.

[2]This is the case when the restriction of the heterotic bundle to the generic elliptic fibre of the heterotic 3-fold splits as a direct sum of line bundles. This condition *does* fail to hold in a series of examples we constructed.

2 The geometric framework of F-theory- Heterotic duality

In the original F-theory paper [7] a duality between the heterotic string on T^2 and the type IIB string on $\mathbb{P}^1$ was convincingly advocated. By giving the axion-dilaton of type IIB a geometric interpretation as a toroidal modulus (which is at least a consistent interpretation due to $SL(2,\mathbb{Z})$ invariance of type IIB string theory), this duality was rephrased in terms of F-theory on an elliptically fibered $K3$ being dual to the heterotic string on T^2.

Once one believes this duality, it is a small step to generalize it to theories with an ever smaller number of noncompact dimensions. One does this by making use of the adiabatic construction of [8]. Namely, if B_H is some large volume manifold then we can use the eight-dimensional F-theory/heterotic duality fiberwise over B_H. The idea here is that, as long as B_H is large, the compactification over $T^2 \rightarrow B_H$ can be approximated with a family of T^2 compacifications, parametrized by B_H. Note, however, that continuing from this rather trivial situation to the case of a base of finite volume is not straightforward and necessitates a proper reformulation of the conjectures. Thus, the heterotic string compactified on T^2 fibered over B_H should be isomorphic to F-theory compactified a $K3$ fibration over B_H, each of whose $K3$ fibers are themselves elliptically fibered over a $\mathbb{P}^1$ base. The total space of this $K3$-fibration is therefore also elliptically fibered over a base B_F, where B_F is itself a $\mathbb{P}^1$ bundle over B_H, i.e. a ruled variety. The duality between F-theory and heterotic models is most easily expressed when each side is presented in Weirstrass form, as we shall discuss below.

Since the heterotic construction also involves specifying a gauge bundle, the data on the heterotic side generally takes the following form. We consider an eliptically fibered n-fold (where $n = 2, 3$) Z over an $n-1$ -dimensional base B_H, whose generic elliptic fiber is denoted by E_H together with the natural projection map $\pi_H : Z \rightarrow B_H$. We then specify 2 (semistable) bundles V and V' that control the representations of the left-moving fermions, thereby specifying the spacetime gauge structure.

As shown in [9] , the correct dual of this is an $(n+1)$-fold X, elliptically fibered over a base B_F by a map $\pi_F : X \rightarrow B_F$, where B_F is a ruled n fold over B_H (simply speaking, a $\mathbb{P}^1$-bundle over B_H). The generic elliptic fibre of X will be denoted by E_F. B_F can be obtained as the projectivisation [3] $B_F := \mathbb{P}(\mathcal{M} \oplus \mathcal{O}_B)$, where the line bundle $\mathcal{M} \rightarrow B_H$ is (partially) determined by the characteristic classes of V, V' as explained in [3]. This specifies

[3]If W is a vector bundle over a base B, then the projectivisation of W is most easily defined in terms of complex geometry, as the variety $\mathbb{P}(W) = W/\mathbb{C}^*$ where $\mathbb{C}^*$ is the natural fiberwise action $\rho : \mathbb{C}^* \rightarrow End(W)$ given by: $\rho(\lambda)(u) := \lambda u, \forall u \in E_b, \forall b \in B$. Intutively, we associate to each fiber W_b of W the projective space $\mathbb{P}(W_b) = W_b/\mathbb{C}^*$ and we glue these together in a $\mathbb{P}^{\mathrm{rank}\, W-1}$-fibration over B in a way specified by the fiberwise character of the action ρ. If $W = \oplus_{a=1..m} L_a$ is a sum of line bundles, then a natural generalization of this construction is to define $\mathbb{P}_{w_1 ... w_m}(W) := W/\rho_{w_1 ... w_m}$, where $\rho_{w_1 ... w_m}$ is the natural fiberwise $\mathbb{C}^*$ action on W given by: $\rho_{w_1 .. w_m}(u_1 \oplus ... \oplus u_m) := \lambda^{w_1} u_1 \oplus ... \oplus \lambda^{w_m} u_m$, the weights $w_1..w_m$ being positive integers. Then $\mathbb{P}_{w_1 ... w_m}(W)$ is a weighted projective space fibration over B_H, which will in general have singularities and is thus only a variety and not a manifold. The case $w_1 = ... = w_m = 1$ leads to the usual projectivisation above. $\mathbb{P}_{w_1 ... w_m}(W)$ is equiped with the *twisting sheaf* $O_{\mathbb{P}_{w_1 ... w_m}(W)}(1)$, which is a line bundle in the case $w_1 = ... = w_m$. $O_{\mathbb{P}_{w_1 ... w_m}(W)}(1)$ has

the $\mathbb{P}^1$-fibration structure of B_F, allowing us to reconstruct B_F from the data of B_H and V, V'. The Kahler moduli of B_H, together with the expectation value of the heterotic dilaton, map to the Kahler moduli of B_F (which are the only Kahler moduli visible in F-theory on X).

The bundle data V, V' are equivalent to pairs (Σ, L), respectively (Σ', L'), where Σ, Σ' are $n-1$-folds which are generically r-branched covers of B_H and L, L' are — generically — line bundles over Σ, Σ'. Σ and L are called the *spectral cover* and the *spectral bundle* of V, as will be discussed in more detail below.

The important point is that this equivalent description of V, V' allows us to organize their moduli in a way which facilitates describing the map to the moduli of the F-theory dual on X. Namely, the complex structure of Z together with the complex structure moduli of Σ, Σ' and the Kahler modulus of the generic elliptic fibre of Z map to the complex structure of X, while the moduli of L map to certain RR moduli of F-theory on X.

To make these statements more precise, one follows [7, 9, 10] in writing both X and Z as Weierstrass models [4]. For this, let K_{B_H} and K_{B_F} be the canonical line bundles of B_H and B_F. (Due to the ruled character of B_F, it is easy to see that we have $K_{B_F}^{-1} = K_{B_H}^{-1} \otimes O_F(2) \otimes \mathcal{M}$, where $O_F(1)$ is the twisting sheaf of $B_F = \mathbb{P}(\mathcal{M} \oplus \mathcal{O}_{B_H})$). Then one constructs the projectivisations: $\mathbb{P}_H := \mathbb{P}_{2,3,1}(K_{B_H}^{-2} \oplus K_{B_H}^{-3} \oplus \mathcal{O}_{B_H})$ (over B_H) and $\mathbb{P}_F := \mathbb{P}_{2,3,1}(K_{B_F}^{-2} \oplus K_{B_F}^{-3} \oplus \mathcal{O}_{B_F})$ (over B_F) where the weights of the natural $\mathbb{C}^*$ actions are indicated by subscripts. If $x \in H^0(K_{B_H}^{-2} \otimes O_{\mathbb{P}_H}(2)), y \in H^0(K_{B_H}^{-3} \otimes O_{\mathbb{P}_H}(3)), z \in H^0(O_{\mathbb{P}_H}(1))$, respectively $X \in H^0(O(K_{B_F}^{-2} \otimes O_{\mathbb{P}_F}(2)), Y \in H^0(K_{B_F}^{-3} \otimes O_{\mathbb{P}_F}(3)), Z \in H^0(O_{\mathbb{P}_F}(1))$ are the homogeneous coordinates of the projectivisations $\mathbb{P}_F, \mathbb{P}_H$, then we can write Z,X as the zero divisors associated to the sections

$$\text{heterotic}: \quad y^2 - x^3 - fxz^4 - gz^6 \in H^0(K_{B_H}^{-6} \otimes O_{P_H}(6)) \tag{1}$$

the property that its restriction to each fibre $O_{\mathbb{P}_{w_1 \dots w_m}}(W)_b \approx \mathbb{WP}^{m-1}_{w_1 \dots w_m}$ is isomorphic with the twisting sheaf $O_{\mathbb{WP}^{m-1}_{w_1 \dots w_m}}(1)$ of $\mathbb{WP}^{m-1}_{w_1 \dots w_m}$. Moreover, there are sections $x_i \in H^0(L_i \otimes O_{\mathbb{P}_{w_1 \dots w_m}(W)}(w_i))$ (called homogeneous coordinates of $\mathbb{P}_{w_1 \dots w_m}(W)$), which restrict on each fibre to usual homogeneous coordinates of $\mathbb{WP}^{m-1}_{w_1 .. w_m}$. The reader can consult [12, 13, 14] for more details on this basic construction in algebraic geometry.

[4] An elliptic fibration $Y \longrightarrow B$ with a section σ will typically have singular fibers even if Y itself is smooth. Some of these singular fibers will simply be elliptic curves with a node (so called type I_1) or with a cusp (type II), but in general there will also be reducible fibres. The Weierstrass model Y_W of Y is obtained from Y by blowing down all components of the reducible fibers which do not meet σ. This will typically introduce extra singularities in Y. Y can be recovered from Y_W by blowing up these singularities (a process called a *small resolution*). In our case, the Kahler parameters associated to small resolutions of X_W are immaterial since they are not physical parameters in F-theory, but the Kahler parameters of small resolutions of Z_W are physical. Therefore, using a Weierstrass model for Z is, strictly speaking, only possible over some locus in the moduli space of the heterotic theory on Z where these moduli are set to zero. On the F-theory side this corresponds to a particular locus in the complex structure moduli space of X. However, throughout this paper, we will only be interested in the case when Z is smooth, so its only degenerate fibers are of types I_1 and/or II, which are irreducible and necessitate no blow-down. In this case, Z is already a Weierstrass model. In other words, we do not allow ourselves to explore the full moduli space of our theories but only a subspace of it.

respectively

$$\mathrm{F-theory:}\quad Y^2 - X^3 - FXZ^4 - GZ^6 \in H^0(K_{B_F}^{-6}\otimes O_{\mathbb{P}_F}(6)), \tag{2}$$

where $f \in H^0(K_{B_H}^{-4}), g \in H^0(K_{B_H}^{-6})$ and $F \in H^0(K_{B_F}^{-4}), G \in H^0(K_{B_F}^{-6})$ are sections specifying the complex structure of Z, X. Since B_F is a $\mathbb{P}^1$ bundle over B_H, one can make an expansion in the coordinate of the $\mathbb{P}^1$ fibre. Geometrically, this is described as follows. Let $u \in H^0(\mathcal{M}\otimes O_F(1)), v \in H^0(\mathcal{O}_{B_H}\otimes O_F(1))$ be the homogeneous coordinates of the projectivisation $B_F = \mathbb{P}(\mathcal{M}\oplus\mathcal{O}_{B_H})$. Then we can expand:

$$F = \sum_{i=0..I} u^i \otimes v^{8-i}\otimes F_i \tag{3}$$

$$G = \sum_{j=0..J} u^j \otimes v^{12-j}\otimes G_j \tag{4}$$

where $F_i \in H^0(K_{B_H}^{-4}\otimes\mathcal{M}^{4-i})$, $G_j \in H^0(K_{B_H}^{-6}\otimes\mathcal{M}^{6-j})$ and I, J are the largest values of i, j for which $H^0(K_{B_H}^{-4}\otimes\mathcal{M}^{4-i})$, respectively $H^0(K_{B_H}^{-4}\otimes\mathcal{M}^{4-i})$, are nonzero.

The work of [9, 10, 3, 4] then shows that the complex structure moduli of Z (which are controlled by f and g) map to the moduli controlled by the 'middle sections' [5] F_4 and G_6, while the moduli of Σ and Σ' map to the moduli of the 'lower sections' F_i, G_j $(i < 4, j < 6)$, respectively, of the 'upper sections' F_i, G_j $(i > 4, j > 6)$ (one can formally extend the sums above up to $i = 8$, respectively $j = 12$, by defining F_i, G_j to be identically zero for $i > I$, respectively for $j > J$; this explains the terminology 'middle sections').

For the sake of the non-expert reader let us specialize this to the simplest case of six-dimensional compactifications [6] with $B_H = \mathbb{P}^1$, for which the choice $\mathcal{M} = O_{\mathbb{P}^1}(m)$ leads to B_F being the Hirzebruch surface F_m. As a toric variety, $F_m = (\mathbb{C}^4 - \mathcal{F})/(\mathbb{C}^*)^2$ where $\mathcal{F}$ is the exceptional set and the holomorphic quotient action is given by $(s,t,u,v) \overset{(\lambda,\mu)\in(\mathbb{C}^*)^2}{\longrightarrow} (\lambda s, \lambda t, \lambda^m\mu u, \mu v)$. The homogeneous coordinates s, t, u, v are thus sections of the line bundles $O(1,0), O(1,0), O(m,1), O(0,1)$, respectively. The coordinate pairs (u,v), (s,t) can be viewed as the homogeneous coordinates of two $\mathbb{P}^1$'s. Since (s,t) are left invariant by the action of μ, F_m as a ruled surface is a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$. The coordinates (s,t) are associated with the $\mathbb{P}^1$ base (denoted by $\mathbb{P}^1_b$) while the coordinates (u,v) are associated with the $\mathbb{P}^1$ fibre (denoted by $\mathbb{P}^1_f$). In this case $\mathbb{P}^1_b$ is nothing other than B_H. Clearly, this $\mathbb{P}^1$-fibration is just the projectivisation $F_m = \mathbb{P}(O_{\mathbb{P}^1_b}(m)\oplus\mathcal{O}_{\mathbb{P}^1_b})$, so that $\mathcal{M} = O_{\mathbb{P}^1_b}(m)$ in this case. The twisting sheaf $O_F(1)$ of this projectivisation can be naturally identified with $O_{F_m}(0,1)$. Thus $O_{\mathbb{P}^1_b}(m)\otimes O_F(1) \approx O_{F_m}(m,1)$, $\mathcal{O}_{\mathbb{P}^1_b}\otimes O_F(1) \approx O_{F_m}(0,1)$ and (u,v) can be naturally identified with homogeneous coordinates of the projectivisation. A canonical divisor of F_m is given by $-(D_u + D_v + D_s + D_t) \sim -2D_v - (m+2)D_s = -(stuv)$ (where D_u, D_v, D_s, D_t are the toric divisors of F_m and $\sim$ denotes linear equivalence), so the anticanonical line bundle

[5] Here we assume that there is enough amplenesss of $K_{B_H}^{-1}$,which is usually the case. In 4 dimensional compactifications,one usually takes B_H to be a Fano 2-fold, so $K_{B_H}^{-1}$ is ample.

[6] For a 4-dimensional example see section 6.1.

is $K_{F_m}^{-1} = O_{F_m}(m+2,2)$. Since B_F is a toric variety, the ambient space of the Weierstrass model of X is itself toric and given by: $\mathbb{P}_F := \mathbb{P}_{2,3,1}(K_{F_m}^{-2} \oplus K_{F_m}^{-3} \oplus \mathcal{O}_{F_m}) \approx (\mathbb{C}^7 - \mathcal{F}')/(\mathbb{C}^*)^3$, with homogeneous coordinates (s,t,u,v,X,Y,Z) and $(\mathbb{C}^*)^3$ action :

$$(s,t,u,v,x,y,z) \stackrel{(\lambda,\mu,\nu)\in(\mathbb{C}^*)^3}{\longrightarrow} (\lambda s, \lambda t, \lambda^m \mu u, \mu v, \lambda^{2m+4}\mu^4\nu^2 X, \lambda^{3m+6}\mu^6\nu^3 Y, \nu Z). \quad (5)$$

By the same reasoning as above, the twisting sheaf of this 'projectivisation' is $O_{\mathbb{P}}(0,0,1)$ and its homogeneous coordinates X,Y,Z are sections of $O_{\mathbb{P}_F}(2m+4,4,2), O_{\mathbb{P}_F}(3m+6,6,3)$ and $O_{\mathbb{P}_F}(0,0,1)$. Then $F \in H^0(O_{\mathbb{P}_F}(4m+8,8,0))$, $G \in H^0(O_{\mathbb{P}_F}(6m+12,12,0))$ can be viewed as polynomials $F = F(s,t,u,v)$, $G = G(s,t,u,v)$ in (s,t,u,v) of multidegrees $(4m+8,8)$, $(6m+12,12)$ under the action of (λ,μ), while $F_i \in H^0(O_{\mathbb{P}^1_b}(m(4-i)+8)) \approx H^0(O_{\mathbb{P}_F}(m(4-i)+8,0,0))$, $G_j \in H^0(O_{\mathbb{P}^1_b}(m(6-j)+12)) \approx H^0(O_{\mathbb{P}_F}(m(12-j)+12,0,0))$ can be identified with polynomials in (s,t) of degrees $m(4-i)+8$, respectively $m(6-j)+12$ under the action of λ. On the other hand, the ambient space of the Weierstrass model of Z is the toric variety $\mathbb{P}_H = \mathbb{P}_{2,3,1}(K_{\mathbb{P}^1_b}^{-2} \oplus K_{\mathbb{P}^1_b}^{-3} \oplus \mathcal{O}_{\mathbb{P}^1_b}) \approx (C^5 - \mathcal{F}'')/(C^*)^2$ with homogeneous coordinates (s,t,x,y,z) and $(C^*)^2$ action :

$$(s,t,x,y,z) \stackrel{(\lambda,\rho)\in(C^*)^2}{\longrightarrow} (\lambda s, \lambda t, \lambda^4\rho^2 x, \lambda^6\rho^3 y, \rho y) \quad (6)$$

and $f \in H^0(O_{\mathbb{P}^1_b}(8)) \approx H^0(O_{\mathbb{P}_H}(8,0)), g \in H^0(O_{\mathbb{P}^1_b}(12)) \approx H^0(O_{\mathbb{P}_H}(12,0))$ can be viewed as homogeneous polynomials $f(s,t), g(s,t)$ of degree 8, respectively 12 under the action of λ. This recovers the more intuitive (albeit less geometric) description of [9, 10].

A precise asymptotic form of the map from Σ, Σ' to the lower/upper sections (in the case when the low energy effective theory has enhanced gauge symmetry) was conjectured in [3], but the precise map between the complex structure moduli of Z and the middle sections is not yet understood. Furthermore, in general, an F-theory compactification on X requires specifying not only the n-fold X but also a certain point ω in a subset $IJ_c(X)$ of the second intermediate jacobian $IJ_2(X)$ of X, giving the corresponding RR moduli of the compactification [3]. In [3] it was argued that the moduli of L, L' map to the moduli of this point $\omega \in IJ_c(X)$, but the precise form of this map is also unknown at present.

Now let us discuss in more detail the correspondence between a stable bundle V over Z with $c_1(V) = 0$ and the pair (Σ, L). Here we assume that $c_2(V)$ and $c_3(V)$ are kept fixed, which gives us a moduli space of stable bundles $\mathcal{M}(0, c_2, c_3)$. [7].

Intuitively, the spectral cover Σ encodes the data of the heterotic Wilson lines over each toroidal fibre. Mathematically, the construction of (Σ, L) is roughly as follows. One limits oneself to bundles V with the property that their restriction to the generic elliptic fibre of Z is a direct sum $V_E = \oplus_{j=1..r} L_j$ of line bundles over E. Note that a 'generic' semistable bundle of degree zero over an elliptic curve will have this property and thus one can argue that the 'generic' stable bundle V over Z will belong to this component of the moduli space. As explained in the appendix, semistability of V_E forces $\mathrm{deg} L_j = 0$ for all j and thus

[7]Note that this moduli space is believed to be stratified in the general case [3, 4]; this stratification may be disentangled by considering more subtle invariants such as the Hilbert polynomials of the bundles involved.

$L_j \approx O_E(q_j - p)$ for some points $q_j \in E$, where p is the distinguished point on E where E touches the section of the elliptic fibration of Z. If $\pi_H(E) = b \in B_H$, this gives a family of points $q_j(b)$ $(j = 1..r)$ with the property that $\pi_H(q_j(b)) = b$. As b varies over B_H, this describes an r-fold cover of B_H which we call Σ, living as a hypersurface in Z. Due to the compactness of Z, this cover will necessarily be branched and the above construction will in general fail over the elliptic fibers of Z lying above the branching locus, which generically is a divisor in B_H. In fact, a correct mathematical construction of Σ reveals that V_E will typically fail to be fully split for E lying above the branching locus, although the corresponding points of Σ can still be obtained by considering the pieces of the associated graded bundle of V_E (see the appendix). We see that, in order to recover Σ from V, all we need, in first approximation, is to understand the splitting behaviour of V_E for a generic elliptic fibre E of Z. This will be accomplished in section 4 by studying the properties of the sections of V_E. More precisely, we will show how one can recover $\Sigma_{B_H - Q}$ where Q is a divisor in B_H above which V_E is not fully split. This leaves us with a certain ambiguity (which is potentially related to interesting physics), a detailed discussion of which will be given elsewhere.

The reason for the appearance of L is more subtle and can only be properly understood in a rigorous construction. The point is that Σ does not contain enough information to recover V and one can see that the missing data are given by a line bundle L over Σ. Intuitively, Σ only fixes the isomorphism class of V_{E_b} for each fiber $E_b = \pi_H^{-1}(b)$ but not the way in which the *concrete* representative of V_{E_b} 'twists' as b varies in B_H in order to form V. By including L, one can show that the correspondence between (Σ, L) and V is one to one. For more details the reader can consult [3, 4, 5, 11][8].

The cohomology class associated to Σ can be computed in terms of the characteristic classes of V as explained in [3]. In general, Σ will be given by the zero divisor of a section $s = \sum_{2i+3j+k=r} a_{ijk} x^i y^j z^k \in H^0(\mathcal{N} \otimes O_Z(n\sigma))$, where $\mathcal{N}$ is a line bundle over B_H and σ is the section of the elliptic fibration of Z. Here $r := \operatorname{rank} V$ and $a_{ijk} \in H^0(\mathcal{N} \otimes K_{B_H}^{-(n-k)})$, the sum being restricted to nonnegative i, j, k. By using the equation of Z, this can always be written in the canonical form:

$$s = \sum_{0 \le e = even \le r} a_e x^{e/2} z^{r-e} + y \sum_{3 \le e = odd \le r} a_e x^{(e-3)/2} z^{r-e} \tag{7}$$

where $a_e := a_{e/2,0,r-e}$ for e even and $a_e := a_{(e-3)/2,1,r-e}$ for e odd. We have $a_e \in H^0(\mathcal{N} \otimes K_{B_H}^e)$ for all e. It was shown in [3] that:

$$c_1(\mathcal{N}) = \pi_{H,*}(c_2(V)) \tag{8}$$

[8]The non-expert reader should be aware that our explanation almost completely sacrificed mathematical rigour in favour of simplicity. For the reader acquainted with [5], let us mention an important difference between the bundles which will appear in our examples in section 5 and the bundles considered there. The most general results of [5] concern bundles of trivial determinant whose restriction to the generic elliptic fibre of Z is not only semistable but also regular – which means that the line bundles L_i appearing in maximal direct sum decomposition of V explained in the appendix are mutually distinct. In particular, this rules out the tangent bundle of a threefold. Most of the 4-dimensional heterotic compactifications in section 5 do not satisfy this regularity condition.

This relation fixes $\mathcal{N}$ completely if $\mathrm{Pic}^0(B_H) = 0$, which is the case, in particular, if B_H is a toric variety. Remember that the dual heterotic n-fold is also characterized by the line bundle $\mathcal{M}$, which specifies B_F as a $\mathbb{P}^1$-fibration over B_H. The conjecture of [3] is that $\mathcal{M} = \mathcal{N} \otimes K_{B_H}^6$. Moreover (concentrating on one of the E_8 factors), if we assume that the gauge symmetry of the effective theory is not completely broken, then a_e should be identified with certain sections related to F_i, G_j which occur naturally once one imposes on X the constraints required by Tate's algorithm [10].

In the case of 6-dimensional compactifications or of 4-dimensional compactifications with B_H a Hirzebruch surface F_k, but Z otherwise generic, one can determine the class of σ more explicitly as explained in [4] [9]. Let us recal the relevant results. Consider a generic stable bundle V with $\det V \approx \mathcal{O}_Z$ over a *smooth* n-fold Z, elliptically fibered over B_H $(n = 2, 3)$. Let $S := [\sigma(B_H)] \in H^2(Z, \mathbb{Z})$ be the cohomology class of $\sigma(B_H)$ and $F \in H^{2n-2}(Z, \mathbb{Z})$ be the cohomology class of the elliptic fibre of Z. Let $[\Sigma] \in H^2(Z, \mathbb{Z})$ be the cohomology class associated to Σ.

In the appendix of [4] it is shown that [10] :

(1) If Z is an elliptically fibered K3 surface with $\mathrm{Pic}(Z) \approx \mathbb{Z}^2$, then :

$$[\Sigma] = \mathrm{rank}\, V \ S + c_2(V)[Z] \ F \tag{9}$$

(2)If Z is an elliptically fibered Calabi-Yau threefold with B_H a Hirzebruch surface F_k (with $a, b \in H^2(B_H, \mathbb{Z})$ the cohomology classes of the $\mathbb{P}^1$-fibre, respectively of the infinity section of F_k) and if the classes $A := \pi_H^*(a)$, $B := \pi_H^*(b)$ and $S := [\sigma(B_H)]$ form a $\mathbb{Z}$-basis of the abelian group $H^2(Z, \mathbb{Z})$ and generate the graded ring $H_c^*(Z) := H^0(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}) \oplus H^4(Z, \mathbb{Z}) \oplus H^6(Z, \mathbb{Z})$, then AB, AS and BS form a $\mathbb{Z}$-basis of the group $H^4(Z, \mathbb{Z})$. Moreover, if $c_2(V) = c_2(V)_{AS}\ AS + c_2(V)_{BS}\ BS + c_2(V)\ AB$ is the associated decomposition of $c_2(V)$, then we have :

$$[\Sigma] = \mathrm{rank}\, V \ S + c_2(V)_{AS}\ A + c_2(V)_{BS}\ B \tag{10}$$

Obviously a generic smooth elliptic K3 surface and a generic smooth Calabi-Yau threefold elliptically fibered over a Hirzebruch surface satisfy the above hypotheses.

Since the spectral covers Σ, Σ' play a key role in the proposed duality map, this naturally leads us to try to understand how they can be explicitly identified. Specifically, since $(0, 2)$ linear sigma models are our most insightful tool for directly constructing heterotic models, we would like to find a procedure for identifying Σ, Σ' in terms of linear sigma model data. To this end, let us briefly recall the linear sigma model construction.

[9]Simplifications occur in this case because of the ruled character of $B_H = F_k$, which implies that Z is itself K3-fibered.

[10]Juxtaposition denotes the cup product in integer cohomology, while $c[Z]$ is the evaluation of a cohomology class $c \in H^{2n}(Z, \mathbb{Z})$ on the fundamental homology class $[Z] \in H_{2n}(Z, \mathbb{Z})$ of Z.

3 $(0,2)$ Models and the Linear Sigma Model Construction

The linear sigma models we consider are abelian supersymmetric gauge theories in two dimensions with $(0,2)$ supersymmetry. Their interest stems from the well-known arguments of [1] to the effect that, if such a model is carefully defined, then it is expected to have an infrared fixed point whose conformal field theory gives a good starting point for a perturbative heterotic compactification.

Beyond a $U(1)^h$ worldsheet gauge symmetry, such a model has a number s of local fermionic symmetries. Associated to these symmetries one has h abelian gauge multiplets and s pairs [11] $(\Sigma_j, \overline{\Sigma_j})(j = 1..s)$ of (nonchiral) complex fermi superfields ('twisted Fermi superfields'). The worldsheet matter content consists of $d+1$ chiral scalar superfields, denoted $\Phi_\rho(\rho = 1..d)$ and P as well as $m+t$ chiral fermi superfields denoted $\Lambda_a(a = 1..m)$ and $\Gamma_i(i = 1..t)$. One also includes spectator fields as discussed in [16]. Each of the matter fields transforms in a representation of $U(1)^h$ characterized by a multicharge $q \in \mathbb{Z}^h$. The only fields which transform nontrivially under the local fermionic symmetries are Γ_i, Λ_a and Σ_j. The transformation laws of Γ_i and Λ_a are controlled by holomorphic functions $E_{0i}^{(j)}(\Phi)(i = 1..t, j = 1..s)$, respectively $E_a^{(j)}(\Phi)(a = 1..m, j = 1..s)$. Beyond the usual gauge kinetic and Fayet-Iliopoulos D and theta terms and the kinetic terms of the matter fields, the action contains a term describing a coupling of the fermionic superfields Λ_a to the twisted fermionic fields Σ_j. It also contains a superpotential which couples Γ_i to holomorphic combinations $G_a(\Phi_1..\Phi_d)$, P to Λ_a and to holomorphic combinations $F_a(\Phi_1...\Phi_d)$, as well as spectator field terms. Our notation for the various $U(1)$ charges is given below :

ϕ_ρ	p	λ_a	γ_i
q_ρ	q_0	$\tilde{q}_a$	$\tilde{q}_{0i}$

$E_a^{(j)}$	$E_{0i}^{(j)}$	F_a	G_i
$\tilde{q}_a$	$\tilde{q}_{0i} - q_0$	$-q_0 - \tilde{q}_a$	$-\tilde{q}_{0i}$

A semiclassical study of these models [15, 17, and references therein] reveals an intricate phase structure for their moduli space which can be explicitly analyzed by methods from toric geometry. Physically, these phases arise by changing the values of the h Fayet-Iliopoulos parameters r_i on which the models depend. Typically one builds such a model so that it admits a Calabi-Yau phase in which all of the r-parameters are positive. By following the assignment of gauge charges for the various fields, one finds out that the moduli space in such a region is described — at least in the limit of large r-parameters — by a nonlinear $(0,2)$ sigma model defined over a Calabi-Yau variety Z realized as a complete intersection in a toric variety. Such a model is specified, in the classical limit, by Z and a vector bundle V over Z. To interpret the data (V, Z) as a starting point for a perturbative string compactification, Z must satisfy the Calabi-Yau condition and V must admit the structure of a (semi-) stable holomorphic vector bundle over Z with $c_1(V) = 0$, $c_2(V) = c_2(T)$. This imposes certain

[11] The notation we use follows the conventions of [15], to which we refer the reader for further details. We denote the lowest component of a superfield by the lower case form of the same letter. Thus, ϕ_ρ is the lowest component of Φ_ρ etc.

constraints on the gauge charges of the original $(0,2)$ linear sigma model, which turn out to be equivalent to the absence of an anomaly for the gauge, left $U(1)$ global and right R-symmetries of that theory. However, the emergence of the stability condition for V is somewhat mysterious in this context, and very hard to test. At present, we do not know the full set of conditions which ensure that a well-defined $(0,2)$ linear sigma model gives rise to a (semi-) stable bundle V.

In practice, a self-consistent way to test this is provided by the analysis of the linear sigma model in other phases. Concretely, such a model will typically also admit a Landau-Ginzburg phase in which it reduces to a supersymmetric Landau-Ginzburg model. By analyzing the corresponding conformal field theory one can extract information about the spectrum [18, 19]. It is believed that inconsistency of the theory at the Landau-Ginzburg point signals a failure of the stability condition for V. One of the main points that we will make in the following is that, when we talk about a 'purely perturbative' linear sigma model, we will require it to be well-behaved at the Landau-Ginzburg point. Note that, since we do not have a good direct way to test semistability of V over Z, we regard this condition as central for having a well-defined perturbative model. Careful implementation of these consistency conditions will have a significant effect on the results we find.

We now explain the mathematical description of the model in the deep Calabi-Yau phase, which — modulo the above — is the only part of the linear $(0,2)$ moduli space directly entering our subsequent discussion. In the limit of large but fixed positive Fayet-Iliopoulos parameters, the moduli space is the simplectic quotient of the minimal set of the bosonic potential via the action of the gauge group $U(1)^h$. Rewriting this as a holomorphic quotient one obtains a complete intersection $G_1(\Phi) = .. = G_t(\Phi)$ in a toric variety $\mathbb{P} = (\mathbb{C}^d - \mathcal{F})/(\mathbb{C}^*)^h$, where $\mathcal{F}$ is the exceptional set. The bosonic fields ϕ_ρ transform as homogeneous coordinates of $\mathbb{P}$. Since in this phase they are restricted to lie in Z, we will however denote the homogeneous coordinates of $\mathbb{P}$ by x_ρ, with the understanding that on Z we have $\phi_\rho = x_\rho$. Analysis of the massless fermionic modes reveals that the gauge data V are given by the cohomology of a monad. Specifically, this means that V is defined by the cohomology of the complex :

$$0 \longrightarrow \oplus_{j=1}^{s} O_Z \xrightarrow{f} \oplus_{a=1}^{m} O_Z(\tilde{q}_a) \xrightarrow{g} O_Z(-q_0) \longrightarrow 0 \tag{11}$$

In this equation, the maps f and g act on the fibers as follows :

$$f_x(\eta_1 ... \eta_s) := \begin{pmatrix} E_1^{(1)}(x) & ... & E_1^{(s)}(x) \\ E_2^{(1)}(x) & ... & E_2^{(s)}(x) \\ ... & ... & ... \\ E_m^{(1)}(x) & ... & E_m^{(s)}(x) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ ... \\ \eta_s \end{pmatrix}, \forall x \in Z, \forall \eta_j \in O_{Z,x} \tag{12}$$

where $E_a^{(j)} \in H^0(O_Z(\tilde{q}_a))$, and

$$g_x(v_1 ... v_m) := \sum_{a=1..m} F_a(x) \otimes v_a \forall x \in Z, \forall v_a \in O_{Z,x}(\tilde{q}_a) \tag{13}$$

where $F_a \in H^0(O_Z(-q_0 - \tilde{q}_a))$. By taking dimensions in (11) one easily sees that we have $m = r + s + 1$ where r :=rank$V$. The sections of V are identified with masless linear combinations of the fermionic coordinates λ_a .

In order to avoid complications, we will choose the defining data so that V is a *bundle* over a nonsingular Calabi-Yau *manifold* Z. As argued in [15], it is possible to consider more general situations, but we will restrict to this case. In particular, this means that we choose the maps f, g in the monad to be transverse.

Analysis of the Landau-Ginzburg phase as in [15] leads to the conclusion that the number s of local fermionic symmetries of the underlying linear sigma model (which is equal to the number of columns $E^{(j)}$ of the matrix above) must be large enough to avoid having massless charge zero fermions at the Landau-Ginzburg point. Mathematically, this is reflected by the fact that, since $c_1(V) = 0$, a necessary condition for stability of V is that $H^0(V) = 0$. That is, the presence of the s twist fields gives a big enough space $H^0(\oplus_{j=1..s}\mathcal{O}_Z)$, which helps to mod out the nontrivial regular sections of V. While this is not a sufficient condition for a well-behaved Landau-Ginzburg theory, presence of a sufficient number of local fermionic symmetries is required for stability of V.

For later reference, we note that the conditions $c_1(V) = c_1(TZ) = 0$ and $c_2(V) = c_2(TZ)$ imply [12] the following constraints on the multicharges :

$$\sum_\rho q_\rho = - \sum_{\alpha=1..t} \tilde{q}_{0\alpha} \tag{14}$$

$$\sum_{a=1..m} \tilde{q}_a = -q_0 \tag{15}$$

$$\sum_{a=1..m} \tilde{q}_a^k \tilde{q}_a^l - q_0^l q_0^k = \sum_\rho q_\rho^k q_\rho^l - \sum_{\alpha=1..t} \tilde{q}_{0\alpha}^k \tilde{q}_{0\alpha}^l \tag{16}$$

Moreover, invariance of the action under the local fermionic symmetries requires :

$$\sum_{a=1..m} E_a^{(j)} F_a = - \sum_{i=1..t} E_{0i}^{(j)} G_i \tag{17}$$

4 Spectral Covers for $(0, 2)$ Linear Sigma Models

4.1 Derivation of the method

We will be interested in the case when Z is smooth, elliptically fibered via a map $\pi_H : Z \to B_H$, and has a nonsingular base B_H. In order to build examples with an 'obvious' fibration structure, one can use the methods of toric geometry. The underlying idea is to build Z inside a toric variety $\mathbb{P}$ which is itself fibered over the base B_H with fibre a compact toric variety $\mathbb{P}_f$. In this case, one also takes the base B_H to be toric. Then Z is realized inside $\mathbb{P}$ as

[12] One can easily see the necessity of these conditions by considering the corresponding Chern classes in the Chow ring $A^*(\mathbb{P})$. The sufficiency of these conditions on charges is not assured unless $\mathbb{P}$ is itslef smooth, but we will neglect this subtlety here.

a 'generalized Weierstrass model' in which case the elliptic fibre is visible by simply freezing the variables corresponding to a point in B_H. The fibres of π_H will then naturally sit inside the fibres $\mathbb{P}_f$ of the fibration of the toric ambient space. The structure of the models involved is very simple and will become apparent in the examples we give in the next section. For all of those examples we will have $\mathbb{P}_f = \mathbb{WP}_{2,3,1}$ and Z will be a hypersurface in $\mathbb{P}$.

Fix a point b in the base B_H and consider the elliptic fibre $\pi^{-1}(b) := E$ above this point. Throughout our discussion we will keep b fixed, treating it as a parameter. We assume that the fibration π_H has a section σ. $\sigma(B_H)$ defines an effective divisor in Z. We denote by $O_Z(\sigma)$ the associated line bundle on Z. If the restriction V_E of V is semistable and of trivial determinant for the generic elliptic fibre E of Z, then V admits a spectral cover description as explained in [3, 4]. Note that this is a nontrivial condition for V and is the main assumption underlying much of the work of [3, 4]. In particular, it is the only case for which a concrete proposal for the map relating heterotic bundle data to F-theory data has been conjectured [3] [13].

Moreover, it can be argued [3, 4] that a 'generic' stable bundle over Z has the property that its restriction to the generic elliptic fibre E of π_H is a fully split semistable vector bundle. Here by fully split we mean that $V|_E$ splits as a direct sum of holomorphic line bundles [14]. While this later assumption is not essential for the validity of our method of computing the cover [6], we will restrict the presentation of this section to bundles satisfying this additional condition. It should be noted, however, that this condition indeed fails to hold in a number of models that we were able to construct.

In general, we denote by W_S the restriction of a vector bundle $W \to \mathbb{P}$ to a sub*manifold* S of $\mathbb{P}$. Assuming that b is a generic point in B_H it folows that

$$V_E := V|_E = \oplus_{i=1}^r L_i \tag{18}$$

where L_i are degree zero holomorphic line bundles over E. Any such line bundle can be represented as $L_i = O(q_i - p)$ where we choose the distinguished point p of E to be given by $p = \sigma(b) \in E$. Now let $V'_E := V_E \otimes O(p)$. V'_E is a degree r semistable vector bundle over E which splits as

$$V'_E = \oplus_{j=1}^r O_E(q_j) \tag{19}$$

The spectral cover associated to the bundle V is essentially the collection of points q_j over every point b in the base. Therefore, we are interested in concretely identifying the points q_j starting from our monad data. By the Riemann-Roch theorem we have $h^0(O(q_j)) =$

[13] Checking that a given bundle V (specified by monad data) has this property is slightly nontrivial in practice and involves a more detailed analysis then we will present here. The full discussion of this point will be given in [6]. We will restrict ourselves to some tangential remarks on this for the examples in the next section.

[14] This is somewhat contrary to the usual mathematical convention in which splitting is considered for the associated $SU(r)$ principal bundle and is thus equivalent to the existence of a filtration by holomorphic subbundles of consecutive dimensions. Since such a filtration is automatic for degree zero semistable vector bundles over (smooth) elliptic curves, this more classical terminology caries no interesting information in our case.

1, $h^1(O(q_j)) = 0$, which shows that, under our assumptions for V and b, $H^0(V'_E)$ is r-dimensional while $H^1(V'_E) = 0$. Moreover, let $s_j \in H^0(O_E(q_j))$ be a meromorphic section whose associated divisor is $(s_j) = q_j$. As this divisor is effective, it follows that s_j is actually regular (holomorphic), having a simple zero at the point q_j and no other zeroes or poles. Since $\dim_{\mathbb{C}} H^0(O_E(q_j)) = 1$, we have $H^0(O_E(q_j)) =< s_j >$ (the $\mathbb{C}$-span of s_j). (This implies in particular that each s_j is unique up to multiplication by a constant complex scalar). It follows that $(s_1...s_r)$ form a basis of the $\mathbb{C}$-vector space $H^0(V'_E)$. Unfortunately, it is hard to identify such sections of V'_E directly. Nevertheless, we now show how one can use any system of generators $(u_1...u_N)$ of the vector space $H^0(V'_E)$ in order to determine the points q_j above. For this, note that there must exist a *constant* matrix A with the property :

$$(u_1...u_N)^t = A(s_1...s_r)^t \tag{20}$$

The matrix A is constant on the given elliptic fibre E. Because $u_1..u_N$ generate $H^0(V'_E)$,we have rank$A = r$. As (20) is a functional equation, we can evaluate it at any point $e \in E$ to obtain :

$$(u_1(e)...u_N(e))^t = A(s_1(e)...s_r(e))^t \tag{21}$$

As rank$A = r$, it easy to see that $\dim_{\mathbb{C}} < u_1(e)...u_N(e) > = \dim_{\mathbb{C}} < s_1(e)...s_r(e) >$ for all $e \in E$, where $< ... >$ denotes the $\mathbb{C}$-vector span of the corresponding set of vectors [15]. Now, it is clear that $\dim_{\mathbb{C}} < s_1(e)...s_r(e) >$ will decrease in discrete jumps precisely for $e \in \{q_1...q_r\}$ and this criterion can be used to find the points q_i. It follows that the (multi)set $q_1...q_r$ can be identified once one posesses any explicit system of generators $(u_1...u_r)$ of $H^0(V'_E)$. Note that the points q_i need not all be distinct. If some subgroups of them coincide (say, $q_1 = q_2 = .. = q_{l_1}, q_{l_1+1} = .. = q_{l_1+l_2}, ..., q_{l_1+..+l_{t-1}} = ... = q_{l_1+...+l_t}$, with $l_1 + ... + l_t = r$) then $\dim_{\mathbb{C}} < \sigma(e)...\sigma_r(e) >$ will decrease by l_j at the point $q_{l_{j-1}+1} = ... = q_{l_{j-1}+l_j}$ and we can still use the information above to completely specify the multiset $q_1...q_r$.

The calculation of the spectral cover is thereby reduced to obtaining a system of generators of $H^0(V'_E)$ from our monad data, and this is a simple exercise in sheaf cohomology. The only slightly subtle point is that, since (21) is only useful if we have a knowledge of the way our sections vary over E, we have to carefully keep track of the *explicit* realization of the maps and cohomology spaces involved. This is possible because all cohomological constructions we use are natural.

We start from the definition (11) of V, which we restrict to E to obtain the following monad, whose cohomology defines the restriction V_E of V to E:

$$0 \longrightarrow \oplus_{j=1}^{s} O_E \xrightarrow{f} \oplus_{a=1}^{m} O_E(\tilde{q}_a) \xrightarrow{g} O_E(-q_0) \longrightarrow 0 \tag{22}$$

[15] Indeed, let $(\epsilon_1(e)...\epsilon_d(e))$ be a basis of $< s_1(e)...s_r(e) >$, where $d = \dim_{\mathbb{C}} < s_1(e)...s_r(e) >$. Then $(s_1(e)...s_r(e))^t = B(e)(\epsilon_1(e)...\epsilon_d(e))^t$, with $B(e)$ a matrix of maximal rank. It follows that : $(u_1(e)...u_N(e))^t = AB(e)(\epsilon_1(e)...\epsilon_d(e))^t$. As A and B have maximal rank, the associated linear operators $\hat{A} \in L(\mathbb{C}^r, \mathbb{C}^n)$ and $\hat{B} \in L(\mathbb{C}^d, \mathbb{C}^N)$ are injective, and so is their composition $\hat{A}\hat{B} = \hat{AB}$, which implies that AB also has maximal rank, equal to r. Therefore $\dim_{\mathbb{C}} < u_1(e)...u_N(e) >= \text{rank}(AB) = d = \dim_{\mathbb{C}} < s_1(e)...s_r(e) >$.

Now twist the restricted monad by the line bundle $O_E(p)$ to obtain:

$$0 \longrightarrow \oplus_{j=1}^{s} O_E(p) \xrightarrow{f} \oplus_{a=1}^{m} O'_E(\tilde{q}_a) \xrightarrow{g} O'_E(-q_0) \longrightarrow 0 \tag{23}$$

where we defined $\mathcal{E}'_E := \mathcal{E}_E \otimes O_E(p), O'_E(\tilde{q}_a) := O_E(\tilde{q}_a) \otimes O_E(p)$ and $O'_E(-q_0) := O_E(-q_0) \otimes O_E(p)$.

It is easy to see that the cohomology of this monad defines a vector bundle which is canonically isomorphic to the twisted bundle $V'_E := V_E \otimes O(p)$. We now rewrite this as a pair of exact sequences:

$$0 \longrightarrow \oplus_{j=1..s} O_E(p) \xrightarrow{f} \oplus_{a=1..m} O'_E(\tilde{q}_a) \xrightarrow{p} \mathcal{E}' \longrightarrow 0 \tag{24}$$

$$0 \longrightarrow V' \xrightarrow{j} \mathcal{E}' \xrightarrow{\tilde{g}} O'(-q_0) \longrightarrow 0 \tag{25}$$

where $\mathcal{E} := \mathrm{coker}(f)$ and $\tilde{g}$ is the map canonically induced by g via descent. Note that $\tilde{g}$ is the unique map making the following diagram commute:

$$\begin{tikzcd} \oplus_{a=1..m} O'_E(\tilde{q}_a) \arrow[d, "p"'] \arrow[dr, "g"] & \\ \mathcal{E}'_E \arrow[r, "\tilde{g}"'] \arrow[ur, phantom, "\bigcirc"] & O'(-q_0) \end{tikzcd} \tag{26}$$

Here j, p act as the natural injection, respectively surjection on each fiber. More precisely, for any $e \in E$, the 'stalk' of p at e is given by the canonical surjection:

$$p_e : \oplus_{a=1..m} O'_{E,e}(\tilde{q}_a) \longrightarrow \mathcal{E}'_{E,e} := \oplus_{a=1..m} O'_{E,e}(\tilde{q}_a) / f_e(\oplus_{j=1..s} O_{E,e}(p)). \tag{27}$$

By using $H^1(O_E(p)) = 0$ and $H^1(V'_E) = 0$ we see that the associated cohomology long sequences collapse to the pair of short exact sequences

$$0 \longrightarrow \oplus_{j=1..s} H^0(O_E(p)) \xrightarrow{f_*} \oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) \xrightarrow{p_*} H^0(\mathcal{E}'_E) \longrightarrow 0 \tag{28}$$

$$0 \longrightarrow H^0(V'_E) \xrightarrow{j_*} H^0(\mathcal{E}'_E) \xrightarrow{\tilde{g}_*} H^0(O'_E(-q_0)) \longrightarrow 0. \tag{29}$$

where f_*,$\tilde{g}_*$,p_*,j_* denote the maps induced in degree zero cohomology. These maps act fiberwise on sections in the natural manner, for example $f_*(\eta)(e) = f_e(\eta(e))$, for any $\eta \in \oplus_{j=1..s} H^0(O_E(p))$ and all $e \in E$. In particular, for all $s \in \oplus_{a=1..m} H^0(O'_E(\tilde{q}_a))$ and all $e \in E$ we have:

$$p_*(s)(e) = p_e(s(e)) = s(e) \bmod \{f_e(\oplus_{j=1..s} O_{E,e}(p))\} = s(e) + f_e(\oplus_{j=1..s} O_{E,e}(p)). \tag{30}$$

The diagram (26) gives $\tilde{g}_* \circ p_* = g_*$. Together with the above sequences, this gives the diagram :

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[d] & & \\
 & & \oplus_{j=1..s} H^0(O_E(p)) \arrow[d, "f_*"] & & \\
 & & \oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) \arrow[d, "p_*"] \arrow[ddr, "g_*"] \arrow[dr, phantom, "\bigcirc"] & & \\
0 \arrow[r] & H^0(V'_E) \arrow[r, "j_*"] & H^0(\mathcal{E}'_E) \arrow[r, "\tilde{g}_*"] \arrow[d] & H^0(O'_E(-q_0)) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}
\tag{31}
$$

From (31) one easily deduces that

$$H^0(V'_E) = p_*(\ker\{\oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) \stackrel{g_*}{\to} H^0(O'_E(-q_0))\}) \tag{32}$$

Moreover, since p_* is surjective, the rank theorem for p_* gives: $\dim_{\mathbb{C}} \oplus_{a=1..m} H^0(O'_E(\tilde{q}_a))$ $= \dim_{\mathbb{C}} H^0(\mathcal{E}'_E) + s$, where we used $\ker p_* = \mathrm{im} f_*$ and injectivity of f_* to deduce that $\mathrm{def} p_* = s$. Taking dimensions in the exact row of the diagram gives $\dim_{\mathbb{C}} H^0(O'_E(\tilde{q}_a)) = r + \dim_{\mathbb{C}} H^0(O'_E(-q_0))$. These relations combined imply :

$$\dim_{\mathbb{C}} \oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) = \dim_{\mathbb{C}} H^0(O'_E(-q_0)) + r + s \tag{33}$$

On the other hand, applying the dimension theorem to the map $p_*|_{ker g_*}$ and noting from the diagram that $\ker p_* \subset \ker g_*, \mathrm{im}(p_*|_{ker g_*}) = p_*(\ker g_*) = H^0(V'_E)$ and $\ker p_* = \mathrm{im} f_*$ gives :

$$N := \dim_{\mathbb{C}} \ker g_* = r + s \tag{34}$$

Note that none of these results requires that V'_E is fully split, so they will hold for any V'_E semistable and of degree zero.

Now suppose that we are able to obtain a basis $(v_1...v_N)$ of the $\mathbb{C}$-vector space :

$$\ker\{\oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) \stackrel{g_*}{\to} H^0(O'_E(-q_0))\} \tag{35}$$

Because p_* is surjective, we are then assured that the sections $u_1 := p_*(v_1)...u_N := p_*(v_N)$ form a system of generators of $H^0(V'_E)$ over $\mathbb{C}$. Now let $w_1(e)..w_s(e)$ be a $\mathbb{C}$-basis of $f_e(\oplus_{j=1..s} O_{E,e}(p))$. Then it is easy to see that [16]:

$$\dim_{\mathbb{C}} < u_1(e)...u_N(e) >= \dim_{\mathbb{C}} < v_1(e)...v_N(e), w_1(e)...w_s(e) > -s \tag{36}$$

and we can identify q_i by looking for jumps of $\dim_{\mathbb{C}} < v_1(e)...v_N(e), w_1(e)...w_s(e) >$. A practical way to do this is to note that all of our vectors belong to $\oplus_{a=1..m} O'_{E,e}(\tilde{q}_a)$, so we

[16] Indeed, let $F_e := < w_1(e)..w_s(e) >= f_e(\oplus_{j=1..s} O_{E,e}(p))$ and let $S_e := < v_1(e)...v_N(e) >$. Then we have $S_e + F_e =< v_1(e)...v_N(e), w_1(e)...w_N(e) >$ and $< u_1(e)...u_N(e) >= p_e(< v_1(e)...v_N(e) >) = p_e(S_e)$. Therefore : $\dim_{\mathbb{C}} < u_1(e)..u_N(e) >= \dim_{\mathbb{C}} p_e(S_e) = \dim_{\mathbb{C}} S_e - \dim_{\mathbb{C}}(\ker p_e \cap S_e) = \dim_{\mathbb{C}} S_e - \dim_{\mathbb{C}}(F_e \cap S_e)$ $= \dim_{\mathbb{C}} S_e - \dim_{\mathbb{C}} F_e - \dim_{\mathbb{C}} S_e + \dim_{\mathbb{C}}(F_e + S_e) = \dim_{\mathbb{C}} < v_1(e)...v_N(e), w_1(e)...w_N(e) > -s$.

may expand them in their components (denoted by upperscript a) along the *one dimensional* vector spaces $O'_{E,e}(\tilde{q}_a)$.

Then $\dim_{\mathbb{C}} < v_1(e)...v_N(e), w_1(e)...w_s(e) >$ can be obtained as the rank of the matrix :

$$\tilde{S}(e) := \begin{pmatrix} v_1^{(1)}(e) & ... & v_N^{(1)}(e) & , w_1^{(1)}(e) & ... & w_s^{(1)}(e) \\ ... & ... & ... & ... & ... & .... \\ v_1^{(m)}(e) & ... & v_N^{(m)}(e) & , w_1^{(m)}(e) & ... & w_s^{(m)}(e) \end{pmatrix} \tag{37}$$

where we define the rank by imploying the minors in the usual way, except that in the definition of determinants we replace multiplication by tensor product. (It is immediate that this definition has the standard properties due to the fact that all components of our vectors live in one-dimensional subspaces, so that one can choose bases of these subspaces to reduce to the usual case of matrices over the field of complex numbers).

Our task will be finished if we can find a family $(w_1(e)...w_s(e))$ of bases of the stalks $f_e(\oplus_{j=1..s} O_{E,e}(p))$ for all $e \in E$. It turns out that this task can be disposed of in the following manner. Consider a local section $\eta \in H^0(U, O_E(p))$ of $O_E(p)$ above an open neighborhood U of p. We can always choose η such that $\eta(e) \neq 0$. As the map $f_e : \oplus_{j=1..s} O_{E,e}(p) \rightarrow \oplus_{a=1..m} O_{E,e} O'(\tilde{q}_a)$ is injective for all $e \in E$, we are then assured that $(w_1(e) := f_e(\eta(e), ..., 0), ..., w_s(e) := f_e(0, ..., \eta(e)))$ forms a $\mathbb{C}$-basis of $f_e(\oplus_{j=1..s} O_{E,e}(p))$. In our case :

$$w_j(e) = (E_1^{(j)}(e) \otimes \eta(e) ... E_m^{(j)}(e) \otimes \eta(e)) \tag{38}$$

which shows that $\eta(e)$ will factor out of all minor determinants involved in the computation of $\text{rank}\tilde{S}(e)$. Thus, we have $\text{rank}\tilde{S}(e) = \text{rank}S(e)$, where :

$$S(e) := \begin{pmatrix} v_1^{(1)}(e) & ... & v_N^{(1)}(e) & , E_1^{(1)}(e) & ... & E_1^{(s)}(e) \\ ... & ... & ... & , ... & ... & .... \\ v_1^{(m)}(e) & ... & v_N^{(m)}(e) & , E_m^{(1)}(e) & ... & E_m^{(s)}(e) \end{pmatrix} \tag{39}$$

and to find the points $q_1...q_r$ it suffices to look for jumps in $\text{rank}S(e)$ as e varies in E.

As is obvious from (21) and (36), we have $\text{rank}S(e) = r + s$ for $e \in E - \{q_j | j = 1..r\}$. To understand how this occurs, note that, in virtue of (32) and of the fact that $\dim_{\mathbb{C}}\ker p_* = \dim_{\mathbb{C}}(\ker g_*) - \dim_{\mathbb{C}} H^0(V'_E) = N - r = s$, there must be s independent linear combinations of $v_1...v_N$, with *constant* coefficients, which give sections of $f(\oplus_{j=1..s} O_E(p))$. Any such combination is of the form:

$$\sum_{l=1..N} \alpha_l v_l(e) = \sum_{j=1..s} \beta_j(e) \nu_j(e) \tag{40}$$

where $\nu_1(e) := f_*(\nu, 0...0), ..., \nu_s(e) := f_*(0, 0, ...\nu)$ and $\nu \in H^0(O_E(p)) - \{0\}$ is arbitrarily chosen. (Indeed, since $f_{*,e}$ is injective and $\nu(e) \neq 0$ for all $e \in E - \{p\}$, we see that $(\nu_1(e)...\nu_s(e))$ form a basis of $f_e(\oplus_{j=1..s} O_{E,e}(p))$ for all $e \in E - \{p\}$. Then a simple argument shows that (40) will hold for all $e \in E$ if we allow $\beta_j(e)$ to be rational functions on E with at most a pole of order 1 at p and no other poles.) Here α_l are *constant* on E. Since

$$\nu_j(e) = (E_1^{(j)}(e) \otimes \nu(e) ... E_m^{(j)}(e) \otimes \nu(e)), \tag{41}$$

we can rewrite (40) as :

$$\sum_{l=1..N} \alpha_l v_l(e) = \sum_{j=1..s} \gamma_j(e) \otimes E^{(j)}(e), \tag{42}$$

where $E^{(j)}(e)$ denotes the column vector $(E_1^{(j)}(e)...E_m^{(j)}(e))$ and where $\gamma_j(e) := \beta_j(e) \otimes \nu(e)$ are *regular* sections of $O_E(p)$. This gives s linear dependences among the columns of $S(e)$, which involve constants coefficients for $v_l(e)$. Therefore, we can use these dependencies to *globally* eliminate s of the sections $v_1..v_N$ over E. One we did that, we are left with a subset $v_{i_1}...v_{i_r}$ which descends to a basis $s_1 := p_*(v_{i_1})...s_r := p_*(v_{i_r})$ of the $\mathbb{C}$-vector space $H^0(V'_E)$ and the matrix $S(e)$ can be reduced to the $(r+s+1) \times (r+s)$ matrix :

$$S_0(e) := \left(\begin{array}{cccccc} v_{i_1}^{(1)}(e) & ... & v_{i_r}^{(1)}(e) & , E_1^{(1)}(e) & ... & E_1^{(s)}(e) \\ ... & ... & ... & , ... & ... & .... \\ v_{i_1}^{(m)}(e) & ... & v_{i_r}^{(m)}(e) & , E_m^{(1)}(e) & ... & E_m^{(s)}(e) \end{array} \right) \tag{43}$$

such that $\mathrm{rank} S(e) = \mathrm{rank} S_0(e)$,$\forall e \in E$. Finding such dependencies can easily be achieved by looking for constant solutions $\alpha_1...\alpha_N$ of the equation :

$$\det \left(\begin{array}{cccc} \sum_{l=1..N} \alpha_l v_l^{(1)}(e) \ , & E_1^{(1)}(e) & ... & E_1^{(s)}(e) \\ ... & ... & ... & ... \\ \sum_{l=1..N} \alpha_l v_l^{(m)}(e) \ , & E_m^{(1)}(e) & ... & E_m^{(s)}(e) \end{array} \right) = 0 \tag{44}$$

Then one finds $v_{i_1}...v_{i_r}$ in the obvious way.

Now let $\Delta_a(e) \in H^0(O_E(r - q_0 - \tilde{q}_a))$ $(a = 1..m)$ be the $(r+s) \times (r+s)$ -minors of $S_0(e)$ obtained by deleting the line (a) of $S_0(e)$). We see that $\mathrm{rank} S_0(e) = r+s-1$ iff $\Delta_a(e) = 0$, $\forall a = 1..m$. We claim that $\Delta_a(e) = G_a(e) \otimes \phi(e)$ where $\phi \in H^0(O_E(r))$. (The easiest way to see this is to note that for a fixed $e \in E$, we can choose local bases of the fibres $O'_{E,e}(\tilde{q}_a)$ and thus reduce to the case of complex numbers. Then we can treat the columns $x_1...x_{r+s}$ of $S_0(e)$ as vectors in the vector space $\mathbb{C}^m = \mathbb{C}^{r+s+1}$. Now we can consider the natural nondegenerate *bilinear* form on $\mathbb{C}^m$ given by $(x, y) = \sum_{a=1..m} x_a y_a$ and the natural mixed product given by the m-linear form $(x^1...x^m) := \epsilon_{i_1...i_m} x^1_{i_1}...x^m_{i_m}$. Associated to these objects there is a natural $(m-1)$ -linear vector product (denoted by $\times$) on $m-1$ ordered sets of vectors. Now consider the vector $\Delta := x_1 \times ... \times x_{r+s}$, whose components coincide with $\Delta_i(e)$. Because all x_i $(i = 1..r+s)$ lies in $\ker(g_e)$,one sees immediately that all x_i are $(.,.)$ -orthogonal on the vector $G := (G_1(e)...G_m(e))$. Therefore, their vector product Δ is colinear to G, and since G is nonzero (as the map g is transverse), we can write $\Delta_a(e) = G_a(e) \otimes \phi(e)$.

Because Δ_a and G_a are regular and because G_a are never all zero on E, this implies that ϕ is a regular section of a line bundle on E, whose degree is then fixed to be r by the degrees of Δ_a and G_a.) As G_a never vanish simultaneously on E, we see that rank$S(e) < r + s$ iff $\phi(e) = 0$. We deduce that the *set* $\{q_1...q_r\}$ coincides with the support of the zero divisor (ϕ).

In conclusion, we can obtain the points $q_1...q_r$ — and thus the spectral cover — provided that we can obtain a basis of $H^0\left[\ker\{\oplus_{a=1..m}H^0(O'_E(\tilde{q}_a)) \stackrel{g_*}{\to} H^0(O'_E(-q_0))\}\right]$. Since one knows the concrete form of g_*, this is straightforward to achieve once one is able to obtain bases for each of the $\mathbb{C}$-vector spaces $H^0(O'_E(\tilde{q}_a)$, respectively $H^0(O'_E(-q_0))$. Such bases can of course be obtained by descending from the ambient space $\mathbb{P}_f$ of E. In favorable cases — as when $\mathbb{P}_f = \mathbb{WP}_{2,3,1}$ — one obtains that the restriction maps $H^0(L) \xrightarrow{p} H^0(L_E)$ for the sheaves involved are surjective. In this case, one has a commutative diagram with *surjective* vertical maps:

$$\begin{array}{ccc} \oplus_{a=1..m}H^0(O'_E(q_a)) & \stackrel{g_*}{\to} & H^0(O'_E(-q_0)) \\ p\downarrow & \bigcirc & \downarrow p' \\ \oplus_{a=1..m}H^0(O'_E(q_a)) & \stackrel{g_*}{\to} & H^0(O'_E(-q_0)) \end{array}$$

which implies that :

$$\ker\{\oplus_{a=1..m}H^0(O'_E(q_a)) \stackrel{g}{\to} H^0(O'_E(-q_0))\} = p(\ker\{\oplus_{a=1..m}H^0(O'_{\mathbb{P}_f}(q_a)) \stackrel{p' \circ g_*}{\to} H^0(O'_E(-q_0))\}) \quad (45)$$

In the examples we consider in section 5, if $\mathbb{P}$ has homogeneous coordinates $x_1..x_n$, then the base B_H is described by some subset of the homogeneous coordinates $(x_1..x_k)$, say, so fixing b is equivalent to fixing these coordinates to some values $b_1..b_k$ (up to the $(\mathbb{C}^*)^\rho$ action). Then $\mathbb{P}_f$ is a toric variety whose homogeneous coordinates can be naturally identified with the restricions of $x_{k+1}...x_n$ to $\mathbb{P}_f$. Thus, the restriction of a line bundle $O(s_1...s_n)$ over $\mathbb{P}$ to a fixed $\mathbb{P}_f$ fibre is nothing but the line bundle $O(s_{k+1}...s_k)$ over $\mathbb{P}_f$. This simple observation can be used to identify the line bundles appearing in (32). Having made these remarks, our task is finished if we are able to explicitly compute a $\mathbb{C}$-basis of the space of sections of a given line bundle $O(s_{k+1}...s_n)$ over $\mathbb{P}_f$. As the description of $\mathbb{P}_f$ as a holomorphic quotient is clear in our case, this task is completely solved by recaling that there is a vector space isomorphism between $H^0(\mathbb{P}_f, O(s_{k+1}...s_n))$ and the subspace of polynomials in $\mathbb{C}[x_{k+1}..x_n]$ which are $(\mathbb{C}^*)^{n-k}$-multihomogeneous of multihomogeneity degree $(s_{k+1}...s_n)$. This reduces the computation of $H^0(\mathbb{P}_f, O(s_{k+1}...s_n))$ to finding all monomials of approprate multidegree, which is an algorithmic problem easily tackled by a computer. In section 4 we carry this out for a number of examples.

4.2 Summary of the procedure

Let E be a smooth elliptic fiber of Z. Assuming that V_E is fully split and semistable, the points where the spectral cover of V intersects E form a multiset $q_1...q_r$. To determine this multiset consider the twisted bundle V'_E defined by the monad :

$$0 \longrightarrow \oplus_{j=1}^{s} O_E(p) \xrightarrow{f} \oplus_{a=1}^{m} O'_E(\tilde{q}_a) \xrightarrow{g} O'_E(-q_0) \longrightarrow 0 \tag{46}$$

Then compute a basis of sections $(v_1...v_N)$ $(N = r+s)$ of the $\mathbb{C}$-vector space:

$$\ker\{\oplus_{a=1..m} H^0(O'_E(\tilde{q}_a)) \xrightarrow{g_*} H^0(O'_E(-q_0))\} \tag{47}$$

and form the matrix of sections S given in (39). The multiset $q_1...q_r$ can be identified as follows. The underlying set $\{q_j|j = 1..r\}$ is the set $Z(S) := \{t \in E|\mathrm{rank}S(t) < r+s\}$. The multiplicity of any $t \in S$ in the multiset $q_1...q_r$ is given by $m_t := r + s - \mathrm{rank}S(t)$. That is, the desired multiset is given by $\{m_t t|t \in S\}$.

Moreover, if one wishes to compute a basis $s_1...s_r$ of sections of V'_E, then one can solve the equation (44) for *constants* $\alpha_1...\alpha_N$. This will give a set of s linear relations with constant coefficients among $v_1...v_N$ (modulo the E's). By eliminating s of the sections $v_1...v_N$ via these relations, one obtains a subset of sections $v_{i_1}...v_{i_r}$ of $\oplus_{a=1..m} H^0(O'_E(\tilde{q}_a))$ whose natural descendants $s_1 := p_*(v_{i_1})...s_r := p_*(v_{i_r})$ form a basis of the vector space $H^0(V'_E)$. If one has already determined such a basis, then one can form the matrix S_0 in equation (43).

Now find all $(r+s)\times(r+s)$ - minors of S. For each line $a \in \{1..m\}$ of S, there is exactly one such minor Δ_a, obtained by deleting the line (a). This gives a set of $m = r+s+1$ such minors, labeled Δ_a, $a = 1..m$. Then there exists a regular section ϕ of $O_E(r)$ such that $\Delta_a = G_a \otimes \phi$ for all $a = 1..m$. The underlying set $\{q_i|i = 1..r\}$ of the multiset $q_1...q_r$ coincides with the support of the zero divisor of the section ϕ. If the support of this divisor is formed of less than r points,then one can deduce the multiplicity of each point $q \in \mathrm{supp}(\phi)$ in the multiset $q_1...q_r$ by computing $\mathrm{rank}S(q)$ at that point.

5 Examples

5.1 Models on a $K3$ surface realized as a hypersurface in a resolution of $\mathbb{WP}^{6,4,1,1}$

5.1.1 A model on the tangent bundle

We first present a model on a $K3$ surface Z realized as a degree $(12,6)$ hypersurface in a resolution $\mathbb{P}$ of $\mathbb{WP}^{6,4,1,1}$. The bundle will simply be its tangent bundle. In the language of $(0,2)$ linear sigma models we have 6 right-moving bosonic fields $(x_1,..x_5,p)$ and 6 left moving fermionic fields $(\lambda_1,..,\lambda_5,\gamma)$ which are charged as follows under the $U(1)^2$ worldsheet gauge group :

x_1	x_2	x_3	x_4	x_5	p
6	4	1	1	0	-12
3	2	0	0	1	-6

λ_1	λ_2	λ_3	λ_4	λ_5	γ
6	4	1	1	0	-12
3	2	0	0	1	-6

Here each line of a table gives the charge under the corresponding $U(1)$ group.

We will choose the following F's and G as a definition of the complex structure of the $K3$ surface and part of the definition of the bundle:

$$\begin{aligned}
G &= x_1^2 + x_2^3 + (x_3^{12} + x_4^{12})x_5^6 \\
F_1 &= 2x_1 \\
F_2 &= 3x_2^2 \\
F_3 &= 12x_3^{11}x_5^6 \\
F_4 &= 12x_4^{11}x_5^6 \\
F_5 &= 6(x_3^{12} + x_4^{12})x_5^5
\end{aligned} \tag{48}$$

We will also need to specify two sets of E's, which are given as follows:

E_1	E_2	E_3	E_4	E_5	E_{01}
0	0	x_3	x_4	$-2x_5$	0
$3x_1$	$2x_2$	0	0	x_5	$-6x_6$

One easily checks that each set of E's satisfies the condition $\sum E_iF_i + E_{01}G = 0$. The transversality conditions are also obeyed.

The fibration of Z is obtained by viewing x_3, x_4 as the base parameters. Then the equation $G = 0$ gives the elliptic fibre as a sextic curve inside of the $WP_{3,2,1}$ fibre (spanned by the homogeneous coordinates x_1, x_2, x_5) of the ambient toric space $\mathbb{P}$. The base of the fibration is a $\mathbb{P}^1$. It is easy to see that the toric divisor (x_5) cuts Z along a section $\sigma = Z \cap (x_5)$ of the elliptic fibration. Therefore, an elliptic fibre E meets σ in the point $p = E \cap (x_5)$ and we have $O_E(p) = O_E(x_5)$, where $O_E(x_5) := O(x_5)|_E = O(0,1)|_E := O_E(0,1)$.

Given the above data, it is now easy to compute the spectral cover via the technique introduced in section 4. In the notation of (46), the twisted bundle $V'_E = V_E \otimes O_E(p)$ is given by the complex :

$$0 \longrightarrow \oplus_{j=1}^2 O_E(p) \xrightarrow{f} O_E(6,4) \oplus O_E(4,3) \oplus O_E(1,1) \oplus O_E(1,1) \oplus O_E(0,2) \xrightarrow{g} O_E(12,6) \longrightarrow 0 \tag{49}$$

where we twisted the defining monad of V_E by $O_E(x_5) = O_E(0,1)$. The first step of the procedure involves finding the kernel of g_*. This is spanned by the columns of the following matrix of sections [17] :

[17]Here (and in all following examples) we decomposed each section of $\ker g_* \subset \oplus_{a=1..m} O'_E(\tilde{q}_a)$ in its components along $O'_E(\tilde{q}_a)$. The lines of the matrix are indexed by $a = 1..m$ in this order, while each column gives the components of a section of $\ker g_*$. The set of these columns forms a basis of $\ker g_*$.

$$K := \begin{bmatrix} \frac{3\,x_1 x_5}{2} & -\frac{3\,{x_2}^2}{2} & 0 & 0 \\ x_2 x_5 & x_1 & 0 & 0 \\ \frac{\left({x_3}^{12}+{x_4}^{12}\right)x_5}{4\,{x_3}^{11}} & 0 & -\frac{{x_4}^{11} x_5}{{x_3}^{11}} & \frac{\left(-{x_3}^{12}-{x_4}^{12}\right)x_5}{2\,{x_3}^{11}} \\ 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & {x_5}^2 \end{bmatrix} \tag{50}$$

It is easy to see that columns 1 and 4 of the above matrix are linearly dependent modulo constants multiplying columns 2,3 and the twisted E's (i.e. columns formed by multiplying the E's by x_5), hence we may remove them in constructing the matrix S_0 as explained above . The matrix S_0 is thus constructed by appending the E's to columns 2 and 3 of K :

$$S_0 := \begin{bmatrix} -\frac{3\,{x_2}^2}{2} & 0 & 0 & 3\,x_1 \\ x_1 & 0 & 0 & 2\,x_2 \\ 0 & -\frac{{x_4}^{11} x_5}{{x_3}^{11}} & x_3 & 0 \\ 0 & x_5 & x_4 & 0 \\ 0 & 0 & -2\,x_5 & x_5 \end{bmatrix} \tag{51}$$

The spectral cover may then be found by dividing the i-th 4x4 minor of S_0 by F_i, leading to the equation $\left[(x_3^{12}+x_4^{12})^2\right] x_5^2 = 0$ (where a denominator given by x_3^{11} was discarded). Further analysis by the methods of [6] shows that V_E is actually F_2, an irreducible stable rank 2 bundle[18]. This agrees with the result of Corollary 6.7 of [5]. Although this is not fully split, it will be shown in [6] that our method for computing the cover still goes through for such a case.

5.1.2 A generalization of the previous model

Let us consider a generalization of the previous model. Specifically, we still consider the tangent bundle of a K3 surface realized in the same ambient space, but now we let the surface be given by a more general degree $(12,6)$ polynomial. The charges of all fields are as before, but now we take :

$$G = x_1^2 + x_2^3 + x_2 x_5^4 f + g x_5^6 \tag{52}$$

$$F_1 = \partial_1 G = 2x_1 \tag{53}$$

$$F_2 = \partial_2 G = 3x_2^2 + x_5^4 f \tag{54}$$

$$F_3 = \partial_3 G = x_2 x_5^4 \partial_3 f + \partial_3 g x_5^6 \tag{55}$$

$$F_4 = \partial_4 G = x_2 x_5^4 \partial_4 f + \partial_4 g x_5^6 \tag{56}$$

[18] This is denoted by I_2 in [5], but we use Atyiah's notation as in the appendix

$$F_5 = \partial_5 G = 4 f x_2 x_5^3 + 6 g x_5^5 \tag{57}$$

where f and g are homogeneous polynomials in (x_3, x_4) of multidegrees $(8,0)$, respectively $(12,0)$. The elliptic fibration degenerates at the zeroes of the discriminant $\Delta = 4f^3 + 27g^2$ of the elliptic fibration. For generic choices of f, g, the resulting $K3$ surface will be smooth. In particular, transversality of $F_1..F_5$ on Z is assured in such a case. Since we desire the tangent bundle of Z, the E's are the same as before. Again each set of E's satisfies $\sum E_i F_i + E_{01} G = 0$ and the fibration structure is similar to that of the previous example. The twisted bundle is given by the same complex.

The kernel of g_* is spanned by the columns of the following matrix of sections :

$$K := \begin{bmatrix} 0 & 0 & x_5 x_1 & -\frac{1}{2}(f x_5^4 + 3x_2^2) \\ -12 f x_5^3 \frac{\mathcal{W}}{\partial_4 \Delta} & -6 f x_5^3 x_3 \frac{\mathcal{W}}{\partial_4 \Delta} & \frac{2}{3} x_5 x_2 + 2 f x_5^3 x_3 \frac{\mathcal{W}}{\partial_4 \Delta} & x_1 \\ x_5 & 0 & 0 & 0 \\ -x_5 \frac{\partial_3 \Delta}{\partial_4 \Delta} & -12 x_5 \frac{\Delta}{\partial_4 \Delta} & 4 x_5 \frac{\Delta}{\partial_4 \Delta} & 0 \\ 9 x_2 \frac{\mathcal{W}}{\partial_4 \Delta} & x_5^2 + \frac{9}{2} x_2 x_3 \frac{\mathcal{W}}{\partial_4 \Delta} & -\frac{3}{2} x_2 x_3 \frac{\mathcal{W}}{\partial_4 \Delta} & 0 \end{bmatrix} \tag{58}$$

where, to simplify formulae, we introduced the Wronskian $\mathcal{W} = \partial_3 g \partial_4 f - \partial_3 f \partial_4 g$. It is easy to see that the second and the third columns are linear combinations of the first column and the twisted E's. The matrix S_0 is thus obtained by appending the E's to columns one and four of K :

$$S = \begin{bmatrix} 0 & 3x_1 & 0 & -\frac{1}{2}(f x_5^4 + 3x_2^2) \\ -12 f x_5^3 \frac{\mathcal{W}}{\partial_4 \Delta} & 2x_2 & 0 & x_1 \\ x_5 & 0 & x_3 & 0 \\ -x_5 \frac{\partial_3 \Delta}{\partial_4 \Delta} & 0 & x_4 & 0 \\ 9 x_2 \frac{\mathcal{W}}{\partial_4 \Delta} & x_5 & -2x_5 & 0 \end{bmatrix} \tag{59}$$

The spectral cover is again found by dividing the i-th 4x4 minor of S by F_i, leading to the equation $x_5^2 \Delta = 0$ (where a denominator given by $\partial_4 \Delta$ was discarded). This is again degenerate.

Note that the coefficient multiplying x_5^2 has zeroes along the discriminant locus of Z. This leads to the conclusion that the spectral cover must include the 24 degenerate elliptic fibers of Z. In particular, this is consistent with the formula (10) for the class of the spectral cover (if one valiantly generalizes it to this degenerate case).

According to the interpretation proposed in [10], such a cover would correspond to heterotic small instantons, but in our case the bundle V is perfectly smooth. If the above statements are correct, it appears that the spectral bundle (or rather the spectral sheaf) must conspire with the degenerate elliptic fibres such that a (generalization of) the pushforward construction yields a well-defined *bundle* (i.e. a locally free sheaf). A more detailed discussion of this will be attempted elsewhere.

5.1.3 A simple deformation of the tangent bundle

As in the previous subsection, we consider a $K3$ surface realized as a degree $(12,6)$ hypersurface in a resolution of the weighted projective space $\mathbb{WP}^{6,4,1,1}$. Let us require that the deformed bundle has the same set of E's as the tangent bundle. This is essentially the simplest possible deformation. Condition (17) severely constrains the allowed F_j. Namely, one must have

$$\begin{aligned}
F_1 &= 2x_1 - \frac{2}{3}hx_2x_5 - kx_5^3 \\
F_2 &= 3x_2^2 + x_5^4(f-a) + hx_1x_5 - mx_2x_5^2 \\
F_3 &= x_2x_5^4p_3 + q_3x_5^6 + r_3x_1x_5^3 + s_3x_2^2x_5^2 \\
F_4 &= x_2x_5^4p_4 + q_4x_5^6 + r_4x_1x_5^3 + s_4x_2^2x_5^2 \\
F_5 &= (4f+2a)x_2x_5^3 + 6gx_5^5 + 3kx_1x_5^2 + 2mx_2^2x_5 \ ,
\end{aligned} \tag{60}$$

where p_3, p_4, q_3, q_4, h and a are polynomials on the base of the appropriate degrees. (17) also implies two relations between them, namely

$$x_3p_3+x_4p_4 = 8f+4a \ , \quad x_3q_3+x_4q_4 = 12g \ , \quad x_3r_3+x_4r_4 = 6k \ , \quad x_3s_3+x_4s_4 = 4m \tag{61}$$

Going through computations similar to the above, one obtains that the spectral curve is given by :

$$\begin{aligned}
\Sigma \ : \ x_5^2 \ &(576\,f^3 + 3888\,g^2 - 32\,h^2mf^2 + 576\,hkf^2 + 144\,m^2f^2 - 288\,h^2fg - 24\,h^3mkf \\
&-144\,am^2f + 144\,hkaf + 120\,hkm^2f - 432\,a^2f + 108\,h^2k^2f + 16\,mh^2af - 72\,h^3kg \\
&-144\,ah^2g - 96\,h^2m^2g + 16\,h^4mg + 1296\,amg - 216\,hmkg + 1944\,k^2g + 144\,m^3g + \\
&16\,mh^2a^2 + 36\,m^3k^2 + 324\,amk^2 - 54\,hk^3m - 48\,ahkm^2 - 144\,a^3 + 243\,k^4 - \\
&72\,hka^2) = 0
\end{aligned} \tag{62}$$

(where a denominator dependent only on the base coordinates was discarded). This is again a degenerate cover.

5.2 Models over a Calabi-Yau threefold realized as a hypersurface in a resolution of $\mathbb{WP}^{9,6,1,1,1}$

5.2.1 A model on the tangent bundle

We first present a model on a Calabi-Yau threefold Z realized as a degree $(18,6)$ hypersurface in a resolution $\mathbb{P}$ of $\mathbb{WP}^{9,6,1,1,1}$. The bundle is the tangent bundle of Z. The $(0,2)$ linear sigma model has 7 right-moving bosonic fields $(x_1,..x_6,p)$ and 7 left moving fermionic fields $(\lambda_1,..,\lambda_6,\gamma)$ with the $U(1)^2$ charges:

x_1	x_2	x_3	x_4	x_5	x_6	p
9	6	1	1	1	0	-18
3	2	0	0	0	1	-6

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	γ
9	6	1	1	1	0	-18
3	2	0	0	0	1	-6

We have :

$$
\begin{aligned}
G &= x_1^2 + x_2^3 + x_2 x_6^4 f + g x_6^6 \\
F_1 &= 2x_1 \\
F_2 &= 3x_2^2 + x_6^4 f \\
F_3 &= \partial_3 f x_2 x_6^4 + \partial_3 g x_6^6 \\
F_4 &= \partial_4 f x_2 x_6^4 + \partial_4 g x_6^6 \\
F_5 &= \partial_5 f x_2 x_6^4 + \partial_5 g x_6^6 \\
F_6 &= 4 f x_2 x_6^3 + 6 g x_6^5 \ ,
\end{aligned}
\tag{63}
$$

where f and g are homogeneous polynomials in (x_3, x_4, x_5) of degrees $(12, 0, 0)$ and $(18, 0, 0)$. The fermionic symmetries are specified by :

E_1	E_2	E_3	E_4	E_5	E_6	E_{01}
0	0	x_3	x_4	x_5	$-3x_6$	0
$3x_1$	$2x_2$	0	0	0	x_6	-6

(64)

The base of the elliptic fibration is a $\mathbb{P}^2$ parameterized by the homogeneous coordinates x_3, x_4, x_5. The elliptic fibre is a sextic in $WP_{3,2,1}$. As before, $\Delta = 4f^3 + 27g^2$ denotes the discriminant of the elliptic fibration. The equation of the cover is $x_6^3 \Delta / \partial_4 \Delta = 0$ (discarding a denominator given by $\partial_4 \Delta$), which is degenerate as well.

5.2.2 Simple deformations of the tangent bundle

As in the K3 case, one can consider the simple deformation of the tangent bundle which preserves the E's. This is given by :

$$
\begin{aligned}
G &= x_1^2 + x_2^3 + x_2 x_6^4 f + g x_6^6 \\
F_1 &= 2x_1 - \frac{2}{3} h x_2 x_6 \\
F_2 &= 3x_2^2 + x_6^4 (f - a) + h x_1 x_6 \\
F_3 &= p_3 x_2 x_6^4 + q_3 x_6^6 \\
F_4 &= p_4 x_2 x_6^4 + q_4 x_6^6 \\
F_5 &= p_5 x_2 x_6^4 + q_6 x_6^6 \\
F_6 &= (4f + 2a) x_2 x_6^3 + 6 g x_6^5 \ ,
\end{aligned}
$$

The functions $p_i,\ q_i$ satisfy the constraints

$$x_3p_3 + x_4p_4 + x_5p_5 = 12f + 6a \quad , \qquad x_3q_3 + x_4q_4 + x_5q_5 = 18g \tag{65}$$

The spectral cover is again degenerate and is given by:

$$\Sigma :\ x_5^3(4f^3 + 27g^2 - 3a^2f - a^3 - h^2ag - 2h^2fg) = 0 \tag{66}$$

(where a denominator given by $p_6q_4 - p_4q_6$ was discarded).

Under the same assumptions as before, the spectral surface would consists of 3 copies of the section of Z and a vertical component, which projects on the base $\mathbb{P}^2$ as a curve γ of degree 36, given by the zero locus of the polynomial $4f^3 + 27g^2 - 3a^2f - a^3 - h^2ag - 2h^2fg$.

So far we've obtained rather 'boring' covers in the sense that they are always merely multiple covers of the base of the heterotic manifold (modulo some extra elliptic fibers). One may wonder if a monad presentation can be found for bundles with a nontrivial spectral cover. We will see that this is indeed the case in the next paragraph; the model we will display is however *not* perturbatively well-defined as a heterotic string theory, due to problems at the LG point.

5.2.3 A perturbatively unsound model on the same hypersurface

We will again consider a degree 18 hypersurface in $\mathbb{WP}^{9,6,1,1,1}$. In the previous example there are stringent constraints on the allowed deformations of the F's because of the requirement of having transverse E's. Here we will relax that requirement and see where it leads us. In doing so, we obtain a model whose spectral cover is nondegenerate but whose LG phase is not well behaved. TIt is commonly believed that in such a case stability of V over Z fails.

Consider a general set of F's and G:

$$\begin{aligned}
F_1 &= 3\,x_2^2 + n_2\,x_2\,x_6^2 + n_4\,x_6^4, \\
F_2 &= 2\,x_1 + p_1\,x_2\,x_6 + p_3\,x_6^3, \\
F_3 &= f_4\,x_2\,x_6^4 + f_6\,x_6^6, \\
F_4 &= g_4\,x_2\,x_6^4 + g_6\,x_6^6, \\
F_5 &= h_4\,x_2\,x_6^4 + h_6\,x_6^6, \\
F_6 &= k_3\,x\,x_6^3 + k_5\,x_6^5, \\
G &= x_1^2 + x_2^3 + f\,x_2\,x_6^4 + g\,x_6^6.
\end{aligned} \tag{67}$$

The polynomials f_4,f_6, g_4, g_6, h_4, h_6, n_2, n_4, p_1, p_3, k_3, k_5, f and g in (67) are taken to have the appropriate degrees and to depend only on the coordinates on the base B_H. This gives a transverse bundle of rank 5 over a smooth threefold, but there are no longer suitable E's in general.

The spectral cover is given by:

$$\Sigma : x_6^3(a_0x_6^2 + a_2x_2) = 0,$$

where

$$a_2 = (-36\,g\,k_3 + 36\,f\,k_5 - 12\,k_5\,n_4) + 4\,k_5\,n_2^2 - 4\,k_3\,n_2\,n_4 - 3\,k_5\,n_2\,p_1^2 + 3\,k_3\,n_4\,p_1^2 + 18\,k_5\,p_1\,p_3 - 9\,k_3\,p_3^2,$$

$$a_0 = (36\,g\,k_5 + 12\,f\,k_3\,n_4 - 4\,k_3\,n_4^2) - 12\,g\,k_3\,n_2 + 4\,k_5\,n_2\,n_4 - 3\,k_5\,n_4\,p_1^2 + 6\,k_3\,n_4\,p_1\,p_3 + 9\,k_5\,p_3^2 - 3\,k_3\,n_2\,p_3^2.$$

Note that this spectral cover is independent of $f_{4,6}$, $g_{4,6}$ and $h_{4,6}$. For the tangent bundle $n_4 \sim f$, $k_3 \sim f$, $k_5 \sim g$ and all other coefficients (except $f_{4,6}$, $g_{4,6}$, $h_{4,6}$) are equal to zero. In that case, the term in parenthesis in a_2 vanishes and the term in parenthesis in a_0 coincide with the discriminant of the elliptic fibration.

We see that relaxing the condition of having fermionic symmetries leads to a nondegenerate spectral cover, at the price of having a poorly defined $(0,2)$ linear sigma model. The situation is very similar to what we found in the case of 6-dimensional compactifications.

5.2.4 General deformations of the E's

In order to convert the perturbatively unsound model of the previous section into a well defined one it is necessary to impose fermionic symmetries. Here we will assume that the fermionic symmetries are the deformations of (64), namely

E_1	E_2	E_3	E_4	E_5	E_6	E_{01}
$6\,x_1 + r\,x_2 \ + q\,x_6^3$	$4\,x_2 + p\,x_6^2$	u	v	w	0	-12
$3\,x_1 + c\,x_2\,x_6 + b\,x_6^3$	$2\,x_2 + a\,x_6^2$	0	0	0	x_6	-6

(68)

where p, r, q, a, b and c are appropriate polynomials in the coordinates of the base.

One can find the general solution of the constraint (17) by writing all the polynomials in (67) and (68) as certain expressions in terms of f, g, k_3, a, b, c. The spectral cover is given by the expression in the previous subsection, with a_0 and a_2 evaluated for these constrained polynomials. Making the relevant substitution one finds that a_2 vanishes identically (thus turning Σ into a degenerate cover) and that a_0 is given by the rather formidable expression :

$$\begin{aligned} a_0 \ = \ & 243\,a^6 - 432\,a^3\,b^2 + 192\,b^4 + 432\,a^4\,b\,c - 384\,a\,b^3\,c - 108\,a^5\,c^2 + 288\,a^2\,b^2\,c^2 - \\ & 96\,a^3\,b\,c^3 + 12\,a^4\,c^4 + 1944\,a^4\,f - 1728\,a\,b^2\,f + 1728\,a^2\,b\,c\,f - 432\,a^3\,c^2\,f + \\ & 3888\,a^2\,f^2 - 3888\,a^3\,g + 3456\,b^2\,g - 3456\,a\,b\,c\,g + 864\,a^2\,c^2\,g - 15552\,a\,f\,g + \\ & 15552\,g^2 - 486\,a^4\,k_3 + 432\,a\,b^2\,k_3 - 432\,a^2\,b\,c\,k_3 + 144\,a^3\,c^2\,k_3 - 32\,b^2\,c^2\,k_3 + \\ & 32\,a\,b\,c^3\,k_3 - 8\,a^2\,c^4\,k_3 - 1944\,a^2\,f\,k_3 + 144\,a\,c^2\,f\,k_3 + 3888\,a\,g\,k_3 - 288\,c^2\,g\,k_3 + \\ & 324\,a^2\,k_3^2 + 96\,b\,c\,k_3^2 - 48\,a\,c^2\,k_3^2 + 432\,f\,k_3^2 - 72\,k_3^3 \end{aligned} \tag{69}$$

This shows rather explicitly how imposition of a sufficient number of fermionic symmetries in the underlying linear sigma model forces the cover to become degenerate. The spectral cover is still given by 3 copies of the zero set of the elliptic fibration and a posible set of vertical components.

5.3 A model over a Calabi-Yau threefold realized as a hypersurface in a resolution of $\mathbb{WP}^{12,8,2,1,1}$

All of the previous models were deformations of the tangent bundle. Since the tangent bundle will generally lead to a degenerate spectral cover, one should attempt to construct models on bundles which are not deformations of TZ. The model we consider here is of this type, being perturbatively well-defined *without* any fermionic symmetries. This is a radical departure from the tangent bundle, but, as we will see, the resulting spectral cover is still degenerate.

The model is constructed over a degree $(24, 12, 6)$ hypersurface in a resolution of $\mathbb{WP}^{12,8,2,1,1}$. The relevant fields are 8 bosonic $(x_1, .., x_7, p)$ and 7 fermionic $(\lambda_1, .., \lambda_6, \gamma)$ under 3 $U(1)$ charges, given by:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	p
12	8	2	1	1	0	0	-24
6	4	1	0	0	1	0	-12
3	2	0	0	0	0	1	-6

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	γ
1	1	2	4	6	8	-22
1	1	0	2	3	4	-11
0	0	0	1	2	3	-6

A suitable choice of polynomials which allow for a transverse bundle over a smooth threefold is given by:

$$\begin{aligned}
G &= \lambda x_1^2 + \mu x_2^3 + \nu x_3^{12} x_7^6 + (\rho_1 x_4^{24} + \rho_2 x_5^{24}) x_6^{12} x_7^6 \\
F_1 &= (a_1 x_4^{21} x_6^{10} + b_1 x_5^{21} x_6^{10}) x_7^6 \\
F_2 &= (a_2 x_4^{21} x_6^{10} + b_2 x_5^{21} x_6^{10}) x_7^6 \\
F_3 &= a_3 x_3^{10} x_6 x_7^6 \\
F_4 &= a_4 x_3^3 x_1 x_7^2 \\
F_5 &= a_5 x_2^2 \\
F_6 &= a_6 x_3^7 x_7^3
\end{aligned}$$

The base is a Hirzebruch surface F_2 with homogeneous coordinates x_3, x_4, x_5, x_6 and the fibre is a sextic in $WP_{3,2,1}$. There are no E's required in this model, and the bundle has rank 5. Computation by the methods of section 4 gives the spectral cover $a_4 a_6^3 x_3^{24} x_7^5 = 0$.

This is again fully degenerate, consisting of 5 copies of the section of the elliptic fibration of Z plus extra vertical components. One can also consider various modifications of this model, by picking other solutions of the transversality constraints. This leads to various families of $(0, 2)$ linear sigma models. For some of these families the restriction of V to the generic elliptic fibre of Z fails to be semistable, while on others the restriction is semistable but leads to a degenerate spectral cover. While we have explored a few of these families, we lacked the computational power to undertake a complete study.

6 Discussion

What are we to make of these results? Both the physical and the mathematical answers appear to depend in part on the dimension of the compactification under study. From a mathematical perspective, the moduli space of (semi-) stable vector bundles over a $K3$ surface differs significantly from that which arises over a Calabi-Yau 3-fold. In the case of the latter, it is believed that the moduli space (for a fixed Calabi-Yau base) is *stratified* into various components. In some of these components, it is expected that the spectral cover is of the degenerate form that we have found. On such loci, the moduli of the bundle over the Calabi-Yau are associated with moduli of the spectral bundle over the spectral cover. This is the situation in which we have found ourselves regarding our $(0,2)$ linear sigma model constructions. On the other hand, the mathematical situation is quite different over a $K3$ surface. The moduli space has no analagous stratification and therefore the generic deformation away from a bundle with a degenerate spectral cover takes us to a bundle with a non-degenerate cover.

As far as our calculations go, this means that if we were to include a generic deformation in $K3$ examples (assuming that such a generic deformation can be torically represented) the spectral cover should split apart. But in our Calabi-Yau 3-fold examples, we would expect — as we have found — that the cover may remain degenerate.

A natural question, though, is why — in both the $K3$ and Calabi-Yau 3-fold cases — have we "landed" at a point with a degenerate spectral cover. We suspect the reason to lie in the fact that the linear sigma model is a physical representation of toric geometry. Toric constructions — although quite useful — are also quite special. The central element of linear $\mathbb{C}^*$ actions might very well account for the degeneracy we find.

From a physical point of view, it is also the case that compactifications on $K3$ surfaces and on Calabi-Yau 3-folds have quite different properties. This is most easily gleaned in the dual F-theory picture in which we are comparing F-theory with six and with four non-compact dimensions, respectively. In four dimensions — and not in six — it was shown by [21] that the F-theory vacuum will be unstable, if the Calabi-Yau 4-fold on which the compactification is done has nonzero Euler number. This is due to the appearance of a nonzero one-point function for the four-form field. This means that the model is inconsistent unless one includes a certain number, I_i, of three-branes to cancel up the tadpole. The worldvolume of such three-branes is taken to fill out the uncompactified spatial directions, in order to preserve the 4-dimensional Poincare invariance of the vacuum. As shown in [21], in the case of a *smooth* Weierstrass model X, one must have:

$$I_i = \chi(X)/24. \tag{70}$$

Under duality, these F-theory three-branes are conjectured to map to I_i heterotic five-branes (called 5-brane 'defects' in what follows) which are wrapped over certain elliptic curves of Z.

Therefore, it would appear that the dual heterotic models for F-theory on a fourfold X with nonzero Euler number necessarily include such 5-branes. In fact, this reasoning is a little too quick, as discussed for the first time in [4]. To understand why, one can start with the case of 6-dimensional compactifications (which is better understood) and use an 'adiabatic argument' to descend to 4 dimensions.

Recall that heterotic compactifications on a K3 surface admit degenerate limits in which 'small instantons' are formed. Such a limit can be understood as a degeneration of the heterotic bundle V to a torsion-free sheaf $\mathcal{E}$. In general, such a sheaf has a singularity locus of codimension at most two, so in the K3 case the sheaf fails to be locally free at a finite number of points. On the other hand, via the Uhlenbeck-Yau theorem, one can think about this in terms of $SU(r)$ anti-self-dual Yang-Mills connections, i.e. instantons on the underlying real 4-manifold. In this case, the degeneration above can be interpreted as the limit when certain instantons collapse to zero size. Such small instantons can be also be interpreted as 5-branes whose worldvolumes fill the 6 uncompactified directions and intersect the K3 surface at the singular points of $\mathcal{E}$.

It is natural to assume that a similar picture holds in 4-dimensional compactifications (over a Calabi-Yau 3-fold Z). More precisely, there will exist degenerations $\mathcal{E}$ of V (presumably to some type of torsion-free sheaves), whose singularity locus will be a subvariety of codimension 2 in Z, i.e. an algebraic curve Γ embedded in Z. There exist at least two interesting cases to consider, namely when Γ is the lift of a curve γ in the base B_H (i.e. the image of a section of $X|_\gamma$) and when Γ coincides with an elliptic fibre of Z. Such singularizations of V can again be interpreted as heterotic compactifications containing 5-branes wrapped over Γ.

To motivate these ideas, one can imagine 'fibering' the 6-dimensional picture above over a base of complex dimension 1. This will give a degeneration of V over a curve of the first type, whose associated 5-branes we will call 'instantonic' 5-branes. On the other hand, it is natural to identify the second type of 5-branes with the 5-brane defects discussed above. Both these types of 5-branes will bring a nontrivial contribution to the anomaly cancellation conditions on the heterotic side, which becomes:

$$c_2(V) + I_d + I_i = c_2(TZ) \tag{71}$$

The inverse process — desingularizing $\mathcal{E}$ to a stable bundle V — can be interpreted as a 5-brane 'dissolving' into a gauge instanton of finite size. If the above picture indeed holds for both types of 5-branes, such a process should be possible not only for instantonic 5-branes, but also for 5-brane defects. On the F-theory side, it should therefore be possible to pass from a situation in which 3-branes are present to a situation with no 3-branes. How is such a process realized ? As argued in [4], this should correspond to the F-theory 3-branes 'dissolving' into some coincident 7-branes, i.e. expanding to become finite size instantons of the worldvolume theory of those 7-branes. It was argued in [4] that a 3-brane can dissolve only on a multiple 7-brane, i.e. a state of at least two 7-branes sitting on top of each other. Such multiple 7-branes are partially wrapped over components of the discriminant locus of X at which the discriminant vanishes in order at least 2; therefore, they will correspond to loci above which X becomes singular. As the 3-branes expand to finite size instantons, part of the associated enhanced gauge group will be broken. In such a situation, therefore, the enhanced gauge symmetry on the F-theory side will be less than that predicted by Tate's algorithm. Similarly, the gauge symmetry will be reduced on the heterotic side since the dual transition of a 5-brane to a finite size instanton increases the holonomy of the heterotic bundle.

We see therefore that one can have F-theory compactifications with $\chi(X) \neq 0$ but with no 3-brane defects. In this situation the tadpole will be canceled by the presence of instantons on the 7-brane and the F-theory consitency condition is modified to :

$$c_2(S) = \chi(X)/24 \tag{72}$$

where S is the gauge bundle over the 7-branes.

Since the spectral cover of V is controlled by the complex structure of X, it follows that the moduli of S must map to the moduli of the spectral bundle W of V. W is a bundle over the spectral cover of V, which generalizes the line bundle $L \to \Sigma$ to the case when Σ has multiple components.

Now, if S is a nonsplit higher rank bundle, then the same shoud be true of W. But the only way that this can happen is if the spectral cover is degenerate so that the line bundles L_i from merging components can themselves merge into a higher rank bundle. Therefore, the heterotic dual of an F-theory model with $\chi(X) \neq 0$, with no 3-branes and with an irreducible instanton connection (more precisely, not fully reducible) on the multiple 7-branes is expected to have a degenerate spectral cover. This is precisely the situation we have found in all of the models we studied.

To understand why this happens for this class of models, remember that we carefully chose our $(0, 2)$ model data such that the heterotic compactifications be consistent as purely perturbative theories (and in particular satisfy $c_2(V) = c_2(TZ)$); it follows that in our case there are no 5-branes on the heterotic side. Therefore, F-theory on X cannot contain 3-branes. In all of our models Z was also chosen to have an 'obvious' elliptic fibration. In the spirit of [3], we therefore expect that their F-theory duals are realizable on fourfolds X presented as Weierstrass models in some toric varieties. Typically such an X admits a desingularization (obtained by varying its complex structure) to a smooth Weierstrass model X_{smooth} realized in the same way. Then a computation similar to that of of [21] will generally imply $\chi(X_{smooth}) > 0$. Thus the F-theory vacuum on X_{smooth} will be forced to contain 3-brane defects. As we singularize to obtain X, the number of these 3-branes can decrease only if some of them dissolve into some multiple 7-branes. Therefore, the only way that F-theory on X will have no 3-brane defects is that all of these 3-branes have dissolved into the 7-branes.

These rather abstract arguments lead us to believe that the reason for obtaining completely degenerate spectral covers is that the condition that Z have an obvious fibration essentially forces X_{smooth} to have a nonzero Euler number and that the vacuum configuration of the gauge theory on the corresponding multiple F-theory 7-branes is given by a gauge connection which is not fully reducible. We now proceed to partially test this hypothesis in our 4-dimensional examples. Since we do not have enough control of the gauge connections on the 7-branes, the best we will be able to do here is to test whether the Euler number of X_{smooth} is nonzero.

6.1 Determination of the dual fourfols

6.1.1 The F-theory duals of the models of section 5.2

Let us first concentrate on the examples of subsection 5.2. The rank of the bundle is 3 and the ten dimensional $E_8 \times E_8$ gauge group is broken to $E_8 \times E_6$.

Having an explicit expression for the spectral cover one can attempt to construct the F-theory dual along the lines of [3]. Although the work of [3] was concerned only with the case of generic spectral covers, let us ignore possible subtleties [19] associated to the degenerate character of the cover and see where direct application of [3] leads us.

In the examples of subsection 5.2 the heterotic base is a $\mathbb{P}^2$, with anticanonical line bundle $K_{B_H}^{-1} = O_{\mathbb{P}^2}(3)$. The toric data is the same for all of the models in that subsection. The heterotic 3-fold is a Weierstrass model embedded in the ambient space $\mathbb{P}_{2,3,1}(O_{B_H}(6) \oplus O_{B_H}(9) \oplus \mathcal{O}_{B_H})$. In the notation of section 2, the fibre variables are $x := x_2$, $y := x_1$, $z := x_6$, which are sections of $K_{B_H}^{-2} \otimes O_{\mathbb{P}_H}(2) \approx O_{\mathbb{P}_H}(6,2)$, $K_{B_H}^{-3} \otimes O_{\mathbb{P}_H}(3) \approx O_{\mathbb{P}_H}(9,3)$ and $O_{\mathbb{P}_H}(1) \approx O_{\mathbb{P}_H}(0,1)$ respectively. Homogeneity constrains the general spectral cover for this toric data to be of the form :

$$a_0 z^3 + a_2 x z + a_3 y = 0 \ , \tag{73}$$

where a_0, a_2 and a_3 are polynomials over B_H of degrees $d, d-6$ and $d-9$. Since our models are purely perturbative we have $c_2(V) = c_2(TZ)$. The precise value of d can be fixed by computing $\pi_{H,*}(c_2(TZ))$ along the lines of [21]. Following the procedure described there one obtains:

$$\pi_{H,*}(c_2(V)) = 12 c_1(TB_H) = 12 c_1(K_{B_H}^{-1}) \tag{74}$$

so that:

$$\mathcal{N} = K_{B_H}^{-12} \tag{75}$$

This relation is valid for a Weierstrass model over an arbitrary toric base B_H. In our case $K_{B_H}^{-1} = K_{\mathbb{P}^2}^{-1} = O_{\mathbb{P}^2}(3)$ and we obtain:

$$\mathcal{N} = O_{\mathbb{P}^2}(36) \tag{76}$$

which fixes $d = 36$. Note that, since $B_H \approx \mathbb{P}^2$ is not a Hirzebruch surface, we cannot apply (10) directly to our model.

Let us first describe the toric data of the F-theory dual. According to section 2, X is a Weierstrass model in the ambient space $\mathbb{P}_F := P_{2,3,1}(K_{B_F}^{-2} \oplus K_{B_F}^{-3} \oplus \mathcal{O}_{B_F})$ where B_F is the ruled 3-fold $\mathbb{P}(\mathcal{M} \oplus O_{B_H})$ with $\mathcal{M}$ a line bundle over B_H. Since $B_H \approx \mathbb{P}^2$, we have $\mathcal{M} = O_{\mathbb{P}^2}(n)$ for some $n \in \mathbb{Z}$. Then $K_{B_F}^{-1} = K_{B_H}^{-1} \otimes O_F(2) \otimes \mathcal{M} = O_{B_H}(n+3, 2)$. The fibre coordinates X, Y, Z are sections of the line bundles $K_{B_F}^{-2} \otimes O_{\mathbb{P}_F}(2) \approx O_{P_F}(2n+6, 4, 2)$, $K_{B_F}^{-3} \otimes O_{\mathbb{P}_F}(3) \approx O_{P_F}(3n+9, 9, 3)$ and $O_{\mathbb{P}_F}(1) \approx O_{P_F}(0,0,1)$. Denoting the homogeneous coordinates of the

[19] Subtleties occur due to the following reason: since we are considering a degenerate cover and since the spectral bundle need not be a direct sum of line bundles over the components, the argument employed in [3] to determine the dual fourfold may have to be modified.

projectivization $B_F = \mathbb{P}(\mathcal{M} \oplus O_{B_H})$ (essentially the homogeneous coordinates of the $\mathbb{P}^1$ fibre of $B_F \to B_H$) by u, v as in section 2, the toric data for the dual fourfold X is :

x_3	x_4	x_5	u	v	X	Y	Z
1	1	1	n	0	$2n+6$	$3n+9$	0
0	0	0	1	1	4	6	0
0	0	0	0	0	2	3	1

X is given by the zero locus of the section $Y^2 - X^3 - FXZ^4 - GZ^6 \in H^0(K_{B_F}^{-6} \otimes O_{\mathbb{P}_F}(6)) = H^0(O_{\mathbb{P}_F}(6n+18, 12, 6))$ where F, G are sections of $K_{B_F}^{-4} = O_{B_F}(4n+12, 8)$ and $K_{B_F}^{-6} = O_{B_F}(6n+18, 12)$. The expansions (3,4) give coefficients F_i, G_j which can be viewed as polynomials over B_H of degrees $(4-i)n+12$, respectively $(6-j)n+18$.

According to [10], to identify F_i, G_j we must first determine the unbroken gauge symmetry of the model, which in our case is $E_6 \times E_8$. Then X should have a section of E_6 singularities (which we can place at $u = 0$) and a section of E_8 singularities (placed at $v = 0$). Via Tate's algorithm [10], this forces all F_i, G_j to be zero except for G_4, G_5, G_6, G_7 and F_3, F_4. It also requires that $G_4 = \tilde{a}_3^2$ for some polynomial a_3 of degree $n+9$ over B_H. Therefore, the defining equation of X takes the form :

$$Y^2 = X^3 + (\tilde{a}_2 u^3 v^5 + F_4 u^4 v^4) X Z^4 + (\tilde{a}_1^2 u^4 v^8 + \tilde{a}_0 u^5 v^7 + G_6 u^6 v^6 + G_7 u^7 v^5) Z^6 \tag{77}$$

where we denoted F_3, G_5 by $\tilde{a}_2, \tilde{a}_0$. In this equation, F_4 and G_6 are controled by the complex structure of Z while $F_3 = \tilde{a}_2 \in H^0(O_{B_H}(n+12))$, $\tilde{a}_3 \in H^0(O_{B_H}(n+9))$ and $G_5 = \tilde{a}_0 \in H^0(O_{B_H}(n+18))$ are controled by the heterotic spectral cover Σ. There is a simple way to determine n directly if one notes that, on the heterotic side, there is no extra matter transforming in a representation of the first E_8. Then the results of [10, 22] imply that G_7 cannot have any zeroes, which fixes $n = 18$. Alternatively, this follows from the conjecture of [3], according to which a_j and $\tilde{a}_j$ should be identified. This fixes the value of n to 18 and partially determines the equation of X.

Now let us return to the perturbatively consistent examples of subsection 5.2. In those cases the spectral cover is degenerate ($a_2 = a_3 = 0$), so that $\tilde{a}_2 = \tilde{a}_3 = 0$. This gives X in the form:

$$Y^2 = X^3 + F_4 u^4 v^4 X Z^4 + (\tilde{a}_0 u^5 v^7 + G_6 u^6 v^6 + G_7 u^7 v^5) Z^6 \tag{78}$$

According to Tate's algorithm, this corresponds to a section of $E_8 \times E_8$ singularities of X. Naive application of the work of [10] would then lead to the conclusion that F-theory on X has an unbroken $E_8 \times E_8$ gauge symmetry. This contradicts the $E_8 \times E_6$ gauge symmetry that we find perturbatively on the heterotic side ! How can we understand this apparent discrepancy ? The only explanation we envisage is the possibility, discussed above, of having an F-theory compactification whose vacuum contains nontrivial instanton configurations of the effective gauge theory of its multiple 7-branes. Such configurations break the effective gauge group from $E_8 \times E_8$ to $E_8 \times E_6$.

To lend more credence to our hypothesis, let us compute the Euler number of the *generic* member X_{gen} of the family of Weierstrass models to which X belongs. Such a member is smooth and given by a Weierstrass equation of the form $Y^2 - X^3 - FXZ^4 - GZ^6 = 0$ in $\mathbb{P}_F$, the polynomials F, G being generic with the same multidegrees as above. The computation of $\chi(X_{gen}) = c_4(X_{gen})[X_{gen}]$ can be achieved by the methods of [21] and gives :

$$\chi(X_{gen}) = 12c_1(c_2 + 30c_1^2)[B] \tag{79}$$

where $c_j := c_j(TB_F)$. In our case TB_F is given by the exact sequence :

$$0 \longrightarrow O_{B_F}^{\oplus 2} \longrightarrow O_{B_F}(1,0)^{\oplus 3} \oplus O_{B_F}(0,1) \oplus O_{B_F}(1,n) \longrightarrow 0 \tag{80}$$

which allows us to express $\chi(X_{gen})$ in terms of triple intersections of the toric divisors of B_F. $H^2(B_F, \mathbb{Z})$ is generated by the classes $\Delta_x = [D_{x_3}] = [D_{x_4}] = [D_{x_5}]$, $\Delta_u := [D_u]$, $\Delta_v := [D_v]$ of the toric divisors $D_{x_i} = (x_i)$, $D_u := (u)$, $D_v := (v)$. We have $c_1(O_{B_F}(1,0)) = \Delta_x$, $c_1(O_{B_F}(n,1)) = \Delta_u$, $c_1(O_{B_F}(0,1)) = \Delta_v$. This gives the relation $\Delta_u = n\Delta_x + \Delta_v$. The triple intersections can be expressed torically as mixed volumes of polytopes and are given by $\Delta_x^3 = 0$, $\Delta_x^2\Delta_v = 1$, $\Delta_x\Delta_v^2 = -n$, $\Delta_v^3 = n^2$. Using this, an easy computation gives the number of 3-branes:

$$\chi(X_{gen})/24 = 30n^2 + 822 \tag{81}$$

which is positive, as expected. For $n = 18$, it is equal to 10542.

6.1.2 The F-theory dual of a purely perturbative model with $B_H = F_n$

The model of subsection 5.3 has $B_H = F_2$. It turns out that the generic member of the family of its F-theory dual also has a positive Euler number. In fact, we now show that this will be the case for any purely perturbative heterotic model with B_H a generalized Hirzebruch surface F_n. For such a model we have $\mathcal{N} = K_{F_n}^{-12} = O_{F_n}(12(n+2), 24)$. The dual is specified by the line bundle $\mathcal{M} = O_{F_n}(m,k)$. Then $B_F = \mathbb{P}(\mathcal{M} \oplus \mathcal{O}_{F_n}) = F_{n,m,k}$ (a generalized Hirzebruch). According to the conjecture of [3], the line bundle $\mathcal{M}$ will be of the form $\mathcal{M} = \mathcal{N} \otimes K_{F_n}^6$. Then $\mathcal{M} = K_{F_n}^{-6} = O_{F_n}(6(n+2), 12))$. Therefore $m = 6(n+2)$ and $k = 12$. The Euler number of a smooth Calabi-Yau Weierstrass model with base $F_{n,m,k}$ was computed in [4] and is given by:

$$\chi(X_{gen})/24 = 732 + 60km - 30k^2n \tag{82}$$

In our case, this gives:

$$\chi(X_{gen})/24 = 9372 \tag{83}$$

which is positive and independent of n.

7 Conclusions

We presented a method for computing the spectral cover associated to heterotic compactifications which are realizable via $(0,2)$ linear sigma models over an elliptically fibered Calabi-Yau manifold.

Contrary to naive expectations, we discovered that the most accessible models give rise to degenerate covers of a rather trivial form. This indicates that in such cases most of the information of the bundle is translated to instanton moduli on multiple F-theory 7-branes, as suggested in [4]. We thus found indirect evidence that a large number of purely perturbative heterotic compactifications are dual to F-theory vacua realized on singular limits of smooth Calabi-Yau fourfolds with nonzero Euler number, containing (in the limit) instanton configurations on multiple 7-branes but no 3-brane defects.

The F-theory encoding of the moduli space of such heterotic models is therefore not completely described by the geometry of the dual 4-fold; rather, a large piece of information is again specified by gauge connections. This opens the door to new 'sub-duality' conjectures, which seem to be intimately related to topological field theories and integrable systems. From a mathematical point of view, such models draw attention to the relevance of degenerate spectral covers and the associated bundles, an issue which as yet has not been intensively studied.

Due to computational difficulties, we avoided considering more general models (with a 'non-obvious' fibration structure). It is an interesting problem to generalize the methods presented here to such situations. It should be noted that elliptically fibered Calabi-Yau fourfolds of zero Euler number are in a certain sense rather rare [21, 23].

As for the story in 6 dimensions, it is important to understand better how the spectral bundle moduli are realized. A detailed analysis seems to require understanding the moduli space of a certain class of sheaves over the nonreduced scheme associated to such a cover. Fully describing the corresponding F-theory duals would require an analysis of certain 'twist' moduli of the effective field theory of partially wrapped multiple 7-branes. We hope to report on these and related problems in a future publication.

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A Some classical results on holomorphic vector bundles over nonsingular elliptic curves

Let E be a nonsingular elliptic curve, together with a distinguished point $p \in E$. Topological vector bundles V over E are classified by the pair $(\mathrm{rank}V, \mathrm{deg}V)$, where $\mathrm{deg}V := \mathrm{deg}(\mathrm{det}V)$. Here $\mathrm{det}V := \Lambda^{\mathrm{rank}V}(V)$ is the determinant bundle of V. In the holomorphic category (which is more 'rigid') the classification is finer.

Consider first the case of line bundles. Let $\mathrm{Pic}^d(E)$ be the set of line bundles of degree d over E. If L is a degree zero holomorphic line bundle over E, then there exists a unique point $q \in E$ such that $L \approx O_E(q-p)$. This gives a p-dependent isomorphism $\mathrm{Pic}^0(E) \approx E$. If $L \in \mathrm{Pic}^d(E)$, then $L \otimes O_E(-dp) \in \mathrm{Pic}^0(E)$ so we can write $L \approx O_E(q+(d-1)p)$ for a uniquely determined $q \in E$. Thus, we have p-dependent isomorphisms $\mathrm{Pic}^d(E) \approx E$ for all $d \in \mathbb{Z}$.

The case of higher rank holomorphic vector bundles is more complicated. First, the Riemann -Roch theorem for an elliptic curve states that for any holomorphic vector bundle V over E we have $\chi(V) := h^0(V) - h^1(V) = \mathrm{deg}V$. Further, for any such V we can consider a maximal decomposition into *holomorphic* subbundles: $V = V_1 \oplus ... \oplus V_k$. This reduces the problem to the classification of holomorphic *indecomposable* vector bundles of fixed rank r and fixed degree d, whose set of isomorphism classes we denote by $\mathcal{E}(r,d)$.

To study this problem it proves useful to first look at the spaces of sections $H^0(V)$ of such bundles. Atiyah [20] proves the that for any $V \in \mathcal{E}(r,d)$ we have $h^0(V) = d$, if $d > 0$, respectively $h^0(V) \in \{0,1\}$, if $d = 0$. Moreover, there exists a unique bundle $F_r \in \mathcal{E}(r,0)$ (up to isomorphism) such that $h^0(F_r) \neq 0$ and we have $h^0(F_r) = 1$. The bundles F_r can be constructed inductively by taking successive nontrivial extensions of O_E by itself. That is, $F_1 := O_E$ and for any $r > 1$, F_r is the unique nonsplit extension of F_{r-1} by O_E. In particular, the bundles F_r have trivial determinant and are semistable.

First consider the case $d = 0$, which is our primary focus in the present paper. Atiyah shows that any indecomposable holomorphic vector bundle $V \in \mathcal{E}(r,0)$ can be written in the form $V = L \otimes F_r$ with $L \in \mathrm{Pic}^0(E)$ a uniquely-determined vector bundle (which obviously satisfies $L^r = \det V$). This shows that $\mathcal{E}(r,0) \approx \mathrm{Pic}^0(E)$. Thus, there is essentially no more 'content' in degree zero indecomposable vector bundles then there is in degree zero line bundles.

The explicit description of $\mathcal{E}(r,d)$ for $d > 0$ is a classical result of Atiyah [20], which states that, given a pair (E,p), any $V \in \mathcal{E}(r,d)$ is of the form $V = L \otimes E(r,d)$, where $E(r,d)$ is a special element of $\mathcal{E}(r,d)$ (defined up to isomorphism) and L is a line bundle of degree zero on E. Here L is defined up to tensoring by an arbitrary holomorphic line bundle of order r/h on E, where $h := \gcd(r,d)$. The bundles $E(r,d)$ are defined (up to isomorphism) by an inductive procedure involving extensions by trivial vector bundles and tensoring with $O(p)$. In particular, one has $E(r,0) = F_r$ for all $r \geq 0$. The bundles $E(r,d)$ have $\mathrm{det}E(r,d) = O(d \times p)$. Using this result, one can show that $\mathcal{E}(r,d)$ is in bijection with E. More precisely, there is a (p-dependent) bijection from $\mathcal{E}(r,d)$ to $\mathcal{E}(h,0)$ and a bijection from $\mathcal{E}(h,0)$ to $\mathrm{Pic}^0(E)$, the last map being given by $V \to L$ where $L \in \mathrm{Pic}^0(E)$

is uniquely determined (up to isomorphism) by $V = L \otimes F_h$, as explained above. Via the usual p-dependent isomorphism $\mathrm{Pic}^0(E) \approx E$, this induces the desired identification. The description of $\mathcal{E}(r,d)$ for $d < 0$ can be obtained by dualizing the above.

If one asks for a suitable moduli space of vector bundles over a Riemann surface of genus g, one finds that it is appropriate to restrict the set of bundles in order to obtain a reasonable result. The correct notion is that of semistable vector bundles. For any holomorphic vector bundle V, define its *normalized degree* $\mu(V)$ by $\mu(V) := \deg V/\operatorname{rank} V$.

Then V is called *semistable* if $\mu(W) \leq \mu(V)$ for all proper subbundles W of V. If strict inequality holds for all such W, then V is called *stable*.

To construct the moduli space $M(r,d)$ of semistable bundles of rank r and degree d over a Riemann surface of genus g, one considers the set $ss(r,d)$ of isomorphism classes of such bundles and divides it by an equivalence relation called S-equivalence. To define this relation, one first shows that any semistable V admits a 'Jordan-Holder' filtration :

$$0 = V_k \subset V_{k-1} \subset ... \subset V_0 = V \tag{84}$$

by semistable subbundles V_j with $\mu(V_j) = \mu(V)$ and with the property that each succesive quotient V_j/V_{j+1} is stable and of normalized degree $\mu(V_j/V_{j+1}) = \mu(V)$. Although such a filtration is not unique, the isomorphism class of the associated graded bundle $gr(V) := \oplus_{j=0..k-1} V_j/V_{j+1}$ does not depend on its choice. If V is stable, then the only such filtration is of the form $0 = V_1 \subset V_0 = V$ and in this case $gr(V) \approx V$

The S-equivalence relation on $ss(r,d)$ is defined by $V_1 \equiv V_2$ iff $gr(V_1) \approx gr(V_2)$. If V_1 is stable, then $V_1 \equiv V_2$ iff $V_1 \approx V_2$.

The desired moduli space is then very roughly given by :

$$M(r,d) := ss(r,d)/\equiv \tag{85}$$

More precisely, $M(r,d)$ is constructed as a coarse moduli space by using Mumford's geometric invariant theory. $M(r,d)$ is a projective variety, whose (closed) points correspond to S-equivalence classes of semistable bundles. There exists an open subset $M^s(r,d)$ of $M(r,d)$, whose points correspond to isomorphism classes of stable bundles. All the points of $M^s(r,d)$ are smooth points of $M(r,d)$ and the converse also holds unless $g = r = 2$ and $d \equiv 0(\mathrm{mod}2)$. Moreover, if r and d are coprime then $M(r,d) = M^s(r,d)$. If $M^s(r,d)$ is nonvoid then $\dim M(r,d) = r^2(g-1)+1$.

Now consider the case of an elliptic curve. Assume $d \geq 0$ and let $h := \gcd(r,d)$. Then the main results of relevance for us can be summarized as follows :

Each S-equivalence class in $ss(r,d)$ contains a unique (up to isomorphism) bundle of the form $V = \oplus_{i=1}^h E_i$ with E_i all stable, of equal rank r/h and equal degree d/h, which is the only member of the class for which $V \approx gr(V)$.

In particular:

(01)If $h > 1$ then there exist no stable bundles of rank r and degree d over E.

(02)If $h = 1$ then every S-equivalence class of semistable bundles over E contains a stable bundle, which is unique in that class up to isomorphism.

(1) $M(r,d)$ is a smooth projective variety of dimension h. Moreover, there exists an isomorphism $M(r,d) \approx \mathrm{Sym}^h(\mathrm{Pic}^0(E)) \approx \mathrm{Sym}^h(E)$, where Sym^h denotes h-th symmetric power. In particular, we have $M(r,d) \approx M(sr,sd)$, for any positive integer s.

(2) If $h=1$, then the map $\det : M(r,d) \to \mathrm{Pic}^d(E)$ is an isomorphism.

For semistable vector bundles of degree zero over E, the filtration (84) takes the form :

$$0 = V_r \subset V_{r-1} \subset ... \subset V_0 = V \tag{86}$$

where $r := \mathrm{rank}V$. In this case, the subbundles V_j have $\mathrm{rank}V_j = r - j$ for all j and the associated graded pieces $L_j := V_j/V_{j+1} (j = 0...r-1)$ are degree zero line bundles on E. This gives $gr(V) = \oplus_{j=0..r-1} L_j$, with $L_j \in \mathrm{Pic}^0(E)$ and the above-mentioned bijection $ss(r,0) \approx Sym^r(\mathrm{Pic}^0(E)) \approx Sym^r(E)$.

Now let V be a rank r and degree zero semistable vector bundle over E and let $V' := V \otimes O_E(p)$ for some $p \in E$. Then V must have a maximal splitting (direct sum decomposition) of the form :

$$V = \sum_{j=1..k} O(q_j - p) \otimes F_{r_j} \tag{87}$$

where $\sum_{j=1..k} r_j = r$ [20]. It is easy to see that the converse is also true, since the direct sum of any finite set of semistable vector bundles of equal normalized degree is semistable ([24],p17,Cor 7). Thus, we have the following simple fact :

Proposition A.1 *Let V be a degree zero holomorphic vector bundle over E. Then the following are equivalent :*

(a) V is semistable

(b) V has a maximal splitting of the form (87)

(c) The twisted bundle $V' := V \otimes O(p)$ has a maximal splitting of the form:

$$V' = \sum_{j=1...k} O(q_j) \otimes F_{r_j} \tag{88}$$

In this case,the multiset $(r_1,q_1)...(r_k,q_k)$ *will be called* the splitting type *of V.*

As $O(q_j) \otimes F_{r_j}$ is indecomposable and of degree r_j, one of the above results shows that $h^0(O(q_j) \otimes F_{r_j}) = r_j$, so that, for V *semistable* and of degree zero, $H^0(V')$ is an r-dimensional $\mathbb{C}$-vector space. Note that this is true indifferent of the precise splitting type of V.

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