# Dualities in Supersymmetric Field Theories

Philip C. Argyres[a]

[a]Newman Laboratory, Cornell University, Ithaca NY 14853, USA

These lectures[1] briefly introduce dualities in four-dimensional quantum field theory, and summarize results found in supersymmetric field theories. The first lecture describes physical aspects of electric-magnetic (EM), strong-weak coupling (S), and infrared (IR) dualities. The second lecture focuses on results and conjectures concerning S-duality in $N=2$ supersymmetric gauge theories. The third lecture discusses IR-dualities and their relation to S-duality in $N=1$ supersymmetric field theories.

## 1. WHAT IS DUALITY?

The word "duality" has been used to describe many things in physics. We will examine those dualities which broadly deal with the quantum equivalence of gauge field theories.

### 1.1. EM-duality

The simplest example of such an equivalence is EM duality, which is apparent classically.

Maxwell's equations with electric and magnetic sources

$$d*F = *j_e, \qquad dF = *j_m, \tag{1}$$

are invariant under the transformation

$$F \leftrightarrow *F, \qquad j_e^\nu \leftrightarrow j_m^\nu, \tag{2}$$

where $*F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

This duality of the classical equations of motion can be shown to hold quantumly as well. To see this, consider a $U(1)$ theory with coupling

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2} \tag{3}$$

(the theta-angle is important in a $U(1)$ theory when there are both electric and magnetic sources). The action can be written

$$S = \int d^4x\, \frac{\tau}{32\pi i}(F + i{*F})^2 + h.c. \tag{4}$$

We compute physical quantities in this theory as a path integral over all gauge potential configurations $\int \mathcal{D}A_\mu e^{iS}$. This can be rewritten as a path integral over field strength configurations as long as we insert the Bianchi identity as a constraint: $\int \mathcal{D}F\mathcal{D}\widetilde{A}e^{iS'}$, where

$$S' = S + \frac{1}{8\pi}\int d^4x\, \widetilde{A}\wedge dF. \tag{5}$$

Here $\widetilde{A}_\mu$ is a Lagrange multiplier enforcing the Bianchi identity, and whose normalization is chosen so that it couples to a monopole with charge one. Performing the Gaussian functional integral over $F$, we find an equivalent action, $\widetilde{S}$, for $\widetilde{A}$:

$$\widetilde{S} = \int d^4x\, \frac{1}{32\pi i}\left(\frac{-1}{\tau}\right)(\widetilde{F} + i{*\widetilde{F}})^2, \tag{6}$$

where $\widetilde{F} = d\widetilde{A}$ is the field strength of $\widetilde{A}$. These steps can be performed just as easily in supersymmetric Maxwell theory.

Thus the free $U(1)$ gauge theory with coupling $\tau$ is quantumly equivalent to another such theory with coupling $-1/\tau$. This is the EM-duality "symmetry". It is not really a symmetry since it acts on the couplings—it is an equivalence between two descriptions of the physics. The EM-duality transformation ($S$) and the invariance of the physics under $2\pi$ shifts of the $\vartheta$-angle ($T$),

$$S:\ \tau \to -1/\tau, \qquad T:\ \tau \to \tau+1, \tag{7}$$

generate the group $SL(2,\mathbf{Z})$ of duality transformations:

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}). \tag{8}$$

Because the dual $U(1)$ potential couples to monopoles with unit strength, we see that the

*presented at the 33rd Karpacz Winter School "Duality: Strings and Fields" (Poland, February 1997).

EM-duality transformation interchanges what we mean by electric and magnetic charges. Under this transformation, a massive dyonic source with electric and magnetic charges $(n_e, n_m)$ in the original description couples to the dual $U(1)$ with charges $(n_m, -n_e)$. The minus sign arises because the square of a duality transformation (2) is charge conjugation. The effect of the $T$ transformation on the charges is $(n_e, n_m) \to (n_e{+}n_m, n_m)$, as follows from the Witten effect [1].

This description of EM-duality might make it seem trivial, since all it deals with is the equivalence of free field theories coupled to classical (massive) sources. Its importance stems from the fact that some interacting field theories have vacua with massless photons and no other massless charged matter. The coupling of the effective low-energy Maxwell theory is determined by the mass of the lightest charged particle. EM-duality is then simply a redundancy in the description of the IR-effective action of this theory which is important to take into account in determining it, as we will see in $N{=}2$ supersymmetric examples in the next lecture.

### 1.2. S-duality

S-duality is the exact quantum equivalence of two apparently (classically) different field theories. This equivalence often takes the form of the identification of a theory with specific deformations of that theory by exactly marginal operators. These deformations can be viewed as transformations on the classical space of couplings of the theory, and the duality group is the group of such transformations. Since the elements of the duality group are supposed to connect equivalent theories, the quantum coupling space is the classical one divided by the action of the duality group.

A famous example [2–4] is the strong-weak coupling duality of $N{=}4$ supersymmetric Yang-Mills under which theories with gauge couplings $g^2$ and $1/g^2$ are identified. There is by now rather compelling evidence for this duality in field theory having to do with the spectrum of BPS saturated states as described in J. Gauntlett's lectures, and with the low-energy effective action [5,6].

In this case the exactly marginal operator corresponds to changing the coupling, since its exact beta-function vanishes. Classically, the coupling can take any positive value, and so the coupling constant space has the topology of an open interval: $(0, \infty)$. Under the S-duality identification, however, we see that the topology is modified quantumly to that of a half-closed interval $(0, 1]$.

Another quantum identification of the coupling space of the $N{=}4$ theory is the angularity of the theta angle: $\vartheta \simeq \vartheta + 2\pi$. The classical parameter space is then the upper-half plane $\mathcal{M}_{cl} = \{\mathrm{Im}\tau > 0\}$, where $\tau \equiv (\vartheta/2\pi) + i(4\pi^2/g^2)$. Combining the theta-angle identification visible at weak coupling with $g^2 \simeq 1/g^2$ gives the quantum coupling space $\mathcal{M} = \mathcal{M}_{cl}/SL(2,\mathbf{Z})$, where $SL(2,\mathbf{Z})$ acts on the coupling as in (8). The topology of $\mathcal{M}$—the fundamental domain of $SL(2,\mathbf{Z})$ with identifications—is that of an open disk. The duality group, $SL(2,\mathbf{Z})$, can be presented as the group with two generators $S$ and $T$ satisfying the relations $S^2 = (ST)^3 = 1$. This group clearly encodes more information than just the topology of $\mathcal{M}$. In particular, the generators $S$ and $ST$ do not act freely on $\mathcal{M}_{cl}$, but have fixed points which are $\mathbf{Z}_2$ and $\mathbf{Z}_3$ orbifold points of $\mathcal{M}$, if we assign $\mathcal{M}_{cl}$ a flat metric. The relations satisfied by these generators then describe the holonomy in $\mathcal{M}$ around these orbifold points.

Since we are free to make coordinate changes on $\mathcal{M}_{cl}$, though the topology of $\mathcal{M}$ certainly has an invariant meaning, it may not be clear that the holonomies around points in $\mathcal{M}$ are physically meaningful. In particular, to define holonomies, one needs a connection.

A natural connection on the coupling space $\mathcal{M}$ does exist [7–9]. The idea is simply that $n$-point correlators of the operators $\mathcal{O}^i$ conjugate to the couplings $\tau_i$ which coordinatize $\mathcal{M}$ transform as rank-$n$ tensors under coordinate changes on $\mathcal{M}$. The derivative of an $n$-point correlator with respect to the coupling $\tau_i$ can be computed in terms of the $(n{+}1)$-point correlator with an insertion of $\mathcal{O}^i$. This insertion must be regularized when it coincides with the space-time position of another operator in the correlator, giving, due to Schwinger terms in the operator product expansion (OPE)

$$\mathcal{O}^i(x)\mathcal{O}^j(0) = c_k^{ij}\delta(x)\mathcal{O}^k(0) + \dots, \qquad (9)$$

a sum of $n$-point correlators with $\mathcal{O}^k$ insertions replacing the original operators. Since all the other terms transform covariantly under coordinate transformations on $\mathcal{M}$, it must be that the $c_k^{ij}$'s transform in such a way as to cancel the non-covariant part from the coordinate derivative $\partial/\partial\tau_i$, *i.e.* as a connection on $\mathcal{M}$. This connection (the $c_k^{ij}$'s) arises as finite counterterms upon subtracting the singular parts of the OPE in a given RG scheme.

It is a complicated business to compute the above connection; moreover the OPE does not in general close on the original conjugate operators, necessitating enlarging $\mathcal{M}$ to include all of theory space. However, in the case at hand, namely four-dimensional supersymmetric field theories with exactly marginal couplings in the superpotential, the situation dramatically simplifies. Because the conjugate operators are exactly marginal, they remain in the theory at the IR fixed point; because they are chiral operators, their OPEs (at the IR fixed point) have no singular parts due to the proportionality of their scaling dimensions to their $R$-charges [10]. Thus the connection vanishes, and $\mathcal{M}$ is locally flat.

The duality group elements which do not act freely on $\mathcal{M}_{cl}$ therefore do indeed encode a physical holonomy of operators parallelly transported in $\mathcal{M}$ around their fixed points. The known (conjectured) S-duality groups for $N$=2 supersymmetric field theories will be discussed in the next lecture.

### 1.3. IR-duality

IR-duality refers to the equivalence of two theories in the infrared, *i.e.* at long distances or low energies. In the context of supersymmetric gauge theories, this is sometimes refered to as "non-Abelian duality" or "$N$=1 duality". However, more generally, IR-dual theories are simply theories that belong to the same RG universality class.

The reason we give a special name to theories in the same universality class is that in the case of supersymmetric gauge theories, the resulting IR-dual pairs have properties reminiscent of the $\tau \leftrightarrow -1/\tau$ transformation in EM and S-dualities. In particular, it is found that if both theories in the IR-dual pair are asymptotically free (AF), then their strong-coupling scales $\Lambda_1$ and $\Lambda_2$ will vary inversely:

$$\Lambda_1^{b_1}\Lambda_2^{b_2} \sim \mu^{b_1+b_2}, \qquad (10)$$

where $b_1$ and $b_2$ are specified constants, and $\mu$ is an arbtrary matching scale.

More strikingly, there are IR-dual pairs one of which is AF and the other IR-free, which have different gauge groups. Restated, this means that certain AF theories which are well-described at short distances in terms of a gauge group $G_1$ give rise to long wave-length physics which is well-described by a different gauge group $G_2$. Furthermore, there is no clear sense in which the gauge bosons of $G_2$ are related to those of $G_1$; in particular, $G_2$ is not simply a subgroup of $G_1$.

We will discuss the evidence for IR-duality and its possible connection to S-duality in the third lecture.

### 1.4. Extensions

So far we have discussed EM-duality in free $U(1)$ gauge theories, S-duality in theories with exactly marginal couplings, and IR-duality of theories with couplings which undergo RG flow. The concepts as introduced have been quite general, and so might be expected to apply more broadly.

In particular, none of these dualities relies in principle on supersymmetry. We formulated EM-duality in Maxwell theory. The description of S-duality as the quantum modification of the topology (and holonomies) of the parameter space of couplings and IR-dualities as universality classes of gauge theories show them to be potentially generic phenomena in field theories. However, at present only supersymmetry gives us the control over the strong coupling dynamics of gauge theories necessary to give evidence for S- and IR-dualities.

Also, nothing in the description of these dualities limits them to four dimensions. Indeed, we have seen extensions of EM-duality to other dimensions in F. Quevedo's lectures. Likewise, analogs of S-duality in supergravity and string theories in various dimensions are the subject of many other lectures at this school.

## 2. N=2 S-DUALITY

This lecture reviews known S-dualities in $N$=2 supersymmetric gauge theories. We start by setting notation with a quick review of $N$=2 superQCD (see the talks of A. Klemm for more details).

There are two $N$=2 supermultiplets which concern us, the vector multiplet and the hypermultiplet. An $N$=2 vector multiplet is made up of an $N$=1 vector multiplet $V$ and an $N$=1 chiral multiplet $\Phi$ in the adjoint. We refer to the complex adjoint scalar component of $\Phi$ as the Higgs field. An $N$=2 vector multiplet is made up of two $N$=1 chiral multiplets $Q$ and $\widetilde{Q}$ transforming in a representation $R$ and $\overline{R}$ respectively, and we refer to their scalar components as quarks.

The only relevant or marginal couplings allowed by $N$=2 supersymmetry are hypermultiplet masses and the gauge coupling $\tau$. The one-loop beta function for $\tau$ is

$$\beta_{1\ell} = -T(Ad) + \sum_i T(R_i) \tag{11}$$

where $T(R)$ is the index of the $R$ representation of the gauge group ($Ad$ is the adjoint representation), and the sum is over hypermultiplets in representations $R_i$. Thus, for example, for $SU(n_c)$ with $n_f$ hypermultiplets in the fundamental representation, $\beta_{1\ell} = -2n_c + n_f$.

To look for S-duality, we want an exactly marginal operator. In the above example, we can arrange for the beta function to vanish by taking $n_f = 2n_c$. However, this is only the one-loop beta function; how can we show that $\beta = 0$ exaxctly?

### 2.1. Exactly marginal operators

A non-renormalization theorem [11,12] states that the beta function is given exactly by its one-loop value if $\beta_{1\ell} \leq 0$. (In IR-free theories where $\beta_{1\ell} > 0$, non-perturbative contributions to the beta function are allowed, and play a crucial role in the exact solution for the low-energy effective action of $N$=2 superQCD [13].) However, since beta functions are RG scheme-dependent, one must ask for which scheme is this non-renormalization theorem valid?

The supersymmetric non-renormalization theorems are always valid in a scheme where the ($N$=1) superpotential is a holomorphic function of the couplings. This is the case for Wilsonian effective actions in which there is no wave-function renormalization—that is, the kinetic terms are not rescaled to their canonical normalization after an RG transformation. Physical beta functions (reflecting the scale-dependence of the physical couplings) are measured in a scheme with wave-function renormalization, though. Translating to such a scheme effectively adds higher-loop and non-perturbative corrections to the one-loop beta-function through the wave function renormalization of fields propagating in the loop. The exact wave-function renormalization is encoded in the anomalous dimensions $\gamma_i$ of the hypermultiplets.

Reasoning along these lines was used in [14] to show that the exact physical beta function is proportional to

$$\beta_\tau \propto \beta_{1\ell} + \sum_i \gamma_i T(R_i). \tag{12}$$

Similarly, for an operator

$$\delta\mathcal{W} = \lambda \prod_i (\Phi_i)^{r_i} \tag{13}$$

in the superpotential (in $N$=1 language), the exact beta-function for its coupling $\lambda$ is

$$\beta_\lambda \propto 3 - \sum_i r_i(1+\gamma_i). \tag{14}$$

Now, following [15], we look for fixed point theories by solving for points in $\tau$-$\lambda$ coupling space where $\beta_\tau = \beta_\lambda = 0$. These are two conditions on two unknown parameters, and so generically have isolated fixed-point solutions. However, if we can arrange that $\beta_\tau \propto \beta_\lambda$, then there is only one independent condition on two unknowns, and so there will typically be a line of fixed points and thus an exactly marginal operator (which moves us along this line).

To apply this strategy for finding exactly marginal operators to $N$=2 theories, consider the $N$=1 superQCD theory with an adjoint chiral superfield $\Phi$ and pairs of chiral "quark" superfields $Q^i$ and $\widetilde{Q}_i$ in representations $R_i$ and $\overline{R}_i$ (*i.e.* the field content of an $N$=2 superQCD theory). In addition to the gauge coupling $\tau$, we add superpotential terms $\mathcal{W} = \sum_i \lambda_i Q^i \Phi \widetilde{Q}_i$. By the global symmetries of this theory, the anomalous dimensions of the quarks and anti-quarks are the same: $\gamma_{Q^i} = \gamma_{\widetilde{Q}_i} \equiv \gamma_i$. From (12) and (13) we then get

$$\begin{aligned} \beta_\tau &\propto (\gamma_\Phi - 1)T(Ad) + \sum_i (1+2\gamma_i)T(R_i), \\ \beta_{\lambda_i} &\propto \gamma_\Phi + 2\gamma_i, \end{aligned} \qquad (15)$$

implying the existence of a fixed line and an exactly marginal operator precisely when the one-loop beta function vanishes, $\sum_i T(R_i) = T(Ad)$, since then $\beta_\tau \propto \sum_i T(R_i)\beta_{\lambda_i}$. In addition, one can check that the fixed line is approximately $\tau = \lambda_i$ at weak coupling which are the $N$=2 supersymmetric values of the couplings.

### 2.2. Low-energy effective actions and S-duality groups

Evidence for S-duality in these scale-invariant $N$=2 theories can be found by looking at the exact low-energy effective action on the Coulomb branch of their moduli spaces.

Classically, the Coulomb branch $\mathcal{M}$ is the manifold of degenerate vacua where the quarks have zero vevs while the Higgs field takes values in the (complexified) Cartan subalgebra of the gauge group. This breaks the gauge group as $G \to U(1)^r$ where $r$ =rank$(G)$. Thus (locally) $\mathcal{M} \simeq \mathbf{C}^r$, and we will denote its coordinates by $\langle\phi\rangle_k$, $k = 1, \ldots, r$.

Quantumly, a non-renormalization theorem implies these flat directions are not lifted. Note that the point $\langle\phi\rangle_k = 0$ will be the scale-invariant (fixed-point) vacuum, but that moving away from it along the Coulomb branch to points with $\langle\phi\rangle_k \neq 0$ spontaneously breaks the scale invariance of the underlying theory and generically flows to the IR-free $U(1)^r$ Maxwell theory. The couplings $\tau_{jk}$ of this low-energy effective $U(1)^r$ theory can receive quantum corrections.

The $U(1)$ $SL(2,\mathbf{Z})$ EM-duality group generalizes straight-forwardly to an $Sp(2r,\mathbf{Z})$ EM-duality group in the $U(1)^r$ theory. (Note that the appearance of this low-energy EM-duality has nothing to do with the S-duality we are looking for!) Seiberg and Witten [13] observed that this EM-duality implies that the effective couplings $\tau_{jk}$ need not be a single-valued function on $\mathcal{M}$ (the Coulomb branch), but can be a section of an $Sp(2r,\mathbf{Z})$ bundle on $\mathcal{M}$. We encode the $Sp(2r,\mathbf{Z})$ "periodicity" of $\tau_{jk}$ geometrically by taking $\tau_{jk}$ to be the complex structure of a $T^{2r}$ torus which varies over $\mathcal{M}$. It is shown in [16] that the total space of $T^{2r}$ fibered over $\mathcal{M}$ is the phase space of an integrable system. Furthermore, in every known example it turns out that this auxiliary torus $T^{2r}$=Jac$(\Sigma_r)$, the Jacobian torus of a genus-$r$ Reimann surface $\Sigma_r$, though no explanation of this fact is known.

The solutions for the low-energy effective action on the Coulomb branch are usually presented in terms of the $\Sigma_r$. No systematic procedure is known for generating these families of Riemann surfaces. The solutions have so far been found either by making simple guesses for the form of the surfaces or for the integrable system, or from type IIB string compactifications as reviewed in A. Klemm's lectures in this school or, most recently, from brane configurations in type IIA string theory or M-theory [17].

The scale-invariant $N$=2 theories whose associated Riemann surfaces were known (before [17]) are the theories with a single $SU$, $SO$, or $Sp$ gauge group and quarks in the fundamental (defining) representation [18,19]. It is found in all these cases that these curves depend on the marginal coupling $\tau$ in such a way that they are invariant under identifications of $\tau$ under a discrete group isomorphic to $\Gamma^0(2) \subset SL(2,\mathbf{Z})$. This is evidence that the S-duality group of all these theories is isomorphic to $\Gamma^0(2)$.

$\Gamma_0(2)$ is the subset of $SL(2,\mathbf{Z})$ matrices with even upper off-diagonal entry, or equivalently, is isomorphic to the subgroup of $SL(2,\mathbf{Z})$ generated by $S$ and $T^2$. The only relation satisfied by these generators is $S^2 = 1$, thus characterizing $\Gamma^0(2)$ more abstractly as the group freely generated by $S$ and $T'$ with the one relation $S^2 = 1$. Since the S-duality group is the group we divide the classical coupling space by, we expect a single point with a $\mathbf{Z}_2$ holonomy in the quantum coupling space at the fixed point of the $S$-transformation's action. Indeed, the fundamental domain of $\Gamma^0(2)$ in the $\tau$ upper half-plane can be taken to be the region $-1 \leq \mathrm{Re}\tau \leq 1$ and $|\tau| \geq 1$. The boundaries of this domain are identified so that there is a $\mathbf{Z}_2$ orbifold point at $\tau = i$ and "cusps" with no holonomy at $\tau = i\infty$ (weak coupling) and $\tau = \pm 1$ (ultra-strong coupling).

An important feature of this duality group is that it does not remove all the ultra-strong coupling points from the quantum coupling space. An important exception is $SU(2)$ $N$=2 superQCD with four fundmental flavors. For this theory it was shown [6] that the low-energy effective theory at the ultra-strong coupling point is in fact equivalent to the weak coupling limit of the same effective action. This is evidence that the S-duality group in this case is enlarged to the full $SL(2,\mathbf{Z})$ group. In the other cases (with higher-rank gauge groups) no weakly coupled description of the ultra-strong couplings points is known. In particular they are not the weak coupling limit of the same theories, so the S-duality groups are not "secretly" enlarged to $SL(2,\mathbf{Z})$, if they are enlarged at all.

### 2.3. Recent developments

Since these lectures were given, the list of conjectured $N$=2 S-duality groups (based on equivalences of exact low-energy effective actions on Coulomb branches) has increased dramatically due to the work of Witten [17]. He argues that 5-brane configurations in M-theory in which two of the dimensions of the 5-brane world-volume form a Riemann surface $\Sigma$ and the other 3+1 are flat give rise to two natural families of $N$=2 four-dimensional superQCD theories. From its embedding in the 11-dimensional space-time of M-theory, $\Sigma$ inherits a metric structure, and, in particular, has metric infinities. For the scale-invariant theories, the values of the couplings can be read off from the relative distances of these metric infinities in two transverse dimensions, say $x^6$ and $x^{10}$, in the 11-dimensional space-time.

The "cylindrical" family of $N$=2 models constructed in this way has gauge group $G = \prod_{i=1}^n SU(k_i)$, $n-1$ hypermultiplets in the $(\mathbf{k_i}, \mathbf{k_{i+1}})$ bi-fundamental representations, as well as $f_i$ hypermultiplets in the fundamental of each $SU(k_i)$ factor. This is a scale-invariant model with $n$ exactly marginal couplings $\tau_i$ if

$$2k_i - k_{i-1} - k_{i+1} = f_i, \qquad k_0 \equiv k_{n+1} \equiv 0. \quad (16)$$

In these models the $x^6$ coordinate is infinite and the $x^{10}$ coordinate lives on a circle making a cylindrical transverse space. $\Sigma$ has $n$+1 metric infinities, and their relative positions on the $x^6$-$x^{10}$ cylinder give the $n$ complex coupling parameters of the model.

This gives a direct description of the quantum space of couplings as the moduli space of complex structures of a cylinder with $n$+1 indistinguishable punctures. The indistinguishablity of the punctures can be seen in an equivalent type IIA string picture by exchanging the positions of any two infinite five-branes. The S-duality group can be extracted from this description of the moduli space, and can easily be seen to be isomorphic to $\Gamma^0(2)$ when $n$=1. The more general case has an S-duality group generated by translations $T_i$ of each puncture around the cylinder, as well as exchanges $S_{i,i+1}$ of pairs of punctures. Dividing the classical coupling space by this group gives rise to a complicated set of orbifold submanifolds and associated holonomies. Also, two punctures colliding corresponds to a point of ultra-strong coupling. In particular, when applied to the $SU(2)$ scale-invariant theory, one only sees in this manner the $\Gamma^0(2)$ S-duality, and not the larger $SL(2,\mathbf{Z})$ group.

The second, "elliptic", family of models arises upon compactifying the $x^6$ direction on a circle. The scale-invariant $N$=2 theories have gauge group $G = SU(k)^n$ with $n$ hypermultiplets in bi-fundamental representations. The Riemann surface has $n$ metric infinities puncturing the $x^6$-$x^{10}$ torus, so the quantum space of couplings is the moduli space of complex structures of a torus with $n$ indistinguishable puntures. When $n$=1 these are the $SU(k)$ $N$=4 theories with S-duality group $SL(2,\mathbf{Z})$. With $n > 1$, however, there are still ultra-strong coupling points in these models which have no known weak-coupling description.

## 3. IR-DUALITY AND N=1 S-DUALITY

This lecture briefly reviews the IR-dualties found in $N$=1 supersymmetric gauge theories and explores the connection between IR-duality and S-duality.

Recall the first example [10] of a series of IR-dual $N{=}1$ theories, which gives the flavor for most of the IR-dual theories found to date. These IR-dual pairs consist of an "electric" theory which is $SU(n_c)$ superQCD with $n_f \geq n_c + 2$ fundamental flavors $Q^i$, $\widetilde{Q}_i$ and no superpotential, and a "magnetic" $SU(\widetilde{n}_c)$ superQCD with $n_f$ fundamental flavors $q_i$, $\widetilde{q}^i$, a gauge singlet field $M^i_j$, and a superpotential $\mathcal{W}_{mag} = (1/\mu)Mq\widetilde{q}$. The two theories have the same IR behavior if

$$\widetilde{n}_c = n_f - n_c. \tag{17}$$

Identifying the magnetic singlet $M$ with a composite meson chiral superfield in the electric theory, $\mu$ in the magnetic superpotential becomes a matching scale between the electric and magnetic theories. If $\Lambda_e$ and $\Lambda_m$ are the strong-coupling scales of the electric and magnetic theories, respectively, then they are related by

$$\Lambda_e^{3n_c-n_f}\Lambda_m^{3\widetilde{n}_c-n_f} \sim \mu^{n_f}. \tag{18}$$

Note that the one-loop beta fuction $\beta_{1\ell} = -3n_c + n_f$. An important feature of this duality is that for $n_f/3 \leq n_c, \widetilde{n}_c \leq 2n_f/3$ both the electric and magnetic theories are AF, and flow to a non-trivial fixed point in the IR. There is a "self-dual" theory at $n_c = \widetilde{n}_c = n_f/2$ where the two gauge groups are the same (though the theories differ by singlet degrees of freedom). Moreover, starting from the self-dual theories, one can flow down to all the other dual pairs by turning on appropriate relevant operators in the superpotential.

Most of these basic features are shared by the many known IR-dual pairs; for a fairly complete (as of early 1997) listing of the known IR-dualities and their properties, see [20]. One striking uniformity is that all known IR-dual series, except for one [21], have self-dual pairs from which all the other dual pairs in a series can be derived by deformation by relevant operators in the superpotential.

The evidence for the IR-equivalence of these theories is based on the equivalence of their chiral ring of operators (*i.e.* the composite massless chiral superfields, more or less) and the equivalence of the theories under a large set of relevant deformations. The matching of the chiral rings means, in practice, that anomalous dimensions of operators at the IR fixed points match, as do the moduli spaces of vacua of the two theories.

Unlike the case of S-duality, which by its very nature involves strong couplings, one can imagine a proof of IR-duality using existing field theory techniques. Recalling that IR-dualities are just universality classes of gauge theories, one needs only show that two theories are connected by an irrelevant deformation to show they are IR-dual. The standard strategy to do this is to find an UV fixed-point theory which flows to both members of the dual pair upon deformation by certain relevant operators. Then one needs to show that the difference between those relevant operators is itself irrelevant. One must be careful, however, since an operator which looks irrelevant near the UV fixed point, may become relevant as it flows to the IR (so-called "dangerously irrelevant operators"). In the supersymmetric case one has some control over this subtlety. At least for non-chiral theories, by a Witten index argument one can often regulate the theory in the IR by putting it in finite volume (so it becomes supersymmetric quantum mechanics) and still keep supersymmetry unbroken so that the vacuum energy is always zero. As long as the operator in question does not change the potential significantly at large field values, the vacuum energy will depend analytically on its coupling, which together with unbroken supersymmetry disallows any crossover in the vacuum state at the coupling is varied. In this way one could show that the operator is indeed irrelevant (or, at worst, exactly marginal) in the IR.

This strategy was pursued in the case of the $SU$ [22] and $SO$ and $Sp$ [23] IR-dual superQCD series. The UV fixed-point theory was taken to be the related AF or scale-invariant $N{=}2$ superQCD. Though it was shown how to flow down in these theories to $N{=}1$ theories with either the electric or magnetic gauge groups, the singlet meson fields of the magnetic theory were not found. It turns out [24] that for all the AF $N{=}2$ theories, the perturbation used to flow down to the $N{=}1$ dual pairs is relevant at their IR fixed points, and so cannot be used to prove IR-duality. In the scale-invariant $N$=2 case, however, these operators are exactly marginal in the IR [15]. Also, the scale-invariant $N$=2 theories flow to the self-dual $N$=1 pairs.

In fact, all the $N$=1 IR self-dual pairs admit exactly marginal deformations corresponding to turning on mass terms for the singlet meson fields in the magnetic theory. The fact that scale-invariant (S-dual) $N$=2 superQCD theories flow to IR self-dual pairs suggests that the IR self-dual theories might be exactly S-dual in their marginal coupling. Indeed, in [15] it was argued that upon turning on a relevant operator causing the S-dual $N$=2 $SU(n_c)$ theory to flow to the $n_f = 2n_c$ $N$=1 theory, the exactly marginal operators of the two theories would be connected by RG flow. Assuming the RG flow maps the parameter spaces of these marginal couplings in a 1-to-1 fashion, the IR (self-)duality of the $N$=1 theory would follow by an action of the S-duality element associated with the $\mathbf{Z}_2$ orbifold point in the $N$=2 coupling constant parameter space.

This argument depends on the assumption that the RG flow maps the marginal coupling space of the UV theory "into" (in the mathematical sense) the IR theory's marginal coupling space. Unfortunately, it is not at all clear why this should be the case. In particular, the relevant operator $\mathcal{O}$ by which we flow down from the $N$=2 theory to the $N$=1 theory will suffer some monodromy $\mathcal{O} \to \mathcal{O}'$ as we transport it around an orbifold point in the $N$=2 coupling constant space. The question becomes whether the operator difference $\Delta\mathcal{O} = \mathcal{O} - \mathcal{O}'$ is itself irrelevant or not. For if it is irrelevant, then the theory and its monodromic image will flow to the same theory in the IR, implying an into mapping of the the $N$=2 to $N$=1 parameter spaces. But if $\Delta\mathcal{O}$ is relevant, then the mapping need not be into, potentially "unwrapping" the S-duality identifications in the IR.

Nevertheless, the striking prevalence of IR self-dual theories and their associated exactly marginal operators suggests that S-duality in these marginal couplings is the explanantion of IR-duality. Perhaps with the richer $N$=2 S-duality groups found in the models with product gauge groups [17], stronger evidence for the existence of $N$=1 S-dualities can be found in the form of an associated "web" of IR-dualities in $N$=1 theories with product gauge groups.

I am grateful to A. Buchel and A. Shapere for useful discussions. The work was supported by NSF grant PHY-9513717.

## REFERENCES


1. E. Witten, *Phys. Lett.* 86B (1979) 283.
2. C. Montonen and D. Olive, *Phys. Lett.* 72B (1977) 117.
3. E. Witten and D. Olive, *Phys. Lett.* 78B (1978) 97.
4. H. Osborn, *Phys. Lett.* 83B (1979) 321.
5. C. Vafa and E. Witten, hep-th/9408074, *Nucl. Phys.* B432 (1994) 3.
6. N. Seiberg and E. Witten, hep-th/9408099, *Nucl. Phys.* B431 (1994) 484.
7. I. Jack and H. Osborn, *Nucl. Phys.* B343 (1990) 647.
8. H. Sonoda, hep-th/9205085, *Nucl. Phys.* B383 (1992) 173; hep-th/9205084, *Nucl. Phys.* B394 (1993) 302.
9. B. Dolan, hep-th/9403070, *Int. J. Mod. Phys.* A10 (1995) 2439.
10. N. Seiberg, hep-th/9411149, *Nucl. Phys.* B435 (1995) 129.
11. N. Seiberg, *Phys. Lett.* 206B (1988) 75.
12. N. Seiberg, hep-ph/9309335, *Phys. Lett.* 318B (1993) 469.
13. N. Seiberg and E. Witten, hep-th/9407087, *Nucl. Phys.* B462 (1994) 19.
14. V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, *Phys. Lett.* 166B (1986) 329; M. Shifman, A. Vainshtein, and V. Zakharov, *Phys. Lett.* 166B (1986) 334; M. Shifman and A. Vainshtein, *Nucl. Phys.* B277 (1986) 456.
15. R. Leigh and M. Strassler, hep-th/9503121, *Nucl. Phys.* B447 (1995) 95.
16. R. Donagi and E. Witten, hep-th/9510101, *Nucl. Phys.* B460 (1996) 299.
17. E. Witten, hep-th/9703166.
18. P.C. Argyres, M.R. Plesser, and A.D. Shapere, hep-th/9505100, *Phys. Rev. Lett.* 75 (1995) 1699.
19. P.C. Argyres and A.D. Shapere, hep-th/9509175, *Nucl. Phys.* B461 (1996) 437.
20. J. Brodie and M. Strassler, hep-th/9611197.

21. P. Pouliot, hep-th/9507018, *Phys. Lett.* 359B (1995) 108; P. Pouliot and M. Strassler, hep-th/9510228, *Phys. Lett.* 370B (1996) 76; hep-th/9602031, *Phys. Lett.* 375B (1996) 175; M. Berkooz, P. Cho, P. Kraus, M. Strassler, hep-th/9705003.
22. P. Argyres, R. Plesser, and N. Seiberg, hep-th/9603042, *Nucl. Phys.* B471 (1996) 159.
23. P. Argyres, R. Plesser, and A. Shapere, hep-th/9608129, *Nucl. Phys.* B483 (1997) 172.
24. P. Argyres and A. Buchel, unpublished.