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Expectation values of boundary fields in the boundary sine-Gordon model

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Abstract

We propose an explicit expression for vacuum expectation values of the boundary field $e^{ia\varphi_B}$ in the boundary sine-Gordon model with zero bulk mass. This expression agrees with known exact results for the boundary free energy and with perturbative calculations.

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In this paper we study the so called boundary sine-Gordon model with zero bulk mass. This is a two-dimensional quantum field theory defined by the Euclidean action

$$\mathcal{A}_{BSG} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \, \Big\{ \big(\partial_y\varphi\big)^2 + \big(\partial_x\varphi\big)^2 \Big\} - 2\mu_B \int_{-\infty}^{\infty} dx \, \cos\big(\beta\varphi(x,0)\big) \ . \qquad (1)$$

Here x, y are coordinates on the Euclidean half-plane $y \geq 0$ and $\varphi(x, y)$ is a scalar field. Interaction is present only at the boundary $y = 0$ and it is controlled by two parameters β and μ_B. This model can be understood as a conformal field theory - a free Bose field with free boundary condition at $y = 0$ - perturbed by a boundary operator $2\cos(\beta\varphi_B)(x)$ of the dimension $\Delta = \beta^2$. Here we use the notation

$$\varphi_B(x) \equiv \varphi(x, 0) \ .$$

Correspondingly, we assume that this boundary operator is normalized according to the following simple asymptotic form of its two-point correlation function

$$\langle\, 2\cos(\beta\varphi_B)(x)\, 2\cos(\beta\varphi_B)(x')\,\rangle \to 2\,|x-x'|^{-2\beta^2} \qquad \text{as} \qquad |x-x'| \to 0 \ . \qquad (2)$$

Under this normalization the parameter μ_B has the dimension $[\, mass\,]^{1-\beta^2}$.

This model attracted much interest recently in connection with the impurity problem in the 1D Luttinger model [1], [2]. Quantum Brownian motion of a one-dimensional particle in a periodic potential is another interesting application of this model [3], [4], [5], [6]. The QFT (1) is integrable [7], [8] and many exact results, in particular concerning static transport properties, have been obtained or conjectured recently [9], [10], [11]. In this paper we propose another exact result for the model (1), the expectation value of the exponential boundary field

$$\langle\, e^{ia\varphi_B}\,\rangle = \left[\, \frac{2^{\beta^2}\pi\,\mu_B}{\Gamma(\beta^2)} \,\right]^{\frac{a^2}{1-\beta^2}} \times$$

$$\exp\left\{ \int_0^{\infty} \frac{dt}{t} \left[\frac{\big(e^t - 1 + e^{t(1-\beta^2)} + e^{-\beta^2 t}\big)\,\sinh^2(a\beta t)}{2\sinh(\beta^2 t)\,\sinh(t)\,\sinh\big((1-\beta^2)t\big)} - a^2\Big(\frac{1}{\sinh\big((1-\beta^2)t\big)} + e^{-t} \Big) \right] \right\}, \qquad (3)$$

with an arbitrary a such that $|\Re e\, 2a| < \beta^{-1}$. In writing (3) we have assumed that the boundary field $e^{ia\varphi_B}(x)$ is normalized in accordance with the short distance limiting form of the two-point function

$$\langle\, e^{ia\varphi_B}(x)\, e^{-ia\varphi_B}(x')\,\rangle \to |x-x'|^{-2a^2} \qquad \text{as} \qquad |x-x'| \to 0 \ , \qquad (4)$$

so that the field $e^{ia\varphi_B}(x)$ has the dimension $[\,mass\,]^{a^2}$. The result (3) is expected to hold in the domain

$$\beta^2 < 1 \ , \tag{5}$$

where the discrete symmetry $\varphi \to \varphi + 2\pi n \beta^{-1} \ \ (n = 0, \pm 1, \pm 2 ...)$ of (1) is spontaneously broken (equivalently, the ground state of the associated quantum Brownian particle is localized) and by $\langle ... \rangle$ in (3) we mean the expectation value over one of the ground states, in which the field $\varphi_B(x)$ is localized near 0.

The model (1) has much in common with the ordinary "bulk" sine-Gordon model

$$\mathcal{A}_{SG} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left\{ \frac{1}{16\pi} \left[\big(\partial_y \phi\big)^2 + \big(\partial_x \phi\big)^2 \right] - 2\mu \, \cos\big(\beta\phi\big) \right\} , \tag{6}$$

which can be thought of as the $c = 1$ conformal field theory perturbed by the "bulk" field $2\cos\big(\beta\phi\big)$. The exact expectation value of the "bulk" operator $e^{ia\phi}$ for the model (6) (again, in the domain (5)), very similar to (3), has been proposed recently in [12]. We would like to note here that the model (6) is related by substitution $\beta = ib, \ \mu \to -\mu$ to the sinh-Gordon model

$$\mathcal{A}_{ShG} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left\{ \frac{1}{16\pi} \left[\big(\partial_y \phi\big)^2 + \big(\partial_x \phi\big)^2 \right] + 2\mu \, \cosh\big(b\phi\big) \right\} . \tag{7}$$

The expression for $\langle\, e^{ia\phi} \,\rangle_{SG}$ in [12] admits (for fixed a) a power series expansion in β^2 with finite radius of convergence. Therefore it is natural to assume that the vacuum expectation values of exponential fields in the sinh-Gordon model can be obtained from the expression in [12] simply by the above continuation $\beta \to ib$. This way one obtains

$$\begin{aligned} \langle\, e^{a\phi} \,\rangle_{ShG} = &\left[\frac{m\,\Gamma\big(\frac{1}{2+2b^2}\big)\,\Gamma\big(1+\frac{b^2}{2+2b^2}\big)}{4\sqrt{\pi}} \right]^{-2a^2} \times \\ &\exp\left\{ \int_0^{\infty} \frac{dt}{t} \left[-\frac{\sinh^2(2abt)}{2\sinh(b^2 t)\,\sinh(t)\,\cosh\big((1+b^2)t\big)} + 2a^2\, e^{-2t} \right] \right\} , \end{aligned} \tag{8}$$

where [13]

$$m = \frac{4\sqrt{\pi}}{\Gamma\big(\frac{1}{2+2b^2}\big)\,\Gamma\big(1+\frac{b^2}{2+2b^2}\big)} \left[-\frac{\mu\pi\Gamma\big(1+b^2\big)}{\Gamma\big(-b^2\big)} \right]^{\frac{1}{2+2b^2}} \tag{9}$$

is the particle mass of the sinh-Gordon model.

Unlike the sine-Gordon model (6) it is not very natural to think of (7) as perturbed free boson conformal field theory. Instead, the model (7) is better understood in terms of the Liouville conformal field theory

$$\mathcal{A}_{Liouv} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left\{ \frac{1}{16\pi} \left[\left(\partial_y \phi\right)^2 + \left(\partial_x \phi\right)^2 \right] + \mu\, e^{b\phi} \right\} \tag{10}$$

perturbed by the operator $e^{-b\phi}$. As is shown in [14] the operators $e^{a\phi}$ in the Liouville theory (10) satisfy the "reflection relation"

$$e^{a\phi}(x,y) = R(a)\ e^{(Q-a)\phi}(x,y)\ , \tag{11}$$

with the function $R(a)$ related in a simple way to the Liouville "reflection amplitude"

$$R\big(Q/2 + iP\big) = S\big(P\big) = -\left(\frac{\pi\mu\,\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{-\frac{2iP}{b}} \frac{\Gamma\big(1 + 2iP/b\big)\,\Gamma\big(1 + 2iPb\big)}{\Gamma\big(1 - 2iP/b\big)\,\Gamma\big(1 - 2iPb\big)} \tag{12}$$

(see [14] for details). Here and below

$$Q = b^{-1} + b\ . \tag{13}$$

It is not difficult to check that the one-point correlation function (8) in the sinh-Gordon model (7) satisfies remarkable relation

$$\langle\, e^{a\phi}\,\rangle_{ShG} = R(a)\ \langle\, e^{(Q-a)\phi}\,\rangle_{ShG} \tag{14}$$

with the same function $R(a)$ as in (12). Although at the moment we do not have completely satisfactory explanation for this phenomenon [1], (14) seems to be a manifestation of an important hidden structure in the sinh-Gordon theory [2] and in fact of more general Affine Toda field theories [3]. Let us note that the relation (14) together with an obvious symmetry

$$\langle\, e^{a\phi}\,\rangle_{ShG} = \langle\, e^{-a\phi}\,\rangle_{ShG} \tag{15}$$

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[1] Although the relation (14) formally holds in the conformal perturbation theory for (7) understood as the Liouville theory perturbed by $e^{-b\phi}$, this conformal perturbation theory by itself does not give a valid definition of the one-point function (8).

[2] For instance, form-factors of the field $e^{a\phi}$ in the sinh-Gordon model proposed in [15] suggest that matrix elements $\langle\, 0\,|\, e^{a\phi}\,|\, \theta_1, ..., \theta_N\,\rangle$ (where $|\, \theta_1, ..., \theta_N\,\rangle$ are multiparticle states) satisfy the relation (14). Then it formally follows that multipoint correlation functions, say $\langle\, e^{a\phi}(x,y)\, e^{a'\phi}(0,0)\,\rangle_{ShG}$, satisfy the relations similar to (14), although this point needs further investigation.

[3] The relations similar to (14) are valid for expectation values of exponential fields in the Affine Toda theories and they can be used to determine these expectation values [16].

determines the expectation value $\langle\, e^{a\phi}\, \rangle_{ShG}$ up to a periodic function so that (8) is a "minimal solution" to the functional equations (14) and (15).

Let us come back to the boundary theory (1). Consider the boundary sinh-Gordon model (BShG) which is obtained from (1) by substitution $\beta = ib$, $\mu_B \to -\mu_B$, just the same way (7) is obtained from (6). Again, BShG can be thought of as the "boundary Liouville theory"

$$\mathcal{A}_{BLiouv} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx \int_0^{\infty} dy\ \Big\{ \big(\partial_y\varphi\big)^2 + \big(\partial_x\varphi\big)^2 \Big\} + \mu_B \int_{-\infty}^{\infty} dx\, e^{b\varphi_B(x)}\ . \tag{16}$$

perturbed by the operator $e^{-b\varphi_B}$. As in the "bulk" Liouville theory, the boundary operators in the conformal field theory (16) satisfy the "reflection relation"

$$e^{a\varphi_B}(x) = R_B(a)\ e^{(Q-a)\varphi_B}(x) \tag{17}$$

with $R_B(a)$ related to the associated "reflection amplitude"

$$R_B\big(Q/2+iP\big) = S_B\big(P\big) = \Big(-\frac{2\pi\,\mu_B}{\Gamma(-b^2)}\Big)^{-\frac{2iP}{b}} \ \Big(\frac{b}{2}\Big)^{2iPQ}\ \frac{\mathcal{G}(-2iP)\,\mathcal{G}^2(Q/2+iP)}{\mathcal{G}(\ 2iP\,)\,\mathcal{G}^2(Q/2-iP)}\ . \tag{18}$$

The function $\mathcal{G}(z)$ here is given by the integral

$$\mathcal{G}(z) = \exp\bigg\{ \int_0^{\infty} \frac{dt}{t}\, \bigg[\frac{e^{-\frac{Q}{2}t} - e^{-zt}}{(1-e^{-bt})(1-e^{-\frac{t}{b}})} + \frac{Q\,(Q/2-z)\,e^{-\frac{Q}{2}t}}{1-e^{-Qt}} + \frac{1}{2}\,(Q/2-z)^2\, e^{-\frac{t}{2}} \bigg] \bigg\} \tag{19}$$

in the domain $\Re e\, z > 0$ and it can be analytically continued into the whole complex plane of z by using the functional relations

$$\begin{aligned} \mathcal{G}(z+b) &= \frac{1}{\sqrt{\pi}}\ 2^{b(z+\frac{Q}{2})-1}\ b^{\frac{1}{2}-bz}\ \Gamma\big(bz\big)\ \mathcal{G}(z)\ , \\ \mathcal{G}\big(z+b^{-1}\big) &= \frac{1}{\sqrt{\pi}}\ 2^{\frac{1}{b}(z+\frac{Q}{2})-1}\ b^{\frac{z}{b}-\frac{1}{2}}\ \Gamma\big(b^{-1}z\big)\ \mathcal{G}(z)\ , \end{aligned} \tag{20}$$

which (19) satisfies. It is easy to show that $\mathcal{G}(z)$ thus defined is an entire function of z with zeroes at $z = -nb - mb^{-1}$ $(n, m = 0, 1, 2, ...)$. The derivation of (17), (18) will be published elsewhere. Now, let us assume that the vacuum expectation value $\langle\, e^{a\varphi_B}\, \rangle_{BShG}$ in the boundary sinh-Gordon model satisfies a "reflection relation" similar to (14), i.e.

$$\langle\, e^{a\varphi_B}\, \rangle_{BShG} = R_B(a)\, \langle\, e^{(Q-a)\varphi_B}\, \rangle_{BShG}\ . \tag{21}$$

It is not difficult to obtain the "minimal" solution to this functional equation which takes into account the symmetry relation analogous to (15). Then this solution can be continued back to pure imaginary $b = -i\beta$ which correspond to the boundary sine-Gordon model (1). This is the way we arrived at (3).

In what follows we give some evidence in support of (3). First, the expectation value of the boundary field $e^{i\beta\varphi_B}$ in the model (1) can be extracted from its specific free energy

$$f_B = -\lim_{L\to\infty} L^{-1}\ \log Z_L\ , \qquad Z_L = \int \big[\mathcal{D}\varphi\big]\, e^{-\mathcal{A}_{BSG}}\ , \tag{22}$$

where L is the size of the system (1) in x direction, namely

$$\langle\, e^{i\beta\varphi_B}\,\rangle = -\frac{1}{2}\,\frac{\partial}{\partial\mu_B}\ f_B(\mu_B)\ . \tag{23}$$

The quantity f_B for the boundary sine-Gordon model is known exactly [9], [11] and (23) gives

$$\langle\, e^{i\beta\varphi_B}\,\rangle = \frac{\Gamma\big(\frac{1-2\beta^2}{2-2\beta^2}\big)\,\Gamma\big(\frac{\beta^2}{2-2\beta^2}\big)}{4\,\pi^{\frac{3}{2}}\,(1-\beta^2)\,\mu_B}\ \left[\frac{2\,\pi\,\mu_B}{\Gamma(\beta^2)}\right]^{\frac{1}{1-\beta^2}}\ . \tag{24}$$

It is easy to check that (3) agrees with (24).

Expanding the expectation value $\langle\, e^{ia\varphi_B}\,\rangle$ into a power series in a^2 one can obtain the expectation values of the polynomial fields $\big(\varphi_B\big)^{2n}$. These in turn admit expansions in power series in β^2. In this way one obtains from (3)

$$\begin{aligned}
\sigma_2 &\equiv \langle\, \varphi_B^2\,\rangle = -2\,\log m_B - 2\,\gamma - \frac{\beta^4}{6}\,\big(\pi^2 - 7\zeta(3)\big) + O\big(\beta^6\big)\ ,\\
\sigma_4 &\equiv \langle\, \varphi_B^4\,\rangle - 3\,\langle\, \varphi_B^2\,\rangle^2 = 4\,\beta^2\,\big(\pi^2 + 7\zeta(3)\big) + 6\,\beta^4\,\big(\pi^2 + 6\zeta(3)\big) + O\big(\beta^6\big)\ ,\\
\sigma_6 &\equiv \langle\, \varphi_B^6\,\rangle - 15\,\langle\, \varphi_B^2\,\rangle\ \langle\, \varphi_B^4\,\rangle + 30\,\langle\, \varphi_B^2\,\rangle^3 = -16\,\beta^4\,\big(\pi^4 + 93\zeta(5)\big) + O\big(\beta^6\big)\,.
\end{aligned} \tag{25}$$

where $\gamma = 0.577216...$ is Euler's constant, $\zeta(s)$ is the Riemann zeta function and we have introduced an auxiliary mass parameter m_B related to f_B as

$$m_B = -2\sin\Big(\frac{\pi\beta^2}{1-\beta^2}\Big)\ f_B\ .$$

On the other hand the coefficients in these expansions can be calculated independently, from the standard perturbation theory for the action (1). In this way one finds [4]

$$\sigma_2 = \lim_{\varepsilon\to 0}\Big[\,E(\varepsilon m_B) + 2\log\varepsilon\,\Big] + \frac{\beta^4}{6}\bigg\{\int_{-\infty}^{\infty}\frac{d\rho}{2\pi}\int_{-\infty}^{\infty}\frac{d\rho'}{2\pi}\ E(\rho)E(\rho')E^3(\rho-\rho') - 2\pi^2\bigg\} + O(\beta^6)\ ,$$
$$\sigma_4 = \beta^2\int_{-\infty}^{\infty}\frac{d\rho}{2\pi}\ E^4(\rho) + \frac{3}{2}\beta^4\ \int_{-\infty}^{\infty}\frac{d\rho}{2\pi}\int_{-\infty}^{\infty}\frac{d\rho'}{2\pi}\ E^2(\rho)E^2(\rho')E^2(\rho-\rho') + O(\beta^6)\ ,$$
$$\sigma_6 = \beta^4\,\bigg\{10\int_{-\infty}^{\infty}\frac{d\rho}{2\pi}\int_{-\infty}^{\infty}\frac{d\rho'}{2\pi}\ E^3(\rho)E^3(\rho')E(\rho-\rho') - \int_{-\infty}^{\infty}\frac{d\rho}{2\pi}\ E^6(\rho)\bigg\} + O(\beta^6)\ , \tag{26}$$

where

$$E(\rho) = \int_{-\infty}^{\infty} d\nu\ \frac{e^{i\nu\rho}}{|\nu|+1}\ . \tag{27}$$

Evaluating the integrals in (26) one finds perfect agreement with (25).

Finally, the following remark is in order. In the limit $\beta^2\to 0$ the expectation value $\langle\, e^{\frac{\omega}{\beta}\varphi_B}\,\rangle$ in the boundary sine-Gordon theory (1) can be described in terms of a particular solution to the classical equations of motion corresponding to the action (1)

$$\big(\partial_\sigma^2+\partial_\tau^2\big)\Phi(\sigma,\tau) = 0 \quad \text{for} \quad \tau>0\ ; \qquad \partial_\tau\Phi(\sigma,\tau)\big|_{\tau=0} = \sin\Phi(\sigma,0)\ , \tag{28}$$

where $\Phi(m_0x, m_0y) = \beta\varphi(x,y)$ and $m_0 = 4\pi\beta^2\mu_B$. Let $\Phi(\sigma,\tau)$ be a function which solves the equations (28) for $\tau\geq 0,\ \sigma^2+\tau^2>0$ and satisfies the following asymptotic conditions

$$\begin{aligned}&\Phi(\sigma,\tau)\to 0 \quad \text{as} \quad \sigma^2+\tau^2\to\infty\ ,\\ &\Phi(\sigma,\tau)\to -\omega\log\big(\sigma^2+\tau^2\big)+C(\omega) \quad \text{as} \quad \sigma^2+\tau^2\to 0\ .\end{aligned} \tag{29}$$

Here $C(\omega)$ is certain constant which in fact is completely determined by the condition that such solution exists. It is easy to show that the boundary value $\Phi_B(\sigma) = \Phi(\sigma,0)$ of this function satisfies the integral equation

$$\Phi_B(\sigma) = \omega\,E(\sigma) + \int_{-\infty}^{\infty}\frac{d\sigma'}{2\pi}\,E(\sigma-\sigma')\big(\,\Phi_B(\sigma') - \sin\Phi_B(\sigma')\,\big)\ , \tag{30}$$

where $E(\sigma)$ is the same function as in (27). Therefore

$$C(\omega) = -2\omega\gamma + \int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}\,E(\sigma)\big(\,\Phi_B(\sigma)-\sin\Phi_B(\sigma)\,\big)\ . \tag{31}$$

[4] Our definition of the composite fields $\big(\varphi_B\big)^{2n}$ which corresponds to the normalization (4) is similar to that in the bulk sine-Gordon theory [12]. For instance $\varphi_B^2(x) = \lim_{\varepsilon\to 0}\big[\,\varphi_B(x)\varphi_B(x+\varepsilon)+2\log\varepsilon\,\big]$.

Now, the expectation value $\langle\, e^{\frac{\omega}{\beta}\varphi_B}\,\rangle$ with fixed ω and $\beta^2 \to 0$ is expressed through the action (1) calculated on the above classical solution $\Phi(\sigma,\tau)$ which in turn can be related to the constant $C(\omega)$

$$\langle\, e^{\frac{\omega}{\beta}\varphi_B}\,\rangle \sim m_0^{-\frac{\omega^2}{\beta^2}}\, \exp\left(\frac{1}{\beta^2}\int_0^{\omega} d\omega'\, C(\omega')\right)\,. \tag{32}$$

Our conjecture (3) allows us to make the following prediction about the constant $C(\omega)$

$$C(\omega) = -2\omega\gamma + \int_0^{\infty}\frac{dt}{t}\, e^{t}\left[\frac{2\omega t - \sin(2\omega t)}{\sinh^2(t)}\right]\,. \tag{33}$$

It would be interesting to check this prediction by solving the equation (30) directly.

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