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Integrability of Coupled Conformal Field Theories

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The massive phase of two–layer integrable systems is studied by means of RSOS restrictions of affine Toda theories. A general classification of all possible integrable perturbations of coupled minimal models is pursued by an analysis of the (extended) Dynkin diagrams. The models considered in most detail are coupled minimal models which interpolate between magnetically coupled Ising models and Heisenberg spin-ladders along the $c<1$ discrete series.

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1. Introduction

Since the works of Belavin, Polyakov and Zamolodchikov on conformally invariant two dimensional systems and integrable deformations thereof[1][2], and Andrews, Baxter and Forrester on related integrable lattice solid–on–solid models[3], much important progress has been reported on the classification of all possible universality classes of two-dimensional statistical models as well as on the complete control of the scaling region nearby (see, for instance [4][5][6]). In particular, the method of exact relativistic scattering[7] and related form factor techniques[8][9] has permitted an exact solution of many models, including for instance the long–standing problem of the two–dimensional Ising model in a magnetic field[2][10]. The techniques of Exact Integrability have recently also been shown to be a powerful tool for providing non-perturbative answers for experimentally important strongly interacting Solid State physics problems[11][12].

In this paper we exhibit a large class of new integrable two-dimensional systems. These are two planar systems (one on top of the other) coupled together by operators which lead to integrable theories (Figure 1).

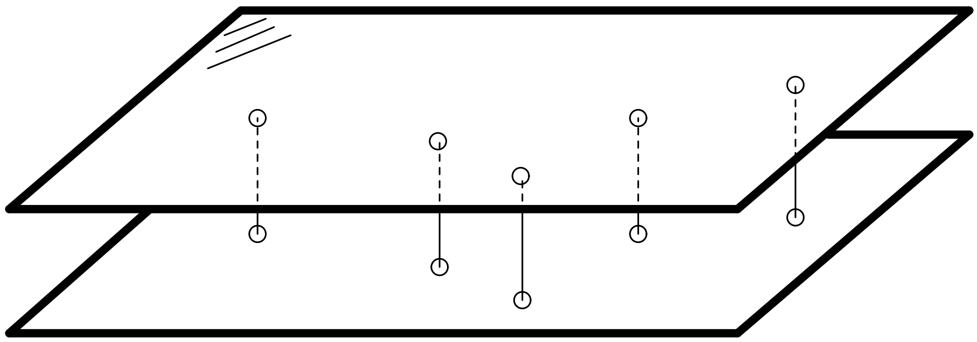

Figure 1. Two coupled two-dimensional models.

Our models are to be thought of as in the same category of so-called spin-ladders (See for instance refs. [13][14] and references therein.) In fact, they are generalization of these systems. The models we treat in most detail are two coupled minimal models, interpolating between two magnetically coupled Ising models and Heisenberg spin-ladders along the $c < 1$ discrete series. (The central charge of the unperturbed models ranges from $c = 1$ to $c = 2$.) These are however only special cases of a much more general class of integrable models which we identify using properties of the (extended) Dynkin diagram of affine Lie algebras. These include: (i) two coupled $SO(2n)$ coset theories, where the central charges of the unperturbed models range from $c = 2$ (two coupled orbifolds) to $c = 2n$ (two coupled $SO(2n)_1$ current algebras), and (ii): four coupled minimal models with unperturbed central charges ranging from $c = 2$ to $c = 4$.

The integrable models studied here are bulk theories which are massive in the infrared. Corresponding integrable massless flows in impurity models are studied by two of us in[15]. These are generalizations of models which have recently attracted much attention in Condensed Matter physics, such as in the context of point contacts in the fractional quantum Hall effect, and Impurities in Quantum Wires (see e.g. [16][11]).

In this paper the emphasis is on the aspects coming from the integrability of the interlayer coupling and on the exact results which follow. We will show, in particular, that the on–shell dynamics of such systems admits a description in terms of an exact scattering theory. The exact scattering amplitudes as well as the exact spectrum of excitations can be computed by employing the RSOS reduction scheme based on the quantum symmetries of the models[17][18]. An important representative of the class of the models analyzed in this paper consists of the two–layer Ising system coupled together by their magnetization operators σ_1 and σ_2. A simple mean–field analysis indicates that in this case the interaction between the two layers drives the system into a massive phase. We will determine the exact dynamics of this model by providing the spectrum of the massive excitations of this model as well as all their S–matrix amplitudes.

The paper is organized as follows: in Section 2 we analyze a particular integrable coupling between two minimal models of conformal field theory (CFT), the latter being regarded as a coset construction on $SU(2)$. In Section 3 we study the integrability of coupled conformal field theories under a more general setting based on (Affine) Toda Field Theory. Finally, in Section 4 the spectrum and the S-matrix of (magnetically) coupled minimal models are worked out explicitly. In Section 5 the conclusions of this work are presented.

2. Coupled Minimal Models and ${}_q d_3^{(2)}$ and ${}_q c_2^{(1)}$ Affine Lie Algebras

Let $\mathcal{C}^{(k)}$ denote the minimal unitary conformal field theory (CFT) with central charge

$$c_k = 1 - \frac{6}{(k+2)(k+3)} \quad , \tag{2.1}$$

$k = 1, 2, \ldots$. These models have local primary fields $\sigma = \Phi_{1,2}$, $\widetilde{\sigma} = \Phi_{2,1}$, $\varepsilon = \Phi_{1,3}$, and $\widetilde{\varepsilon} = \Phi_{3,1}$ with scaling dimension:

$$
\begin{aligned}
\dim(\sigma) = 2\Delta_\sigma &= \frac{1}{2}\left(\frac{k}{k+3}\right) \qquad & \dim(\widetilde{\sigma}) = 2\Delta_{\widetilde{\sigma}} &= \frac{1}{2}\left(\frac{k+5}{k+2}\right) \\
\dim(\varepsilon) = 2\Delta_\varepsilon &= 2\left(\frac{k+1}{k+3}\right) \qquad & \dim(\widetilde{\varepsilon}) = 2\Delta_{\widetilde{\varepsilon}} &= 2\left(\frac{k+4}{k+2}\right).
\end{aligned}
\tag{2.2}
$$

(Here, dim refers to the sum of left and right conformal dimension.) We define four infinite series of models $\mathcal{M}_k^\sigma, \mathcal{M}_k^{\widetilde{\sigma}}$, $\mathcal{M}_k^\varepsilon$ and $\mathcal{M}_k^{\widetilde{\varepsilon}}$ by coupling two copies of $\mathcal{C}^{(k)}$ via the operators $\sigma, \widetilde{\sigma}$, ε, $\widetilde{\varepsilon}$. This is described by an action which perturbs the tensor product of the two CFT's:

$$
\mathcal{A} = \mathcal{A}_{\mathcal{C}_1^{(k)}} + \mathcal{A}_{\mathcal{C}_2^{(k)}} + \lambda \int d^2x \ \Phi_1 \Phi_2,
\tag{2.3}
$$

where the subscripts refer to copy 1 or 2 of $\mathcal{C}^{(k)}$, and $\Phi = \sigma, \widetilde{\sigma}, \varepsilon$ or $\widetilde{\varepsilon}$.

The models $\mathcal{M}_k^\sigma$ and $\mathcal{M}_k^{\widetilde{\sigma}}$ are characterized by relevant perturbations for all k. The models $\mathcal{M}_k^\varepsilon, \mathcal{M}_k^{\widetilde{\varepsilon}}$ on the other hand are irrelevant perturbations, except for $\mathcal{M}_1^\varepsilon$ (from the dimensions (2.2), one sees in this case that $\mathcal{M}_1^{\widetilde{\sigma}} = \mathcal{M}_1^\varepsilon$). The latter is a strictly marginal perturbation corresponding to the Ashkin-Teller model: we have then a line of fixed points described by the coupling constant λ. With the appropriate choice of sign of λ, the models $\mathcal{M}_k^{\sigma,\widetilde{\sigma}}$ are massive field theories. The other models perhaps describe the infrared limit of an integrable flow from a model with higher central charge in the ultraviolet. We are only concerned in this paper with the massive models, however we will continue to point out where the models $\mathcal{M}_k^{\varepsilon,\widetilde{\varepsilon}}$ reside in the algebraic classification. Also, this information may be useful for coupled non-unitary minimal models.

One approach to integrable perturbations of minimal models and other coset conformal field theories is based on quantum group restrictions of affine Toda theories[19][20][17][18]. Remarkably, the same approach can be applied here to classify the possible integrable perturbations of coupled minimal models. Let us see how this can be achieved.

It is well known that the minimal models $\mathcal{C}^{(k)}$ of conformal field theory have a description in terms of a scalar field with background charge [21]. Thus two copies of $\mathcal{C}^{(k)}$ can be represented with two scalar fields ϕ_1, ϕ_2, each with the appropriate background charge to give the requisite central charge and the conformal spectrum. In the affine-Toda theory approach to perturbed conformal field theory, one starts with a Toda theory on a finite Lie group g, then identifies the perturbation with an affine extension of g to $\widehat{g}$. For our problem, the conformal field theory $\mathcal{C}^{(k)} \otimes \mathcal{C}^{(k)}$ is represented with two scalar fields, thus the rank of $\widehat{g}$ must be three. The other requirement of $\widehat{g}$ is that when the root associated with the perturbation is omitted, the resulting non-affine Toda theory must be an $su(2) \otimes su(2)$ Toda theory in order to represent $\mathcal{C}^{(k)} \otimes \mathcal{C}^{(k)}$. Otherwise stated, if two coupled minimal models can be described by a quantum group restriction of the affine Toda theory $\widehat{g}$, then the Dynkin diagram of $\widehat{g}$ must contain 3 nodes, and the removal of one node must leave two decoupled nodes with roots of the same length. Referring to the known classification of affine Lie algebras [22], the only possibilities are $c_2^{(1)}$ and $d_3^{(2)}$. Neither of these are simply laced. The Dynkin diagrams for these algebras are shown in Figure 2. Removing the middle node leaves $su(2) \otimes su(2)$, thus it is the middle node that will be associated with the perturbation. This is in contrast to the usual application of affine Toda theory to perturbed coset theories, where there the extended affine root α_0 is associated with the perturbation.

Figure 2. Dynkin diagrams for the algebras $c_2^{(1)}$ and $d_3^{(2)}$ respectively.

The affine Toda theories associated with the Dynkin diagrams of Fig. 2 are defined by the action

$$\mathcal{A} = \frac{1}{4\pi} \int d^2x \left(\frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \lambda \sum_{\alpha_j} e^{-i\beta \vec{\alpha}_j \cdot \vec{\phi}} \right) \quad , \tag{2.4}$$

where here $\vec{\phi} = (\phi_1, \phi_2)$, $\alpha_j \in \{\alpha_0, \alpha_1, \alpha_2\}$ are simple roots of the affine algebra, and β is a coupling. For a general affine Lie algebra $\widehat{g}$, $\vec{\phi}$ has rank$(\widehat{g}) - 1$ components and the sum runs over all simple roots of $\widehat{g}$.

For $c_2^{(1)}$, one can chose $\vec{\alpha}_0^2 = \vec{\alpha}_2^2 = 2$, $\vec{\alpha}_1^2 = 1$. The Cartan matrix $K_{ij} = 2\vec{\alpha}_i \cdot \vec{\alpha}_j / \alpha_j^2$ is

$$K = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad . \tag{2.5}$$

We will also need $\vec{\alpha}_0 \cdot \vec{\alpha}_1 = \vec{\alpha}_1 \cdot \vec{\alpha}_2 = -1$.

The algebra $d_3^{(2)}$ is the dual of $c_2^{(1)}$ under the transformation $\vec{\alpha} \to 2\vec{\alpha}/\alpha^2$. This duality is the usual one that exchanges the orientation of the arrows of the Dynkin diagram and takes K into its transpose. For $d_3^{(2)}$, one then has $\vec{\alpha}_0^2 = \vec{\alpha}_2^2 = 2$, $\vec{\alpha}_1^2 = 4$ and $\vec{\alpha}_0 \cdot \vec{\alpha}_1 = \vec{\alpha}_2 \cdot \vec{\alpha}_1 = -2$.

We identify the α_0 and α_2 terms in the Toda potential with the conformal field theory $\mathcal{C}^{(k)} \otimes \mathcal{C}^{(k)}$, which requires these operators to have left and right conformal dimension equal to 1. This can be accomplished by turning on a background charge $\vec{\gamma}$ with modified energy momentum tensor

$$T = -\frac{1}{2}\partial_z \vec{\phi} \cdot \partial_z \vec{\phi} + i\sqrt{2}\vec{\gamma} \cdot \partial_z^2 \vec{\phi} \quad . \tag{2.6}$$

We take $\vec{\gamma} = \gamma(\vec{\alpha}_0 + \vec{\alpha}_2)$, which leads to the central charge $c = 2(1 - 48\gamma^2)$. Identifying $c = 2c_k$, one fixes the parameter γ to be

$$\gamma = \frac{1}{\sqrt{8(k+2)(k+3)}} \quad . \tag{2.7}$$

The chiral dimension of the exponential operators are then given by

$$\Delta \left(e^{-i\beta \vec{\alpha} \cdot \vec{\phi}} \right) = \beta^2 \vec{\alpha}^2 / 2 + \sqrt{2} \beta \vec{\alpha} \cdot \vec{\gamma} \quad . \tag{2.8}$$

Imposing that $e^{-i\beta\vec{\alpha}_0 \cdot \vec{\phi}}$ and $e^{-i\beta\vec{\alpha}_2 \cdot \vec{\phi}}$ have dimension 1 leads to the equation

$$1 = \beta^2 + 2\sqrt{2}\beta\gamma \quad , \tag{2.9}$$

with two solutions:

$$\beta_+ = \sqrt{\frac{k+2}{k+3}} \quad , \qquad \beta_- = -\sqrt{\frac{k+3}{k+2}} \quad . \tag{2.10}$$

Finally, once β and γ are fixed we can identify the chiral dimension of the perturbation as $\Delta_{\rm pert} = \Delta(e^{-i\beta\vec{\alpha}_1 \cdot \vec{\phi}})$. For $c_2^{(1)}$ one finds $\Delta_{\rm pert} = 2\Delta_\sigma$ for β_+ and $\Delta_{\rm pert} = 2\Delta_{\widetilde{\sigma}}$ for β_-. For $d_3^{(2)}$ one finds $\Delta_{\rm pert} = 2\Delta_\varepsilon$ for β_+ and $\Delta_{\rm pert} = 2\Delta_{\widetilde{\varepsilon}}$ for β_-. We summarize these results by listing below the model and its associated affine Toda theory and coupling:

$$\begin{aligned}
\mathcal{M}_k^{\sigma} \ &: \quad c_2^{(1)} \text{ affine Toda with } \beta = \beta_+ \quad ; \\
\mathcal{M}_k^{\widetilde{\sigma}} \ &: \quad c_2^{(1)} \text{ affine Toda with } \beta = \beta_- \quad ; \\
\mathcal{M}_k^{\varepsilon} \ &: \quad d_3^{(2)} \text{ affine Toda with } \beta = \beta_+ \quad ; \\
\mathcal{M}_k^{\widetilde{\sigma}} \ &: \quad d_3^{(2)} \text{ affine Toda with } \beta = \beta_- \quad .
\end{aligned} \tag{2.11}$$

With this identification, the spectrum and S-matrices of the models can be obtained as quantum group restrictions of the affine Toda theory. One must bear in mind that the affine Toda theory based on $\widehat{g}$ possesses the dual quantum affine symmetry ${}_q\widehat{g}^\vee$[23][24] with

$$q = e^{-i\pi/\beta^2}. \tag{2.12}$$

Thus the (restricted) quantum symmetries of the models are as follows:

$$\begin{aligned}
\mathcal{M}_k^{\sigma} &: \quad {}_q d_3^{(2)} \text{ symmetry}, \quad q = -e^{-i\pi/(k+2)} \quad ; \\
\mathcal{M}_k^{\widetilde{\sigma}} &: \quad {}_q d_3^{(2)} \text{ symmetry}, \quad q = -e^{+i\pi/(k+3)} \quad ; \\
\mathcal{M}_k^{\varepsilon} &: \quad {}_q c_2^{(1)} \text{ symmetry}, \quad q = -e^{-i\pi/(k+2)} \quad ; \\
\mathcal{M}_k^{\widetilde{\varepsilon}} &: \quad {}_q c_2^{(1)} \text{ symmetry}, \quad q = -e^{+i\pi/(k+3)} \quad .
\end{aligned} \tag{2.13}$$

Vaysburd first established the integrability of the models $\mathcal{M}_k^{\sigma,\widetilde{\sigma}}$ directly as perturbations of cosets [25], by using the counting arguments of Zamolodchikov [2]. The CFT $\mathcal{C}^{(k)}$ can be formulated as the coset $\mathcal{C}^{(k)} = SU(2)_k \otimes SU(2)_1/SU(2)_{k+1}$, where $SU(2)_k$ is the $SU(2)$ current algebra at level k[26]. Using the fact that $SU(2)_k \otimes SU(2)_k = SO(4)_k$, one has

$$\mathcal{C}^{(k)} \otimes \mathcal{C}^{(k)} = \frac{SO(4)_k \otimes SO(4)_1}{SO(4)_{k+1}} \quad . \tag{2.14}$$

Thus, the models $\mathcal{M}_k^{\sigma,\widetilde{\sigma}}$ can be formulated as perturbations of the $SO(4)$ cosets by operators of dimension $2 \cdot \dim(\sigma, \widetilde{\sigma})$. These coset perturbations are not the generic ones which are integrable for arbitrary Lie algebras where the perturbing field is associated with the adjoint representation [18], and in the affine Toda approach are associated with the affine root $\vec{\alpha}_0$; rather the perturbing fields here are associated with the vector representation. As explained in [25], the latter corresponds to a different way of affinizing $SO(4)$ to yield the affine algebras $d_3^{(2)}, c_2^{(1)}$.

3. General Scheme and Other Examples

3.1. Affine Toda Theories for Coupled Conformal Field Theories

The construction of the last section is just an example of a more general one for studying integrability of coupled conformal field theories based on affine Toda theories. Let $\widehat{g}$ denote an affine Lie algebra and $\{\alpha(\widehat{g})\}$ its simple roots, $\{\vec{\alpha}_0, \vec{\alpha}_1, ..., \vec{\alpha}_r\}$. In the Dynkin diagram of $\widehat{g}$, we identify one node and its associated root as the perturbation and denote this root as $\vec{\alpha}_{pert} \in \{\alpha(\widehat{g})\}$. We further require that upon removing the node $\vec{\alpha}_{pert}$ we are left with two decoupled Dynkin diagrams representing $g_1 \oplus g_2$, where g_1 and g_2 are finite dimensional, simply laced Lie algebras. By chosing the background charges appropriately, the conformal field theory corresponds to two decoupled conformal Toda theories based on g_1 and g_2, and because these are simply laced, these can represent the coset theories of the g_1 and g_2 current algebras [27][28]. The perturbation term $\exp(-i\beta\vec{\alpha}_{pert}\cdot\vec{\phi})$ is the one which couples the two conformal field theories, and its dimension is fixed once the background charge is fixed.

Normally, one choses $\vec{\alpha}_{pert}=\vec{\alpha}_0$, which is the negative of the highest root, and always occurs at an end of the Dynkin diagram. This well-known case describes the perturbation of a single coset theory since here $g_1=g$ and g_2 is empty. In this case, the background charges require that one begin with the S-matrices of the unrestricted Toda theory in the homogeneous gradation, since it is in this gradation that the ${}_qg$ invariance is manifest (See e.g. [24].). For the new cases we are considering, the background charges are different, and one must first transform the S-matrices to the appropriate gradation where the ${}_qg_1\oplus{}_qg_2$ symmetry is manifested, before doing the restriction. As far as the spectrum and S-matrices are concerned, this is the main dynamical difference between models with $\vec{\alpha}_{pert}=\vec{\alpha}_0$ and $\vec{\alpha}_{pert}\neq\vec{\alpha}_0$. We now discuss two examples of this construction.

3.2. Coupled $SO(2n)$ Cosets and ${}_qd_{2n}^{(1)}$ Affine Algebras

Let us begin with the Toda theory based on the affine algebra $d_{2n}^{(1)}$, which is the standard affinization of $d_{2n}=so(4n)$. Its Dynkin diagram is shown in Figure 3. If one removes the central node on the string, the diagram decouples into two $d_n=so(2n)$ Lie algebras. Thus, if we identify $\vec{\alpha}_{pert}=\vec{\alpha}_n$, the $d_{2n}^{(1)}$ affine Toda theory can be used to describe two coupled $so(2n)$ cosets. We denote by $\{\vec{\alpha}^{(1,2)}(d_n)\}$ the simple roots for copies 1 and 2 of $so(2n)$ so that

$$\{\vec{\alpha}(d_{2n}^{(1)})\}=\{\vec{\alpha}^{(1)}(d_n)\}+\{\vec{\alpha}^{(2)}(d_n)\}+\vec{\alpha}_n \quad . \tag{3.1}$$

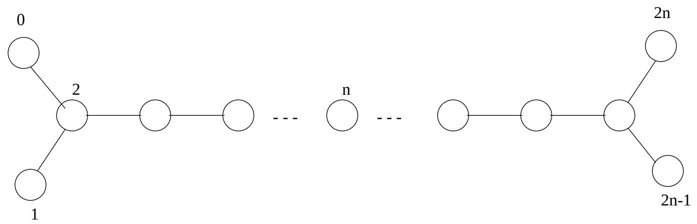

Figure 3. Dynkin diagram for the affine algebra $d_{2n}^{(1)}$.

Let $\mathcal{C}^{(k)}$ denote the coset CFT

$$\mathcal{C}_n^{(k)} = \frac{SO(2n)_k \otimes SO(2n)_1}{SO(2n)_{k+1}} \quad , \tag{3.2}$$

with central charge

$$c_k^n = n \left(1 - \frac{h^*(h^*+1)}{(k+h^*)(k+h^*+1)}\right) \quad , \tag{3.3}$$

where the dual Coxeter number of $so(2n)$ is $h^* = 2n-2$. The $d_{2n}^{(1)}$ affine Toda theory contains $2n$ scalar fields combined into the vector $\vec{\phi}$. We let the energy momentum tensor take the form (2.14), with central charge $c = 2n - 24\vec{\gamma}^2$. The background charge $\vec{\gamma}$ is chosen such as to represent two decoupled $\mathcal{C}_n^{(k)}$ theories,

$$\vec{\gamma} = 2\gamma\left(\vec{\rho}_1 + \vec{\rho}_2\right), \tag{3.4}$$

where $\vec{\rho}_{1,2}$ are the Weyl vectors for copies 1 and 2 of $so(2n)$, namely, $\vec{\rho}_{1,2} = \sum_{i=1}^n \vec{\mu}_i^{(1,2)}$, where $\vec{\mu}_i^{(1,2)} \cdot \vec{\alpha}_j^{(1,2)} = \delta_{ij}$. This implies

$$\vec{\rho}_{1,2} = \sum_{i=1}^n K_{ij}^{-1} \vec{\alpha}_j^{(1,2)} \quad , \tag{3.5}$$

where K is the Cartan matrix of $so(2n)$. The Weyl vectors satisfy

$$\vec{\rho}_1 \cdot \vec{\rho}_2 = 0, \qquad (\vec{\rho}_1)^2 = (\vec{\rho}_2)^2 = nh^*(h^*+1)/12 \quad . \tag{3.6}$$

Identifying c with $2c_k^n$, one requires

$$\gamma = \frac{1}{\sqrt{8(k+h^*)(k+h^*+1)}} \quad . \tag{3.7}$$

Next we require that the terms in the Toda potential $\exp(-i\beta\vec{\alpha}\cdot\vec{\phi})$ with $\vec{\alpha}$ a simple root of either copy of $so(2n)$ to have conformal dimension equal to 1. This gives the equation (2.9), with solutions

$$\beta_+ = \sqrt{\frac{k+h^*}{k+h^*+1}} \quad , \qquad \beta_- = -\sqrt{\frac{k+h^*+1}{k+h^*}} \quad . \tag{3.8}$$

The dimension of the perturbation follows from (2.8) and $(\vec{\rho}_1+\vec{\rho}_2)\cdot\vec{\alpha}_{pert} = -h^*$:

$$\begin{aligned} \Delta_{pert} = \Delta_{pert}^+ &= \frac{k}{k+h^*+1} \quad , \qquad \text{for} \quad \beta = \beta_+ \quad ; \\ \Delta_{pert} = \Delta_{pert}^- &= \frac{k+2h^*+1}{k+h^*} \quad , \qquad \text{for} \quad \beta = \beta_- \quad . \end{aligned} \tag{3.9}$$

Let us now interpret these models. The CFT $\mathcal{C}_n^{(k)}$ has two primary fields Φ^e and Φ^h, which are associated with vector representations of $so(2n)$, with chiral scaling dimension

$$\Delta(\Phi^h) = \frac{1}{2}\ \Delta^+_{pert} \quad , \qquad\qquad \Delta(\Phi^e) = \frac{1}{2}\ \Delta^-_{pert} \quad . \tag{3.10}$$

It was shown by Vaysburd that the following perturbations of a single copy of $\mathcal{C}_n^{(k)}$ are integrable:

$$\mathcal{A}^{e,h} = \mathcal{A}_{\mathcal{C}_n^{(k)}} + \lambda \int d^2x\ \Phi^{e,h}. \tag{3.11}$$

From (3.10) one sees that our models correspond to two $\mathcal{C}_n^{(k)}$ theories coupled by these operators:

$$\mathcal{A} = \mathcal{A}_{\mathcal{C}_n^{(k)}\otimes\mathcal{C}_n^{(k)}} + \lambda \int d^2x\ \Phi_1^{e,h}\ \Phi_2^{e,h} \quad . \tag{3.12}$$

To summarize, two coupled $so(2n)$ cosets defined by the action (3.12) can be solved by a quantum group restriction of the ${}_qd_{2n}^{(1)}$ affine Toda theory with $q = \exp(-i\pi/\beta_+^2)$ or $q = \exp(-i\pi/\beta_-^2)$.

3.3. Four Coupled Minimal Models and ${}_qd_4^{(1)}$

The construction of the last section is special for $d_4^{(1)}$ since here removing the node $\vec{\alpha}_{pert}$ leaves four decoupled $su(2)$ nodes. From our general approach, we expect that this case corresponds to *four* coupled minimal models.

Let $\vec{\alpha}_i, i = 0, .., 4$ denote the simple roots of $d_4^{(1)}$. The central node is $\vec{\alpha}_{pert} = \vec{\alpha}_2$. We now chose

$$\vec{\gamma} = \gamma \sum_{i\neq 2} \vec{\alpha}_i, \tag{3.13}$$

where γ is the same as in (2.7). This leads to $c = 4c_k$ where c_k is the central charge (2.1) of the $k-th$ minimal model $\mathcal{C}^{(k)}$. In order for each node $\vec{\alpha}_i, i \neq 2$ to represent a single copy of the minimal model $\mathcal{C}^{(k)}$, one requires β to be β_+ or β_- as defined in (2.10).

The dimension of the perturbation is

$$\Delta_{pert} = \beta^2 + \sqrt{2}\beta\vec{\alpha}_2 \cdot \vec{\gamma} = \beta^2 - 4\sqrt{2}\beta\gamma \quad , \tag{3.14}$$

and one finds

$$\begin{aligned} \Delta_{pert} &= 4\Delta_\sigma \quad , \qquad \text{for} \quad \beta = \beta_+ \quad ; \\ \Delta_{pert} &= 4\Delta_{\tilde{\sigma}} \quad , \qquad \text{for} \quad \beta = \beta_- \quad . \end{aligned} \tag{3.15}$$

Thus, the appropriate quantum group restriction of the $d_4^{(1)}$ affine Toda theory with $q = \exp(-i\pi/\beta^2)$ describes a model of four minimal conformal models all coupled at one point via the operators $\sigma, \widetilde{\sigma}$. For $\beta = \beta_+$, the action is given by

$$\mathcal{A} = \sum_{i=1}^{4} \mathcal{A}_{\mathcal{C}_i^{(k)}} + \lambda \int d^2x \; \sigma_1\sigma_2\sigma_3\sigma_4, \tag{3.16}$$

where the subscripts refer to which copy of $\mathcal{C}^{(k)}$.

An interesting case is $k = 1$, which corresponds to four coupled Ising models. They can be grouped into two pairs, each pair with $c = 1$. We can bosonize each pair with scalar fields ϕ_1 and ϕ_2. Then the action (3.16) can be expressed in this case as

$$\mathcal{A} = \frac{1}{4\pi} \int d^2x \left(\sum_{i=1,2} \frac{1}{2} (\partial\phi_i)^2 + \lambda \cos(\phi_1/2) \cos(\phi_2/2) \right) \quad . \tag{3.17}$$

Since the interaction can be written as $\cos((\phi_1 + \phi_2)/2) + \cos((\phi_1 - \phi_2)/2)$, one sees that this corresponds to two decoupled sine-Gordon models each at $\beta^2/8\pi = 1/4$.[2]

Another example of this kind is furnished by the $e_6^{(1)}$ affine Toda theory. Here the central node couples *three* $su(3)$ coset CFT's.

4. Spectrum and S-matrices for Coupled Minimal Models

For the remainder of this paper we will be concerned only with the models $\mathcal{M}_k^\sigma$. In ref.[29] ${}_q d_3^{(2)}$ invariant S-matrices were constructed[3]. These are S-matrices in the unrestricted (vertex) form for the fundamental multiplets of solitons. There are two such fundamental multiplets which transform in the 4–dimensional vector $\{4\}$ and in the 6–dimensional adjoint $\{6\}$ representations of $SO(4)_q$. The mass ratio of the two fundamental multiplets of solitons is[4]

$$\frac{M_{\{6\}}}{M_{\{4\}}} = 2\cos\left(\frac{\pi}{k+6}\right) \quad . \tag{4.1}$$

In addition to these fundamental solitons there are scalar bound states and excited solitons depending on k [29]. As previously explained, the models $\mathcal{M}_k^\sigma$ are described by quantum group (RSOS) restrictions of these S-matrices. The RSOS spectrum proposed in [25] appears incomplete however. This will be evident below where we consider the Ising case at $k = 1$.

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[2] Here β is normalized in the usual convention where $\beta^2/8\pi = 1/2$ is the free fermion point.

[3] We remark that one has the identifications $d_3^{(2)} = a_3^{(2)}$ and $b_2^{(1)} \equiv c_2^{(1)}$

[4] The parameters ω, λ introduced in [29] take the values $\omega = 1/(k+2)$, $\lambda = (k+6)/4(k+2)$.

4.1. Ising Case and Relation to Sine-Gordon at $\beta^2/8\pi = 1/8$

The model $\mathcal{M}_1^{\sigma}$ can be described by the action

$$\mathcal{A} = \mathcal{A}_{\mathrm{Ising}_1} + \mathcal{A}_{\mathrm{Ising}_2} + \lambda \int d^2x \ \sigma_1\sigma_2 \quad , \tag{4.2}$$

where $\sigma_{1,2}$ are the spin fields in copies $\mathrm{Ising}_{1,2}$. We will refer to the model (4.2) as Ising_h^2. The presence of the coupling constant λ destroys the critical fluctuations of the two individual models and the resulting system has a tendency to acquire a net magnetization: its spectrum becomes then massive[5]. It is easy to predict the existence of kink excitations in the spectrum: in fact, there are two degenerate ground states of the system (4.2), one where both systems have a positive total magnetization the other where the total magnetization is negative. The two ground states are related each other by the Z_2 symmetry[6] $\sigma_1 \to -\sigma_1; \sigma_2 \to -\sigma_2$ and therefore there will be kink (antikink) excitations K_{+-} (K_{-+}) interpolating asymptotically between them (Figure 4). However, multi-kink configurations can only be constructed in terms of a string of kink strictly followed by an antikink: $|...K_{+-}K_{-+}K_{+-}...\rangle$. This means that the kinks of this system should behave actually like ordinary particles, as will be indeed confirmed by the analysis which follows.

A simple argument relates the model (4.2) to the sine-Gordon theory at the reflectionless point $\beta^2/8\pi = 1/8$. The sine-Gordon theory $SG_{\beta^2/8\pi}$ is defined by the action

$$\mathcal{A} = \frac{1}{4\pi} \int d^2x \ \left[\frac{1}{2}\left(\partial\phi\right)^2 + \Lambda \cos\widehat{\beta}\phi\right] , \tag{4.3}$$

where $\widehat{\beta} = \beta/\sqrt{4\pi}$. From the Ising Majorana fermions $\psi_{1,2}$, we can form a Dirac fermion $\psi_{\pm} = \psi_1 \pm i\psi_2$. This is a $c = 1$ CFT which can be bosonized by means of the formula $\psi_{\pm} = e^{\pm i\phi_L}$, where ϕ_L is the left-moving component of ϕ. The operator $\sigma_1\sigma_2$ has dimension 1/4, thus in the bosonized description it corresponds to $\cos(\phi/2)$, which corresponds to $\beta^2/8\pi = 1/8$. Let us refer to the latter theory as $SG_{1/8}$.

The above simple argument is not strictly correct since it ignores the fact that the conformal field theory of $\mathrm{Ising}_1 \otimes \mathrm{Ising}_2$, is not identical to that of a free Dirac fermion.

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[5] In the following λ is assumed to be positive. However, all the following conclusions hold independently from the sign of λ since the sign of the coupling constant can be altered by changing the sign of one of the magnetization operators, say the one with index 1: $\lambda \to -\lambda$; $\sigma_1 \to -\sigma_1$. $\mathcal{A}_1$ is left invariant under this transformation since is the action of the critical point.

[6] The system presents another Z_2 symmetry related to the exchange of the labels $1 \leftrightarrow 2$.

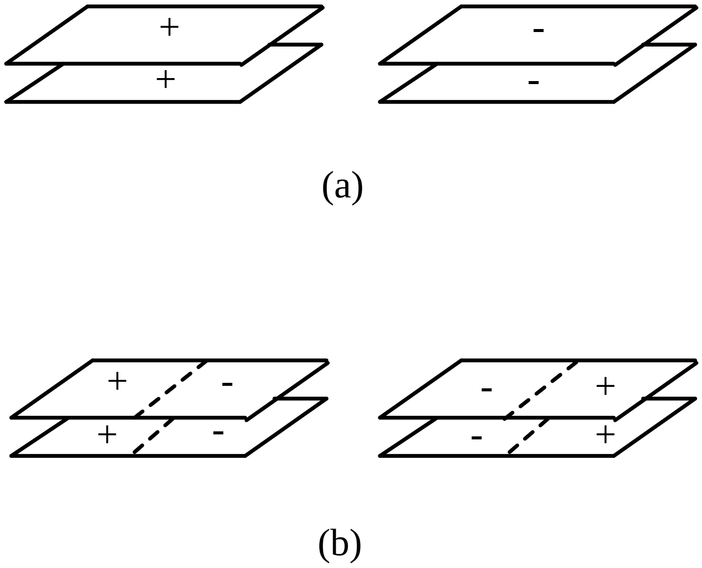

Figure 4. Ground states (a) kink and antikink excitations.

Rather, it is an orbifold model at $R_{orb} = 1$, whereas the Dirac theory is a scalar field compactified on a circle with $R_{circle} = 1$ [30]. More generally, consider the theory $SG_{\beta^2/8\pi}$. The potential in (4.3) has the symmetry $\phi \to \phi + 2\pi/\widehat{\beta}$, $\phi \to -\phi$. Thus the potential preserves the orbifold symmetries at $R_{orb} = 1/\widehat{\beta}$, and starting from $SG_{\beta^2/8\pi}$ one can easily define a perturbed orbifold version of it at this radius. For $\beta^2/8\pi = 1/8$, $R_{orb} = 2$ and the resulting theory is the $D_8^{(1)}$ theory [31][32]. The latter only differs from $SG_{\beta^2/8\pi}$ by some signs in the S-matrices.

It turns out that the $SG_{\beta^2/8\pi}$ theory is closely related to a second perturbed orbifold CFT at the different radius $\widetilde{R}_{orb} = R_{orb}/2$, and this is what corresponds to Ising_h^2. To see this, first redefine $\phi = \pi/2 - \widetilde{\phi}$, so that the potential in (4.3) becomes $V(\widetilde{\phi}) = \sin\widehat{\beta}\widetilde{\phi}$. The potential now satisfies

$$V(\widetilde{\phi} + 2\pi\widetilde{R}_{orb}) = -V(\widetilde{\phi}) = V(-\widetilde{\phi}). \tag{4.4}$$

Since in an orbifold, $\widetilde{\phi} \sim -\widetilde{\phi}$, we see that the potential preserves the orbifold symmetry at $\widetilde{R}_{orb} = R_{orb}/2 = \sqrt{\pi/\beta}$. For $\beta^2/8\pi = 1/8$, this corresponds to $\widetilde{R}_{orb} = 1$.

We will resolve the distinction of Ising_h^2 from $SG_{1/8}$ and $D_8^{(1)}$ by appealing to the formulation of the last section based on ${}_q d_3^{(2)}$. As we will show below, the spectrum is the same as for $SG_{1/8}$, and the S-matrices again differ only by some signs. The final result can be anticipated more simply as follows. Let s_1, s_2 denote the excitations corresponding to the $SG_{1/8}$ solitons, with mass m_s. Their S-matrices are

$$S_{s_1 s_1} = S_{s_2 s_2} = \sigma' \; S_{s_1 s_2} = -\widetilde{\sigma} \; F_{1/7}(\theta) F_{2/7}(\theta) F_{3/7}(\theta), \tag{4.5}$$

where

$$F_\alpha(\theta) \equiv \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)} \;, \tag{4.6}$$

and θ is the rapidity variable, $E = m \cosh \theta$. The $SG_{1/8}$ model corresponds to $(\sigma' = 1, \widetilde{\sigma} = 1)$, whereas for the $D_8^{(1)}$ model $(\sigma' = -1, \widetilde{\sigma} = -1)$. We claim that the Ising_h^2 model corresponds to the 3rd possibility

$$\mathrm{Ising}_h^2 : \qquad \sigma' = 1, \widetilde{\sigma} = -1. \tag{4.7}$$

on the basis of the following argument. In the $SG_{1/8}$ model, there is a $U(1)$ symmetry under which s_1 and s_2 are charge conjugate states. Crossing symmetry then implies $S_{s_1 s_1} = S_{s_1 s_2}$, i.e. $\sigma' = 1$. The breathers of the SG model are $s_1 - s_2$ bound states, which implies a positive imaginary residue in the corresponding poles of $S_{s_1 s_2}$, and this fixes $\widetilde{\sigma} = 1$. For the $D_8^{(1)}$ model on the other hand, since it is a perturbation of an orbifold theory, the $U(1)$ symmetry is broken to Z_2, and this allows $\sigma' = -1$ since s_1, s_2 are no longer charge conjugated particles; the sign $\widetilde{\sigma}$ implies that the breathers continue to be $s_1 - s_2$ bound states. For the choice (4.7), the first breather is neither a $s_1 - s_1$, $s_2 - s_2$, nor $s_1 - s_2$ bound state, because none of these S-matrices have a positive imaginary residue. We will show how this arises from the ${}_q d_3^{(2)}$ description.

The remaining S-matrices of the Ising_h^2 model are the same as for $SG_{1/8}$. There are 6 neutral excitations with mass

$$m_a = m_1 \frac{\sin \frac{a\pi}{14}}{\sin \frac{\pi}{14}} \quad , \qquad m_s = m_1 \frac{1}{2 \sin \frac{\pi}{14}}, \qquad a = 1, 2, .., 6 \tag{4.8}$$

and exact S–matrix amplitudes given by

$$\begin{aligned} S_{ab}(\theta) &= \left(\frac{|a-b|}{14} \right) \left[\prod_{k=1}^{\min(a,b)-1} \left(\frac{|a-b| + 2k}{14} \right) \right]^2 \left(\frac{a+b}{14} \right) \; ; \\ S_{a s_1}(\theta) &= S_{a s_2}(\theta) = (-1)^a \prod_{k=0}^{a-1} \left(\frac{7 - a + 2k}{14} \right) \;, \end{aligned} \tag{4.9}$$

where we have used $(\alpha) \equiv F_\alpha(\theta)$. Note that, as anticipated at the beginning of this section, the kinks of this system behave indeed like ordinary particles, since their S–matrix can be entirely written in terms of the simple functions (4.6).

Now let us describe how the above result follows from the RSOS restriction of ${}_q d_3^{(2)}$. We denote the relevant $SO(4)_q$ representations as $\{0\}, \{4\}, \{6\}$ for the singlet, vector, and adjoint representations, respectively. The unrestricted S-matrix for the $\{4\}$ can be written as [29][33]

$$S_{\{4\}\{4\}}(\theta) = \mathcal{F}(\theta)\ \tau_{21}\ \widehat{R}_{\{4\}\{4\}}(x,q)\ \tau_{12}^{-1}, \tag{4.10}$$

where $\mathcal{F}$ is a scalar factor, $\widehat{R}_{\{4\}\{4\}}(x,q)$ is the R-matrix for ${}_q d_3^{(2)}$ multiplied by the permutation operator $\mathbf{P}$, and τ_{12} is a gauge transformation. The R-matrix has the explicit form

$$\check{R}_{\{4\}\{4\}} = \check{P}_{\{9\}} + \left(\frac{1-xq^2}{x-q^2}\right)\check{P}_{\{6\}} + \left(\frac{1+xq}{x+q}\right)\check{P}_{\{0\}} \quad , \tag{4.11}$$

where $\mathbf{P}\check{P}_\rho$ is a projector onto the $SO(4)_q$ representation ρ, and $x = \exp\left((k+6)\theta/(k+2)\right)$. The scalar factor is given by

$$\mathcal{F}(\theta) = \frac{G_1(\theta)G_{1-k/2}(\theta)}{G_0(\theta)G_{-k/2}(\theta)} \ , \tag{4.12}$$

where

$$G_\alpha(\theta) = \prod_{j=1}^{\infty} \frac{\Gamma\left(\frac{k+6}{k+2}\left(j-\frac{i\theta}{2\pi}\right)-\frac{\alpha}{k+2}\right)\Gamma\left(\frac{k+6}{k+2}\left(j-\frac{i\theta}{2\pi}\right)-\frac{4-\alpha}{k+2}\right)}{\Gamma\left(\frac{k+6}{k+2}\left(j+\frac{i\theta}{2\pi}\right)-\frac{\alpha}{k+2}\right)\Gamma\left(\frac{k+6}{k+2}\left(j+\frac{i\theta}{2\pi}\right)-\frac{4-\alpha}{k+2}\right)} \ . \tag{4.13}$$

For the Ising_h^2 case, one must restrict the model at the root of unity $q = -e^{i\pi/3}$, where $x = e^{7\theta/3}$. Specializing the formula(4.12), one finds

$$\mathcal{F}(\theta) = \left(\frac{x+q}{xq+1}\right) F_{1/7}(\theta) \ . \tag{4.14}$$

To perform the quantum group restriction we must examine the fusion ring of $SO(4)_q$ at the above q. Recalling that $SO(4) = SU(2) \otimes SU(2)$, let us label the $SO(4)_q$ representations as $(j, \tilde{j})$, where j denotes the spin j representation of SU(2) with dimension $2j+1$, and similarly for $\tilde{j}$. The fundamental spinorial representations of $SO(4)$ are the $(0,1/2), (1/2,0)$; there are no fundamental multiplets of solitons in these representations. One also has $\{0\} = (0,0)$, $\{4\} = (1/2,1/2)$, $\{6\} = (0,1) \oplus (1,0)$, and $\{9\} = (1,1)$. At this root of unity, the $SU(2)$ fusion ring has a maximum spin $j = 1/2$, and $(0)(1/2) = (1/2); (1/2)(1/2) = (0)$, where (j) is an $su(2)$ spin j representation. Since the $\{6\}$ of $SO(4)_q$ requires $j = 1$, it is projected out of the spectrum. This leaves only the RSOS restriction of the $\{4\}$, which is frozen and then behave like a scalar particle[7]. The restriction leaves only the $\check{P}_{\{0\}}$ term in $\check{R}_{\{4\}\{4\}}$. Letting '1' denote the RSOS restriction of the fundamental soliton $\{4\}$, one then obtains the scalar S-matrix

$$S_{11}(\theta) = F_{1/7}(\theta). \tag{4.15}$$

This is the S-matrix for the lightest sine-Gordon breather of $SG_{1/8}$. Let m_1 be the mass of this particle. Then closing the bootstrap for this particle leads to a total of six particles with masses and S-matrices given in (4.8)(4.9). This is the spectrum proposed in [25].

In the above analysis it is easy to overlook additional particles for the following reason. Though the $\{6\}$ is projected out, any $\{6\} - \{6\}$ bound states which are scalars survive the restriction. These are the particles denoted as the ${}_q d_3^{(2)}$ breathers $B_1^{(2)}, B_2^{(2)}$ in [29][8]. According to [29], the mass of the particle $B_1^{(2)}$ is given by

$$M_{B_1^{(2)}} = 2m_1 \cos(\pi/7) \sin(3\pi/14). \tag{4.16}$$

The mass of this particle can be identified with that of the $SG_{1/8}$ soliton due the identity $4\cos(\pi/7)\sin(3\pi/14) = 1/\sin(\pi/14)$ (the latter identity is only valid due to the 14-th roots of unity involved). One can also check that the S-matrices involving the particle $B_1^{(2)}$ computed in [29] indeed correspond to those in $SG_{1/8}$ with the assignment of signs (4.7). Similarly, the particle $B_2^{(2)}$ is identified with the 4-th $SG_{1/8}$ breather.

So far we have the 6 breathers and one soliton of the $SG_{1/8}$ theory. A second soliton can be seen as necessary for the following reasons. The S-matrix for the scattering of the 1st and 6th $SG_{1/8}$ breathers is $S_{16} = F_{1/2}F_{5/14}$. The factor $F_{1/2}$ has a double pole at $\theta = i\pi/2$. This corresponds to a "bound state" of mass $M^2 = m_1^2 + m_6^2 = (2m_s)^2$, i.e. to a state right at the threshold of a 2-soliton state. The fact that this pole indeed corresponds to a 2–soliton state is easily verified by checking that the S-matrices for this "bound state" with a particle a, as computed from the bootstrap, is equal to $(S_{sa})^2$. Further reasons for this double degeneracy will be given in the general case ahead.

Due to the signs in (4.7), the bound state structure of the model Ising_h^2 is different from those of $SG_{1/8}$ and $D_8^{(1)}$. For the Ising_h^2 model, closing the bootstrap starting from

[7] The same freezing of degrees of freedom occurs when one restricts the SG S-matrix to obtain the energy perturbation of the Ising model.

[8] The breather $B_1^{(1)}$ is already included as the second $SG_{1/8}$ breather.

the solitons s_1 and s_2, and requiring a positive imaginary residue, leads to the 2nd, 4th, and 6th breathers. The odd breathers arise by closing the bootstrap starting from the 1st breather, which is viewed as a fundamental particle. We remark that the sign differences of the S-matrices in (4.5) do not change the TBA analysis of the ultraviolet central charge, which therefore reproduces correctly $c = 1$ for all three cases, $SG_{1/8}$, $D_8^{(1)}$ and Ising_h^2.

4.2. General Case of $\mathcal{M}_k^\sigma$

The fundamental solitons in the $\{4\}$ and $\{6\}$ of $SO(4)_q$ become RSOS kinks $K_{\rho_2\rho_1}^{\{4\}}$ and $K_{\rho_2\rho_1}^{\{6\}}$ with RSOS indices ρ_2, ρ_1 labeling representations of $SO(4)_q$. The kinks $K^{\{6\}}$ are $K^{\{4\}}$ bound states occurring at the bootstrap fusion pole $\theta = 2i\pi/(k+6)$. As in section 3, we use the decomposition $SO(4) = SU(2) \otimes SU(2)$ to label $SO(4)_q$ representations as $(j, \tilde{j})$, where $j, \tilde{j} \in \mathcal{Z} + 1/2$ are $SU(2)$ spins. The selection rule on the kink $K_{\rho_2\rho_1}^{\rho_0}$ is that the representation ρ_2 must appear in the tensor product $\rho_1 \times \rho_0$ within the fusion ring of $SO(4)_q$. Hence, we will need the $SU(2)_q$ fusion ring at $q = q^{(h)} = -\exp(i\pi/(k+2))$:

$$(j_1) \times (j_2) = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} (j), \tag{4.17}$$

with $j \leq k/2$.

Since $\{4\} = (1/2, 1/2)$, the $\{4\}$ fundamental solitons become the RSOS kinks:

$$K^{\{4\}}_{(j_2\tilde{j}_2)(j_1\tilde{j}_1)}(\theta), \qquad j_2 \in j_1 \times 1/2, \ \tilde{j}_2 \in \tilde{j}_1 \times 1/2. \tag{4.18}$$

Similarly, since $\{6\} = (0,1) \oplus (1,0)$ there are two kinds of RSOS $\{6\}$ kinks:

$$\begin{aligned} &K^{\{6\}}_{(j_2\tilde{j}_2)(j_1\tilde{j}_1)}(\theta), \qquad j_2 \in j_1 \times 1, \ \tilde{j}_2 = \tilde{j}_1 \\ &\tilde{K}^{\{6\}}_{(j_2\tilde{j}_2)(j_1\tilde{j}_1)}(\theta), \qquad j_2 = j_1, \ \tilde{j}_2 \in \tilde{j}_1 \times 1. \end{aligned} \tag{4.19}$$

The mass ratio of $K^{\{6\}}, \tilde{K}^{\{6\}}$ to $K^{\{4\}}$ is given in (4.1).

In addition to the above kinks there are breathers, which are scalar kink-kink bound states. Let $B_p^{(\{4\})}$, $p = 1, 2, ...$ denote the $K^{\{4\}} - K^{\{4\}}$ bound state breathers, and $B_p^{\{6\}}$ the $K^{\{6\}} - K^{\{6\}}$ breathers. From the results in [29], one can reach the following conclusions. As k increases one enters a repulsive regime wherein most breathers become unbound and disappear from the spectrum. The $B_1^{\{4\}}, B_2^{\{6\}}$ breathers occur at the fusion pole $\theta = i\pi(2-k)/(k+6)$, whereas the $B_1^{\{6\}}$ breather occurs at $4i\pi/(k+6)$. When $k = 2$, for the mass of these breathers we have $M(B_1^{([4])}) = 2M_{\{4\}}$ and $M(B_2^{\{6\}}) = 2M_{\{6\}}$, thus $k=2$ is the threshold value for these breather state and they disappear. The only remaining breather for all $k \geq 2$ is $B_1^{\{6\}}$, which we denote simply as B, with a mass given by

$$M_B = 4M_{\{4\}} \cos\left(\frac{\pi}{k+6}\right) \cos\left(\frac{2\pi}{k+6}\right) \ . \tag{4.20}$$

The S-matrix for this breather is

$$S_{BB}(\theta) = F_{\frac{k+2}{k+6}}(\theta) F_{\frac{k+4}{k+6}}(\theta) F_{\frac{k}{k+6}}(\theta) \ . \tag{4.21}$$

Since the breather B can arise as a bound state of either $K^{\{6\}}$ or of $\widetilde{K}^{\{6\}}$, we believe this breather is doubly degenerate; certainly it is doubly degenerate in the Ising case where it is the $SG_{1/8}$ soliton. The threshold for the disappearance of the B breather, i.e. when $M_B = 2M_{\{6\}}$, occurs at $k = \infty$.

Let us now come back to the problem of the S–matrix of the kink states. The S-matrices of the kink states are characterized by the exchange relation:

$$K^{\{4\}}_{(j_3\widetilde{j}_3)(j_2\widetilde{j}_2)}(\theta_2) K^{\{4\}}_{(j_2\widetilde{j}_2)(j_1\widetilde{j}_1)}(\theta_1) = \sum_{(j_4\widetilde{j}_4)} S^{(j_4\widetilde{j}_4)(j_1\widetilde{j}_1)}_{(j_3\widetilde{j}_3)(j_2\widetilde{j}_2)}(\theta_2 - \theta_1)\ K^{\{4\}}_{(j_3\widetilde{j}_3)(j_4\widetilde{j}_4)}(\theta_1) K^{\{4\}}_{(j_4\widetilde{j}_4)(j_1\widetilde{j}_1)}(\theta_2) \tag{4.22}$$

and similarly for the scattering involving $K^{\{6\}}$. The S-matrix in (4.22) follows from (4.10) with $x = \exp(4\lambda\theta)$, $q = -\exp(i\pi\omega)$ where we have defined $\lambda = (k+6)/4(k+2)$ and $\omega = 1/(k+2)$:

$$S^{(j_4\widetilde{j}_4)(j_1\widetilde{j}_1)}_{(j_3\widetilde{j}_3)(j_2\widetilde{j}_2)}(\theta) = \mathcal{F}(\theta) \big[P_{\{9\}} - \frac{\sinh(2\lambda\theta + i\pi\omega)}{\sinh(2\lambda\theta - i\pi\omega)} P_{\{6\}} + \frac{\cosh(2\lambda\theta + i\pi\omega/2)}{\cosh(2\lambda\theta - i\pi\omega/2)} P_{\{0\}} \big]^{(j_4\widetilde{j}_4)(j_1\widetilde{j}_1)}_{(j_3\widetilde{j}_3)(j_2\widetilde{j}_2)} \tag{4.23}$$

with $\mathcal{F}$ the same as in (4.12), and with the projectors in RSOS form. The latter form of the projectors can be expressed in terms of $q-6j$ symbols, as we now describe. Clearly one has

$$P_{(j,\widetilde{j})} = P_j \widetilde{P}_{\widetilde{j}} \ , \tag{4.24}$$

where P_j is the projector onto the spin j representation of $SU(2)_q$ in the tensor product space $1/2 \times 1/2$, and similarly for the second copy $\widetilde{P}_{\widetilde{j}}$. One also needs

$$P_{\{0\}} = P_0 \widetilde{P}_0, \qquad P_{\{6\}} = P_0 \widetilde{P}_1 + P_1 \widetilde{P}_0, \qquad P_{\{9\}} = P_1 \widetilde{P}_1 \ . \tag{4.25}$$

The projectors P_j in unrestricted vertex form have matrix elements expressed in terms of q-Clebsch-Gordon coefficients:

$$\langle 1/2, m_3; 1/2, m_4 | P_j | 1/2, m_1; 1/2, m_2 \rangle = \sum_m \langle 1/2, m_3; 1/2, m_4 | j, m \rangle_q \langle j, m | 1/2, m_1; 1/2, m_2 \rangle_q \tag{4.26}$$

Going to the RSOS basis, and using the identity

$$\begin{aligned}
&(\langle j, m | j_1, m_1; j_{23}, m_{23} \rangle_q) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_2 3 \end{Bmatrix}_q \\
= &\sum_{\substack{m_2, m_3; \\ m_2 + m_3 = m - m_1}} \langle j_{23}, m_{23} | j_2, m_2; j_3, m_3 \rangle_q \langle j, m | j_{12}, m_{12}; j_3, m_3 \rangle_q \langle j_{12}, m_{12} | j_1, m_1; j_2, m_2 \rangle_q
\end{aligned} \tag{4.27}$$

one obtains the simple result [34][35]

$$(P_j)^{j_4 j_1}_{j_3 j_2} = \begin{Bmatrix} 1/2 & 1/2 & j \\ j_3 & j_1 & j_4 \end{Bmatrix}_q \begin{Bmatrix} j_3 & 1/2 & j_2 \\ 1/2 & j_1 & j \end{Bmatrix}_q . \tag{4.28}$$

The $q-6j$ symbols can be found in [35][18]. The complete S-matrix follows from (4.23)(4.24) and (4.25), along with the evident relation

$$\left(P_{(j,\widetilde{j})} \right)^{(j_4 \widetilde{j}_4)(j_1 \widetilde{j}_1)}_{(j_3 \widetilde{j}_3)(j_2 \widetilde{j}_2)} = (P_j)^{j_4 j_1}_{j_3 j_2} \left(\widetilde{P}_{\widetilde{j}} \right)^{\widetilde{j}_4 \widetilde{j}_1}_{\widetilde{j}_3 \widetilde{j}_2} . \tag{4.29}$$

The analog of the formula (4.11) involving $\{6\}$ fundamental solitons is unknown. However, the kinks $K^{\{6\}}, \widetilde{K}^{\{6\}}$ are bound states of the kinks $K^{\{4\}}$ occurring at the fusion pole $\theta = 2i\pi/(k+6)$. Therefore, the S-matrices involving the $\{6\}$-kinks can in principle be computed from bootstrap fusion[9].

4.3. The $k = \infty$ limit

When $k = \infty$, the model $\mathcal{M}^\sigma_k$ corresponds to two level-1 $SU(2)$ current algebras coupled via their primary field in the spin 1/2 representation of dimension 1/4. Denoting the latter by $\mathbf{\Phi}^{1/2}$, the action (2.3) becomes

$$\mathcal{A} = \mathcal{A}_{su(2)_1} + \mathcal{A}_{su(2)_2} + \lambda \int d^2x \; \mathbf{\Phi}_1^{1/2} \mathbf{\Phi}_2^{1/2}, \tag{4.30}$$

[9] The spectrum proposed in [25] does not contain the kinks $K^{\{6\}}, \widetilde{K}^{\{6\}}$ nor the breather B. Also, the conjectured S-matrices for the $K^{\{4\}}$ kinks were constructed by borrowing RSOS solutions of the Yang-Baxter equation which define certain lattice statistical mechanics models in [36]. We have not checked if they agree with the S-matrices constructed here.

where $su(2)_{1,2}$ refers to the copies 1 and 2 of the current algebra. Above,

$$\mathbf{\Phi}_{1,2}^{1/2} = \sum_{m=\pm 1/2} \phi_{1,2}^{(1/2,m)} \overline{\phi}_{1,2}^{(1/2,-m)}, \tag{4.31}$$

where $\phi^{(1/2,m)}$, and $\overline{\phi}^{(1/2,m)}$ are the left and right moving factors.

The current algebras can each be bosonized with a scalar field $\varphi_{1,2}$. The primary fields have the representation:

$$\phi_{1,2}^{(1/2,m)} = e^{i\sqrt{2}m\varphi_{1,2}}, \qquad \overline{\phi}_{1,2}^{(1/2,m)} = e^{-i\sqrt{2}m\overline{\varphi}_{1,2}} . \tag{4.32}$$

Thus,

$$\mathbf{\Phi}_1^{1/2}\mathbf{\Phi}_2^{1/2} = \left(e^{i\eta_1/\sqrt{2}} + e^{-i\eta_1/\sqrt{2}}\right)\left(e^{i\eta_2/\sqrt{2}} + e^{-i\eta_2/\sqrt{2}}\right) , \tag{4.33}$$

where $\eta_{1,2} = \varphi_{1,2} + \overline{\varphi}_{1,2}$ are local scalar fields. Finally, the interaction can be expressed in terms of fermion bilinears:

$$\mathbf{\Phi}_1^{1/2}\mathbf{\Phi}_2^{1/2} = \psi_+\overline{\psi}_- + \overline{\psi}_+\psi_- + \psi'_+\overline{\psi'}_- + \overline{\psi}'_+\psi'_-, \tag{4.34}$$

with

$$\psi_+\overline{\psi}_- = e^{i(\eta_1+\eta_2)/\sqrt{2}} \quad , \quad \psi'_+\overline{\psi}'_- = e^{i(\eta_1-\eta_2)/\sqrt{2}} . \tag{4.35}$$

Since the fermions are complex, combined together they correspond to 4 real fermions. The interaction simply gives each real fermion the same mass. Thus, as $k \to \infty$, the model $\mathcal{M}_k^\sigma$ becomes the free field theory of 4 real massive fermions with $SO(4)$ symmetry. This result is closely related to the lattice model results obtained in[14][10].

The above result arises from the restricted ${}_q d_3^{(2)}$ symmetry in the following way. Firstly, as $k \to \infty$, in the $(j, \widetilde{j})$ labeling of $SO(4)$ representations, we have $j_{\max} = \infty$; this implies that the RSOS S-matrices are unrestricted (SOS) and by a change of basis can be brought back to unrestricted vertex form. Secondly, since $M_{\{6\}} = 2M_{\{4\}}$, $k = \infty$ is the threshold for the disappearance of the fundamental solitons of mass $M_{\{6\}}$. Also, $M_B = 2M_{\{6\}}$, so that the breather B also disappears. This leaves a 4-plet of solitons transforming under the undeformed vector of $SO(4)$, and these are the 4 real fermions. Finally, since $q \to -1$, the ${}_q d_3^{(2)}$ is undeformed. It is known that $2n$ free massive fermions has an undeformed $a_{2n-1}^{(2)}$ symmetry algebra[37][38], thus the S-matrices above must become free as $k \to \infty$.

[10] For the lattice model considered in [14] the $SO(4)$ symmetry is broken to $\mathcal{Z}_2 \times SU(2)$ which leads to a triplet and a singlet of fermions of different mass.

5. Conclusions

In this paper we have studied the on–shell dynamics of coupled conformal field theories under the constraint of the integrability for the inter-layer coupling. A general framework for this kind of models is provided by the reduction of Affine Toda Field Theory associated with particular Dynkin diagrams. These are the Dynkin diagrams of the affine algebra $\widehat{g}$ which have the property that upon removing one of its nodes, one is left with two decoupled Dynkin diagrams g_1 and g_2 of finite dimensional simply laced Lie algebras: the latter represent then the Lie algebras from which tensor product of two minimal models is constructed using the coset construction. The removed node, on the other hand, specifies a particular integrable coupling between these two minimal models. An interesting model of this class is represented by the two–layer Ising model coupled by the magnetic operators: this model has been analyzed both in terms of the RSOS restriction of ${}_q d_3^{(2)}$ as well as in terms of a bosonization scheme related to the Sine-Gordon model. It would be interesting to pursue further the analysis of this model as well as of the others by computing their form factors and their correlation functions: quantities particularly interesting in this respect would be the correlators involving operators living on the two different planes, as for instance the correlator $\langle \sigma_1(x) \sigma_2(y) \rangle$ for the two–layer Ising model. Also it is clear that there are other interesting examples of the Dynkin gymnastics used in this paper. Finally, at a speculative level, one might ask if our integrable analysis of two (or four) coupled minimal models, discussed in this paper, could be some first step in understanding N coupled conformal field theories. If this could be done systematically, one may perhaps hope to be able to learn something about three dimensional theories.

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