CNLS 97/1500
TECHNION-PH 97-06

$1/N_c$ Expansion for Excited Baryons

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(July 27, 1997)

Abstract

We derive consistency conditions which constrain the possible form of the strong couplings of the excited baryons to the pions. The consistency conditions follow from requiring the pion-excited baryon scattering amplitudes to satisfy the large-N_c Witten counting rules and are analogous to consistency conditions used by Dashen, Jenkins and Manohar and others for s-wave baryons. The consistency conditions are explicitly solved, giving the most general allowed form of the strong vertices for excited baryons in the large-N_c limit. We show that the solutions to the large-N_c consistency conditions coincide with the predictions of the nonrelativistic quark model for these states, extending the results previously obtained for the s-wave baryons. The $1/N_c$ corrections to these predictions are studied in the quark model with arbitrary number of colors N_c.

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I. INTRODUCTION

The successes of the nonrelativistic quark model (NRQM) in describing the baryon spectroscopy and couplings [1] have remained for a long time something of a mystery. Recent work by several groups [2, 3, 4, 5, 6, 7, 8, 9], most notably by Dashen, Jenkins and Manohar (see also earlier related work in [10]) helped to shed light on this problem and clarify the relation of the NRQM to QCD. These works showed that the predictions of the NRQM for low-lying s-wave baryons follow from QCD in the large-N_c limit [11] as a consequence of the counting rules of Witten [12, 13, 14] for pion-baryon scattering amplitudes. In this way they have been able to derive consistency conditions which constrain the mass splittings, pion couplings and magnetic moments of ground-state baryons up to order $O(1/N_c)$ in the $1/N_c$ expansion.

The nonrelativistic quark model has been used to describe also the properties of the orbitally excited baryons. The realization of the fact that these states can be accounted for in the quark model has been one of the first significant arguments in its favor [15]. Later works applied the quark model to explaining the phenomenology of the strong decays of the $L = 1$ baryons to the ground state baryons. The measured decay widths have been found to be well described by a fit to the quark model predictions [16, 17, 18]. When supplemented with dynamic assumptions, the quark model can be also used to make more detailed predictions about the mass spectrum and decay properties of these states [19, 20, 21].

In addition to the quark model, various other approaches have been employed to describe the orbital excitations of baryons. Among them the Skyrme model, which is closely related to the large-N_c approximation, has been used to construct these states as bound states of a soliton and a meson [22, 23, 24, 25, 26]. A bag model description of these states has been given in [27]. The properties of the negative parity baryons have been investigated also with the help of the QCD sum rules in [28, 29, 30]. More recently, in [31] the structure of the mass spectrum of the excited baryons has been studied using an effective Hamiltonian motivated by large-N_c arguments.

Following the recent progress in understanding the predictions of the quark model for ground state baryons, some effort has been also directed into explaining the analogous predictions for the excited baryons sector. Thus, in [18] the data on the strong decays of these states have been used to test the idea that the large-N_c limit might provide an explanation for the validity of the quark model description. The authors of [18] adopted a Hartree description with the number of quarks in the baryon fixed to its physical value $N_c = 3$. The large-N_c expansion has been implemented at the level of operators mediating the strong decays, which can be classified according to their order in $1/N_c$. A fit to the experimental data on strong decays of the $L = 1$ baryons in the **70** of SU(6) gave the result that the naive quark model, containing only one-body operators, reproduces the experimental data to a good precision. On the other hand, two-body operators which could contribute to same order in $1/N_c$ as those kept in the quark model, appear to be suppressed in Nature for reasons seemingly unrelated to the large-N_c expansion. From this, the authors of [18] concluded that there might be more than large-N_c to the success of the quark model relations.

In this paper we study the strong pion couplings of the orbitally excited baryons, both light and heavy, in the large-N_c limit using as input constraints on pion-baryon scattering amplitudes following from the counting rules of Witten. This approach is closer in spirit to the one used in [2, 3] by Dashen, Jenkins and Manohar. We derive in this way consistency conditions which constrain the possible form of the strong coupling vertices, which are then solved explicitly. Our final conclusion is that the model-independent results obtained from solving the consistency conditions are the same as those following from the quark model in the large-N_c limit, thus extending the statements of [2, 3, 4, 5, 6] to the excited baryons' sector. We stress that our results do not conflict with the conclusions of [18]. Rather, the findings of [18] can be formulated in the light of our results as a puzzle: why does the quark model work better than it should?

Our paper is structured as follows. We begin by introducing in Section II the spectrum of the orbital excitations and constructing its generalization to the large-N_c limit. The structure of these states is more complex than for the case of the s-wave baryons. We introduce the concept of P-spin to deal with the mixed symmetry spin-flavor states and point out an additional problem connected with the appearance of spurious unphysical states in the $N_c > 3$ case. Section III contains the derivation of the consistency conditions for strong coupling vertices. These arise from a mismatch between the scaling power with N_c of the meson-baryon vertices and the Witten scaling law for the meson-baryon scattering amplitudes. The consistency conditions are explicitly solved in Section III giving the most general solution for S-, P- and D-wave pion couplings in the large-N_c limit. We show in Section IV that the solutions to the consistency conditions actually coincide with the predictions of the constituent quark model in the large-N_c limit. The orbital excitations are first explicitly constructed in the quark model with u and d quarks only and arbitrary number of colors N_c. Armed with these wavefunctions, we develop the machinery necessary to compute the strong coupling vertices of these states. A by-product of this quark model calculation is a determination of the large-N_c scaling law of the decay vertices, which exhibits a surprising dependence on the symmetry type of the excited state involved. We conclude in Section V with a summary and outlook on our results. Appendix A contains an exact calculation in the quark model of the strong coupling vertices for an arbitrary value of N_c. These results are used to examine the structure of the $1/N_c$ corrections to the large-N_c relations obtained in Section III. Appendix B presents the quark model calculation of the strong couplings among excited states transforming under the mixed symmetry representation of SU(4).

II. SPIN-FLAVOR STRUCTURE OF THE EXCITED BARYONS

In the large-N_c limit, the s-wave baryons containing only u, d quarks form $I = J$ towers of degenerate states. Both possibilities $I = J = 1/2, 3/2, \cdots$ and $I = J = 0, 1, \cdots$ are of physical significance, the former corresponding to the light baryons and the latter to baryons with one heavy quark (in this case J is to be interpreted as the angular momentum of the light degrees of freedom). Baryons with strangeness can be also incorporated in the large-N_c limit as separate towers of states, each labeled by a quantum number K related to the number of strange quarks as $K = \frac{1}{2}n_s$. For each K tower, the spin J and isospin I take values restricted by the condition $|I - J| \leq K$.

This picture is precisely the same as the one predicted by the NRQM with SU(4) spin-flavor symmetry. In NRQM language the s-wave baryons have orbital wavefunctions which are completely symmetrical under permutations of two quarks. This constrains their spin-isospin wavefunction to transform also under the completely symmetric representation of

SU(4), which contains the I, J values given above. Spin- and flavor-independence of the interquark forces in the NRQM is responsible for the degeneracy of all these states. Fig.1 shows the Young diagram of the totally symmetric representation of SU(4) for $N_c = 3$ and its extension to the case of arbitrary N_c.

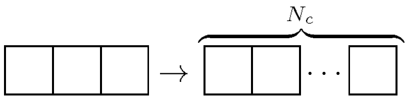

Fig.1. Young tableaux for the SU(4) representation of the s-wave baryons for $N_c = 3$ and in the large-N_c limit.

The spectrum of the p-wave baryons has a more complicated structure. The spin-flavor wavefunction of the light baryons has mixed symmetry, transforming for $N_c = 3$ as a **70** under SU(6) and as a **20** under SU(4). To keep our results as general as possible and to avoid some ambiguities connected with the identification of the large-N_c states with physical states, we will not assume SU(3) symmetry. Just as in the case of the s-wave baryons, we will divide the p-wave states into sectors with well-defined strangeness and assume only isospin symmetry. We extrapolate the mixed symmetry representation from $N_c = 3$ to the large-N_c case as shown in Fig.2.

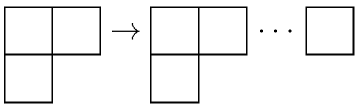

Fig.2. SU(4) representations for light p-wave baryons, for $N_c = 3$ and in the large-N_c limit.

Under the isospin-spin SU(2)$\times$SU(2) group this representation splits into (I, S) representations which satisfy $|I - S| \leq 1$ (except for $I = S = N_c/2$ which is only contained in the totally symmetric representation).

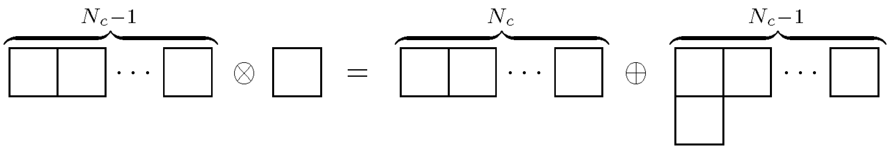

Fig.3. Product of SU(4) representations used in the text for the determination of the (I, S) content of the mixed symmetry representation.

This can be proven by considering the product of SU(4) representations shown in Fig.3 and its decomposition into irreducible representations of $SU(2)_{spin} \times SU(2)_{isospin}$. For definiteness we will take N_c to be odd, although the argument is equally valid also for even values of N_c. The isospin-spin (I, S) content of the product of representations on the l.h.s. can be obtained from the corresponding product

$$\left\{(0,0)\,,(1,1)\,,\cdots(\frac{N_c-1}{2},\frac{N_c-1}{2})\right\} \otimes (\frac{1}{2},\frac{1}{2}) \tag{2.1}$$

and includes all representations of the form $(i \pm \frac{1}{2}, i \pm \frac{1}{2})$ with $i = 1, \cdots, (N_c - 1)/2$. All the representations with $I \neq S$ occur with multiplicity 1. The respresentations with $(I, S) = (i + \frac{1}{2}, i + \frac{1}{2}), (i - \frac{1}{2}, i - \frac{1}{2})$ appear twice, except for $(I, S) = (N_c/2, N_c/2)$ which appears only once. On the other hand, the symmetric representation on the r.h.s. of Fig.3 contains only the $I = S$ representations described above, but with unit multiplicity. Subtracting them from the (I, S) representations on the l.h.s. of Fig.3 we are left with the representation content mentioned above for the mixed symmetry SU(4) representation. This can be further checked by comparing the dimensionality of the SU(4) representation given by the Young diagram in Fig.2 for arbitrary N_c with the sum of the dimensions of the (I, S) representations described above

$$\dim = \frac{1}{2}(N_c - 1)(N_c + 1)(N_c + 2) = \sum_{n=2}^{N_c - 1} \left[n^2 + 2n(n+2) \right] . \tag{2.2}$$

The total baryon spin J is given by $\vec{J} = \vec{S} + \vec{L}$ with $L = 1$. The lowest-lying observed p-wave light baryons containing only u, d quarks are listed in Table 1 together with their quantum numbers in the quark model [32]. Note that the states $(I, S) = (3/2, 3/2)$ which would be present in the large-N_c limit are forbidden in the $N_c = 3$ case for the reason mentioned above.

State	(I, J^P)	$L_{2I,2J}$	(I, S)	$(SU(3), SU(2))$
N(1535)	$(\frac{1}{2}, \frac{1}{2}^-)$	S_{11}	$(\frac{1}{2}, \frac{1}{2})$	$(\mathbf{8}, \mathbf{2})$
N(1520)	$(\frac{1}{2}, \frac{3}{2}^-)$	D_{13}		
N(1650)	$(\frac{1}{2}, \frac{1}{2}^-)$	S_{11}	$(\frac{1}{2}, \frac{3}{2})$	$(\mathbf{8}, \mathbf{4})$
N(1700)	$(\frac{1}{2}, \frac{3}{2}^-)$	D_{13}		
N(1675)	$(\frac{1}{2}, \frac{5}{2}^-)$	D_{15}		
$\Delta(1620)$	$(\frac{3}{2}, \frac{1}{2}^-)$	S_{31}	$(\frac{3}{2}, \frac{1}{2})$	$(\mathbf{10}, \mathbf{2})$
$\Delta(1700)$	$(\frac{3}{2}, \frac{3}{2}^-)$	D_{33}		

Table 1. The p-wave light baryons containing only u, d quarks and their quantum numbers.

It is not difficult to introduce also strangeness in this picture. Because the strange quark is now different from the other $N_c - 1$ quarks in the baryon, the Pauli principle constrains only the symmetry properties of the wavefunction for the latter. In this case both SU(4) representations shown in Figs.1 and 2 are possible. We show in Table 2 the lowest-lying observed and expected p-wave hyperons with one strange quark together with their quark model quantum numbers. For example, the states with $(I, S) = (1, 3/2)$ in Table 2 are completely symmetric under a permutation of the u, d quarks, whereas the states $(I, S) = (0, 3/2)$ are antisymmetric under the same transformation (for $N_c = 3$ the mixed symmetry state is in fact antisymmetric). The symmetric representation corresponds to **10** and the antisymmetric one to **6** of SU(4). The other states in Table 2 are mixtures of both representations.

To construct the analogs of these states in the large-N_c limit, it is convenient to introduce two vectors $\vec{K}$ and $\vec{P}$, which will be called K-spin and P-spin respectively. The K-spin counts the number of strange quarks as described above and takes the value 1/2 for hyperons with one s quark [5]. The P-spin labels the type of permutational symmetry of the u, d quarks' wavefunction in the baryon and is equal to 0 for the symmetric representation and to 1 for the mixed symmetric representation.

With these definitions the total quark spin S of a p-wave baryon takes all the values compatible with

$$\vec{S} = \vec{I} + \vec{K} + \vec{P} \,. \tag{2.3}$$

In addition to this, an exclusion rule must be imposed for $P = 1$, forbidding the following symmetric states

$$|\vec{S} - \vec{K}\,| = I = \frac{N_c}{2} - K \,. \tag{2.4}$$

This exclusion rule is operative only at the top of the large-N_c towers and therefore can be neglected in the large-N_c limit. One should keep however in mind the fact that new unphysical states are introduced in the large-N_c limit which would be otherwise forbidden by this rule.

The classification of the states into symmetric and mixed representations is even more transparent for the p-wave baryons with one heavy quark. In the heavy mass limit the spin and parity of the light degrees of freedom become good quantum numbers. Furthermore, in the NRQM the total spin of the light quarks S_ℓ is also conserved and can be used together with the isospin to identify the permutational symmetry of the state.

State	(I, J^P)	(I, S)	$(SU(3), SU(2))$
$\Lambda(1405)$	$(0, \frac{1}{2}^-)$	$(0, \frac{1}{2})$	$(\mathbf{1}, \mathbf{2})$
$\Lambda(1520)$	$(0, \frac{3}{2}^-)$		
$\Lambda(1670)$	$(0, \frac{1}{2}^-)$	$(0, \frac{1}{2})$	$(\mathbf{1}, \mathbf{2})$
$\Lambda(1690)$	$(0, \frac{3}{2}^-)$		
$\Sigma(1620)$	$(1, \frac{1}{2}^-)$	$(1, \frac{1}{2})$	
$\Sigma(1670)$	$(1, \frac{3}{2}^-)$		
$\Lambda(1800)$	$(0, \frac{1}{2}^-)$	$(0, \frac{3}{2})$	$(\mathbf{8}, \mathbf{4})$
$\Lambda(?)$	$(0, \frac{3}{2}^-)$		
$\Lambda(1830)$	$(0, \frac{5}{2}^-)$		
$\Sigma(1750)$	$(1, \frac{1}{2}^-)$	$(1, \frac{3}{2})$	
$\Sigma(?)$	$(1, \frac{3}{2}^-)$		
$\Sigma(1775)$	$(1, \frac{5}{2}^-)$		
$\Sigma(?)$	$(1, \frac{1}{2}^-)$	$(1, \frac{1}{2})$	$(\mathbf{10}, \mathbf{2})$
$\Sigma(?)$	$(1, \frac{3}{2}^-)$		

Table 2. The p-wave hyperons containing one strange quark and their quantum numbers.

Thus, in the large-N_c limit the symmetric representation will give rise to an $I = S_\ell$ tower of states, in analogy to the situation for the light s-wave baryons (with the total spin of the light quarks S_ℓ taking the place of the total spin J). The mixed symmetry representation will generate also a tower with $|I - S_\ell| \leq 1$, as in the case of the light p-wave baryons. From this the states with $I = S_\ell = (N_c - 1)/2$ will have to be excluded. The total heavy baryon spin J will be given in the general case including also strangeness by

$$\vec{J} = \vec{I} + \vec{S}_\ell + \vec{S}_Q + \vec{K} + \vec{P} + \vec{L} \tag{2.5}$$

with $S_Q = 1/2$ the heavy quark spin.

We emphasize that the use of quark model quantum numbers such as S, S_ℓ, S_Q, etc. does not imply any dynamical assumption on our part and is made with the sole purpose of counting states. All our main results below will be obtained without any assumption of whether these quantities are conserved or not. We use the NRQM just as a convenient language which serves to guide our intuition about the spin and flavor structure of the states of interest.

In the next Section we will study the strong couplings of the excited baryons in the large-N_c limit.

III. CONSISTENCY CONDITIONS FOR EXCITED BARYONS

We will obtain constraints on the pion couplings of the excited baryons by studying both elastic pion scattering on these states and inelastic scattering among s-wave and excited states. The results will follow from a set of consistency conditions, derived by requiring the total scattering amplitude to satisfy large-N_c counting rules [12, 13, 14]. We start by reviewing the large-N_c scaling properties of the different couplings which will be needed.

Pions couple to baryons with a strength proportional to the matrix element of the axial current taken between the corresponding states. In the case of the s-wave baryons this matrix element was parametrized in [2, 3, 5] as

$$\langle J', m', \alpha' | \bar{q}\gamma^i \gamma_5 \frac{1}{2}\tau^a q | J, m, \alpha\rangle = N_c g(X) \langle J', m', \alpha' | X^{ia} | J, m, \alpha\rangle \tag{3.1}$$

with X^{ia} an irreducible tensor operator of spin and isospin 1 and $g(X)$ a reduced matrix element of order 1 in the large-N_c limit. X^{ia} has a large-N_c expansion of the form $X^{ia} = X_0^{ia} + X_1^{ia}/N_c + \cdots$. The matrix element (3.1) grows linearly with N_c because the axial current couples to each of the N_c quarks in the baryon.

We will use a similar parametrization for the matrix element of the axial current taken between two excited baryons

$$\langle J', I'; m', \alpha' | \bar{q}\gamma^i \gamma_5 \frac{1}{2}\tau^a q | J, I; m, \alpha\rangle = N_c g(Z) \langle J', I'; m', \alpha' | Z^{ia} | J, I; m, \alpha\rangle \tag{3.2}$$

where Z^{ia} is again an irreducible tensor operator with $J = I = 1$. This matrix element grows also with N_c for the same reason as in the preceding case.

On the other hand, the axial current matrix elements taken between s-wave and p-wave baryons grow slower than N_c. We parametrize the matrix elements of the time and space components of the axial current as

$$\langle J', m', \alpha' | \bar{q}\gamma^0\gamma_5 \frac{1}{2}\tau^a q | J, I; m, \alpha\rangle = N_c^\kappa g(Y) \langle J', m', \alpha' | Y^a | J, I; m, \alpha\rangle \tag{3.3}$$

$$\langle J', m', \alpha' | \bar{q}\gamma^i\gamma_5 \frac{1}{2}\tau^a q | J, I; m, \alpha\rangle = N_c^\kappa g(Q) q^j \langle J', m', \alpha' | Q^{ij,a} | J, I; m, \alpha\rangle \tag{3.4}$$

with q^μ the momentum of the current. In the quark model the scaling power κ is equal to 1/2 for p-wave baryons transforming under the completely symmetric representation of SU(4) and 0 for baryons transforming according to the mixed symmetry representation of SU(4). This will be proved in Section IV. Y^a is a tensor operator with spin 0 and isospin 1 and $Q^{ij,a}$ has spin 2 and isospin 1 ($Q^{ij,a} = Q^{ji,a}$, $Q^{ii,a} = 0$). The operators Z, Y, Q have expansions in powers of $1/N_c$ of the same form as X.

The pion coupling to the states appearing in (3.1-3.4) is obtained by dividing these matrix elements by the pion decay constant $f_\pi = O(\sqrt{N_c})$ [2, 3, 5, 10]. The consistency conditions of DJM were obtained by considering pion scattering on s-wave baryons $\pi^a(q) + B \to \pi^b(p) + B'$ [2, 3, 5]. The leading contribution to this amplitude arises from two tree graphs with the pions coupling in either order and is given by

$$\mathcal{T} = \frac{N_c^2 g^2(X)}{f_\pi^2} \cdot \frac{q^i p^j}{E(\vec{q})} \left(X_0^{jb\dagger} X_0^{ia} - X_0^{ia} X_0^{jb\dagger} \right) . \tag{3.5}$$

This scattering amplitude is of order N_c, in apparent contradiction with the large-N_c counting rules of Witten according to which it should be of order 1. One concludes therefore that one has

$$\left[X_0^{jb\dagger}, X_0^{ia} \right] = 0 . \tag{3.6}$$

This is the leading order consistency condition of DJM [2, 3, 5, 10].

Taking as target a p-wave baryon, the above reasoning can be extended immediately to the couplings Z, for which one obtains the analogous condition

$$\left[Z_0^{jb\dagger}, Z_0^{ia} \right] = 0 . \tag{3.7}$$

The operators Z^{ia} act only on the space of the p-wave states which are degenerate among themselves and have vanishing matrix elements between p-wave states of different mass.

We would like next to derive consistency conditions involving the couplings Y and Q. In order to do so we consider the scattering amplitude for the process $\pi^a(q)+$ (p-wave) $\to \pi^b(p)+$ (s-wave). The mass splitting between s-wave and p-wave states is of order 1 in the large-N_c limit, so that the initial and final pions will not have the same energy. Adding together the contributions of the diagrams with intermediate s-wave and p-wave baryons we obtain for this case

$$\begin{aligned} \mathcal{T} = \frac{N_c^{1+\kappa} g(Y)}{f_\pi^2} & \left\{ -\frac{p^i E(\vec{q})}{E(\vec{p})} \left(g(X) X^{ia} Y^{b\dagger} - g(Z) Y^{b\dagger} Z^{ia} \right) \right. \\ & \left. + \frac{q^i E(\vec{p})}{E(\vec{q})} \left(g(X) X^{ib\dagger} Y^a - g(Z) Y^a Z^{ib\dagger} \right) \right\} \\ + \frac{N_c^{1+\kappa} g(Q)}{f_\pi^2} & \left\{ -\frac{p^i q^j q^k}{E(\vec{p})} \left(g(X) X^{ia} Q^{jk,b\dagger} - g(Z) Q^{jk,b\dagger} Z^{ia} \right) \right. \\ & \left. + \frac{q^k p^i p^j}{E(\vec{q})} \left(g(X) X^{kb\dagger} Q^{ij,a} - g(Z) Q^{ij,a} Z^{kb\dagger} \right) \right\} . \end{aligned} \tag{3.8}$$

This scattering amplitude is apparently of order $\kappa \geq 0$ which again violates the counting rules of Witten, according to which it should be at most of order $N_c^{-1/2}$ [13]. This requires all the independent kinematical structures to vanish to leading order

$$g(X)X_0^{ia}Y_0^{b\dagger} - g(Z)Y_0^{b\dagger}Z_0^{ia} = 0\,, \quad g(X)X_0^{ib\dagger}Y_0^{a} - g(Z)Y_0^{a}Z_0^{ib\dagger} = 0 \tag{3.9}$$

$$g(X)X_0^{ia}Q_0^{jk,b\dagger} - g(Z)Q_0^{jk,b\dagger}Z_0^{ia} = 0\,, \quad g(X)X_0^{kb\dagger}Q_0^{ij,a} - g(Z)Q_0^{ij,a}Z_0^{kb\dagger} = 0\,. \tag{3.10}$$

All of our conclusions about the pion couplings of the excited baryons in the large-N_c limit will follow from the set of consistency conditions (3.7,3.9,3.10). In the present paper we restrict ourselves to the leading order in the large-N_c expansion. Therefore, to simplify the notation, we will drop the index 0 on the coupling operators throughout in the following.

A. Consistency condition for Z

The consistency condition for Z^{ia} (3.7) is completely identical in form to the one for X^{ia} (3.6) which has been studied in detail in [2, 5]. These authors showed that X^{ia} forms, together with the generators of the spin-isospin SU(2)×SU(2) group J^i, I^a a contracted SU(4) algebra. Every possible solution for X^{ia} corresponds to a particular irreducible representation of this algebra. The most general irreducible representation can be labeled by a spin vector $\vec{\Delta}$, in terms of which the basis states of the representation are constructed as $\vec{J} = \vec{I} + \vec{\Delta}$.

In principle it would be possible to take over the results of [5] for X^{ia} and write down directly the matrix elements of Z^{ia}. We will prefer however to construct the solution for Z^{ia} by using a NRQM-inspired ansatz. Besides reproducing the result of [5], this approach has the advantage of suggesting a method for obtaining the solution of the consistency conditions (3.9,3.10). In retrospect, this will furnish also a proof of the validity of the NRQM predictions for excited baryons in the large-N_c limit.

We begin by parametrizing the matrix elements of Z^{ia} taken between states belonging to Δ- and Δ'-towers respectively as

$$\begin{aligned}&\langle J', I'; m', \alpha'|Z^{ia}|J, I; m, \alpha\rangle = \\ &\quad (-1)^{J+I-\Delta}\sqrt{(2I+1)(2J+1)}Z(J', I'; J, I)\langle J', m'|J, 1; m, i\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,.\end{aligned} \tag{3.11}$$

The notation adopted anticipates a result to be proven below, according to which Z only connects towers with $\Delta = \Delta'$. The reduced matrix element $Z(J', I'; J, I)$ depends on the common value of Δ, although for the sake of simplicity this is not made explicit. The normalization coefficient is chosen such that the reduced matrix element is symmetric under a permutation of the initial and final indices $Z(J', I'; J, I) = Z(J, I; J', I')$.

The consistency condition (3.7) can be used to obtain constraints on the reduced matrix elements $Z(J', I'; J, I)$. For this it will be sandwiched between two general states $\langle J', I'; m', \alpha'| \cdots |J, I; m, \alpha\rangle$ and a complete set of intermediate states is inserted. We obtain

$$\begin{aligned}\sum_{J_1 I_1 m_1 \alpha_1} &\langle J', I'; m', \alpha'|Z^{jb\dagger}|J_1, I_1; m_1, \alpha_1\rangle\langle J_1, I_1; m_1, \alpha_1|Z^{ia}|J, I; m, \alpha\rangle \\ &- (Z^{jb\dagger} \leftrightarrow Z^{ia}) = 0\,.\end{aligned} \tag{3.12}$$

This equation can be projected, as in [2], onto the channel with total angular momentum H and isospin K by multiplying it with

$$\langle H',h'|J',1;m',j\rangle\langle H,h|J,1;m,i\rangle\langle K',\eta'|I',1;\alpha',b\rangle\langle K,\eta|I,1;\alpha,a\rangle \tag{3.13}$$

and summing over $m,m',i,j,\alpha,\alpha',a,b$. The resulting consistency condition takes the form

$$\begin{aligned}\sum_{J_1,I_1}(2J_1+1)(2I_1+1)\left\{\begin{array}{ccc} J & 1 & H \\ J' & 1 & J_1\end{array}\right\}\left\{\begin{array}{ccc} I & 1 & K \\ I' & 1 & I_1\end{array}\right\}Z(J',I';J_1,I_1)Z(J_1,I_1;J,I) \\ = (-)^{2(J'+I')}Z(H,K;J',I')Z(H,K;J,I)\,.\end{aligned} \tag{3.14}$$

We will try to guess the solution of this consistency condition by using as guidance the nonrelativistic quark model. Once found, the solution will be seen to be unique by using for example numerical solution of the consistency condition (3.14) or the method of the induced representations [5].

Let us consider for simplicity the case of baryons without strange quarks. Also, let us first assume that the total baryon spin is given by $\vec{J}=\vec{I}+\vec{L}$, which is to say that the baryon will be regarded as containing a "core" of u,d quarks transforming under the symmetric representation of SU(4). The "core" spin S is therefore equal to its isospin I. In addition to this, the orbital angular momentum $\vec{L}$ is added to make up the total spin $\vec{J}$. This corresponds to the case of a heavy baryon transforming under the symmetric representation of SU(4), provided that J is interpreted as the angular momentum of the light degrees of freedom.

The basis states can be easily constructed and are given by

$$|I,L;J,m,\alpha\rangle=\sum_{m_S m_L}|I,L;m_S,m_L,\alpha\rangle\langle J,m|I,L;m_S,m_L\rangle\,. \tag{3.15}$$

The current Z^{ia} becomes in the quark model

$$Z^{ia}\to\sigma^i\otimes\tau^a \tag{3.16}$$

where σ^i acts only on the spin of the u,d quarks $\vec{S}$ and τ^a acts only on the isospin $\vec{I}$.

Therefore the matrix element of Z^{ia} between the states (3.15) can be expressed as

$$\begin{aligned}&\langle I',L';J',m',\alpha'|Z^{ia}|I,L;J,m,\alpha\rangle= \\ &\sum_{m_S m_L m'_S m'_L}\langle I',L';m'_S,m'_L,\alpha'|\sigma^i\otimes\tau^a|I,L;m_S,m_L,\alpha\rangle\langle J',m'|I',L';m'_S,m'_L\rangle\langle J,m|I,L;m_S,m_L\rangle\,.\end{aligned} \tag{3.17}$$

The matrix element in the basis $|I,L;m_S,m_L,\alpha\rangle$ can be parametrized with the help of the Wigner-Eckart theorem in terms of a new reduced matrix element $Z(I',I)$

$$\begin{aligned}&\langle I',L';m'_S,m'_L,\alpha'|\sigma^i\otimes\tau^a|I,L;m_S,m_L,\alpha\rangle= \\ &\frac{1}{2I'+1}Z(I',I)\langle I',m'_S|I,1;m_S,i\rangle\delta_{LL'}\delta_{m_Lm'_L}\langle I',\alpha'|I,1;\alpha,a\rangle\,.\end{aligned} \tag{3.18}$$

With this normalization the reduced matrix element is symmetric $Z(I',I)=Z(I,I')$.

Inserting this expression in (3.17) it is possible to compute the matrix element of Z^{ia} taken between $|I,L;J,m,\alpha\rangle$ states. Comparing with the parametrization (3.11) we obtain the following connection between $Z(J'I',JI)$ and $Z(I',I)$

$$(-)^{J+I-\Delta}\sqrt{(2J+1)(2I+1)}Z(J'I',JI) = \tag{3.19}$$
$$Z(I',I)\delta_{LL'}\delta_{L\Delta}(-)^{2J'+J-I'-\Delta+1}\sqrt{\frac{2J+1}{2I'+1}}\left\{\begin{array}{ccc} I' & 1 & I \\ J & L & J' \end{array}\right\}.$$

We can find a consistency condition for $Z(I',I)$ by inserting this expression into (3.14). The sum over J_1 can be performed explicitly and we find

$$(2K+1)\sum_{I_1}(-)^{-2I_1}\left\{\begin{array}{ccc} I & 1 & I_1 \\ I' & 1 & K \end{array}\right\}\left\{\begin{array}{ccc} I & 1 & I_1 \\ \Delta & J' & I' \\ J & H & 1 \end{array}\right\}Z(I',I_1)Z(I_1,I) \tag{3.20}$$
$$= (-)^{2I'-2K}\left\{\begin{array}{ccc} K & 1 & I' \\ J' & \Delta & H \end{array}\right\}\left\{\begin{array}{ccc} K & 1 & I \\ J & \Delta & H \end{array}\right\}Z(K,I')Z(K,I).$$

It is easy to see, by making use of the relation (Eq.(6.4.8) in [33])

$$\sum_{\mu}(2\mu+1)\left\{\begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{array}\right\}\left\{\begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{array}\right\} = (-)^{2\lambda}\left\{\begin{array}{ccc} j_{21} & j_{22} & j_{23} \\ j_{12} & \lambda & j_{32} \end{array}\right\}\left\{\begin{array}{ccc} j_{31} & j_{32} & j_{33} \\ \lambda & j_{11} & j_{21} \end{array}\right\}. \tag{3.21}$$

that this equation is satisfied by the solution $Z(I',I) = \sqrt{(2I+1)(2I'+1)}$ (up to a constant which can be absorbed into $g(Z)$).

We obtain in this way the result

$$Z(J',I';J,I) = (-)^{-I+I'+1}\left\{\begin{array}{ccc} I' & 1 & I \\ J & L & J' \end{array}\right\}\delta_{\Delta\Delta'}\delta_{L\Delta}. \tag{3.22}$$

We consider next the slightly more complicated case of the baryons transforming under the mixed symmetry representation of SU(4). This is relevant for the light baryons containing only u,d quarks. In this case, the total spin of the excited baryon is given by $\vec{J} = \vec{I} + \vec{P} + \vec{L}$. There is an important difference in the application of the quark model to this situation, connected with the fact that σ^i in (3.16) acts on the spins of the u,d quarks. The total spin of the u,d quarks $\vec{S} = \vec{I} + \vec{P}$ is not equal to I as before. Therefore the natural set of states for doing the quark model calculation is $|(IP)S,L;J,m,\alpha\rangle$.

On the other hand, we would like to classify the states in (3.11) according to the value of the spin vector $\vec{\Delta}$, such that $\vec{I} + \vec{\Delta} = \vec{J}$. This requires a different coupling of the vectors $\vec{I},\vec{P},\vec{L}$: $|I,(PL)\Delta;J,m,\alpha\rangle$. The connection between these two sets of states is a well-known recoupling problem in the theory of angular momentum and is given by Eq.(6.1.5) in [33]

$$|I,(PL)\Delta;J,m,\alpha\rangle = \tag{3.23}$$
$$(-)^{-I-P-L-J}\sum_{S}\sqrt{(2S+1)(2\Delta+1)}\left\{\begin{array}{ccc} I & P & S \\ L & J & \Delta \end{array}\right\}|(IP)S,L;J,m,\alpha\rangle.$$

The matrix element of Z^{ia} taken between the $|(IP)S,L;J,m\rangle$ states can be written as

$$\langle(I'P')S',L';J',m',\alpha'|Z^{ia}|(IP)S,L;J,m,\alpha\rangle = \tag{3.24}$$
$$\sum_{m_S m_L m'_S m'_L}\langle I'S'L';m'_S,m'_L,\alpha'|\sigma^i\otimes\tau^a|ISL;m_S,m_L,\alpha\rangle$$
$$\times\langle J',m'|S',L';m'_S,m'_L\rangle\langle J,m|S,L;m_S,m_L\rangle.$$

An application of the Wigner-Eckart theorem gives

$$\langle S'I'L'; m'_S, m'_L, \alpha'|\sigma^i \otimes \tau^a|SIL; m_S, m_L, \alpha\rangle = \frac{1}{\sqrt{(2S'+1)(2I'+1)}} Z(S'I', SI)\delta_{LL'}\delta_{m_L m'_L}\langle S', m'_S|S, 1; m_S, i\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,, \tag{3.25}$$

with $Z(S'I', SI)$ a new reduced matrix element. With this choice for the normalization factor, it transforms under a permutation of the initial and final indices as

$$Z(S'I', SI) = (-)^{S+I-S'-I'} Z(SI, S'I')\,. \tag{3.26}$$

It is easy to compute now the matrix element of Z^{ia} between the $|I, (PL)\Delta; J, m, \alpha\rangle$ states by inserting (3.25) into (3.24) and using the expansion (3.23). We obtain

$$\begin{aligned}
&\langle I', (P'L')\Delta'; J', m', \alpha'|Z^{ia}|I, (PL)\Delta; J, m, \alpha\rangle = (-)^{-I'-P'-L'-J'-I-P-L-J} \\
&\times \sum_{SS'} \sqrt{\frac{(2S+1)(2\Delta+1)(2\Delta'+1)}{2I'+1}} \delta_{LL'} \left\{ \begin{array}{ccc} I' & P' & S' \\ L' & J' & \Delta' \end{array} \right\} \left\{ \begin{array}{ccc} I & P & S \\ L & J & \Delta \end{array} \right\} Z(S'I', SI) \\
&\times \sum_{m_S m_L m'_S} \langle S', m'_S|S, 1; m_S, i\rangle\langle J', m'|S', L; m'_S, m_L\rangle\langle J, m|S, L; m_S, m_L\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,.
\end{aligned} \tag{3.27}$$

Let us pause for one moment and compare this expression with (3.11). One can see that the isospin CG coefficient is the same on the r.h.s. of these two relations. We extract $Z(J'I', JI)$ by multiplying both equations with $\langle J', m'|J, 1; m, i\rangle$ and summing over (m, i). The resulting sum over 4 CG coefficients can be expressed as a $6j$ symbol. We obtain finally

$$\begin{aligned}
&(-)^{J+I-\Delta}\sqrt{(2J+1)(2I+1)} Z(J', I'; J, I) = \\
&\qquad (-)^{-I'+J'-I-P'-P-L+1} \sqrt{\frac{(2J+1)(2\Delta+1)(2\Delta'+1)}{2I'+1}} \\
&\quad \times \sum_{SS'} (-)^{-S'} \sqrt{(2S+1)(2S'+1)} \left\{ \begin{array}{ccc} I' & P' & S' \\ L & J' & \Delta' \end{array} \right\} \left\{ \begin{array}{ccc} I & P & S \\ L & J & \Delta \end{array} \right\} \left\{ \begin{array}{ccc} S' & 1 & S \\ J & L & J' \end{array} \right\} Z(S'I', SI)\,.
\end{aligned} \tag{3.28}$$

We insert the following ansatz for the reduced matrix element $Z(S'I', SI)$ (inspired by (3.22) with the identification $(ILJ) \to (IPS)$)

$$Z(S'I', SI) = (-)^{-S-I'} \sqrt{(2S+1)(2S'+1)(2I+1)(2I'+1)} \left\{ \begin{array}{ccc} I & P & S \\ S' & 1 & I' \end{array} \right\} \tag{3.29}$$

which has the required symmetry property (3.26).

If we assume that $P = P'$ we can perform the sums over S and S' in (3.28) with the help of the identity (Eq.(C.35e) in [34])

$$\sum_x (-)^\phi (2x+1) \left\{ \begin{array}{ccc} a & b & x \\ c & d & g \end{array} \right\} \left\{ \begin{array}{ccc} c & d & x \\ e & f & h \end{array} \right\} \left\{ \begin{array}{ccc} e & f & x \\ b & a & j \end{array} \right\} = \left\{ \begin{array}{ccc} g & h & j \\ e & a & d \end{array} \right\} \left\{ \begin{array}{ccc} g & h & j \\ f & b & c \end{array} \right\}\,, \tag{3.30}$$

with $\phi = a + b + c + d + e + f + g + h + x + j$.

The final result for $Z(J'I', JI)$ is

$$Z(J', I'; J, I) = (-)^{I-2J+I'+P} \left\{ \begin{array}{ccc} I' & 1 & I \\ J & \Delta & J' \end{array} \right\} \delta_{\Delta\Delta'} \delta_{PP'} \,. \tag{3.31}$$

For $P \neq P'$ this reduced matrix element vanishes because the operator (3.16) is totally symmetric and the initial and final states have different permutational symmetry in the spin-flavor of the N_c quarks[1].

One can see that, in spite of their quite different detailed structure, both cases considered lead to the same answer (3.22) and (3.31), which also coincides with the result obtained by [5] for the case of X^{ia}. The two most important properties of this solution are now apparent:

- The excited states can be classified in towers of states labeled by a spin vector Δ such that $|J - I| \leq \Delta$. To enforce the cancellation of the leading N_c-dependence among the different intermediate states, expressed by the consistency condition, all the members of a Δ-tower must be degenerate among themselves.
- Pions do not couple towers of excited states with different values of Δ, as the corresponding nondiagonal matrix elements of Z^{ia} vanish.

The second property will be useful in the study of the consistency conditions for Y and Q (3.9,3.10), as it allows us to consider the couplings of each Δ-tower at a time. We turn now to the first of them, the coupling Y responsible for S-wave pion couplings between p- and s-wave baryons.

B. Consistency condition for Y

We parametrize the matrix elements of the Y^a operator as

$$\underset{\text{(s-wave)}}{\langle J', I'; m', \alpha'|} Y^a \underset{\text{(p-wave)}}{|J, I; m, \alpha\rangle} = (-)^{I+I'}\sqrt{2I+1}\, Y(J'I', JI)\delta_{JJ'}\delta_{mm'}\langle I', \alpha'|I, 1; \alpha, a\rangle \,, \tag{3.32}$$

$$\underset{\text{(p-wave)}}{\langle J', I'; m', \alpha'|} Y^a \underset{\text{(s-wave)}}{|J, I; m, \alpha\rangle} = (-)^{2I}\sqrt{2I+1}\, \bar{Y}(J'I', JI)\delta_{JJ'}\delta_{mm'}\langle I', \alpha'|I, 1; \alpha, a\rangle \,. \tag{3.33}$$

With this choice for the normalization coefficients we have $Y(J'I', JI) = \bar{Y}(JI, J'I')$. The same definitions (3.32) and (3.33) will be used for transitions between other orbital excitations.

We proceed next in complete analogy to the derivation of the consistency condition for Z^{ia} (3.14). The relation (3.7) is sandwiched between states belonging to Δ'- and Δ-towers respectively

$$\langle J', I'; m', \alpha' \text{ (s-wave)}| r X^{ia} Y^{b\dagger} - Y^{b\dagger} Z^{ia} |J, I; m, \alpha \text{ (p-wave)}\rangle = 0 \,. \tag{3.34}$$

[1] In the quark model this matrix element receives a nonvanishing contribution starting at order $(v/c)^2$ in the nonrelativistic expansion [38]. However, a study of these effects would take us beyond the model-independent framework of the present work.

We denoted here $r = g(X)/g(Z)$. Then a complete set of intermediate states is inserted between each two operators. The necessary matrix elements of X and Z are expressed with the help of the general result (3.22). The resulting equation is finally projected onto the particular channel with total spin-isospin (H, K). We obtain in this way the consistency condition

$$r\sum_{I_1}(2I_1+1)\left\{\begin{array}{ccc} I & 1 & K \\ I' & 1 & I_1 \end{array}\right\}\left\{\begin{array}{ccc} I_1 & 1 & I' \\ H & \Delta' & J \end{array}\right\} Y(JI_1, JI) \tag{3.35}$$
$$= (-)^{-I-K+\Delta'-\Delta}\left\{\begin{array}{ccc} I & 1 & K \\ H & \Delta & J \end{array}\right\} Y(HI', HK)\,.$$

In addition to determining the structure of the reduced matrix element $Y(J'I', JI)$, this relation will fix also the value of the ratio r.

It is straightforward to check that the solution of the consistency condition (3.35) is given by

$$Y(JI', JI) = (-)^{I'+J+\Delta'}\left\{\begin{array}{ccc} I' & 1 & I \\ \Delta & J & \Delta' \end{array}\right\}, \tag{3.36}$$

provided that $r = 1$. After substituting this solution in (3.35) the sum over I_1 can be done with the help of the identity (3.30).

In particular, for decays into s-wave baryons containing only u, d quarks, we obtain the solution

$$Y(J, JI) = (-)^{2J}\left\{\begin{array}{ccc} J & 1 & I \\ \Delta & J & 0 \end{array}\right\} = (-)^{1-J+I}\delta_{\Delta 1}\frac{1}{\sqrt{3(2J+1)}}\,. \tag{3.37}$$

Let us trace again how the same result arises in the quark model. The quark model counterpart of the operator Y^a is

$$Y^a \to \sum_{i,j}\langle 0|1,1;j,i\rangle \sigma^j r^i \otimes \tau^a = \frac{1}{\sqrt{3}}\sum_j (-)^{1-j}\sigma^j r^{-j}\otimes\tau^a\,, \tag{3.38}$$

where the light quark operator σ^j acts only on the spins, r^i acts on the orbital degrees of freedom and τ^a acts on the isospins.

We consider first the coupling of a p-wave state transforming under the symmetric representation of SU(4). The matrix element (3.32) is written in the quark model as

$$\langle J; m, \alpha'|Y^a|J, I; m, \alpha\rangle = \sum_{m_S m_L}\langle J; m, \alpha'|Y^a|I, L; m_S, m_L, \alpha\rangle\langle J, m|I, L; m_S, m_L\rangle\,. \tag{3.39}$$

The Wigner-Eckart theorem can be used to parametrize the matrix element of the operator on the r.h.s. of (3.38) in terms of a new reduced matrix element $\mathcal{T}(J, I)$

$$\langle J; m, \alpha'|\sigma^j r^i \otimes \tau^a|I, L; m_S, m_L, \alpha\rangle = \tag{3.40}$$
$$\frac{1}{2J+1}\mathcal{T}(J, I)\langle J, m|I, 1; m_S, j\rangle\langle 0|L, 1; m_L, i\rangle\langle J, \alpha'|I, 1; \alpha, a\rangle\,.$$

Inserting this expression into (3.39) we obtain the result

$$\begin{aligned}
&\langle J; m, \alpha' | Y^a | J, I; m, \alpha \rangle && (3.41)\\
&= \frac{1}{3(2J+1)} \mathcal{T}(J,I) \sum_{m_S m_L} \langle J, m | I, 1; m_S, m_L \rangle \langle J, m | I, L; m_S, m_L \rangle \langle J, \alpha' | I, 1; \alpha, a \rangle \delta_{L1}\\
&= \frac{1}{3(2J+1)} \mathcal{T}(J,I) \langle J, \alpha' | I, 1; \alpha, a \rangle \delta_{L1} \,,
\end{aligned}$$

which can be compared with the defining matrix element of Y^a (3.32). Taking into account the fact that for the quark model states considered $\Delta = L$, we find the following expression for $Y(J, JI)$ in terms of the quark model reduced matrix element $\mathcal{T}(J,I)$

$$Y(J, JI) = (-)^{-J-I} \frac{1}{3(2J+1)\sqrt{2I+1}} \delta_{\Delta 1} \mathcal{T}(J,I) \,. \tag{3.42}$$

It will be shown in Section IV.D by an explicit calculation in the quark model that the reduced matrix $\mathcal{T}(J,I)$ is given in the large-N_c limit, up to a numerical factor, by

$$\mathcal{T}(J,I) = (-)^{2I+1} \sqrt{3(2J+1)(2I+1)} \,. \tag{3.43}$$

This leads to the same expression (3.37) for $Y(J, JI)$ as the model-independent approach based on the consistency conditions.

It is possible to generalize this argument by keeping the orbital angular momentum of the quark model state arbitrary L, L'. They are only constrained by the requirement of parity conservation $\pi(-)^L = \pi'(-)^{L'}$. The relevant quark model matrix element can be parametrized in this case as

$$\begin{aligned}
&\langle I', L'; m'_S, m'_L, \alpha' | \sigma^j r^i \otimes \tau^a | I, L; m_S, m_L, \alpha \rangle = && (3.44)\\
&\frac{1}{(2I'+1)\sqrt{2L'+1}} \mathcal{T}(I'L', IL) \langle I', m'_S | I, 1; m_S, j \rangle \langle L', m'_L | L, 1; m_L, i \rangle \langle I', \alpha' | I, 1; \alpha, a \rangle \,,
\end{aligned}$$

with $\mathcal{T}(I'L', IL)$ another reduced matrix element. We assumed again that the spin-flavor wavefunction of the u, d quarks in the baryon is completely symmetric.

With the help of this relation it is possible to compute the matrix element of Y^a between eigenstates of the total spin

$$\begin{aligned}
&\langle J, I'; m, \alpha' | Y^a | J, I; m, \alpha \rangle = \frac{1}{(2I'+1)\sqrt{2L'+1}} \mathcal{T}(I'L', IL) \times && (3.45)\\
&\quad \sum \langle 0 | 1, 1; j, i \rangle \langle J, m | I', L'; m'_S, m'_L \rangle \langle J, m | I, L; m_S, m_L \rangle\\
&\quad\quad \times \langle I', m'_S | I, 1; m_S, j \rangle \langle L', m'_L | L, 1; m_L, i \rangle \langle I', \alpha' | I, 1; \alpha, a \rangle\\
&\quad = \frac{1}{\sqrt{3(2I'+1)}} \mathcal{T}(I'L', IL) (-)^{1-I-J-L'} \left\{ \begin{array}{ccc} I & L & J \\ L' & I' & 1 \end{array} \right\} \langle I', \alpha' | I, 1; \alpha, a \rangle \,.
\end{aligned}$$

This has the same structure as the model-independent solution (3.36), which is reproduced provided one takes

$$\mathcal{T}(I'L', IL) = (-)^{1+2I'} \sqrt{3(2I+1)(2I'+1)} \,. \tag{3.46}$$

The phase can be equivalently rewritten as $1 + 2I' = 1 + 2I$ which gives an expression identical to (3.43).

A similar result is obtained also for the case of the excited baryons whose spin-flavor wavefunction transforms according to the mixed representation of SU(4). The relevant matrix element of Y^a can be expressed, with the help of the recoupling relation (3.23) in terms of the matrix element

$$\langle J'I'L'; m', \alpha'|Y^a|(IP)S, L; J, m, \alpha\rangle = \tag{3.47}$$
$$\sum_{m_S m_L m'_S m'_L} \langle I', L'; m'_S, m'_L, \alpha'|Y^a|(IP)S, L; m_S, m_L, \alpha\rangle\langle J, m|S, L; m_S, m_L\rangle\langle J', m'|I', L'; m'_S, m'_L\rangle\,.$$

The matrix element on the r.h.s. can be expressed with the help of (3.38) in terms of a new reduced matrix element $\mathcal{T}(I'L', SIL)$ defined by

$$\langle I', L'; m'_S, m'_L, \alpha'|\sigma^j r^i \otimes \tau^a|(IP)S, L; m_S, m_L, \alpha\rangle \tag{3.48}$$
$$= \frac{1}{(2I'+1)\sqrt{2L'+1}}\mathcal{T}(I'L', SIL)\langle I', m'_S|S, 1; m_S, j\rangle\langle L', m'_L|L, 1; m_L, i\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,.$$

Inserting this relation into (3.47) we find for the matrix element of Y^a in the $|(IP)S, L; J, m, \alpha\rangle$ basis

$$\langle J'I'L'; m', \alpha'|Y^a|(IP)S, L; J, m, \alpha\rangle = \delta_{JJ'}\delta_{mm'}\langle I'\alpha'|I1; \alpha, a\rangle \tag{3.49}$$
$$\times\frac{1}{\sqrt{3(2I'+1)}}(-)^{1+L'-S-J}\left\{\begin{array}{ccc} L & L' & 1 \\ I' & S & J \end{array}\right\}\mathcal{T}(I'L', SIL)\,.$$

Next we transform to the $|I, (PL)\Delta; J, m, \alpha\rangle$ basis with the help of the recoupling relation (3.23). We adopt the following ansatz for the quark model matrix element $\mathcal{T}(I'L', SIL)$

$$\mathcal{T}(I'L', SIL) = (-)^{-I+I'}\sqrt{(2I+1)(2I'+1)(2S+1)}\left\{\begin{array}{ccc} I & 1 & S \\ 1 & I' & 1 \end{array}\right\}\mathcal{I}(L', L) \tag{3.50}$$

with $\mathcal{I}(L', L)$ an arbitrary function of its arguments[2]. This will be derived in Section IV.D by explicit calculation in the quark model in the large-N_c limit. Inserting (3.50) into (3.23) we can perform the sum over S with the help of (3.30). The final result for the matrix element of Y^a has the form, with $P = 1$,

$$Y(J'I', JI) = \delta_{JJ'}c(LL'\Delta)(-)^{I'+J+\Delta'}\left\{\begin{array}{ccc} \Delta & 1 & \Delta' \\ I' & J & I \end{array}\right\}\delta_{\Delta' L'} \tag{3.51}$$

with $c(LL'\Delta)$ a numerical coefficient given by

$$c(LL'\Delta) = \frac{1}{\sqrt{3}}\sqrt{2\Delta+1}(-)^{-\Delta-L'-1}\left\{\begin{array}{ccc} \Delta & 1 & L' \\ 1 & L & 1 \end{array}\right\}\,. \tag{3.52}$$

The result (3.51) can be seen to coincide with the general solution of the consistency condition for Y (3.36).

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[2]For simplicity we will omit $\mathcal{I}(L', L)$ throughout in the following.

C. Consistency condition for Q

The operator $Q^{ij,a}$ parametrizes pion coupling in a D-wave. It will prove convenient to define a modified operator Q^{ka} with only one index $k=-2,-1,0,1,2$, by

$$Q^{ka} = \sum_{ij} \langle 2,k|1,1;i,j\rangle Q^{ij,a}\,. \tag{3.53}$$

It is easy to see that Q^{ka} satisfies the same consistency condition (3.10) as $Q^{ij,a}$.

We introduce reduced matrix elements associated with this operator, defined by

$$\begin{aligned} &\langle J',I';m',\alpha'\ (\text{s-wave})|Q^{ka}|J,I;m,\alpha\ (\text{p-wave})\rangle = & (3.54)\\ &\quad (-)^{J+I+J'+I'}\sqrt{(2J+1)(2I+1)}Q(J'I',JI)\langle J',m'|J,2;m,k\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\,, \\ &\langle J',I';m',\alpha'\ (\text{p-wave})|Q^{ka}|J,I;m,\alpha\ (\text{s-wave})\rangle = & (3.55)\\ &\quad (-)^{2J+2I}\sqrt{(2J+1)(2I+1)}\bar{Q}(J'I',JI)\langle J',m'|J,2;m,k\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\,. \end{aligned}$$

As usual, the choice for the normalization coefficients is made such that $Q(J'I',JI) = \bar{Q}(JI,J'I')$. The same definitions (3.54) and (3.55) apply to transitions between other orbital excitations.

We derive a consistency condition for $Q(JI,J'I')$ by taking the following matrix element of the relation (3.10)

$$\langle J',I';m',\alpha'\ (\text{s-wave})|rX^{ia}Q^{jb\dagger} - Q^{jb\dagger}Z^{ia}|J,I;m,\alpha\ (\text{p-wave})\rangle = 0\,. \tag{3.56}$$

We insert a complete set of intermediate states between the two operators and project this relation onto the particular channel with total spin-isospin (H,K) by multiplication with

$$\langle H',h'|J',2;m',j\rangle\langle H,h|J,1;m,i\rangle\langle K',\eta'|I',1;\alpha',b\rangle\langle K,\eta|I,1;\alpha,a\rangle\,. \tag{3.57}$$

We obtain finally the set of constraints

$$\begin{aligned} r\sum_{J_1I_1}(-)^{-J'+J_1}(2J_1+1)(2I_1+1)\left\{\begin{array}{ccc} I' & 1 & I_1 \\ J_1 & \Delta' & J' \end{array}\right\}\left\{\begin{array}{ccc} J & 1 & H \\ J' & 2 & J_1 \end{array}\right\}\left\{\begin{array}{ccc} I & 1 & K \\ I' & 1 & I_1 \end{array}\right\}Q(J_1I_1,JI) \quad (3.58)\\ = (-)^{2H-I-K+\Delta'-\Delta+1}\left\{\begin{array}{ccc} K & 1 & I \\ J & \Delta & H \end{array}\right\}Q(J'I',HK)\,. \end{aligned}$$

We quote directly the solution of this consistency condition. We will attempt later to make it plausible using a quark model construction. The most general solution can be written as a sum of $9j$ symbols of the form

$$Q(J'I',JI) = \sum_{y=1,2,3} c_y \left\{\begin{array}{ccc} \Delta' & I' & J' \\ \Delta & I & J \\ y & 1 & 2 \end{array}\right\} \tag{3.59}$$

which satisfies (3.58) provided that $r=1$. In particular, for final s-wave states containing only u,d quarks one has $\Delta'=0$, $J'=I'$ and the $9j$ symbols reduce to $6j$ symbols

$$Q(J', JI) = c_1 \frac{(-)^{J+J'}}{\sqrt{3(2J'+1)}} \left\{ \begin{array}{ccc} 2 & J & J' \\ I & 1 & 1 \end{array} \right\} \delta_{\Delta 1} + c_2 \frac{(-)^{J+J'+1}}{\sqrt{5(2J'+1)}} \left\{ \begin{array}{ccc} 2 & J & J' \\ I & 1 & 2 \end{array} \right\} \delta_{\Delta 2} \,. \tag{3.60}$$

It is not completely straightforward to check that (3.58) is indeed satisfied by (3.59). Therefore it might be useful to sketch the steps of this derivation. First, the $9j$ symbols on the l.h.s. are written as a sum over 3 $6j$ symbols with the help of Eq.(6.4.3) in [33]

$$\left\{ \begin{array}{ccc} \Delta' & I_1 & J_1 \\ \Delta & I & J \\ y & 1 & 2 \end{array} \right\} = \sum_x (-)^{2x}(2x+1) \left\{ \begin{array}{ccc} \Delta' & \Delta & y \\ 1 & 2 & x \end{array} \right\} \left\{ \begin{array}{ccc} I_1 & I & 1 \\ \Delta & x & J \end{array} \right\} \left\{ \begin{array}{ccc} J_1 & J & 2 \\ x & \Delta' & I_1 \end{array} \right\} . \tag{3.61}$$

This allows the sum over J_1 to be performed with the help of (3.30)

$$\begin{aligned} &\sum_{J_1} (-)^{-J'+J_1}(2J_1+1) \left\{ \begin{array}{ccc} J' & 1 & J_1 \\ I_1 & \Delta' & I' \end{array} \right\} \left\{ \begin{array}{ccc} I_1 & \Delta' & J_1 \\ 2 & J & x \end{array} \right\} \left\{ \begin{array}{ccc} 2 & J & J_1 \\ 1 & J' & H \end{array} \right\} \\ &\qquad = (-)^{\phi_1} \left\{ \begin{array}{ccc} I' & x & H \\ 2 & J' & \Delta' \end{array} \right\} \left\{ \begin{array}{ccc} I' & x & H \\ J & 1 & I_1 \end{array} \right\} \end{aligned} \tag{3.62}$$

with $\phi_1 = -2J' + 1 + I_1 + I' - \Delta' - J - x - H$. Next, the sum over I_1 can be done, also with the help of (3.30). As a result, the l.h.s. of (3.58) takes the form

$$r(-)^{\phi_2} \sum_x (-)^{2(x+\Delta')}(2x+1) \left\{ \begin{array}{ccc} \Delta' & \Delta & y \\ 1 & 2 & x \end{array} \right\} \left\{ \begin{array}{ccc} I' & K & 1 \\ \Delta & x & H \end{array} \right\} \left\{ \begin{array}{ccc} J' & H & 2 \\ x & \Delta' & I' \end{array} \right\} \times \left\{ \begin{array}{ccc} K & H & \Delta \\ J & I & 1 \end{array} \right\} . \tag{3.63}$$

with $\phi_2 = 2H - \Delta - \Delta' - K - I + 1$. We added a phase factor identically equal to 1 under the summation sign, which allows the x-sum to be performed with the help of a relation analogous to (3.61). The result is a $9j$ symbol identical to the one on the r.h.s. of (3.58). It is easy to check that also the total phase factor and the remaining $6j$ symbol are the same as the ones on the r.h.s. of (3.58), which proves the validity of (3.59). The solution (3.58) satisfies the consistency condition for Q regardless of the value of y, which can take therefore all values compatible with the nonvanishing of the $9j$ symbol in which it appears.

We will try now to make the result (3.59) plausible, by examining the structure of this coupling in the quark model. The operator Q^{ka} is given in the quark model by

$$Q^{ka} \to \sum_{ij} \langle 2, k | 1, 1; j, i \rangle \sigma^j r^i \otimes \tau^a \tag{3.64}$$

with the same σ^j, r^i, τ^a as in (3.38). Let us consider first baryons containing only u, d quarks and whose flavor-spin wavefunction transforms under the symmetric representation of SU(4). We will keep the orbital angular momenta of the initial and final states L, L' completely general, subject only to the requirement of parity conservation $\pi(-)^L = \pi'(-)^{L'}$.

The matrix element of the quark model operators on the r.h.s. of (3.64) between eigenstates of $\vec{S}$ and $\vec{L}$ has been already parametrized in (3.44) in terms of the reduced matrix element $\mathcal{T}(I'L', IL)$. The matrix element of Q^{ka} between eigenstates of $\vec{J} = \vec{I} + \vec{L}$ can be easily obtained as

$$\begin{aligned} \langle J', I'; m', \alpha' | Q^{ka} | J, I; m, \alpha \rangle = &\sum_{ijm_S m_L m'_S m'_L} \langle 2, k | 1, 1; j, i \rangle \langle J', m' | I', L'; m'_S, m'_L \rangle \\ &\times \langle J, m | I, L; m_S, m_L \rangle \langle I', L'; m'_S, m'_L, \alpha' | \sigma^j r^i \otimes \tau^a | I, L; m_S, m_L, \alpha \rangle \,. \end{aligned} \tag{3.65}$$

Comparing with (3.54) we see that it is possible to extract $Q(J'I',JI)$ by multiplying the r.h.s. with $\langle J',m'|J,2;m,k\rangle$ and summing over m,k. The resulting sum over 6 CG coefficients can be written in terms of a $9j$ symbol by using Eq.(6.4.4) in [33]. We obtain finally

$$\begin{aligned}(-)^{J+I+J'+I'}Q(J'I',JI)\sqrt{(2J+1)(2I+1)} \\ = (-)^{-I-J+I'+J'+L+L'}\sqrt{5\frac{2J+1}{2I'+1}}\mathcal{T}(I'L',IL)\left\{\begin{array}{ccc} L' & I' & J' \\ L & I & J \\ 1 & 1 & 2 \end{array}\right\}.\end{aligned} \tag{3.66}$$

For the quark model states considered one has $\Delta = L$, so that the $9j$ symbol corresponds to $y = 1$ in (3.59). Requiring equality with the model-independent solution (3.59) of the consistency condition for Q gives for the quark model reduced matrix element $\mathcal{T}(I'L',IL)$ the expression

$$\mathcal{T}(I'L',IL) = (-)^{L+L'}\frac{1}{\sqrt{5}}\sqrt{(2I'+1)(2I+1)}. \tag{3.67}$$

This agrees, up to an unimportant overall coefficient, with the result (3.46) obtained from considering the matrix element of the s-wave operator Y^a.

The $9j$-symbol with $y = 2$ in (3.59) arises when considering initial states transforming under the mixed symmetry representation of SU(4). The calculation for this case proceeds in close analogy to the one for the Y^a operator. First we compute the matrix element of Q^{ka} in the $|(IP)S,L;J,m,\alpha\rangle$ basis with the help of the relation

$$\begin{aligned}&\langle J'I'L';m',\alpha'|Q^{ka}|(IP)S,L;J,m,\alpha\rangle = \\ &\sum_{m_S m_L m'_S m'_L}\langle I',L';m'_S,m'_L,\alpha'|Q^{ka}|(IP)S,L;m_S,m_L,\alpha\rangle\langle J,m|S,L;m_S,m_L\rangle\langle J',m'|I',L';m'_S,m'_L\rangle\end{aligned} \tag{3.68}$$

followed by the application of (3.64,3.48). We obtain

$$\begin{aligned}\langle J'I'L';m',\alpha'|Q^{ka}|(IP)S,L;J,m,\alpha\rangle = \langle J'm'|J2;mk\rangle\langle I'\alpha'|I1;\alpha a\rangle \\ \times\sqrt{5\frac{2J+1}{2I'+1}}\left\{\begin{array}{ccc} S & 1 & I' \\ L & 1 & L' \\ J & 2 & J' \end{array}\right\}\mathcal{T}(I'L',SIL).\end{aligned} \tag{3.69}$$

We are eventually interested in the matrix elements of Q^{ka} in the basis $|I,(PL)\Delta;J,m,\alpha\rangle$. Using the recoupling relation (3.23) we get

$$\begin{aligned}\langle J'I'L';m',\alpha'|Q^{ka}|I,(PL)\Delta;J,m,\alpha\rangle = \langle J'm'|J2;mk\rangle\langle I'\alpha'|I1;\alpha a\rangle \\ \times\sqrt{5(2\Delta+1)(2I+1)(2J+1)}(-)^{-L-J+2J'-I'+1} \\ \times\sum_S(-)^{-2S}(2S+1)\left\{\begin{array}{ccc} I & 1 & S \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ I & I' & S \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & I' \\ L & 1 & L' \\ J & 2 & J' \end{array}\right\}\end{aligned} \tag{3.70}$$

where we used the ansatz (3.50) for $\mathcal{T}(I'L',SIL)$. To do the sum over S we first combine the two $6j$ symbols with the help of (3.30) such that S appears only in one $6j$ symbol

$$\left\{\begin{matrix} I & 1 & S \\ 1 & I' & 1 \end{matrix}\right\}\left\{\begin{matrix} I & 1 & S \\ L & J & \Delta \end{matrix}\right\} = \sum_x (-)^{\phi}(2x+1)\left\{\begin{matrix} I' & J & x \\ \Delta & 1 & I \end{matrix}\right\}\left\{\begin{matrix} \Delta & 1 & x \\ 1 & L & 1 \end{matrix}\right\}\left\{\begin{matrix} 1 & L & x \\ J & I' & S \end{matrix}\right\} \tag{3.71}$$

with $\phi = I' + J + \Delta + L + I + S + 1 + x$. Now the sum over S can be performed using (3.21)

$$\sum_S (2S+1)\left\{\begin{matrix} 1 & I' & S \\ J & L & x \end{matrix}\right\}\left\{\begin{matrix} 1 & I' & S \\ 2 & J' & J \\ 1 & L' & L \end{matrix}\right\} = (-)^{2x}\left\{\begin{matrix} 2 & J' & J \\ I' & x & L' \end{matrix}\right\}\left\{\begin{matrix} 1 & L' & L \\ x & 1 & 2 \end{matrix}\right\}. \tag{3.72}$$

We obtain for the matrix element of Q^{ka} in the $|I,(PL)\Delta;J,m,\alpha\rangle$ basis the following expression containing a sum over 4 $6j$ symbols

$$\begin{aligned} \langle J'I'L';m',\alpha'|Q^{ka}|I,(PL)\Delta;J,m,\alpha\rangle &= \langle J'm'|J2;mk\rangle\langle I'\alpha'|I1;\alpha a\rangle \\ &\times\sqrt{5(2\Delta+1)(2I+1)(2J+1)}(-)^{-I-J'-I'+\Delta+J+L'+L} \\ &\times\sum_x (-)^{-x}(2x+1)\left\{\begin{matrix} I' & J & x \\ \Delta & 1 & I \end{matrix}\right\}\left\{\begin{matrix} \Delta & 1 & x \\ 1 & L & 1 \end{matrix}\right\}\left\{\begin{matrix} 2 & L' & x \\ I' & J & J' \end{matrix}\right\}\left\{\begin{matrix} 2 & L' & x \\ L & 1 & 1 \end{matrix}\right\}. \end{aligned} \tag{3.73}$$

This can be put in a form resembling (3.59) by first combining the second and fourth $6j$ symbols with (3.30)

$$\left\{\begin{matrix} 1 & L & x \\ \Delta & 1 & 1 \end{matrix}\right\}\left\{\begin{matrix} 1 & L & x \\ L' & 2 & 1 \end{matrix}\right\} = \sum_{y=1,2} (-)^{\phi'}(2y+1)\left\{\begin{matrix} 1 & 2 & y \\ 1 & 1 & 1 \end{matrix}\right\}\left\{\begin{matrix} 1 & 1 & y \\ \Delta & L' & L \end{matrix}\right\}\left\{\begin{matrix} \Delta & L' & y \\ 2 & 1 & x \end{matrix}\right\} \tag{3.74}$$

with $\phi' = \Delta + L - L' - x + y$. The sum over x can be now done in terms of a $9j$ symbol similar to those in (3.59)

$$\sum_x (-)^{2x}(2x+1)\left\{\begin{matrix} I' & J & x \\ \Delta & 1 & I \end{matrix}\right\}\left\{\begin{matrix} \Delta & 1 & x \\ 2 & L' & y \end{matrix}\right\}\left\{\begin{matrix} L' & 2 & x \\ J & I' & J' \end{matrix}\right\} = \left\{\begin{matrix} L' & I' & J' \\ \Delta & I & J \\ y & 1 & 2 \end{matrix}\right\}. \tag{3.75}$$

When inserted into (3.73) this gives a result for the reduced matrix element $Q(J'I',JI)$ of the same form as (3.59)

$$Q(J'I',JI) = \sum_{y=1,2} c_y(LL'\Delta)\left\{\begin{matrix} L' & I' & J' \\ \Delta & I & J \\ y & 1 & 2 \end{matrix}\right\} \tag{3.76}$$

with coefficients c_y given by

$$c_y(LL'\Delta) = \sqrt{5(2\Delta+1)}(-)^{-2J'+y}(2y+1)\left\{\begin{matrix} 1 & 2 & y \\ 1 & 1 & 1 \end{matrix}\right\}\left\{\begin{matrix} 1 & 1 & y \\ \Delta & L' & L \end{matrix}\right\}. \tag{3.77}$$

Finally, the most general solution for $Q(J'I',JI)$ containing also $9j$ symbols with $y=3$ is obtained if one considers transitions among two states with mixed symmetry. This situation is not very relevant from a phenomenological point of view so that its discussion is relegated to Appendix B.

IV. QUARK MODEL MATRIX ELEMENTS

A. Symmetric states

In this Section we compute the reduced matrix elements of the operator $\sigma^i \otimes \tau^a$ on quark model states with arbitrary number of colors. It will be seen that in the limit $N_c \to \infty$ these reduced matrix elements coincide with those required by the consistency conditions discussed in Section III.

We start by computing the reduced matrix element $Z(I', I)$ defined by

$$\begin{aligned} \langle I', L'; m'_S, m'_L, \alpha' | \sigma^i \otimes \tau^a | I, L; m_S, m_L, \alpha\rangle = \\ \frac{1}{2I'+1} Z(I', I) \langle I', m'_S | I, 1; m_S, i\rangle \delta_{LL'} \delta_{m_L m'_L} \langle I', \alpha' | I, 1; \alpha, a\rangle . \end{aligned} \tag{4.1}$$

The states on the l.h.s. transform under the completely symmetric representation of SU(4). For simplicity we will take them to contain only u- and d-type quarks, although additional quark flavors can be included in a straightforward way. In the quark model with N_c colors they are given by

$$\begin{aligned} |I, m, \alpha\rangle &= \sum_i \langle I, m | \frac{N_u}{2}, \frac{N_d}{2}; i, m-i\rangle \mathcal{S} | \frac{N_u}{2}, i\rangle_u | \frac{N_d}{2}, m-i\rangle_d \\ &= \sum_i \langle I, m | \frac{N_u}{2}, \frac{N_d}{2}; i, m-i\rangle \mathcal{S} (u\uparrow)^{\frac{N_u}{2}+i} (u\downarrow)^{\frac{N_u}{2}-i} (d\uparrow)^{\frac{N_d}{2}+m-i} (d\downarrow)^{\frac{N_d}{2}-m+i} , \end{aligned} \tag{4.2}$$

with

$$N_u = \frac{N_c}{2} + \alpha , \qquad N_d = \frac{N_c}{2} - \alpha \tag{4.3}$$

the numbers of u and d quarks, respectively, in the baryon state. The symbol $\mathcal{S}$ means complete symmetrization under permutation of all quarks. The explicit form of the wavefunction (4.2) has been given without proof in [35] and a particular case was previously considered in [36]. For a simple method of computing matrix elements in the quark model with N_c colors see [37]. In the following, for completeness of the presentation we give a detailed derivation of (4.2).

Proof. Any completely symmetric state of N_c quarks can be constructed as a linear combination of symmetrized products of one-quark states

$$\mathcal{S}(n_1, n_2, n_3, n_4) = \frac{1}{\sqrt{\mathcal{N}}} \left((u\uparrow)^{n_1} (u\downarrow)^{n_2} (d\uparrow)^{n_3} (d\downarrow)^{n_4} + \text{ permutations} \right) \tag{4.4}$$

with

$$\mathcal{N} = \frac{(n_1+n_2+n_3+n_4)!}{n_1! n_2! n_3! n_4!} . \tag{4.5}$$

It is easy to see that the action of spin and isospin operators on these states is given by

$$\sum_i \sigma_+^i \mathcal{S}(n_1,n_2,n_3,n_4) = \sqrt{n_2(n_1+1)}\mathcal{S}(n_1+1,n_2-1,n_3,n_4) \tag{4.6}$$
$$+\sqrt{n_4(n_3+1)}\mathcal{S}(n_1,n_2,n_3+1,n_4-1)$$
$$\sum_i \sigma_-^i \mathcal{S}(n_1,n_2,n_3,n_4) = \sqrt{n_1(n_2+1)}\mathcal{S}(n_1-1,n_2+1,n_3,n_4) \tag{4.7}$$
$$+\sqrt{n_3(n_4+1)}\mathcal{S}(n_1,n_2,n_3-1,n_4+1)$$
$$\sum_i \sigma_z^i \mathcal{S}(n_1,n_2,n_3,n_4) = (n_1-n_2+n_3-n_4)\mathcal{S}(n_1,n_2,n_3,n_4) \tag{4.8}$$
$$\sum_i \tau_+^i \mathcal{S}(n_1,n_2,n_3,n_4) = \sqrt{n_3(n_1+1)}\mathcal{S}(n_1+1,n_2,n_3-1,n_4) \tag{4.9}$$
$$+\sqrt{n_4(n_2+1)}\mathcal{S}(n_1,n_2+1,n_3,n_4-1)$$
$$\sum_i \tau_-^i \mathcal{S}(n_1,n_2,n_3,n_4) = \sqrt{n_1(n_3+1)}\mathcal{S}(n_1-1,n_2,n_3+1,n_4) \tag{4.10}$$
$$+\sqrt{n_2(n_4+1)}\mathcal{S}(n_1,n_2-1,n_3,n_4+1) \tag{4.11}$$
$$\sum_i \tau_z^i \mathcal{S}(n_1,n_2,n_3,n_4) = (n_1+n_2-n_3-n_4)\mathcal{S}(n_1,n_2,n_3,n_4)\,.$$

It will prove more convenient to express the arguments of the symmetrized products of one-quark states in terms of four angular momentum-like variables defined as

$$n_1 = j_1 + m_1 \tag{4.12}$$
$$n_2 = j_1 - m_1 \tag{4.13}$$
$$n_3 = j_2 + m_2 \tag{4.14}$$
$$n_4 = j_2 - m_2\,. \tag{4.15}$$

In terms of these variables, the action of the spin and isospin operators can be expressed as

$$\sum_i \sigma_+^i \mathcal{S}(j_1,j_2,m_1,m_2) = \sqrt{(j_1-m_1)(j_1+m_1+1)}\mathcal{S}(j_1,j_2,m_1+1,m_2) \tag{4.16}$$
$$+\sqrt{(j_2-m_2)(j_2+m_2+1)}\mathcal{S}(j_1,j_2,m_1,m_2+1)$$
$$\sum_i \sigma_-^i \mathcal{S}(j_1,j_2,m_1,m_2) = \sqrt{(j_1+m_1)(j_1-m_1+1)}\mathcal{S}(j_1,j_2,m_1-1,m_2) \tag{4.17}$$
$$+\sqrt{(j_2+m_2)(j_2-m_2+1)}\mathcal{S}(j_1,j_2,m_1,m_2-1)$$
$$\sum_i \sigma_z^i \mathcal{S}(j_1,j_2,m_1,m_2) = 2(m_1+m_2)\mathcal{S}(j_1,j_2,m_1,m_2) \tag{4.18}$$
$$\sum_i \tau_+^i \mathcal{S}(j_1,j_2,m_1,m_2) = \sqrt{(j_2+m_2)(j_1+m_1+1)}\mathcal{S}(j_1+\frac{1}{2},j_2-\frac{1}{2},m_1+\frac{1}{2},m_2-\frac{1}{2}) \tag{4.19}$$
$$+\sqrt{(j_2-m_2)(j_1-m_1+1)}\mathcal{S}(j_1+\frac{1}{2},j_2-\frac{1}{2},m_1-\frac{1}{2},m_2+\frac{1}{2})$$
$$\sum_i \tau_-^i \mathcal{S}(j_1,j_2,m_1,m_2) = \sqrt{(j_1+m_1)(j_2+m_2+1)}\mathcal{S}(j_1-\frac{1}{2},j_2+\frac{1}{2},m_1-\frac{1}{2},m_2+\frac{1}{2}) \tag{4.20}$$
$$+\sqrt{(j_1-m_1)(j_2-m_2+1)}\mathcal{S}(j_1-\frac{1}{2},j_2+\frac{1}{2},m_1+\frac{1}{2},m_2-\frac{1}{2})$$

$$\sum_i \tau_z^i \mathcal{S}(j_1,j_2,m_1,m_2) = 2(j_1-j_2)\mathcal{S}(j_1,j_2,m_1,m_2)\,. \tag{4.21}$$

A state of well-defined spin is constructed by taking appropriate linear combinations of symmetrized products of one-particle states

$$|I,m,\alpha\rangle = \sum_{m_1,m_2} c(m_1,m_2,m,j_1,j_2)\mathcal{S}(j_1,j_2,m_1,m_2) \tag{4.22}$$

with $m_1+m_2=m$. The quantum numbers of the state fix j_1 and j_2 through the conditions

$$\alpha = j_1 - j_2 \tag{4.23}$$

$$N_c = 2(j_1+j_2) \tag{4.24}$$

which give (4.3) with $j_1 = N_u/2$ and $j_2 = N_d/2$.

The coefficients c in (4.22) can be determined by requiring the states $|I,m,\alpha\rangle$ to satisfy the relations

$$J_\pm|I,m,\alpha\rangle = \sqrt{(I\mp m)(I\pm m+1)}|I,m\pm 1,\alpha\rangle\,. \tag{4.25}$$

Inserting the expansion (4.22) one finds, with the help of (4.16-4.21), the following recursion relations among the coefficients c

$$\begin{aligned}&\sqrt{(I-m)(I+m+1)}c(m_1,m_2,m+1,j_1,j_2) \\ &\quad = \sqrt{(j_1+m_1)(j_1-m_1+1)}c(m_1-1,m_2,m,j_1,j_2) \\ &\quad +\sqrt{(j_2+m_2)(j_2-m_2+1)}c(m_1,m_2-1,m,j_1,j_2)\end{aligned} \tag{4.26}$$

$$\begin{aligned}&\sqrt{(I+m)(I-m+1)}c(m_1,m_2,m-1,j_1,j_2) \\ &\quad = \sqrt{(j_1-m_1)(j_1+m_1+1)}c(m_1+1,m_2,m,j_1,j_2) \\ &\quad +\sqrt{(j_2-m_2)(j_2+m_2+1)}c(m_1,m_2+1,m,j_1,j_2)\,.\end{aligned} \tag{4.27}$$

These relations can be seen to coincide with the familiar recursion relations for the Clebsch-Gordan coefficients, with the identification

$$c(m_1,m_2,m,j_1,j_2) = \langle I,m|j_1,j_2;m_1,m_2\rangle\,. \tag{4.28}$$

It is known that these recursion relations fix uniquely the CG coefficients up to an overall phase. To complete our proof of (4.2) we still have to show that this state is also an eigenstate of $\vec{I}^{\,2}$, with the same eigenvalue as $\vec{J}^{\,2}$. This can be done by comparing the action of $\vec{J}^{\,2}$ on the state (4.2) with that of $\vec{I}^{\,2}$. We obtain

$$\begin{aligned}\vec{J}^{\,2}|I,m,\alpha\rangle &= \left(\frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ + J_z^2\right)|I,m,\alpha\rangle \\ &= \sum_{m_1,m_2} c(m_1,m_2,m,j_1,j_2)\left\{(j_1(j_1+1)+j_2(j_2+1)+2m_1m_2)\,\mathcal{S}(j_1,j_2,m_1,m_2)\right. \\ &+ \sqrt{(j_1+m_1)(j_1-m_1+1)(j_2-m_2)(j_2+m_2+1)}\mathcal{S}(j_1,j_2,m_1-1,m_2+1) \\ &+ \left.\sqrt{(j_1-m_1)(j_1+m_1+1)(j_2+m_2)(j_2-m_2+1)}\mathcal{S}(j_1,j_2,m_1+1,m_2-1)\right\}\end{aligned} \tag{4.29}$$

which also coincides with the result of applying $\vec{I}^{\,2}$ on the same state.

The knowledge of the states (4.2) can be used to calculate the matrix element (4.1). We will choose for this calculation the spherical component $(i,a) = (0,0)$ of the current in (4.1). The corresponding quark model operator can be written as a sum over N_c one-quark operators

$$\sigma^0 \otimes \tau^0 = \sum_{i=1}^{N_c} \sigma_3^i \tau_3^i \,. \tag{4.30}$$

Because of the symmetry property of $Z(I', I)$, there are only two independent quantities to calculate: $Z(I,I)$ and $Z(I,I-1)$. We obtain for them the results

$$Z(I,I) = (2I+1)(N_c+2) \tag{4.31}$$
$$Z(I,I-1) = \sqrt{(2I-1)(2I+1)}\sqrt{(N_c+2+2I)(N_c+2-2I)} \,. \tag{4.32}$$

In order to obtain $Z(I,I)$ we consider the following matrix element of the type (4.1)

$$\langle I,I,I|\sum_{i=1}^{N_c} \sigma_3^i \tau_3^i |I,I,I\rangle = \frac{I}{(I+1)(2I+1)} Z(I,I) \,. \tag{4.33}$$

The quark model matrix element on the l.h.s. can be computed with the help of the wave-function (4.2) with the result

$$\begin{aligned} \langle I,I,I|\sum_{i=1}^{N_c} \sigma_3^i \tau_3^i |I,I,I\rangle &= \sum_m |\langle I,I|\frac{N_u}{2},\frac{N_d}{2};m,I-m\rangle|^2 (4m-2I) \\ &= \frac{N_u(N_u+2) - N_d(N_d+2)}{2(I+1)} \,. \end{aligned} \tag{4.34}$$

N_u, N_d are given by (4.3). Inserting this expression into (4.33) one obtains the result (4.31) for $Z(I,I)$.

For $Z(I,I-1)$ we consider the matrix element

$$\langle I,I-1,I-1|\sum_{i=1}^{N_c} \sigma_3^i \tau_3^i |I-1,I-1,I-1\rangle = \frac{1}{I(2I+1)} Z(I,I-1) \,. \tag{4.35}$$

The quark model matrix element can be computed with the result

$$\begin{aligned} &\langle I,I-1,I-1|\sum_{i=1}^{N_c} \sigma_3^i \tau_3^i |I-1,I-1,I-1\rangle \\ &= 4\sum_m m \langle I,I-1|\frac{N_u}{2},\frac{N_d}{2};m,I-1-m\rangle\langle I-1,I-1|\frac{N_u}{2},\frac{N_d}{2};m,I-1-m\rangle \\ &= \frac{2}{I}\sqrt{\frac{2I-1}{2I+1}\left(\frac{N_c}{2}+1+I\right)\left(\frac{N_c}{2}+1-I\right)} \,. \end{aligned} \tag{4.36}$$

Comparing with (4.35) gives immediately the result (4.32).

The results (4.31), (4.32) can be put into a common form

$$Z(I',I) = \sqrt{(2I'+1)(2I+1)}\sqrt{(N_c+2)^2-(I'-I)^2(I'+I+1)^2} \qquad (4.37)$$
$$= (N_c+2)\sqrt{(2I'+1)(2I+1)} + \mathcal{O}(1/N_c)\,.$$

We have made here apparent the fact that the corrections to the lowest-order result come only at sub-subleading order in $1/N_c$. This is an illustration, on the example of the quark model, of a model-independent result obtained by Dashen and Manohar [2] using the counting rules for pion-baryon scattering.

B. Mixed symmetry states

In this Section we construct quark model states whose spin-flavor wavefunctions transform under the mixed symmetry representation of SU(4) shown in Fig.2. They can be built using the procedure described in Sect.II, by adding one extra quark to a symmetric state of N_c-1 quarks. We write the state obtained by adding the j^{th} quark to a symmetric state of N_c-1 quarks with spin and isospin i, as

$$|SI,m,\alpha\rangle_j = \sum_{m_1m_2\alpha_1\alpha_2} \langle S,m|i,\frac{1}{2};m_1,m_2\rangle\langle I,\alpha|i,\frac{1}{2};\alpha_1,\alpha_2\rangle|i,m_1,\alpha_1\rangle\otimes|\frac{1}{2},m_2,\alpha_2\rangle_j\,. \quad (4.38)$$

The states of mixed symmetry under SU(4) must be antisymmetric under permutations of the two quarks corresponding to the first column of the Young diagram. There are $N_c(N_c-1)/2$ ways to choose such a pair of quarks, but not all the states obtained in this way will be linearly independent. In fact there are only N_c-1 independent states with mixed symmetry, and we will choose them such that they are antisymmetric under a permutation of the first quark with any of the remaining N_c-1 quarks in the baryon. The corresponding spin-flavor wavefunction will be denoted as

$$|SI,m,\alpha\rangle_{[j,1]} = \frac{1}{\sqrt{2}}\left(|SI,m,\alpha\rangle_j - |SI,m,\alpha\rangle_1\right)\,, \qquad j=2,3,\dots,N_c\,. \quad (4.39)$$

The space part of the wavefunction must transform also under the mixed symmetry representation of the permutation group, corresponding to the same Young diagram as in Fig.2. There are again N_c-1 linearly independent wavefunctions, which can be chosen to be antisymmetric under a permutation of the j^{th} and 1^{st} quarks. Their generic form is

$$|L,m_L\rangle_{[j,1]} = \frac{1}{\sqrt{2}}\left(\psi(r_j)Y_{Lm_L}(\hat r_j)\phi(r_1) - \psi(r_1)Y_{Lm_L}(\hat r_1)\phi(r_j)\right)\phi_S(r_2,\cdots,r_{j-1},r_{j+1},\cdots,r_{N_c}) \quad (4.40)$$

with $\phi_S(r_2,\cdots,r_{j-1},r_{j+1},\cdots,r_{N_c})$ a symmetric function of its arguments. In (4.40), we have assumed that the orbital angular momentum is carried by a single quark. This is strictly true only for the lowest orbital excitations.

It is easy to combine now the spatial and the spin-flavor parts into a completely symmetric wavefunction of well-defined spin and isospin. Our final result for such a quark model state is

$$|JI,m,\alpha\rangle = \sum_{m_S,m_L}\langle J,m|S,L;m_S,m_L\rangle|SIL,m_S,m_L,\alpha\rangle \quad (4.41)$$

with

$$|SIL,m_S,m_L,\alpha\rangle = (-)^{\psi(SIi)}\frac{1}{\sqrt{N_c-1}}\sum_{j=2}^{N_c}|SI,m_S,\alpha\rangle_{[j,1]}\otimes|L,m_L\rangle_{[j,1]}\,. \tag{4.42}$$

The phase of these states $\psi(SIi)$ will be chosen later for convenience.

These states have a peculiar normalization, due to the fact that the spatial wavefunctions (4.40) with $j\neq j'$ are not orthogonal. They satisfy instead

$${}_{[j',1]}\langle L,m'_L|L,m_L\rangle_{[j,1]} = \frac{1}{2}(\delta_{jj'}+1)\delta_{m_L m'_L}\mathcal{I} \tag{4.43}$$

with $\mathcal{I}$ an overlap integral. Using this expression we obtain the following exact result for the norm of the states (4.42)

$$\begin{aligned}\langle S'I'1,m'_S,m'_L,\alpha'|SI1,m_S,m_L,\alpha\rangle &= \delta_{SS'}\delta_{m_S m'_S}\delta_{m_L m'_L}\delta_{II'}\delta_{\alpha\alpha'}\frac{N_c+2}{4}\mathcal{I} \\ \times\Bigg\{3(2i+1)\left\{\begin{matrix} S & I & 1\\ \frac12 & \frac12 & i\end{matrix}\right\}^2 &- \frac{2(2i+1)}{N_c-1}\left\{\begin{matrix} S & I & 0\\ \frac12 & \frac12 & i\end{matrix}\right\}^2 \\ &+\frac{1}{2(N_c-1)}\left[\frac{5}{2}+2i(i+1)-S(S+1)-I(I+1)\right]\Bigg\}\,.\end{aligned} \tag{4.44}$$

The derivation of this relation will be presented in some detail, as it illustrates a few techniques useful in dealing with the mixed symmetry states. We start by computing the scalar product of two direct product states

$$\begin{aligned}{}_{j'}\langle S'I',m',\alpha'|SI,m,\alpha\rangle_j &= \sum\langle S',m'|i',\frac12;m'_1,m'_2\rangle\langle I',\alpha'|i',\frac12;\alpha'_1,\alpha'_2\rangle\langle S,m|i,\frac12;m_1,m_2\rangle \\ &\times\langle I,\alpha|i,\frac12;\alpha_1,\alpha_2\rangle_{j'}\langle\frac12,m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|i,m_1,\alpha_1\rangle\otimes|\frac12,m_2,\alpha_2\rangle_j\,.\end{aligned} \tag{4.45}$$

The matrix element on the r.h.s. can be written as

$$\begin{aligned}&{}_{j'}\langle\frac12,m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|i,m_1,\alpha_1\rangle\otimes|\frac12,m_2,\alpha_2\rangle_j \\ &= {}_j\langle\frac12,m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|P_{jj'}|i,m_1,\alpha_1\rangle\otimes|\frac12,m_2,\alpha_2\rangle_j\end{aligned} \tag{4.46}$$

where

$$P_{jj'} = \frac14\left(1+\vec\sigma_j\cdot\vec\sigma_{j'}\right)\left(1+\vec\tau_j\cdot\vec\tau_{j'}\right) \tag{4.47}$$

is an operator which exchanges the spins and isospins of the j,j' quarks. We obtain in this way

$$\begin{aligned}&{}_{j'}\langle\frac12,m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|i,m_1,\alpha_1\rangle\otimes|\frac12,m_2,\alpha_2\rangle_j \\ &= \frac14\,{}_j\langle\frac12,m'_2,\alpha'_2|\frac12,m_2,\alpha_2\rangle_j\langle i',m'_1,\alpha'_1|i,m_1,\alpha_1\rangle\end{aligned} \tag{4.48}$$

$$+\frac{1}{4}\sum_k(-)^k\,{}_j\langle\frac{1}{2},m_2',\alpha_2'|\sigma_j^{-k}|\frac{1}{2},m_2,\alpha_2\rangle_j\langle i',m_1',\alpha_1'|\sigma_{j'}^k|i,m_1,\alpha_1\rangle$$
$$+\frac{1}{4}\sum_b(-)^b\,{}_j\langle\frac{1}{2},m_2',\alpha_2'|\tau_j^{-b}|\frac{1}{2},m_2,\alpha_2\rangle_j\langle i',m_1',\alpha_1'|\tau_{j'}^b|i,m_1,\alpha_1\rangle$$
$$+\frac{1}{4}\sum_{k,b}(-)^{k+b}\,{}_j\langle\frac{1}{2},m_2',\alpha_2'|\sigma_j^{-k}\tau_j^{-b}|\frac{1}{2},m_2,\alpha_2\rangle_j\langle i',m_1',\alpha_1'|\sigma_{j'}^k\tau_{j'}^b|i,m_1,\alpha_1\rangle\,.$$

The matrix elements on the one-quark states are computed easily with the results

$${}_j\langle\frac{1}{2},m_2',\alpha_2'|\frac{1}{2},m_2,\alpha_2\rangle_j=\delta_{m_2m_2'}\delta_{\alpha_2\alpha_2'} \tag{4.49}$$
$${}_j\langle\frac{1}{2},m_2',\alpha_2'|\sigma_j^k|\frac{1}{2},m_2,\alpha_2\rangle_j=\sqrt{3}\langle\frac{1}{2},m_2'|\frac{1}{2},1;m_2,k\rangle\delta_{\alpha_2\alpha_2'} \tag{4.50}$$
$${}_j\langle\frac{1}{2},m_2',\alpha_2'|\tau_j^b|\frac{1}{2},m_2,\alpha_2\rangle_j=\sqrt{3}\langle\frac{1}{2},\alpha_2'|\frac{1}{2},1;\alpha_2,b\rangle\delta_{m_2m_2'} \tag{4.51}$$
$${}_j\langle\frac{1}{2},m_2',\alpha_2'|\sigma_j^k\tau_j^b|\frac{1}{2},m_2,\alpha_2\rangle_j=3\langle\frac{1}{2},m_2'|\frac{1}{2},1;m_2,k\rangle\langle\frac{1}{2},\alpha_2'|\frac{1}{2},1;\alpha_2,b\rangle\,. \tag{4.52}$$

The matrix elements of the one-quark operators taken on symmetric states containing N_c-1 quarks can be obtained with the help of the wavefunction (4.2) of these states. For example, the matrix element of σ_j^k is parametrized as

$$\langle i',m',\alpha'|\sigma_j^k|i,m,\alpha\rangle=F(i)\delta_{ii'}\delta_{\alpha\alpha'}\langle i,m'|i,1;m,k\rangle\,. \tag{4.53}$$

The state $|i,m,\alpha\rangle$ has the explicit form

$$|i,m,\alpha\rangle=\sum_k\langle i,m|j_1,j_2;k,m-k\rangle\mathcal{S}_{N_c-1}(j_1+k,j_1-k,j_2+m-k,j_2-m+k) \tag{4.54}$$

with $j_1=N_u/2$, $j_2=N_d/2$ and $N_{u,d}=(N_c-1)/2\pm\alpha$. Next, we single out the quark j by using the relation

$$\begin{aligned}&\mathcal{S}_{N_c}(n_1,n_2,n_3,n_4)= \\ &\quad\sqrt{\frac{n_1}{N_c}}(u\uparrow)_j\mathcal{S}_{N_c-1}(n_1-1,n_2,n_3,n_4)+\sqrt{\frac{n_2}{N_c}}(u\downarrow)_j\mathcal{S}_{N_c-1}(n_1,n_2-1,n_3,n_4)\\ &\quad+\sqrt{\frac{n_3}{N_c}}(d\uparrow)_j\mathcal{S}_{N_c-1}(n_1,n_2,n_3-1,n_4)+\sqrt{\frac{n_4}{N_c}}(d\downarrow)_j\mathcal{S}_{N_c-1}(n_1,n_2,n_3,n_4-1)\,.\end{aligned} \tag{4.55}$$

The reduced matrix element $F(i)$ can be computed by taking the spherical component $k=0$ in (4.53). The matrix element on the l.h.s. of this relation can be written with the help of (4.55) as

$$\begin{aligned}\langle i,m,\alpha|\sigma_j^0|i,m,\alpha\rangle&=\sum_k|\langle i,m|j_1,j_2;k,m-k\rangle|^2\\ &\times\left\{\frac{j_1+k}{N_c-1}-\frac{j_1-k}{N_c-1}+\frac{j_2+m-k}{N_c-1}-\frac{j_2-m+k}{N_c-1}\right\}=\frac{2m}{N_c-1}\,.\end{aligned} \tag{4.56}$$

Comparing with (4.53) we obtain

$$F(i) = \frac{2}{N_c - 1}\sqrt{i(i+1)}\,. \tag{4.57}$$

In a completely analogous way we write the other needed matrix elements as

$$\langle i', m', \alpha' | \tau_j^a | i, m, \alpha\rangle = G(i)\delta_{ii'}\delta_{mm'}\langle i, \alpha' | i, 1; \alpha, a\rangle \tag{4.58}$$

$$\langle i', m', \alpha' | \sigma_j^k \tau_j^a | i, m, \alpha\rangle = \sqrt{\frac{2i+1}{2i'+1}} H(i', i)\langle i', m' | i, 1; m, k\rangle\langle i', \alpha' | i, 1; \alpha, a\rangle\,. \tag{4.59}$$

The corresponding reduced matrix elements can be computed with the results

$$G(i) = \frac{2}{N_c - 1}\sqrt{i(i+1)} \tag{4.60}$$

$$H(i,i) = \frac{N_c + 1}{N_c - 1} \tag{4.61}$$

$$H(i, i-1) = \frac{2}{N_c - 1}\sqrt{\left(\frac{N_c - 1}{2} + i + 1\right)\left(\frac{N_c - 1}{2} - i + 1\right)} \tag{4.62}$$

$$H(i', i) = 1 + \frac{2}{N_c} + \mathcal{O}(1/N_c^2)\,. \tag{4.63}$$

We note from these results that only the unit operator 1 and $\sigma_j^k \tau_j^a$ give leading contributions to (4.45) in the large-N_c limit. Inserting the individual expressions for the matrix elements into (4.45) we obtain

$$\begin{aligned} {}_{j'}\langle S'I'i, m'\alpha' | SIi, m\alpha\rangle_j &= \delta_{SS'}\delta_{mm'}\delta_{II'}\delta_{\alpha\alpha'} \\ \times\frac{1}{4}\Bigg[1 &+ \sqrt{6(2i+1)}F(i)(-)^{i+\frac{1}{2}+S}\begin{Bmatrix} \frac{1}{2} & i & S \\ i & \frac{1}{2} & 1 \end{Bmatrix} + \sqrt{6(2i+1)}G(i)(-)^{i+\frac{1}{2}+I}\begin{Bmatrix} \frac{1}{2} & i & I \\ i & \frac{1}{2} & 1 \end{Bmatrix} \\ &+6(-)^{2i+1+S+I}(2i+1)H(i,i)\begin{Bmatrix} S & i & \frac{1}{2} \\ 1 & \frac{1}{2} & i \end{Bmatrix}\begin{Bmatrix} I & i & \frac{1}{2} \\ 1 & \frac{1}{2} & i \end{Bmatrix}\Bigg]\,. \end{aligned} \tag{4.64}$$

The product of two $6j$-symbols can be transformed with the help of the identity (3.30) into the form

$$\begin{Bmatrix} S & i & \frac{1}{2} \\ 1 & \frac{1}{2} & i \end{Bmatrix}\begin{Bmatrix} I & i & \frac{1}{2} \\ 1 & \frac{1}{2} & i \end{Bmatrix} = \frac{1}{2}(-)^{S+I+1+2i}\left[\begin{Bmatrix} S & I & 0 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}^2 - \begin{Bmatrix} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}^2\right]\,. \tag{4.65}$$

Furthermore, the second $6j$-symbol on the r.h.s. can be eliminated by using the relation

$$\sum_{x=0,1}(2x+1)\begin{Bmatrix} I & S & x \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}^2 = \frac{1}{2i+1}\,. \tag{4.66}$$

We obtain finally for the scalar product of tensor product states (4.64) the simple result

$$\begin{aligned} {}_{j'}\langle S'I', m', \alpha' | SI, m, \alpha\rangle_j &= \delta_{SS'}\delta_{mm'}\delta_{II'}\delta_{\alpha\alpha'} \\ \times\Bigg\{(2i+1)\frac{N_c+1}{N_c-1}\begin{Bmatrix} S & I & 0 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}^2 &- \frac{1}{2(N_c-1)}\left[\frac{5}{2} + 2i(i+1) - S(S+1) - I(I+1)\right]\Bigg\}\,. \end{aligned} \tag{4.67}$$

This result only holds if the two external quarks are different $j \neq j'$. If they are identical, only the first term in (4.48) contributes (without the factor 1/4). This gives

$$
{}_j\langle S'I', m', \alpha' | SI, m, \alpha \rangle_j = \delta_{SS'}\delta_{mm'}\delta_{II'}\delta_{\alpha\alpha'} \,. \tag{4.68}
$$

We can use (4.68) and (4.67) to compute the norm of the states $|SIL, m_S, m_L, \alpha\rangle$. With the help of the definition (4.42), it can be written as

$$
\begin{aligned}
&\langle S'I'1, m'_S, m'_L, \alpha' | SI1, m_S, m_L, \alpha \rangle = \qquad (4.69)\\
&= \frac{1}{N_c - 1} \sum_{j,j'=2}^{N_c} {}_{[j',1]}\langle S'I', m'_S, \alpha' | SI, m_S, \alpha \rangle_{[j,1]} \, {}_{[j',1]}\langle 1, m'_L | 1, m_L \rangle_{[j,1]} \\
&= \frac{N_c + 2}{4} \delta_{m_L m'_L} \mathcal{I} \left({}_j\langle S'I', m', \alpha' | SI, m, \alpha \rangle_j - {}_{j'}\langle S'I', m', \alpha' | SI, m, \alpha \rangle_j \right) ,
\end{aligned}
$$

where we used (4.39) and (4.43). To bring this into the final form (4.44) we only need to insert the expressions (4.68) and (4.67) for the scalar products on the r.h.s. and simplify the resulting expression with the help of (4.66).

C. Matrix elements of Z^{ka} on mixed symmetry states

In this Section we will compute the matrix element (3.25) of Z^{ka} taken between quark model states with mixed symmetry. It will be shown that the ansatz for $Z(S'I', SI)$ introduced in Sect.III.A can in fact be obtained by an explicit calculation in the quark model.

We parametrize the matrix element of Z^{ka} between the quark model states (4.42) as

$$
\begin{aligned}
&\langle S'I'1; m'_S, m'_L, \alpha' | \sum_{n=1}^{N_c} \sigma_n^k \tau_n^a | SI1; m_S, m_L, \alpha \rangle = \qquad (4.70)\\
&\frac{1}{\sqrt{(2S'+1)(2I'+1)}} Z(S'I', SI) \delta_{m_L m'_L} \langle S', m'_S | S, 1; m_S, k \rangle \langle I', \alpha' | I, 1; \alpha, a \rangle \,.
\end{aligned}
$$

We obtain for the reduced matrix element the following result

$$
\begin{aligned}
Z(S'I', SI) &= \frac{3}{4} N_c (N_c + 2) \sqrt{(2i+1)(2i'+1)} \sqrt{(2S+1)(2S'+1)(2I+1)(2I'+1)} \qquad (4.71)\\
&\times (-)^{i'-i+S'+I'+\psi(SIi)+\psi(S'I'i')} \left\{ \begin{matrix} 1 & S' & S \\ 1 & I & I' \end{matrix} \right\} \left\{ \begin{matrix} i & S & \frac{1}{2} \\ 1 & \frac{1}{2} & I \end{matrix} \right\} \left\{ \begin{matrix} i' & S' & \frac{1}{2} \\ 1 & \frac{1}{2} & I' \end{matrix} \right\} \mathcal{I} \,.
\end{aligned}
$$

This has to be divided with the square roots of the norms of the initial and final states (4.44). To leading order in N_c the result takes exactly the form (3.29) provided the phase $\psi(SIi)$ of the quark model states (4.42) is chosen as

$$
\psi(SIi) = i + I + \frac{1}{2} \,. \tag{4.72}
$$

The derivation of (4.71) proceeds in close analogy to the computation of the norm of the mixed symmetry states. First, we express the matrix element (4.70) of Z^{ka} in terms of matrix elements on direct product states as

$$\langle S'I'1;m'_S,m'_L,\alpha'|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|SI1;m_S,m_L,\alpha\rangle \tag{4.73}$$
$$= (-)^{\psi(SIi)+\psi(S'I'i')}\frac{1}{N_c-1}\sum_{j,j'=2}^{N_c} {}_{[j',1]}\langle S'I',m'_S,\alpha'|\sum_{n=1}^{N_c}\sigma_n^j\tau_n^a|SI,m_S,\alpha\rangle_{[j,1]}\ {}_{[j',1]}\langle 1,m'_L|1,m_L\rangle_{[j,1]}$$
$$= (-)^{\psi(SIi)+\psi(S'I'i')}\frac{N_c+2}{4}\delta_{m_Lm'_L}\mathcal{I}\left(Z_1-Z_2\right)\ ,$$

where we denoted the diagonal and nondiagonal matrix elements of Z^{ia} on direct product states by

$$Z_1 = {}_j\langle S'I',m',\alpha'|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|SI,m,\alpha\rangle_j \tag{4.74}$$
$$Z_2 = {}_{j'}\langle S'I',m',\alpha'|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|SI,m,\alpha\rangle_j\,. \tag{4.75}$$

The nondiagonal matrix element on direct product states ($j'\neq j$) can be transformed into a diagonal one with the help of the exchange operator (4.47)

$$\begin{aligned}&{}_{j'}\langle\frac{1}{2},m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle\otimes|\frac{1}{2},m_2,\alpha_2\rangle_j \\ &={}_j\ \langle\frac{1}{2},m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|P_{jj'}\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle\otimes|\frac{1}{2},m_2,\alpha_2\rangle_j\end{aligned} \tag{4.76}$$

This expression can be computed by expanding the $P_{jj'}$ operator and inserting a complete set of intermediate states.

$$\begin{aligned}&{}_{j'}\langle\frac{1}{2},m'_2,\alpha'_2|\otimes\langle i',m'_1,\alpha'_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle\otimes|\frac{1}{2},m_2,\alpha_2\rangle_j = \\ &\frac{1}{4}\,{}_j\langle\frac{1}{2},m'_2,\alpha'_2|\frac{1}{2},m_2,\alpha_2\rangle_j\langle i',m'_1,\alpha'_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle \\ &+\frac{1}{4}\sum_l(-)^l\,{}_j\langle\frac{1}{2},m'_2,\alpha'_2|\sigma_j^{-l}|\frac{1}{2},m_2,\alpha_2\rangle_j \\ &\qquad\times\sum_{m''_1}\langle i',m'_1,\alpha'_1|\sigma_{j'}^l|i',m''_1,\alpha'_1\rangle\langle i',m''_1,\alpha'_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle \\ &+\frac{1}{4}\sum_b(-)^b\,{}_j\langle\frac{1}{2},m'_2,\alpha'_2|\tau_j^{-b}|\frac{1}{2},m_2,\alpha_2\rangle_j \\ &\qquad\times\sum_{\alpha''_1}\langle i',m'_1,\alpha'_1|\tau_{j'}^b|i',m'_1,\alpha''_1\rangle\langle i',m'_1,\alpha''_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle \\ &+\frac{1}{4}\sum_{l,b}(-)^{l+b}\,{}_j\langle\frac{1}{2},m'_2,\alpha'_2|\sigma_j^{-l}\tau_j^{-b}|\frac{1}{2},m_2,\alpha_2\rangle_j \\ &\qquad\times\sum_{i'',m''_1,\alpha''_1}\langle i',m'_1,\alpha'_1|\sigma_{j'}^l\tau_{j'}^b|i'',m''_1,\alpha''_1\rangle\langle i'',m''_1,\alpha''_1|\sum_{n=1}^{N_c}\sigma_n^k\tau_n^a|i,m_1,\alpha_1\rangle\,.\end{aligned} \tag{4.77}$$

Only completely symmetric states of $N_c - 1$ quarks contribute to the sum over intermediate states since both the operator and the initial state in the last matrix elements of each term are symmetric under permutations of any quarks. One notes that keeping only the first and the last term in this relation is sufficient to obtain the large-N_c limit of this matrix element[3]. Furthermore, in the sum over quarks in Z^{ka} one can omit the term acting on the j'^{th} quark, as this will only change the result by an amount nonleading in N_c. This allows us to compute these matrix elements by using the results of Sec.IV.A. Putting all pieces together one obtains for the matrix element of Z^{ka} between quark-model states with well-defined spin and isospin (S, I) the following result

$$
\begin{aligned}
&{}_{j'}\langle S'I', m'_S, \alpha'|Z^{ka}|SI, m_S, \alpha\rangle_j = \qquad (4.78)\\
&\quad N_c\sqrt{(2i+1)(2i'+1)}\sqrt{(2S+1)(2I+1)}\langle S', m'_S|S,1;m_S,k\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\\
&\quad\times\left[\frac{1}{4}(-)^{-2i'+I+S-1}\left\{\begin{array}{ccc} 1 & S & S' \\ \frac{1}{2} & i' & i \end{array}\right\}\left\{\begin{array}{ccc} 1 & I & I' \\ \frac{1}{2} & i' & i \end{array}\right\}\right.\\
&\quad\left.+\frac{3}{2}(-)^{I+I'+S+S'}\sum_{i''}(2i''+1)\left\{\begin{array}{ccc} i'' & \frac{1}{2} & S' \\ \frac{1}{2} & i' & 1 \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & S' \\ i'' & \frac{1}{2} & i \end{array}\right\}\left\{\begin{array}{ccc} i'' & \frac{1}{2} & I' \\ \frac{1}{2} & i' & 1 \end{array}\right\}\left\{\begin{array}{ccc} I & 1 & I' \\ i'' & \frac{1}{2} & i \end{array}\right\}\right].
\end{aligned}
$$

Each of the last two lines corresponds to the contributions of the first and fourth terms in (4.77), respectively. They can be transformed into the following form by a repeated application of (3.30)

$$
\begin{aligned}
&\left\{\begin{array}{ccc} 1 & S & S' \\ \frac{1}{2} & i' & i \end{array}\right\}\left\{\begin{array}{ccc} 1 & I & I' \\ \frac{1}{2} & i' & i \end{array}\right\} = (-)^{i+i'+I+S+I'+S'} \qquad (4.79)\\
&\quad\times\left[\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 0 \end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 0 & I \end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 0 \end{array}\right\} - 3\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 1 \end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 1 & I \end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 1 \end{array}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i''}(2i''+1)\left\{\begin{array}{ccc} i'' & \frac{1}{2} & S' \\ \frac{1}{2} & i' & 1 \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & S' \\ i'' & \frac{1}{2} & i \end{array}\right\}\left\{\begin{array}{ccc} i'' & \frac{1}{2} & I' \\ \frac{1}{2} & i' & 1 \end{array}\right\}\left\{\begin{array}{ccc} I & 1 & I' \\ i'' & \frac{1}{2} & i \end{array}\right\} = (-)^{1+i'-i+S+I} \qquad (4.80)\\
&\quad\times\frac{1}{2}\left[\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 0 \end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 0 & I \end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 0 \end{array}\right\} + \left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 1 \end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 1 & I \end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 1 \end{array}\right\}\right].
\end{aligned}
$$

Inserting these expressions into (4.78) we obtain the following result for the nondiagonal matrix element of Z^{ka} between direct product states

$$
\begin{aligned}
Z_2 &= N_c\sqrt{(2i+1)(2i'+1)}\sqrt{(2S+1)(2I+1)}\langle S', m'_S|S,1;m_S,k\rangle\langle I',\alpha'|I,1;\alpha,a\rangle \qquad (4.81)\\
&\quad\times(-)^{1+i'-i+S'+I'}\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 0 \end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 0 & I \end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 0 \end{array}\right\}.
\end{aligned}
$$

For the diagonal case, only the first term in (4.77) survives (without the factor of 1/4). Using (4.79) we can write for this case

---

[3]The exact result for arbitrary N_c is presented in the Appendix A.

$$
\begin{aligned}
Z_1 = (-)^{1-i'+i+S'+I'} & \qquad\qquad (4.82)\\
& \times N_c\sqrt{(2i+1)(2i'+1)}\sqrt{(2S+1)(2I+1)}\langle S', m'_S|S,1;m_S,k\rangle\langle I',\alpha'|I,1;\alpha,a\rangle \\
& \times\left[\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 0\end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 0 & I\end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 0\end{array}\right\} - 3\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 1\end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 1 & I\end{array}\right\}\left\{\begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 1\end{array}\right\}\right] .
\end{aligned}
$$

Inserting (4.81) and (4.82) into (4.73) and using the definition of $Z(S'I', SI)$ (4.70) gives the final result for the matrix element of Z (4.71).

We will present in the following an alternative method of calculating the matrix element of a current between states with mixed symmetry. Besides reproducing the result (4.71), this method has the advantage of simplifying very much the computation of transition matrix elements between excited and ground state baryons, to be discussed in the next Section. We start by writing the matrix element of the current Z^{ka} taken between two states (4.42) as

$$
\begin{aligned}
\langle S'I'|Z^{ka}|SI\rangle = (-)^{\psi(SIi)+\psi'(S'I'i')}\frac{1}{2(N_c-1)} & \qquad (4.83)\\
\times\left\{\sum_{jj'=2}^{N_c} {}_{[j',1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]} + \sum_{j=2}^{N_c} {}_{[j,1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]}\right\}\mathcal{I} .
\end{aligned}
$$

We consider the two terms of this relation in turn. The first sum can be written as

$$
\sum_{jj'=2}^{N_c} {}_{[j',1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]} = {}_1\langle S'I'|\sum_{j'=2}^{N_c} A[j',1]Z^{ka}\sum_{j=2}^{N_c} A[j,1]|SI\rangle_1 , \qquad (4.84)
$$

where

$$
A[j,1] = \frac{1}{\sqrt{2}}(1-P_{j1}) \qquad (4.85)
$$

is the antisymmetrization operator for quarks $[j,1]$ and P_{j1} has been defined in (4.47). The spin states defined in (4.39) can be written in terms of it as $|SI\rangle_{[i,1]} = -A[i,1]|SI\rangle_1$.

An important relation we will use extensively in the following expresses the result of symmetrizing a direct product state $|SI\rangle_1$ under a permutation of any two quarks

$$
\Pi|SI,m\alpha\rangle_1 = (1+P_{12}+\cdots+P_{1N_c})\,|SI,m\alpha\rangle_1 = \delta_{SI}B(Ii)|I,m,\alpha\rangle \qquad (4.86)
$$

with $|I,m,\alpha\rangle$ the completely symmetric state constructed in Section IV.A. The normalization constant $B(Ii)$ can be computed by taking the norm of the both sides of this relation. We obtain

$$
\begin{aligned}
\delta_{SI}B^2(Ii) &= \sum_{nn'=1}^{N_c} {}_{n'}\langle SI,m\alpha|SI,m\alpha\rangle_n \qquad (4.87)\\
&= N_c + N_c(N_c-1)_{n'}\langle SI,m\alpha|SI,m\alpha\rangle_n . \qquad (n\neq n')
\end{aligned}
$$

The nondiagonal matrix element appearing on the r.h.s. has been calculated previously and is given by (4.67). We obtain finally

$$B^2(Ii) = N_c(N_c+1)(2i+1)\left\{\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ I & S & i \end{array}\right\}^2 - \frac{N_c}{2}\left[\frac{1}{2} + 2i(i+1) - 2I(I+1)\right]. \tag{4.88}$$

It will be shown below that the phase of $B(Ii)$ can be chosen such that the leading term in N_c is positive.

The sums over the antisymmetrization operators in (4.84) can be written in terms of the complete symmetrization operator Π as

$$\sum_{j=2}^{N_c} A[j,1] = \frac{1}{\sqrt{2}}(N_c - \Pi). \tag{4.89}$$

We will need also the following matrix element

$$\begin{aligned} N_c\, {}_1\langle S'I'|\Pi Z^{ka}|SI\rangle_1 &= {}_1\langle S'I'|\Pi Z^{ka}[1 + P_{12}^2 + \cdots + P_{1N_c}^2]|SI\rangle_1 \\ &= {}_1\langle S'I'|\Pi Z^{ka}\Pi|SI\rangle_1 . \end{aligned} \tag{4.90}$$

We used in the first line the property of the permutation operator $P_{ij}^2 = 1$. The second equality is obtained by writing $\Pi Z^{ka} P_{1j}^2 = \Pi P_{1j} Z^{ka} P_{1j}$. When acting to the left on ${}_1\langle S'I'|$, this gives

$${}_1\langle S'I'|\Pi P_{1j} = {}_1\langle S'I'|\Pi \tag{4.91}$$

since ${}_1\langle S'I'|\Pi$ is completely symmetric under any permutation of the N_c quarks. This completes the proof of (4.90).

The relations (4.89) and (4.90) allow us to express the sum of matrix elements (4.84) as

$$\begin{aligned} \sum_{jj'=2}^{N_c} {}_{[j',1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]} &= \frac{1}{2}N_c^2\, {}_1\langle S'I'|Z^{ka}|SI\rangle_1 - \frac{1}{2}\, {}_1\langle S'I'|\Pi Z^{ka}\Pi|SI\rangle_1 \\ &= \frac{1}{2}N_c^2\, {}_1\langle S'I'|Z^{ka}|SI\rangle_1 - \frac{1}{2}\delta_{SI}\delta_{S'I'}B(Ii)B(I'i')\langle S'=I'|Z^{ka}|S=I\rangle . \end{aligned} \tag{4.92}$$

The second term in (4.83) can be computed in an analogous way. We note for this the following useful properties of the antisymmetrization operator $A[j,1]$.

1. $A[j,1]$ commutes with Z^{ka}

   $$[A[j,1]\,, Z^{ka}] = 0\,. \tag{4.93}$$

   This follows from the fact that Z^{ka} is completely symmetric under a permutation of two quarks and commutes therefore with the P operator (4.47).

2. The square of $A[j,1]$ is given by

   $$A[j,1]^2 = \sqrt{2}A[j,1]\,. \tag{4.94}$$

With the help of these relations and (4.89) we can write

$$\sum_{j=2}^{N_c} {}_{[j,1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]} = \sum_{j=2}^{N_c} {}_1\langle S'I'|A[j,1]Z^{ka}A[j,1]|SI\rangle_1 \tag{4.95}$$
$$= {}_1\langle S'I'|Z^{ka}\sum_{j=2}^{N_c} A[j,1]^2|SI\rangle_1$$
$$= {}_1\langle S'I'|Z^{ka}\left(N_c - \Pi\right)|SI\rangle_1 \,.$$

Using a relation similar to (4.90) for the second term, this equation can be put into the form

$$\sum_{j=2}^{N_c} {}_{[j,1]}\langle S'I'|Z^{ka}|SI\rangle_{[j,1]} = \tag{4.96}$$
$$N_c\, {}_1\langle S'I'|Z^{ka}|SI\rangle_1 - \frac{1}{N_c}\delta_{SI}\delta_{S'I'}B(Ii)B(I'i')\langle S'=I'|Z^{ka}|S=I\rangle \,.$$

Combining the two results (4.92) and (4.96) gives the following general expression for the matrix element of the current Z^{ka} taken between two mixed symmetry states

$$\langle S'I'|Z^{ka}|SI\rangle = (-)^{\psi(SIi)+\psi'(S'I'i')} \tag{4.97}$$
$$\times\frac{N_c(N_c+2)}{4(N_c-1)}\mathcal{I}\left\{ {}_1\langle S'I'|Z^{ka}|SI\rangle_1 - \frac{1}{N_c^2}\delta_{SI}\delta_{S'I'}B(Ii)B(I'i')\langle S'=I'|Z^{ka}|S=I\rangle\right\} .$$

We are now in a position to compute the phase of the normalization constant $B(Ii)$. This can be done by comparing the two expressions (4.73) and (4.97) for the matrix element $\langle Z^{ia}\rangle$. We obtain in this way the following exact relation

$$\frac{1}{N_c^2}\delta_{SI}\delta_{S'I'}B(Ii)B(I'i')\langle S'=I'|Z^{ka}|S=I\rangle = Z_2 + \frac{1}{N_c}(Z_1 - Z_2)\,. \tag{4.98}$$

Using (4.81) for Z_2 one finds to leading order in N_c

$$(N_c+2)\sqrt{\frac{2I+1}{2I'+1}}B(Ii)B(I'i') = N_c^3\sqrt{(2i+1)(2i'+1)}\sqrt{(2S+1)(2I+1)} \tag{4.99}$$
$$\times(-)^{1+i'-i+S'+I'}\left\{\begin{array}{ccc} S' & S & 1 \\ I & I' & 0\end{array}\right\}\left\{\begin{array}{ccc} S & i & \frac12 \\ \frac12 & 0 & I\end{array}\right\}\left\{\begin{array}{ccc} S' & \frac12 & i' \\ \frac12 & I' & 0\end{array}\right\} = N_c^3\frac{\sqrt{(2i+1)(2i'+1)}}{2(2I'+1)}\delta_{IS}\delta_{I'S'}\,.$$

From this follows that $B(Ii)$ can be chosen to be positive for all values of its arguments.

It is easy to see now with the help of (4.82) and (4.99) that (4.97) gives, to leading order in N_c, the same result for $\langle S'I'|Z^{ka}|SI\rangle$ as (4.71).

D. Matrix elements of Y^a and Q^{ka} in the quark model

As already mentioned in Section III, the matrix elements of the operators Y^a and Q^{ka} in the quark model can be reduced to those of the operator $\sum_{n=1}^{N_c} r_n^i\sigma_n^j\tau_n^a$. Here r_n, σ_n, τ_n are vector operators acting on the orbital, spin and isospin degrees of freedom of the n^{th}

quark respectively. In this Section we prove that the quark model reproduces, in the large-N_c limit, the results (3.43) and (3.50) expected from the model-independent treatment of Sections III.B and III.C.

We consider first the transitions from an excited baryon state transforming under the symmetric representation of SU(4) to another symmetric baryon state. For generality we leave the orbital momenta of the initial and final states completely arbitrary L, L'. The dependence on the spin-isospin quantum numbers is contained in the reduced matrix element $\mathcal{T}(I', I)$ defined by

$$\langle I'L', m_S' m_L' \alpha' | \sum_{n=1}^{N_c} r_n^i \sigma_n^j \tau_n^a | IL, m_S m_L \alpha \rangle = \frac{1}{(2I'+1)\sqrt{2L'+1}} \mathcal{T}(I', I) \mathcal{I}(L', L) \qquad (4.100)$$
$$\times \langle I', m_S' | I, 1; m_S, j \rangle \langle I', \alpha' | I, 1; \alpha, a \rangle \langle L', m_L' | L, 1; m_L, i \rangle \,.$$

We will restrict our considerations to baryon states for which all the orbital angular momentum is carried by one quark at a time. This is strictly true only for the lowest orbital excitations. In Hartree language the spatial part of the wavefunction for these states has the form

$$|L, m_L\rangle = \frac{1}{\sqrt{N_c}} \sum_{p=1}^{N_c} \phi(r_1)\phi(r_2) \cdots \psi_{L,m_L}(\vec{r}_p) \cdots \phi(r_{N_c}) \qquad (4.101)$$

where $\phi(r)$ is a s-wave one-particle wavefunction and $\psi_{L,m_L}(\vec{r})$ carries angular momentum (L, m_L). The spatial part of the matrix element (4.100) can be written in terms of the matrix element

$$\langle L', m_L' | r_n^i | L, m_L \rangle = \frac{1}{N_c} \frac{1}{\sqrt{2L'+1}} \mathcal{I}(L', L) \langle L', m_L' | L, 1; m_L, i \rangle \,, \qquad (4.102)$$

with $\mathcal{I}(L', L)$ an overlap integral of order N_c^0. The case $L' = 0$ of a s-wave baryon in the final state is special, as the scaling law with N_c is different

$$\langle 0 | r_n^i | L, m_L \rangle = \frac{1}{\sqrt{N_c}} \mathcal{I} \delta_{L1} \langle 0 | L, 1; m_L, i \rangle \,, \qquad (4.103)$$

For both these cases the matrix element of r_n^i is independent of n due to the symmetry of the wavefunction under any permutation of two quarks. Therefore the spin-isospin part of the matrix element (4.100) decouples completely from the spatial part and is given exactly by the formula (4.37) for the ground state baryons. We obtain in this way for the reduced matrix element $\mathcal{T}(I', I)$

$$\mathcal{T}(I', I) = \begin{cases} \frac{N_c+2}{N_c}\sqrt{(2I+1)(2I'+1)}\,, & L' \neq 0 \\ \frac{N_c+2}{\sqrt{N_c}}\sqrt{(2I+1)(2I'+1)}\,, & L' = 0\,. \end{cases} \qquad (4.104)$$

which can be seen to coincide, up to an unimportant phase and numerical factor, with the result (3.43) anticipated in Section III.

We consider next the case of an excited baryon transforming under the mixed symmetry representation of SU(4) in the initial state. The final state corresponds to the completely symmetric representation. We write the matrix element relevant for this case as

$$\langle I'L', m'm'_L\alpha'|\sum_{n=1}^{N_c} r^i_n\sigma^j_n\tau^a_n|SIL, m_S m_L\alpha\rangle = \frac{1}{(2I'+1)\sqrt{2L'+1}}\mathcal{T}(I', SI)\mathcal{I}(L', L) \tag{4.105}$$
$$\times\langle I', m'|S, 1; m_S, j\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\langle L', m'_L|L, 1; m_L, i\rangle\,.$$

The scaling law with N_c of the spatial part of this matrix element is again different, depending on whether $L' \neq 0$ or $L' = 0$. Both cases can be considered together by writing it as

$$\langle L', m'_L|r^i_n|SIL, m_S m_L\alpha\rangle = \frac{N_c^\kappa(-)^{\psi(SIi)}}{\sqrt{2N_c(N_c-1)}}\sum_{k,k'=1}^{N_c}(\delta_{k'n}\delta_{nk} - \delta_{k'n}\delta_{n1}) \tag{4.106}$$
$$\times\frac{1}{\sqrt{2L'+1}}\mathcal{I}(L', L)\langle L', m'_L|L1, m_L, i\rangle|SI, m_S\alpha\rangle_{[k,1]}$$
$$= \frac{N_c^\kappa(-)^{\psi(SIi)}}{\sqrt{2N_c(N_c-1)}}\frac{1}{\sqrt{2L'+1}}\mathcal{I}(L', L)\langle L', m'_L|L1, m_L, i\rangle\left\{|SI, m_S\alpha\rangle_{[n,1]} - \delta_{n1}\sum_{k=2}^{N_c}|SI, m_S\alpha\rangle_{[k,1]}\right\}\,.$$

Here $\kappa = 1/2$ for $L' = 0$ and $\kappa = 0$ for $L' \neq 0$. Adding the spin-isospin part of the operator and summing over the N_c quarks in the baryon gives

$$\langle I'L', m'm'_L\alpha'|\sum_{n=1}^{N_c} r^i_n\sigma^j_n\tau^a_n|SIL, m_S m_L\alpha\rangle = \tag{4.107}$$
$$\frac{N_c^\kappa(-)^{\psi(SIi)}}{\sqrt{2N_c(N_c-1)}}\frac{1}{\sqrt{2L'+1}}\mathcal{I}(L', L)\langle L', m'_L|L1, m_L, i\rangle$$
$$\times\left\{\sum_{n=1}^{N_c}\langle I', m'\alpha'|\sigma^j_n\tau^a_n|SI, m_S\alpha\rangle_{[n,1]} - \langle I', m'\alpha'|\sigma^j_1\tau^a_1\sum_{k=2}^{N_c}|SI, m_S\alpha\rangle_{[k,1]}\right\}\,.$$

The first term in the braces can be written as

$$\sum_{n=1}^{N_c}\langle I', m'\alpha'|\sigma^j_n\tau^a_n|SI, m_S\alpha\rangle_{[n,1]} = \tag{4.108}$$
$$\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{N_c}\langle I', m'\alpha'|\sigma^j_n\tau^a_n|SI, m_S\alpha\rangle_n - \langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n|SI, m_S\alpha\rangle_1\right) =$$
$$\frac{1}{\sqrt{2}}\left(N_c\langle I', m'\alpha'|\sigma^j_1\tau^a_1|SI, m_S\alpha\rangle_1 - \frac{1}{N_c}\delta_{SI}B(Ii)\langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n|I, m_S\alpha\rangle\right)\,.$$

In the second line we used the identity

$$N_c\langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n|SI, m_S\alpha\rangle_1 = \langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n(1 + P^2_{12} + \cdots + P^2_{1N_c})|SI, m_S\alpha\rangle_1 \tag{4.109}$$
$$= \sum_{k=1}^{N_c}\langle I', m'\alpha'|P_{1k}\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n P_{1k}|SI, m_S\alpha\rangle_1 = \langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma^j_n\tau^a_n\Pi|SI, m_S\alpha\rangle_1$$

followed by the application of the relation (4.86). In (4.109) we have defined $P_{11} = 1$.

The second term in (4.107) can be put into the following form through an application of (4.89) and (4.86)

$$\langle I', m'\alpha'|\sigma_1^j\tau_1^a \sum_{k=2}^{N_c} |SI, m_S\alpha\rangle_{[k,1]} = \tag{4.110}$$
$$\frac{1}{\sqrt{2}}\left(\delta_{SI}B(Ii)\langle I', m'\alpha'|\sigma_1^j\tau_1^a|I, m_S, \alpha\rangle - N_c\langle I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m_S\alpha\rangle_1\right) .$$

Combining (4.108) and (4.110) together we obtain the following general formula for the matrix element (4.105)

$$\langle I'L', m'm'_L\alpha'|\sum_{n=1}^{N_c} r_n^i\sigma_n^j\tau_n^a|SIL, m_Sm_L\alpha\rangle = \tag{4.111}$$
$$\frac{N_c^\kappa(-)^{\psi(SIi)}}{\sqrt{N_c(N_c-1)}}\frac{1}{\sqrt{2L'+1}}\mathcal{I}(L', L)\langle L', m'_L|L1, m_L, i\rangle$$
$$\times\left\{N_c\langle I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m_S\alpha\rangle_1 - \frac{1}{N_c}\delta_{SI}B(Ii)\langle I', m'\alpha'|\sum_{n=1}^{N_c}\sigma_n^j\tau_n^a|I, m_S, \alpha\rangle\right\} .$$

The second term in (4.111) is already known from our analysis of the symmetric states in Section IV.A. The first matrix element is new. In the following we present the details of its calculation.

Using (4.86) one can write

$$\langle I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m_S\alpha\rangle_1 = \frac{1}{B(I'i')}\,{}_1\langle I'I', m'\alpha'|\sum_{k=1}^{N_c} P_{1k}\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 . \tag{4.112}$$

A typical term of the sum over k has the form

$${}_1\langle I'I', m'\alpha'|P_{1k}\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 = \tag{4.113}$$
$$\frac{1}{4}\,{}_1\langle I'I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 + \frac{1}{4}\,{}_1\langle I'I', m'\alpha'|(\vec{\sigma}_1\cdot\vec{\sigma}_k)(\vec{\tau}_1\cdot\vec{\tau}_k)\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 + \mathcal{O}(1/N_c)$$

where we used the definition of the P operator (4.47) and the fact that $F(i)$ and $G(i)$ computed in Section IV.B are nonleading in $1/N_c$. The first matrix element is easily calculated with the result

$${}_1\langle I'I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 = \langle I', m'|S, 1; m, j\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle \tag{4.114}$$
$$\times 6\delta_{ii'}\sqrt{(2S+1)(2I+1)}\begin{Bmatrix} 1 & S & I' \\ i & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2}\end{Bmatrix} .$$

The second matrix element can be reduced to quantities already known by simplifying the products of two spin and isospin Pauli matrices with the help of the identity

$$\sigma_a\sigma_b = -\sqrt{3}\langle 0|11; ab\rangle\, 1 - \sqrt{2}\sum_c\langle 1c|11; ab\rangle\sigma_c . \tag{4.115}$$

We obtain in this way

$$
\begin{aligned}
&{}_1\langle I'I', m'\alpha'|(\vec\sigma_1\cdot\vec\sigma_k)(\vec\tau_1\cdot\vec\tau_k)\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 = \\
&\quad \langle I', m'|S,1;m,j\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\sqrt{(2i+1)(2i'+1)(2S+1)(2I+1)} \\
&\quad \times\left[(-)^{-i+\frac12+S}\left\{\begin{array}{ccc} S & 1 & I' \\ i' & \frac12 & i\end{array}\right\} + 6(-)^{-S-I'}\left\{\begin{array}{ccc} \frac12 & 1 & \frac12 \\ i & 1 & i' \\ S & 1 & I'\end{array}\right\}\right] \\
&\quad \times\left[(-)^{-i+\frac12+I}\left\{\begin{array}{ccc} I & 1 & I' \\ i' & \frac12 & i\end{array}\right\} + 6(-)^{-I-I'}\left\{\begin{array}{ccc} \frac12 & 1 & \frac12 \\ i & 1 & i' \\ I & 1 & I'\end{array}\right\}\right] \\
&= 36\langle I', m'|S,1;m,j\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\sqrt{(2i+1)(2i'+1)(2S+1)(2I+1)} \\
&\times\left\{\begin{array}{ccc} i & i' & 1 \\ \frac12 & \frac12 & I'\end{array}\right\}^2\left\{\begin{array}{ccc} S & I' & 1 \\ \frac12 & \frac12 & i\end{array}\right\}\left\{\begin{array}{ccc} I & I' & 1 \\ \frac12 & \frac12 & i\end{array}\right\}.
\end{aligned} \tag{4.116}
$$

We used in the second equality the identity

$$
\left\{\begin{array}{ccc} \frac12 & \frac12 & 1 \\ i & i' & 1 \\ S & I' & 1\end{array}\right\} = \left\{\begin{array}{ccc} i & i' & 1 \\ \frac12 & \frac12 & I'\end{array}\right\}\left\{\begin{array}{ccc} S & I' & 1 \\ \frac12 & \frac12 & i\end{array}\right\} + \frac16(-)^{i'+S+\frac32}\left\{\begin{array}{ccc} i & i' & 1 \\ I' & S & \frac12\end{array}\right\} \tag{4.117}
$$

and a similar one with $S \to I$, which can be obtained from (3.21) by taking $\lambda = 1/2$ and the j's are the same as in the $9j$ symbol on the l.h.s.

Combining (4.114) and (4.116) we find the following result for the matrix element (4.113)

$$
\begin{aligned}
&{}_1\langle I'I', m'\alpha'|P_{1k}\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1 = \\
&\quad \sqrt{(2i+1)(2i'+1)(2I+1)(2S+1)}\langle I', m'|S,1;m,j\rangle\langle I',\alpha'|I,1;\alpha,a\rangle \\
&\quad \times 3\frac{1}{2I'+1}\left\{\begin{array}{ccc} S & I' & 1 \\ \frac12 & \frac12 & i\end{array}\right\}\left\{\begin{array}{ccc} I & I' & 1 \\ \frac12 & \frac12 & i\end{array}\right\},
\end{aligned} \tag{4.118}
$$

where we have rewritten the $\delta_{ii'}$ symbol in (4.114) as

$$
\delta_{ii'} = 2\sqrt{(2i+1)(2i'+1)}\left\{\begin{array}{ccc} i & i' & 0 \\ \frac12 & \frac12 & I'\end{array}\right\}^2 \tag{4.119}
$$

and added the two terms with the help of the identity

$$
\sum_{x=0,1}(2x+1)\left\{\begin{array}{ccc} i & i' & x \\ \frac12 & \frac12 & I'\end{array}\right\}^2 = \frac{1}{2I'+1}\,. \tag{4.120}
$$

Next we use (3.30) to write the product of 2 $6j$-symbols in (4.118) as

$$
\begin{aligned}
\left\{\begin{array}{ccc} \frac12 & I' & i \\ S & \frac12 & 1\end{array}\right\}\left\{\begin{array}{ccc} \frac12 & I' & i \\ I & \frac12 & 1\end{array}\right\} &= \frac{\delta_{SI}}{6\sqrt{(2S+1)(2I+1)}} \\
&\quad +(-)^{-\frac12+S+I+I'+i}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I & I'\end{array}\right\}\left\{\begin{array}{ccc} S & I & 1 \\ \frac12 & \frac12 & i\end{array}\right\}.
\end{aligned} \tag{4.121}
$$

The sum over k in (4.112) is dominated by the terms with $k \neq 1$, of which there are $N_c - 1$. Neglecting the contribution of the $k = 1$ term, we obtain for the matrix element (4.112) to leading order in N_c

$$\begin{aligned}\langle I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m_s\alpha\rangle_1 &= 3\sqrt{2(2i+1)}\sqrt{\frac{(2S+1)(2I+1)}{2I'+1}} \\ &\times\langle I', m'|S,1;m,j\rangle\langle I',\alpha'|I,1;\alpha,a\rangle \\ &\times\left\{\frac{\delta_{SI}}{6\sqrt{(2S+1)(2I+1)}} + (-)^{-\frac{1}{2}+S+I+I'+i}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I & I' \end{array}\right\}\left\{\begin{array}{ccc} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{array}\right\}\right\}.\end{aligned} \tag{4.122}$$

When inserted into (4.111), the term proportional to δ_{SI} will be cancelled exactly by the second term in (4.111). As a result, we obtain for the reduced matrix element $\mathcal{T}(I', SI)$ taken between the unnormalized quark model states

$$\begin{aligned}\mathcal{T}(I', SI) = N_c^\kappa 3\sqrt{2(2i+1)}\sqrt{(2S+1)(2I+1)(2I'+1)}(-)^{-\frac{1}{2}+S+I+I'+i}(-)^{\psi(SIi)} \\ \times\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I & I' \end{array}\right\}\left\{\begin{array}{ccc} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{array}\right\}.\end{aligned} \tag{4.123}$$

The physical value of this reduced matrix element is obtained by dividing with the square root of the norm of the initial state (4.44). This gives

$$\begin{aligned}[\mathcal{T}(I', SI)]_{norm} = N_c^{\kappa-1/2}2\sqrt{6}\sqrt{(2S+1)(2I+1)(2I'+1)}(-)^{-\frac{1}{2}+S+I+I'+i}(-)^{\phi(SIi)+\psi(SIi)} \\ \times\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I & I' \end{array}\right\}.\end{aligned} \tag{4.124}$$

Finally we insert here $\phi(SIi) = 1 + I + S$ the phase of the $6j$-symbol appearing in the formula for the norm (4.44) and $\psi(SIi) = i + I + \frac{1}{2}$ the phase of the mixed symmetry state, which gives for the total phase $(-)^{-I+I'}$. Thus (4.124) can be seen to coincide exactly, up to a numerical factor, with the expression (3.50) expected from the model-independent treatment of Section III.B.

An important by-product of this calculation is the large-N_c scaling law of the transition matrix elements into final states in an s-wave. We obtain that the matrix elements of Y^a and Q^{ka} from an initial state with mixed symmetry scales like N_c^0. On the other hand, the same matrix elements with a symmetric excited state in the initial state scale as $N_c^{\frac{1}{2}}$. This dependence of the scaling law on the symmetry type of the excited state is a new feature, unnoticed previously. As discussed in Section III, for both cases the scaling law for the total scattering amplitude is sufficiently restrictive to allow the derivation of useful consistency conditions. In spite of their different N_c scaling, the solutions for these matrix elements have the same dependence on spin and flavor quantum numbers.

The quark model computations of Appendix A illustrate another important asymmetry between the symmetric and the mixed symmetry states. The $1/N_c$ corrections to the large-N_c results for coupling ratios vanish for the former [2] but not for the latter. Such dependence on the symmetry properties of these states raises the question of how to distinguish states with different permutational symmetry beyond the framework of the quark model.

The exact large-N_c scaling law for matrix elements of Y and Q following from the calculations of this Section is strictly correct only for the case of the baryons made of heavy quarks, for which the constituent quark picture is known to be exactly valid. Our results following from the consistency conditions discussed in Sect.III rest on the assumption that no important changes occur as the quarks become light and that the modified scaling law corresponding to this situation still allows the derivation of consistency conditions. While this assumption seems plausible and is similar to smoothness arguments commonly used in other large-N_c studies [12, 9], it is important to keep it in mind as one of the vulnerable points of an analysis of this type.

V. CONCLUSIONS AND OUTLOOK

We have studied the strong couplings of the excited baryons in the large-N_c limit with the help of consistency conditions on pion-baryon scattering amplitudes. This method is similar to the one used by Dashen, Jenkins and Manohar [2, 3, 5, 6] in their analysis of the strong couplings of the s-wave baryons. In extending their analysis to the excited baryons' sector one has to deal with additional complications, related to the more complex structure of the spectrum of these states.

The consistency conditions are very effective in constraining the large-N_c spin-isospin dependence of the strong vertices of these states, especially for the S-wave pion coupling, which is completely determined in terms of just one unknown constant. The allowed form of the strong vertices turns out to be exactly the same as the one following from the constituent quark model. In addition to constraining the structure of the strong vertex, the consistency conditions predict also the equality of the pion couplings to excited and to s-wave baryons respectively. This is again what is expected from the constituent quark model.

Our findings extend therefore the results obtained in [2, 3, 5, 6] for the strong couplings of the s-wave baryons and give a natural explanation for the successes of the quark model when applied to strong decays of the excited baryons [1, 16, 17, 18] in terms of the large-N_c expansion. For example, this lends additional support to some predictions made recently for strong decays of excited heavy baryons [38, 39] with the help of the quark model. However, as discussed in Appendix A, the quark model predictions for ratios of strong couplings for these states cannot be expected to hold to the same accuracy as in the s-wave sector, as these ratios are not in general protected against $1/N_c$ corrections. The exact results in Appendix A provide a specific framework to study quantitatively how good the large-N_c approximation is by examining their complete N_c dependence as N_c varies from the physical value $N_c = 3$ to infinity.

The results of the present paper can be expanded in a number of directions. We recall that our analysis has only assumed isospin symmetry. Thus, one can attempt to incorporate SU(3) with some amount of symmetry breaking, by studying consistency conditions following from large-N_c counting rules for kaon-baryon scattering amplitudes [5, 6]. In this way one should be able to relate the strong couplings of different towers of states with different strangeness quantum numbers, which in our present analysis are left completely unrelated. Second, we have only discussed excited states transforming under the symmetric and mixed symmetric representations of SU(4). It is known that excited states exist which transform also under the antisymmetric representation. Extending our analysis to this case should be completely straightforward. Finally, a similar analysis could be performed for the electromagnetic couplings of the excited baryons, with the help of consistency conditions for photon-baryon scattering amplitudes. For the s-wave baryons such constraints on the magnetic moments have been worked out in [4]. We plan to return to some of these problems in a future publication.

ACKNOWLEDGMENTS

The research of D.P. was supported by the Ministry of Science and the Arts of Israel. The work of T.M.Y. was supported in part by the National Science Foundation.

APPENDIX A: QUARK MODEL MATRIX ELEMENTS FOR ARBITRARY N_C

In this Appendix we give, for completeness, the full expressions for reduced matrix elements in the quark model with arbitrary number of colors N_c. The results presented in the main text are obtained from these expressions by keeping only the leading terms in N_c. We take advantage of our ability to derive exact relations for the quark model matrix elements to study the $1/N_c$ corrections to the large-N_c predictions. By examining a few simple particular cases we conclude that the results obtained in Section III in the large-N_c limit will receive, in general, $1/N_c$ corrections.

1. $Z(S'I', SI)$

We begin by giving the result for the matrix element $Z(S'I', SI)$ defined by

$$\langle S'I'L'; m'_S, m'_L, \alpha'| \sum_{n=1}^{N_c} \sigma_n^k \tau_n^a |SIL; m_S, m_L, \alpha\rangle = \tag{A1}$$
$$\frac{1}{\sqrt{(2S'+1)(2I'+1)}} Z(S'I', SI)\delta_{LL'}\delta_{m_L m'_L}\langle S', m'_S|S, 1; m_S, k\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,.$$

According to (4.73) this matrix element is completely determined in terms of the diagonal and the nondiagonal matrix elements of the current on direct product states. These will be characterized by two quantities z_1, z_2 defined by

$$Z_1 = {}_j\langle S'I', m', \alpha'| \sum_{n=1}^{N_c} \sigma_n^k \tau_n^a |SI, m, \alpha\rangle_j \tag{A2}$$
$$= \sqrt{(2S+1)(2I+1)} z_1 \langle S', m'|S, 1; m, k\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle$$
$$Z_2 = {}_{j'}\langle S'I', m', \alpha'| \sum_{n=1}^{N_c} \sigma_n^k \tau_n^a |SI, m, \alpha\rangle_j \tag{A3}$$
$$= \sqrt{(2S+1)(2I+1)} z_2 \langle S', m'|S, 1; m, k\rangle\langle I', \alpha'|I, 1; \alpha, a\rangle\,.$$

The reduced matrix element $Z(S'I', SI)$ (taken between unnormalized quark model states) is expressed in terms of z_1 and z_2 as

$$Z(S'I', SI) = (-)^{\psi(S'I'i')+\psi(SIi)}\sqrt{(2S+1)(2I+1)(2S'+1)(2I'+1)}\frac{N_c+2}{4}(z_1 - z_2)\,. \tag{A4}$$

We obtain for the diagonal matrix element the simple result

$$z_1 = \sqrt{(2i+1)(2i'+1)} z(i', i)(-)^{-1+2i'+S+I} \left\{ \begin{array}{ccc} 1 & S & S' \\ \frac12 & i' & i \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ \frac12 & i' & i \end{array} \right\} \tag{A5}$$
$$+ 6\delta_{ii'}(-)^{1-2i-S'-I'} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac12 & \frac12 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac12 & \frac12 \end{array} \right\},$$

with

$$z(i',i) = \sqrt{(N_c+1)^2 - (i'-i)^2(i'+i+1)^2} = N_c + 1 + \mathcal{O}(1/N_c)\,. \tag{A6}$$

The nondiagonal matrix element z_2 can be written as a sum over the four terms into which it can be decomposed with the help of (4.77)

$$z_2 = \frac{1}{4}(T_1 + T_2 + T_3 + T_4)\,. \tag{A7}$$

We find

$$T_1 = z_1 \tag{A8}$$

$$T_2 = \sqrt{6(2i+1)(2i'+1)^2}F(i')z(i',i)(-)^{-\frac{1}{2}+2i+i'+I+S+S'} \tag{A9}$$
$$\times \left\{ \begin{array}{ccc} i' & \frac{1}{2} & S' \\ \frac{1}{2} & i' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ \frac{1}{2} & i' & i \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ \frac{1}{2} & i' & i \end{array} \right\}$$
$$- 6\sqrt{6}F(i)\delta_{ii'}\sqrt{2i'+1}(-)^{\frac{1}{2}-i-I'} \left\{ \begin{array}{ccc} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & S' \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\}$$

$$T_3 = \sqrt{6(2i+1)(2i'+1)^2}F(i')z(i',i)(-)^{-\frac{1}{2}+2i+i'+I+S+I'} \tag{A10}$$
$$\times \left\{ \begin{array}{ccc} i' & \frac{1}{2} & I' \\ \frac{1}{2} & i' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ \frac{1}{2} & i' & i \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ \frac{1}{2} & i' & i \end{array} \right\}$$
$$- 6\sqrt{6}F(i)\delta_{ii'}\sqrt{2i'+1}(-)^{\frac{1}{2}-i-S'} \left\{ \begin{array}{ccc} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\}$$

$$T_4 = 6\sqrt{(2i+1)(2i'+1)}(-)^{S+I+S'+I'} \tag{A11}$$
$$\times \sum_{i''} \frac{z(i',i'')z(i'',i)}{N_c-1}(2i''+1) \left\{ \begin{array}{ccc} i'' & \frac{1}{2} & S' \\ \frac{1}{2} & i' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} S & 1 & S' \\ i'' & \frac{1}{2} & i \end{array} \right\} \left\{ \begin{array}{ccc} i'' & \frac{1}{2} & I' \\ \frac{1}{2} & i' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} I & 1 & I' \\ i'' & \frac{1}{2} & i \end{array} \right\}$$
$$+ 36\frac{z(i',i)}{N_c-1}\sqrt{(2i+1)(2i'+1)} \left\{ \begin{array}{ccc} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & S' \end{array} \right\} \left\{ \begin{array}{ccc} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\} \left\{ \begin{array}{ccc} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{array} \right\} \left\{ \begin{array}{ccc} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{array} \right\}.$$

The expression for z_2 greatly simplifies if only terms of order 1 are kept, in addition to the leading ones of order N_c, due to the fact that in this approximation the $z(i',i)$ factors are constants. This allows the sum over i'' to be performed with the help of (4.80). We obtain

$$\frac{z_1}{\sqrt{(2i+1)(2i'+1)}} = (N_c+1)(-)^{\phi_0}[\{6j_0\}^3 - 3\{6j_1\}^3] \tag{A12}$$
$$+12 \left\{ \begin{array}{ccc} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & S' \end{array} \right\} \left\{ \begin{array}{ccc} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} + \mathcal{O}(N_c^{-1})$$

$$\frac{z_2}{\sqrt{(2i+1)(2i'+1)}} = (N_c+2)(-)^{\phi_0}\{6j_0\}^3 + 3(-)^{\phi_0}\{6j_1\}^3 \tag{A13}$$
$$+3\frac{\delta_{S'I'}}{2I'+1} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\}$$
$$-\frac{1}{2}(-)^{\phi_0}[\frac{1}{2} + 2i'(i'+1) - S'(S'+1) - I'(I'+1)][\{6j_0\}^3 - 3\{6j_1\}^3] + \mathcal{O}(N_c^{-1})$$

with $\phi_0 = 1 + i - i' + I' + S'$. We denoted here the products of $6j$ symbols encountered in Section III

$$\{6j_{0(1)}\}^3 = \left\{ \begin{array}{ccc} S' & S & 1 \\ I & I' & 0(1) \end{array} \right\} \left\{ \begin{array}{ccc} S & i & \frac{1}{2} \\ \frac{1}{2} & 0(1) & I \end{array} \right\} \left\{ \begin{array}{ccc} S' & \frac{1}{2} & i' \\ \frac{1}{2} & I' & 0(1) \end{array} \right\} . \tag{A14}$$

The difference of z_1 and z_2 can be finally written as

$$\begin{aligned} \frac{z_1 - z_2}{\sqrt{(2i+1)(2i'+1)}} &= -3(N_c+2)(-)^{\phi_0}\{6j_1\}^3 - (-)^{\phi_0}\{6j_0\}^3 \\ &+\frac{1}{2}(-)^{\phi_0}[\frac{1}{2} + 2i'(i'+1) - S'(S'+1) - I'(I'+1)][\{6j_0\}^3 - 3\{6j_1\}^3] \\ &-18(-)^{i+i'+I'+S'} \left\{ \begin{array}{ccc} S' & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{array} \right\} \left\{ \begin{array}{ccc} S' & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{array} \right\} \left\{ \begin{array}{ccc} 1 & S & S' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{array} \right\} + \mathcal{O}(N_c^{-1}) . \end{aligned} \tag{A15}$$

The alternative method presented in Section IV.C can be also used to give an exact expression for the reduced matrix element of Z^{ia}. We find from (4.97) the following result

$$\begin{aligned} Z(S'I', SI) &= (-)^{\psi(S'I'i')+\psi(SIi)}\sqrt{(2S+1)(2I+1)(2S'+1)(2I'+1)}\frac{N_c(N_c+2)}{4(N_c-1)} \\ &\times \left(z_1 - \frac{1}{N_c^2}B(Ii)B(I'i')\frac{Z(I',I)}{(2I+1)(2I'+1)}\delta_{SI}\delta_{S'I'} \right) , \end{aligned} \tag{A16}$$

where $Z(I',I)$ has been defined in (4.37) and $B(Ii)$ is given by (4.88). We have checked explicitly that both methods lead to the same answer for $Z(S'I',SI)$ up to the next-to-leading order in $1/N_c$. We notice that (A16) does not involve any summation over intermediate state quantum numbers.

The leading order term in (A15) is written as proportional to $N_c + 2$, which was seen to give the correct result to two orders in the $1/N_c$ expansion for the case of the symmetric baryons. It is natural to ask whether a similar result holds also for the reduced matrix element $Z(S'I',SI)$. In the following we will argue that no result of comparable simplicity can be obtained for the $1/N_c$ corrections to this quantity. Strictly speaking this still does not prove that there are nonvanishing $1/N_c$ corrections to $Z(J'I',JI)$ (which is the true physical coupling with a meaning beyond the quark model) which is related to $Z(S'I',SI)$ by (3.28). It is still conceivable that the $1/N_c$ corrections to $Z(S'I',SI)$ add up to zero when inserted into (3.28), although we have not been able to prove it.

We will consider for simplicity the case when the quantum numbers of the initial and final states satisfy

$$S \neq I\,, S' \neq I' \tag{A17}$$

and examine the structure of the $1/N_c$ corrections in the following two particular cases: a) $S = S'\,, I = I'$ and b) $S = I'\,, I = S'$. This constrains i, i' to be equal: $i = i'$.

The norm of a state satisfying (A17) can be obtained from (4.44) and is given exactly by

$$\langle SI|SI\rangle = \frac{N_c+2}{4}\frac{N_c}{N_c-1} . \tag{A18}$$

This will have to be divided out from the quantity on the r.h.s. of (A15). We obtain

$$\frac{1}{\sqrt{\langle SIi|SIi\rangle\langle S'I'i'|S'I'i'\rangle}}\frac{N_c+2}{4}(z_1-z_2)=\sqrt{(2i+1)(2i'+1)}\Big\{-3(N_c+2)(-)^{\phi_0}\{6j_1\}^3 \tag{A19}$$
$$-18(-)^{i+i'+I'+S'}\left\{\begin{matrix} S' & I' & 1\\ \frac12 & \frac12 & i\end{matrix}\right\}\left\{\begin{matrix} S' & I' & 1\\ \frac12 & \frac12 & i'\end{matrix}\right\}\left\{\begin{matrix} 1 & S & S'\\ i & \frac12 & \frac12\end{matrix}\right\}\left\{\begin{matrix} 1 & I & I'\\ i & \frac12 & \frac12\end{matrix}\right\}+3(-)^{\phi_0}\{6j_1\}^3\Big\}\,.$$

Next, we note that in the limit (A17) the product of 4 $6j$ symbols on the r.h.s. can be written as

$$(-)^{i+i'+I'+S'}\left\{\begin{matrix} S' & I' & 1\\ \frac12 & \frac12 & i\end{matrix}\right\}\left\{\begin{matrix} S' & I' & 1\\ \frac12 & \frac12 & i'\end{matrix}\right\}\left\{\begin{matrix} 1 & S & S'\\ i & \frac12 & \frac12\end{matrix}\right\}\left\{\begin{matrix} 1 & I & I'\\ i & \frac12 & \frac12\end{matrix}\right\} \tag{A20}$$
$$=\left\{\begin{matrix} \frac{1}{36(2i+1)^2}\sqrt{\frac{(2i-1)(2i+3)}{i(i+1)}} & \text{case a)}\\ \frac{1}{9(2i+1)^2} & \text{case b)}\end{matrix}\right.$$

On the other hand, the leading order term is proportional to $\{6j_1\}^3$. From (4.79) we obtain in the limit (A17)

$$\{6j_1\}^3=(-)^{1+2i}\frac{1}{3}\left\{\begin{matrix} 1 & S & S'\\ \frac12 & i' & i\end{matrix}\right\}\left\{\begin{matrix} 1 & I & I'\\ \frac12 & i' & i\end{matrix}\right\}=\left\{\begin{matrix} (-)^{2i}\frac{1}{3(2i+1)^2}\sqrt{\frac{(2i-1)(2i+3)}{2i(2i+2)}} & \text{case a)}\\ (-)^{1+2i}\frac{1}{6(2i+1)^2 i(i+1)} & \text{case b)}\end{matrix}\right. \tag{A21}$$

One can see that for case b) the terms of order 1 in (A19) do not have the same structure as the leading term of order N_c and therefore cannot be generally absorbed into a rescaling of the latter.

2. $\mathcal{T}(I', SI)$

Next we present the exact calculation of the reduced matrix element $\mathcal{T}(I', SI)$ defined by

$$\langle I'L', m'm'_L\alpha'|\sum_{n=1}^{N_c} r_n^i\sigma_n^j\tau_n^a|SIL, m_S m_L\alpha\rangle=\frac{1}{(2I'+1)\sqrt{2L'+1}}\mathcal{T}(I',SI)\mathcal{I}(L',L) \tag{A22}$$
$$\times\langle I',m'|S,1;m_S,j\rangle\langle I',\alpha'|I,1;\alpha,a\rangle\langle L',m'_L|L,1;m_L,i\rangle\,,$$

relevant for the transitions from orbital excitations with mixed symmetry to symmetric states. We calculate this matrix element starting from the general formula (4.111) and proceeding along the same steps as in Section IV.D. The first term in (4.111) can be expressed with the help of (4.112) in terms of the two matrix elements ($k\neq 1$)

$${}_1\langle I'I', m'\alpha'|\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1=\sqrt{(2S+1)(2I+1)}t_1\langle I',m'|S1;m,j\rangle\langle I',\alpha'|I1;\alpha,a\rangle \tag{A23}$$
$${}_1\langle I'I', m'\alpha'|P_{1k}\sigma_1^j\tau_1^a|SI, m\alpha\rangle_1=\sqrt{(2S+1)(2I+1)}t_2\langle I',m'|S1;m,j\rangle\langle I',\alpha'|I1;\alpha,a\rangle\,. \tag{A24}$$

Expanding the permutation operator P_{1k} and evaluating the resulting matrix elements with the help of the results of Section IV.B we obtain the following exact expressions for the coefficients t_1, t_2

$$t_1 = 6\delta_{ii'} \begin{Bmatrix} 1 & S & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \tag{A25}$$

$$\begin{aligned} t_2 = {} & \frac{3}{2}\delta_{ii'} \begin{Bmatrix} 1 & S & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \\ & - 3\sqrt{\frac{3}{2}} F(i)\delta_{ii'}\sqrt{2i'+1}(-)^{I'-\frac{1}{2}+i} \begin{Bmatrix} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix} \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \\ & - 3\sqrt{\frac{3}{2}} G(i)\delta_{ii'}\sqrt{2i'+1}(-)^{I'-\frac{1}{2}+i} \begin{Bmatrix} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix} \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} 1 & I & I' \\ i & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \\ & + 9H(i',i)\sqrt{(2i+1)(2i'+1)} \begin{Bmatrix} i & i' & 1 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix}^2 \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}. \end{aligned} \tag{A26}$$

The result for t_2 simplifies considerably when only the terms of order 1 and $1/N_c$ are kept

$$\begin{aligned} & \frac{t_2}{\sqrt{(2i+1)(2i'+1)}} = \\ & \left\{ \frac{3}{2I'+1} + \frac{1}{N_c}\frac{6}{2I'+1} - \frac{6}{N_c}\left[\frac{5}{2} + 2i'(i'+1) - 2I'(I'+1)\right] \begin{Bmatrix} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix}^2 \right\} \\ & \times \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} + \mathcal{O}(N_c^{-2}). \end{aligned} \tag{A27}$$

The matrix element on the r.h.s. of (4.112) is proportional to the combination

$$\begin{aligned} t_1 + (N_c - 1)t_2 = {} & \sqrt{(2i+1)(2i'+1)} \\ & \times \left[\frac{3}{2I'+1}N_c + \frac{3}{2I'+1} - 6\left[\frac{1}{2} + 2i'(i'+1) - 2I'(I'+1)\right] \begin{Bmatrix} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix}^2 \right] \\ & \times \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} + \mathcal{O}(N_c^{-1}). \end{aligned} \tag{A28}$$

After dividing this expression with $B(I'i')$ we find for the first matrix element in (4.111) the following result

$$\begin{aligned} & \langle I', m'\alpha' | \sigma_1^j \tau_1^a | SI, m_S\alpha\rangle_1 = \\ & \quad \sqrt{(2S+1)(2I+1)}\langle I', m'|S1; m, j\rangle \langle I', \alpha'|I1; \alpha, a\rangle \; 3\sqrt{2}\sqrt{\frac{2i+1}{2I'+1}} \\ & \times \left\{ 1 + \frac{1}{2N_c} + \frac{2I'+1}{3N_c}\left[\frac{1}{2} + 2i'(i'+1) - 2I'(I'+1)\right] \left(\frac{3}{2(2i'+1)} - 6\begin{Bmatrix} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{Bmatrix}^2 \right) \right\} \\ & \times \begin{Bmatrix} S & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} + \mathcal{O}(N_c^{-1}). \end{aligned} \tag{A29}$$

The second matrix element in (4.111) is given by

$$\delta_{SI} B(Ii) \langle I', m'\alpha' | \sum_{n=1}^{N_c} \sigma_n^j \tau_n^a | I, m_S, \alpha\rangle = \tag{A30}$$

$$\delta_{SI} N_c^2 \sqrt{(2S+1)(2I+1)} \langle I', m' | S1; m, j\rangle \langle I', \alpha' | I1; \alpha, a\rangle \; 3\sqrt{2}\sqrt{\frac{2i+1}{2I'+1}}$$

$$\times \frac{1}{6(2I+1)} \left\{ 1 + \frac{5}{2N_c} - \frac{2I+1}{2N_c(2i+1)} \left[\frac{1}{2} + 2i(i+1) - 2I(I+1) \right] \right\} .$$

The result for the reduced matrix element $\mathcal{T}(I', SI)$ valid to next-to-leading order in $1/N_c$ is obtained by inserting (A29) and (A30) into (4.111) and making use of (4.121) for the product of $6j$ symbols in (A29).

Let us examine closer the structure of the $1/N_c$ corrections to the leading order result for $\mathcal{T}(I', SI)$ on the simple particular case when $S \neq I$. After dividing with the norm of the initial state (A18) we find for this case

$$\mathcal{T}(I', SI) = N_c^{\kappa - \frac{1}{2}} 2\sqrt{6}\sqrt{(2S+1)(2I+1)(2I'+1)} (-)^{-I+I'} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ S & I & I' \end{array} \right\} \tag{A31}$$

$$\times \left\{ 1 - \frac{1}{2N_c} + \frac{2I'+1}{3N_c} \left[\frac{1}{2} + 2i'(i'+1) - 2I'(I'+1) \right] \left(\frac{3}{2(2i'+1)} - 6 \left\{ \begin{array}{ccc} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\}^2 \right) \right\} .$$

The last term in the braces has the explicit expression

$$\frac{2I'+1}{3N_c} \left[\frac{1}{2} + 2i'(i'+1) - 2I'(I'+1) \right] \left(\frac{3}{2(2i'+1)} - 6 \left\{ \begin{array}{ccc} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\}^2 \right) \tag{A32}$$

$$= \frac{2I'+1}{2N_c} (-)^{\frac{1}{2} - i' + I'} (1 - 2\delta_{ii'})$$

which shows that it cannot be absorbed into a rescaling of the leading order term. This quark model calculation suggests therefore that the ratios of the Y and Q couplings of the mixed symmetry states predicted by the consistency conditions in the large-N_c limit will receive nontrivial $1/N_c$ corrections.

APPENDIX B: TRANSITION MATRIX ELEMENTS BETWEEN STATES WITH MIXED SYMMETRY

We present in this Appendix the computation in the quark model of the matrix elements of Y^a and Q^{ka} between excited baryon states with mixed symmetry. This quantity is phenomenologically relevant for strong decays of positive-parity excited baryons into negative-parity states in the **70**. We include this calculation here merely for the sake of completeness and because the result provides an explicit realization for the most general solution of the consistency condition for $Q(J'I', JI)$ (3.59).

We start by computing the quark model reduced matrix element $\mathcal{T}(S'I', SI)$ defined by

$$\begin{aligned}\langle S'I'L', m'_S m'_L \alpha' | \sum_{n=1}^{N_c} r_n^i \sigma_n^j \tau_n^a | SIL, m_S m_L \alpha\rangle &= \frac{1}{\sqrt{(2S'+1)(2I'+1)(2L'+1)}} \qquad \text{(B1)}\\ \times \mathcal{T}(S'I', SI)\mathcal{I}(L', L)\langle S', m'_S|S1; m_S, j\rangle\langle L', m'_L|L1; m_L, i\rangle &\langle I', \alpha'|I1; \alpha, a\rangle\,.\end{aligned}$$

We proceed in close analogy to the calculation of $\mathcal{T}(I', SI)$ in Section IV.D. First we take the matrix element of the spatial part of the operator, which is parametrized by the overlap integral $\mathcal{I}(L', L)$

$$\begin{aligned}&\langle S'I'L', m'_S m'_L \alpha' | r_n^i \sigma_n^j \tau_n^a | SIL, m_S m_L \alpha\rangle \qquad \text{(B2)}\\ &= \frac{(-)^{\psi(S'I'i')+\psi(SIi)}}{2(N_c-1)} \sum_{k,k'=2}^{N_c} {}_{[k',1]}\langle S'I', m'_S\alpha'|\sigma_n^j \tau_n^a|SI, m_S\alpha\rangle_{[k,1]}\\ &\times \frac{1}{\sqrt{2L'+1}}\langle L', m'_L|L1; m_L, i\rangle \mathcal{I}(L', L)[\delta_{kn}\delta_{kk'} + \delta_{n1}]\,.\end{aligned}$$

After summing over the contributions of the N_c quarks to the transition operator we obtain the following general expression for the reduced matrix element $\mathcal{T}(S'I', SI)$

$$\begin{aligned}\frac{1}{\sqrt{(2S'+1)(2I'+1)}}\mathcal{T}(S'I', SI)\langle S', m'_S|S1; m_S, j\rangle\langle I', \alpha'|I1; \alpha, a\rangle &= \frac{(-)^{\psi(S'I'i')+\psi(SIi)}}{2(N_c-1)} \qquad \text{(B3)}\\ \times\left\{\sum_{n=1}^{N_c} {}_{[n,1]}\langle S'I', m'_S\alpha'|\sigma_n^j\tau_n^a|SI, m_S\alpha\rangle_{[n,1]} + \sum_{k'=2}^{N_c} {}_{[k',1]}\langle S'I', m'_S\alpha'|\sigma_1^j\tau_1^a \sum_{k=2}^{N_c}|SI, m_S\alpha\rangle_{[k,1]}\right\}.&\end{aligned}$$

The first term in the braces is of order N_c and is therefore suppressed relative to the second one, which is of order N_c^2. In this section we work only to leading order in N_c so we keep only the contribution of the second term. It can be computed by expressing the sums over k, k' with the help of (4.89)

$$\begin{aligned}&\sum_{k'=2}^{N_c} {}_{[k',1]}\langle S'I', m'_S\alpha'|\sigma_1^j\tau_1^a \sum_{k=2}^{N_c}|SI, m_S\alpha\rangle_{[k,1]} = \qquad \text{(B4)}\\ &\frac{1}{2}\Big\{\delta_{SI}\delta_{S'I'}B(Ii)B(I'i')\langle I', m'_S, \alpha'|\sigma_1^j\tau_1^a|I, m_S, \alpha\rangle - N_c^2\, {}_1\langle S'I'; m'_S, \alpha'|P_{1k}\sigma_1^j\tau_1^a|SI; m_S, \alpha\rangle_1\\ &\quad - N_c^2\, {}_1\langle S'I'; m'_S, \alpha'|\sigma_1^j\tau_1^a P_{1k}|SI; m_S, \alpha\rangle_1 + N_c^2\, {}_1\langle S'I'; m'_S, \alpha'|\sigma_1^j\tau_1^a|SI; m_S, \alpha\rangle_1\Big\} + \mathcal{O}(N_c)\,.\end{aligned}$$

Each of the terms on the r.h.s. can be evaluated using the methods of Sec.IV.D. We obtain for the reduced matrix element $\mathcal{T}(S'I', SI)$

$$\frac{1}{\sqrt{(2S'+1)(2I'+1)}}\mathcal{T}(S'I',SI) = \tag{B5}$$
$$(-)^{\psi(S'I'i')+\psi(SIi)}\frac{N_c^2}{4(N_c-1)}\sqrt{(2i+1)(2i'+1)(2S+1)(2I+1)}$$
$$\times\left\{\frac{\delta_{SI}\delta_{S'I'}}{2(2I+1)(2I'+1)} - 3\frac{\delta_{S'I'}}{2I'+1}\begin{Bmatrix} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}\begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}\right.$$
$$-3(-)^{-I-S-I'-S'}\frac{\delta_{SI}}{2I+1}\begin{Bmatrix} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}$$
$$\left.+12(-)^{-I-S-I'-S'}\begin{Bmatrix} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & I \end{Bmatrix}\begin{Bmatrix} i & i' & 0 \\ \frac{1}{2} & \frac{1}{2} & S \end{Bmatrix}\begin{Bmatrix} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\right\}.$$

The first two terms can be combined together by using (4.121) for the product of two $6j$-symbols in the second term. The last two terms can be also written together such that we obtain for the total sum of the four terms in the braces

$$\{\cdots\} = \begin{Bmatrix} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}\left[-3(-)^{-\frac{1}{2}+S+I+I'+i}\frac{\delta_{S'I'}}{2I'+1}\begin{Bmatrix} 1 & 1 & 1 \\ S & I & I' \end{Bmatrix}\right. \tag{B6}$$
$$\left.+18(-)^{1+i+i'-I'-S'}\begin{Bmatrix} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\right].$$

The product of three $6j$-symbols on the r.h.s. can be transformed into the following form by repeated application of (4.121)

$$\begin{Bmatrix} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} S & S' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\begin{Bmatrix} I & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix} = \tag{B7}$$
$$\begin{Bmatrix} S' & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}\left[\frac{\delta_{SI'}}{6(2I'+1)} + (-)^{2S+S'+I}\begin{Bmatrix} 1 & 1 & 1 \\ S & I' & I \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 1 \\ I' & S' & S \end{Bmatrix}\right]$$
$$+\frac{\delta_{S'I'}}{6(2I'+1)}(-)^{-\frac{1}{2}+S+S'+I+i'}\begin{Bmatrix} 1 & 1 & 1 \\ S & I' & I \end{Bmatrix}.$$

When inserted into (B6), the last term in this relation exactly cancels the first term in (B6). We obtain in this way the following expression for the reduced matrix element $\mathcal{T}(S'I',SI)$ taken between unnormalized mixed symmetry states

$$\mathcal{T}(S'I',SI) = \frac{9}{2}N_c(-)^{\psi(S'I'i')+\psi(SIi)}\sqrt{(2i+1)(2i'+1)(2S+1)(2I+1)(2S'+1)(2I'+1)} \tag{B8}$$
$$\times(-)^{1+i+i'-I'-S'}\begin{Bmatrix} S & I & 1 \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix}\begin{Bmatrix} S' & I' & 1 \\ \frac{1}{2} & \frac{1}{2} & i' \end{Bmatrix}$$
$$\times\left[\frac{\delta_{SI'}}{6(2I'+1)} + (-)^{2S+S'+I}\begin{Bmatrix} 1 & 1 & 1 \\ S & I' & I \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 1 \\ I' & S' & S \end{Bmatrix}\right].$$

The physical value of $\mathcal{T}(S'I',SI)$ is obtained after dividing this expression with the squared roots of the norms for the initial and final states (4.44). Inserting the appropriate phases of the mixed symmetry states $\psi(SIi) = \frac{1}{2} + I + i$ we obtain our final result

$$[\mathcal{T}(S'I',SI)]_{norm} = 6(-)^{S+2I+I'}\sqrt{(2S+1)(2I+1)(2S'+1)(2I'+1)} \tag{B9}$$
$$\times\left[\frac{\delta_{SI'}}{6(2I'+1)} + (-)^{2S+S'+I}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I' & I \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ I' & S' & S \end{array}\right\}\right].$$

This will be used in the following to compute the matrix elements of Y^a and Q^{ka} between states with mixed symmetry.

1. Matrix elements of Y^a

The matrix element of Y^a takes its simplest form in the $|(IP)S,L;J,m,\alpha\rangle$ basis, where it is directly proportional to $\mathcal{T}(S'I',SI)$

$$\langle (I'P)S',L';J',m',\alpha'|Y^a|(IP)S,L;J,m,\alpha\rangle = \tag{B10}$$
$$\sqrt{\frac{2J+1}{2I'+1}}\mathcal{T}(S'I',SI)\delta_{JJ'}\delta_{mm'}\left\{\begin{array}{ccc} S & 1 & S' \\ L & 1 & L' \\ J & 0 & J' \end{array}\right\}\langle I',\alpha|I1;\alpha,a\rangle.$$

The $9j$-symbol with one value of 0 can be reduced to a $6j$-symbol. We decided to write it in this form, as it allows us to read off the results from the corresponding expressions for Q^{ka} given below by making the replacement $2\to 0$ in the Wigner symbols.

We are interested finally in the matrix elements of Y^a in the $|I,(PL)\Delta;J,m,\alpha\rangle$ basis, which is reached through the recoupling relation (3.23). With $P=P'=1$, we have

$$\langle I',(P'L')\Delta';J',m',\alpha'|Y^a|I,(PL)\Delta;J,m,\alpha\rangle = \tag{B11}$$
$$(-)^{-I-L-J-I'-L'-J'}\sqrt{\frac{2J+1}{2I'+1}}\delta_{JJ'}\delta_{mm'}\langle I',\alpha|I1;\alpha,a\rangle$$
$$\times\sum_{SS'}\sqrt{(2S+1)(2S'+1)(2\Delta+1)(2\Delta'+1)}\left\{\begin{array}{ccc} I & 1 & S \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} I' & 1 & S' \\ L' & J' & \Delta' \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & S' \\ L & 1 & L' \\ J & 0 & J' \end{array}\right\}\mathcal{T}(S'I',SI).$$

All what is left to do is insert here the result of the quark model calculation of $\mathcal{T}(S'I',SI)$ (B9) and perform the summations over S,S'.

We write the total result for the reduced matrix element $Y(JI',JI)$ as

$$Y(JI',JI) = [Y(JI',JI)]_1 + [Y(JI',JI)]_2 \tag{B12}$$

where $[Y(JI',JI)]_{1,2}$ stand for the contributions of the two terms in $\mathcal{T}(S'I',SI)$ (B9). We find

$$[Y(JI',JI)]_1 = (-)^{1+2J}\sqrt{2\Delta+1}\delta_{\Delta'L}\left\{\begin{array}{ccc} L & \Delta' & 0 \\ 1 & 1 & L' \end{array}\right\}\times(-)^{J+I'+\Delta'}\left\{\begin{array}{ccc} I & 1 & I' \\ \Delta' & J & \Delta \end{array}\right\} \tag{B13}$$

$$[Y(JI',JI)]_2 = 2\sqrt{3}(-)^{2J+\Delta+\Delta'}\sqrt{(2\Delta+1)(2\Delta'+1)}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ L & \Delta' & L' \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ \Delta' & \Delta & L \end{array}\right\} \tag{B14}$$
$$\times(-)^{J+I'+\Delta'}\left\{\begin{array}{ccc} I & 1 & I' \\ \Delta' & J & \Delta \end{array}\right\}.$$

Their sum can be seen to have the same form as the model-independent solution of the corresponding consistency condition (3.36).

2. Matrix elements of Q^{ka}

The matrix element of Q^{ka} is given, in the $|(IP)S,L;J,m,\alpha\rangle$ basis, by an expression similar to (B10)

$$\langle (I'P)S',L';J',m',\alpha'|Q^{ka}|(IP)S,L;J,m,\alpha\rangle = \tag{B15}$$
$$\sqrt{5\frac{2J+1}{2I'+1}}\mathcal{T}(S'I',SI)\left\{\begin{array}{ccc} S & 1 & S' \\ L & 1 & L' \\ J & 2 & J' \end{array}\right\}\langle J',m'|J2;m,k\rangle\langle I',\alpha|I1;\alpha,a\rangle\,.$$

This can be transformed to the $|I,(PL)\Delta;J,m,\alpha\rangle$ basis with the help of the recoupling relation (3.23). Again with $P=P'=1$, we have

$$\langle I',(P'L')\Delta';J',m',\alpha'|Q^{ka}|I,(PL)\Delta;J,m,\alpha\rangle = \tag{B16}$$
$$(-)^{-I-L-J-I'-L'-J'}\sqrt{5\frac{2J+1}{2I'+1}}\langle J',m'|J2;m,k\rangle\langle I',\alpha|I1;\alpha,a\rangle$$
$$\times\sum_{SS'}\sqrt{(2S+1)(2S'+1)(2\Delta+1)(2\Delta'+1)}\left\{\begin{array}{ccc} I & 1 & S \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} I' & 1 & S' \\ L' & J' & \Delta' \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & S' \\ L & 1 & L' \\ J & 2 & J' \end{array}\right\}\mathcal{T}(S'I',SI)\,.$$

In the following we consider the contributions of the two terms in $\mathcal{T}(S'I',SI)$ (B9) to this relation in turn. For the first term the summation over S is trivial and amounts to the substitution $S\to I'$. The remaining sum over S' can be readily done by using (3.21), which gives for the contribution of this term to $Q(J'I',JI)$

$$[Q(J'I',JI)]_1 = (-)^{1+2\Delta'}\sqrt{5(2\Delta+1)(2\Delta'+1)}\left\{\begin{array}{ccc} I & 1 & I' \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} L & \Delta' & 2 \\ 1 & 1 & L' \end{array}\right\}\left\{\begin{array}{ccc} J & J' & 2 \\ \Delta' & L & I' \end{array}\right\}\,. \tag{B17}$$

This can be put into a form similar to (3.59) by expressing the product of the first and last $6j$-symbols with the help of (3.21)

$$\left\{\begin{array}{ccc} I & 1 & I' \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} J & J' & 2 \\ \Delta' & L & I' \end{array}\right\} = \sum_{y=1,2,3}(2y+1)\left\{\begin{array}{ccc} 1 & 2 & y \\ \Delta' & \Delta & L \end{array}\right\}\left\{\begin{array}{ccc} \Delta' & I' & J' \\ \Delta & I & J \\ y & 1 & 2 \end{array}\right\}\,. \tag{B18}$$

The contribution of the second term is proportional to the double sum over S,S'

$$I_{SS'} = \sum_{SS'}(-)^{S+S'+I+I'}(2S+1)(2S'+1)\left\{\begin{array}{ccc} I & 1 & S \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} I' & 1 & S' \\ L' & J' & \Delta' \end{array}\right\} \tag{B19}$$
$$\times\left\{\begin{array}{ccc} 1 & 1 & 1 \\ S & I' & I \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ I' & S' & S \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & S' \\ L & 1 & L' \\ J & 2 & J' \end{array}\right\}\,.$$

The summations over S and S' are analogous to the sum over S encountered in Sec.III.C in Eq.(3.70) and can be performed along similar lines. A slight generalization of the sum over S in (3.70) gives the identity

$$\sum_S (-)^{2S}(2S+1)\left\{\begin{array}{ccc} I & 1 & S \\ L & J & \Delta \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & 1 \\ I & I' & S \end{array}\right\}\left\{\begin{array}{ccc} S & 1 & I' \\ L & y & L' \\ J & 2 & J' \end{array}\right\} = \tag{B20}$$

$$(-)^{I+J'+L+1} \sum_{z=1,2,3} (-)^z (2z+1)\left\{\begin{array}{ccc} y & 1 & z \\ 1 & 2 & 1 \end{array}\right\}\left\{\begin{array}{ccc} y & 1 & z \\ \Delta & L' & L \end{array}\right\}\left\{\begin{array}{ccc} L' & I' & J' \\ \Delta & I & J \\ z & 1 & 2 \end{array}\right\}$$

where y can take the values $y = 1, 2, 3$. Applying (B20) twice we obtain

$$I_{SS'} = \sum_{z=1,2,3} (-)^z (2z+1) \sum_{y=1,2} (-)^y (2y+1)\left\{\begin{array}{ccc} 1 & 1 & y \\ 1 & 2 & 1 \end{array}\right\}\left\{\begin{array}{ccc} 1 & 1 & y \\ \Delta' & L & L' \end{array}\right\} \tag{B21}$$

$$\times\left\{\begin{array}{ccc} y & 1 & z \\ 1 & 2 & 1 \end{array}\right\}\left\{\begin{array}{ccc} y & 1 & z \\ \Delta & \Delta' & L \end{array}\right\}\left\{\begin{array}{ccc} \Delta' & I' & J' \\ \Delta & I & J \\ z & 1 & 2 \end{array}\right\}.$$

The contribution of the second term in (B9) to the reduced matrix element $Q(J'I', JI)$ is given, in terms of $I_{SS'}$, by

$$[Q(J'I', JI)]_2 = 6(-)^{L+L'}\sqrt{5(2\Delta+1)(2\Delta'+1)}\, I_{SS'}\,. \tag{B22}$$

The total expression for $Q(J'I', JI)$ is given by

$$Q(J'I', JI) = [Q(J'I', JI)]_1 + [Q(J'I', JI)]_2 \tag{B23}$$

which can be seen to have the form of the general solution (3.59).

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