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Couplings In Asymmetric Orbifolds and Grand Unified String Models

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Abstract

Using the bosonic supercurrent (or covariant lattice) formalism, we review how to compute scattering amplitudes in asymmetric orbifold string models. This method is particularly useful for calculating scattering of multiple asymmetrically twisted string states, where the twisted states are rewritten as ordinary momentum states. We show how to reconstruct some of the 3-family grand unified string models in this formalism, and identify the quantum numbers of the massless states in their spectra. The discrete symmetries of these models are rather intricate. The superpotentials for the 3-family E_6 model and a closely related $SO(10)$ model are discussed in some detail. The forms of the superpotentials of the two 3-family $SU(6)$ models (with asymptotically-free hidden sectors $SU(3)$ and $SU(2) \otimes SU(2)$) are also presented.

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I. INTRODUCTION

A phenomenologically interesting string model must contain the standard model of strong and electroweak interactions in its low energy effective field theory limit. Many of such realizations are constructed as orbifolds, both symmetric [1] and asymmetric [2], of the 4-dimensional heterotic string. Recently, 3-family grand unified string models were constructed via asymmetric orbifolds [3]. Analyses of these models would require the determination of the couplings of states in their spectra. The main goal of the present work is to provide a prescription for calculating their correlation functions and scattering amplitudes. In the process, the quantum numbers of the massless states are found and the selection rules of their couplings are determined.

The prescription for calculating any correlation function in orbifold conformal field theory is given in Ref [4]. However, the actual calculations can be quite non-trivial [5], when the couplings of twisted string states are involved. The problem becomes even more difficult in asymmetric orbifold models, where one typically encounters some ambiguities which are not easily resolved.

In contrast to symmetric orbifolds, where the original lattice of the 6 compactified dimensions (*e.g.*, the compactification radii) can be arbitrary, consistency of asymmetric orbifolds imposes strong constraints on the allowed lattices, typically with enhanced (discrete or local) symmetry. This enhanced symmetry allows us to treat twists as shifts in the momentum lattices, in the so-called bosonic supercurrent (or covariant lattice) formalism [6,7]. Twisted states in an asymmetric orbifold now become ordinary momentum states in this bosonic supercurrent formalism; so their quantum numbers are straightforward to identify and calculating their correlation functions becomes relatively easy.

In this paper, we shall first review this bosonic supercurrent formalism, and discuss briefly its rules for model-building. This formalism is a generalization of the free fermionic string model construction [8]. So their rules for model-building are quite similar. Next we discuss the prescription for calculating the correlation functions, in particular for the twist states. All the couplings can be determined from the correlation functions, or the scattering amplitudes. For clarification, we shall apply this approach to the original $\mathbf{Z}_3$ symmetric (at special radii) and asymmetric models [1,2]. Then we shall discuss the construction of higher-level string models. Our goal is to apply this formalism to determine the couplings in the 3-family grand unified models constructed recently. After we explain the prescription for calculating the couplings, the form of the superpotential for a number of these models will be presented. Their phenomenological implications will be discussed elsewhere.

Within the framework of conformal (free) field theory and orbifolds, we find only one 3-family E_6 model with a non-abelian hidden sector [3]. Here, we give its construction in the bosonic supercurrent formalism, where we see that its discrete symmetry and the charge assignments in the spectrum are quite non-trivial. There are two orbifold constructions of this model, where some of the twisted states in one orbifold appear as untwisted states in the other orbifold. The supercurrents in these two constructions are rather different in the bosonic supercurrent formalism; however, the form of the superpotential (at least at tree-level) turns out to be the same. The same procedure to obtain the forms of superpotentials is also applied to the two interesting 3-family $SU(6)$ models, one with an $SU(3)$, and the other with an $SU(2) \otimes SU(2)$, asymptotically-free hidden sector.

Without much work, we can obtain the superpotential for the 3-family grand unified models (or standard-like models) that can be obtained by giving the E_6 adjoint Higgs an appropriate vacuum expectation value. We illustrate this with the 3-family $SO(10)$ model that can be obtained this way, and which was constructed as an orbifold before.

The plan is as follows: some preliminary discussions are given in section II, where the basic idea, the advantage, and the issues of the bosonic supercurrent formalism are reviewed. The orbifold rules for level-1 models are discussed in section III and that for higher-level models are discussed in section IV. Here, the prescription for calculating correlation functions is also discussed. Since the rules for model-building are given in the light-cone gauge, we explain how to calculate the scatterings in the covariant gauge. Section V contains a discussion of the original $\mathbf{Z}_3$ symmetric (at special radii) and asymmetric models. The construction of the 3-family E_6 grand unified model in this formalism is discussed in Section VI, where some terms in its superpotential is also presented. The $SO(10)$ model is discussed in Section VII. Section VIII discusses the $SU(6)$ models. The quantum numbers of the massless spectra of these models and the form of the relevant supercurrents are given in the tables. Section IX contains some concluding remarks.

II. PRELIMINARIES

To be specific, let us consider 4-dimensional heterotic string models within the conformal field theory (CFT) framework, where free fields are used. Consider such a model with a Lorentzian lattice $\Gamma^{6,22}$. An orbifold is realized via modding out this lattice by a point group P, *i.e.*, a group of discrete rotations, or twists. We shall restrict ourselves to Abelian twists only. Let $X(\overline{z})$ be one of the right-moving complex chiral bosons in the 6 compactified dimensions in $\Gamma^{6,22}$. In terms of 2 real bosons, $X = (X_1 + iX_2)/\sqrt{2}$. For a $\mathbf{Z}_N$ twist, in the neighborhood of a twist field located at the origin, $X(\overline{z})$ undergoes a phase rotation

$$\partial X(\overline{z}e^{-2\pi i}) = \exp(2\pi i k/N)\partial X(\overline{z}) \ , \tag{1}$$

which is called the monodromy of X. (Note that k is an integer.) The basic twist field $\sigma(\overline{z})$ has conformal weight $h = k(1 - k/N)/2N$. It twists X by $\exp(2\pi ik/N)$ and its complex conjugate $\overline{X}$ by $\exp(-2\pi ik/N)$, *i.e.*, their operator product expansions (OPEs) are [4]

$$\begin{aligned} i\partial X(\overline{z})\sigma(\overline{w}) &= (\overline{z} - \overline{w})^{-(1-k/N)}\tau(\overline{w}) + \ldots \ , \\ i\partial\overline{X}(\overline{z})\sigma(\overline{w}) &= (\overline{z} - \overline{w})^{-k/N}\tau'(\overline{w}) + \ldots \ , \end{aligned} \tag{2}$$

where τ and τ' are excited twist fields.

Suppose we can rewrite the $i\partial X$ as exponentials of a pair of boson fields ϕ_1 and ϕ_2,

$$i\partial X(\overline{z}) = \exp(ie\cdot\phi(\overline{z})) + \ldots \ . \tag{3}$$

Here e is a 2-dimensional vector and the conformal weight $h = 1$ condition requires $e^2 = 2$. Then, the phase rotation of ∂X in Eq. (1) becomes a shift in ϕ

$$\phi(\overline{z}e^{-2\pi i}) = \phi(\overline{z}) + 2\pi u \ , \tag{4}$$

where $e\cdot u = k/N$; that is, a twist on ∂X becomes a shift in ϕ. To recover the correct OPEs of ∂X and $\partial\overline{X}$, it turns out that the ∂X must be written as a linear combination of such exponential terms. The proper monodromy condition of $i\partial X$ then implies that ϕ must be compactified in some lattice. ϕ provides a particularly useful basis if the supercurrent, as well as the twist fields such as σ and τ, can also be written in terms of ordinary momentum states. This is the basic idea of the bosonic supercurrent formalism [6].

Let us see more explicitly how this conversion of twists to shifts can be realized, and the advantages of this approach. In this paper, we are mostly interested in $\mathbf{Z}_2$ and $\mathbf{Z}_3$ twists. For $N=3$ in Eq. (1), the lattice must have a $\mathbf{Z}_3$ symmetry, so the $U(1)^2$ symmetry enhances to a $SU(3)$ symmetry. Here, ∂X_1 and ∂X_2 are the Cartan generators and the six root generators are given by $e^{\pm i e_\alpha \cdot X(\overline{z})} c(\pm e_\alpha)$, $\alpha=1,2,3$, where the 2-dimensional vectors e_1 and e_2 are the simple roots of $SU(3)$ and we define $e_3=-e_1-e_2$. Note that $e_\alpha \cdot e_\beta = 2$ for $\alpha=\beta$, and -1 otherwise. For convenience, we shall not always explicitly display the cocycle operator $c(\pm e_\alpha)$: its presence is understood.

In the standard orbifold formalism, the supercurrent for the right-movers can be written as

$$T_F = \frac{i}{2}\psi \partial X + \text{h.c.} + \ldots \ , \tag{5}$$

where ψ is a world-sheet complex fermion. Since each $e^{\pm i e_\alpha \cdot X}$ has conformal weight 1, same as the $i\partial X$, we may choose to rewrite the supercurrent in terms of the $SU(3)$ roots. The choice is constrained by the condition that they must obey the same OPEs as $i\partial X$ and $i\partial\overline{X}$. For any Lie algebra, one can always find such a subalgebra by rotating the Cartan generators to the root system of the algebra. The new $U(1)$ generators will always be a linear combination of the root generators. For $SU(3)$, the choice is unique,

$$i\partial X = \frac{1}{\sqrt{3}} \sum_\alpha e^{-i e_\alpha \cdot \phi} \ , \quad i\partial\overline{X} = \frac{1}{\sqrt{3}} \sum_\alpha e^{i e_\alpha \cdot \phi} \ . \tag{6}$$

Now, there are three twist fields σ_α in the $\mathbf{Z}_3$ twisted sector. In this basis, it is easy to check that,

$$\begin{aligned} i\partial X(\overline{z})\sigma_\alpha(\overline{w}) &= (\overline{z}-\overline{w})^{-2/3}\tau_\alpha(\overline{w}) + \ldots \ , \\ i\partial\overline{X}(\overline{z})\sigma_\alpha(\overline{w}) &= (\overline{z}-\overline{w})^{-1/3}\tau'_\alpha(\overline{w}) + \ldots \ , \end{aligned} \tag{7}$$

and these OPEs agree with Eq. (2) for $N=3$, where

$$\sigma_\alpha = e^{i e_\alpha \phi/3} \ , \quad \tau_\alpha = \frac{1}{\sqrt{3}} e^{-2 i e_\alpha \phi/3} \ , \quad \tau'_\alpha = \frac{1}{\sqrt{3}} \sum_{\beta\neq\alpha} e^{i(e_\alpha/3+e_\beta)\phi} \ . \tag{8}$$

Note that τ'_α is a linear combination of the corresponding vertex operators for *two* states with the same highest weight.

The usefulness of the bosonic supercurrent formalism is now clear. The untwisted states lie in the original lattice while the twist states lie in the shifted lattice. Since they are all expressed in terms of ordinary momentum states, their normalizations and degeneracies, as well as their OPEs with each other, are easy to calculate.

For a $\mathbf{Z}_2$ twist, it is sometimes convenient to decompose X into 2 real bosons, X_1 and X_2. If the compactification radii of the bosons are 1, we can use the fermion-boson equivalence in CFT to rewrite each X_i as a complex fermion. So, for orbifolds with only $\mathbf{Z}_2$ twists, the internal parts of the supercurrent can be rewritten entirely into world-sheet fermions. This is the free fermionic string model construction [8]. In the next section, we shall give the rules for model-building using this bosonic supercurrent formalism. Since some of the 3-family grand unified models can be constructed with the bosonic supercurrent formalism, it is natural to use this formalism to calculate the couplings in these models.

In the examples with only level-1 current algebras, we shall use the above supercurrent with (6) involving the $SU(3)$ lattice. For the 3-family grand unified models, the supercurrents are more complicated. Consistency with the transformation of specific twists to shifts in a given orbifold model essentially fixes the supercurrent. As mentioned earlier, there are two equivalent constructions of the 3-family E_6 model. In one construction, namely, the $E1$ model, the supercurrent involves a E_6 lattice and its $SU(3)^2$ sub-lattice; while, in the other construction of the same model, namely, the $E2$ model, the supercurrent involves the appropriate $SU(2)^4$, $SU(6)$ and $SU(3)^2$ sub-lattices of E_6.

III. MODEL-BUILDING RULES

In this section we give the rules for constructing Abelian asymmetric orbifolds using the bosonic supercurrent formalism. The rules that we present here are less general than the ones in Ref [3] because we confine our attention to a more limited class of orbifolds. Nevertheless, some of the three-family grand unified models found recently [3] can be constructed within the present framework. Moreover, this formalism is particularly useful in computing couplings and identifying quantum numbers (both local and discrete) of the physical states. Throughout the paper, we consider only heterotic strings compactified to four space-time dimensions.

A. Framework

In this subsection we set up the framework for the remainder of this section. In the light -cone gauge which we adopt, we have the following world-sheet degrees of freedom: one right-moving complex boson X^0 (which along with its left-moving counterpart corresponds to two transverse space-time coordinates); three right-moving complex bosons $X^{1,2,3}$ (corresponding to six internal coordinates); four right-moving complex fermions ψ^a, $a = 0,1,2,3$ (ψ^0 is the world-sheet superpartner of $(X^0)^\dagger$, whereas $\psi^{1,2,3}$ are the world-sheet superpartners of $(X^{1,2,3})^\dagger$); two left-moving real bosons $\mathcal{X}^\mu_L$, $\mu = 1,2$ (these are the left-moving counterparts of the two real bosons $\mathcal{X}^\mu_R$ corresponding to X^0 via $X^0 = (\mathcal{X}^1_R + i\mathcal{X}^2_R)/\sqrt{2})$; 22 left-moving real bosons φ^A, $A = 1,...,22$ (corresponding to twenty-two internal coordinates). Before orbifolding, the corresponding string model has $N = 4$ space-time supersymmetry and the internal momenta span an even self-dual Lorentzian lattice $\Gamma^{6,22}$. The underlying conformal field theory of the internal degrees of freedom is given by $G_L \otimes G_R$ where G_L and G_R are the left- and right-moving level-1 Kac-Moody algebras with central charges $c_L = 22$ and $c_R = 9$. The right-moving Kac-Moody algebra consists of two factors, *i.e.*, $G_R = \mathcal{G}_R \otimes SO(6)_1$ where $\mathcal{G}_R$ is a level 1 Kac-Moody algebra (with central charge 6) corresponding to the right-moving part of the lattice $\Gamma^{6,22}$ and $SO(6)_1$ (with central charge 3) comes from the right-moving fermions. After orbifolding the underlying conformal field theory becomes

$(G'_L \otimes \mathcal{C}_L) \otimes (G'_R \otimes \mathcal{C}_R)$. Here G'_L and G'_R are left- and right-moving Kac-Moody algebras, and $\mathcal{C}_L$ and $\mathcal{C}_R$ are certain cosets that arise in the breakings $G_L \equiv \mathcal{G}_L \supset \mathcal{G}'_L$ and $\mathcal{G}_R \supset \mathcal{G}'_R$, respectively. (Note that since we are considering Abelian orbifolds, the $SO(6)$ subgroup of G_R breaks to a level-1 subgroup, and the corresponding coset is trivial.) In this section, we restrict ourselves to the cases where after orbifolding, the left- and right-moving Kac-Moody algebras are realized at level 1 and the cosets are trivial. The rules for higher level models will be discussed in the next section.

It is convenient to organize the string states into sectors labeled by the monodromies of the string degrees of freedom. Consider the right-movers

$$\begin{aligned} \psi^a(\overline{z}e^{-2\pi i}) &= \exp(-2\pi i V_i^a)\psi^a(\overline{z}) \ , \\ \partial X^a(\overline{z}e^{-2\pi i}) &= \exp(-2\pi i T_i^a)\partial X^a(\overline{z}) \ . \end{aligned} \tag{9}$$

Here we note that T_i^0 must always be zero as $\mathcal{X}^\mu(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = \mathcal{X}^\mu(z,\overline{z})$ since $\mathcal{X}^\mu(z,\overline{z}) = \mathcal{X}^\mu_L(z) + \mathcal{X}^\mu_R(\overline{z})$ correspond to space-time coordinates. Let us define $s_i \equiv V_i^0$. The sectors with $s_i \in \mathbf{Z}$ give rise to the space-time bosons, whereas the sectors with $s_i \in \mathbf{Z}+1/2$ give rise to space-time fermions. The monodromy of the supercurrent

$$T_F = \frac{i}{2}\sum_{a=0}^{3} \psi^a \partial X^a + \text{h.c.} \ . \tag{10}$$

is given by s_i, *i.e.*, $T_F(\overline{z}e^{-2\pi i}) = \exp(-2\pi i s_i) T_F(\overline{z})$. This, in particular, implies the *triplet* constraint on the supercurrent:

$$V_i^a + T_i^a = s_i \ (\text{mod } 1) \ , \quad a = 0,1,2,3 \ . \tag{11}$$

The twists on ψ^a can be written as shifts if we bosonize the complex fermions:

$$\begin{aligned} \psi^a &= \exp(i\rho^a) \ , \\ \psi^{a\dagger} &= \exp(-i\rho^a) \ . \end{aligned} \tag{12}$$

This latter form will be useful when we rewrite the *triplet* constraint for the bosonic supercurrent.

Because of the worldsheet $N=2$ superconformal field theory (SCFT) (which is necessary for $N=1$ space-time supersymmetry), there is a conserved right moving $U(1)$ current

$$Y(\overline{z}) = i\sum_{a=1}^{3} \rho^a(\overline{z}) \ . \tag{13}$$

Therefore the supercurrent can be divided into two pieces $T_F(\overline{z}) = T_F^+(\overline{z}) + T_F^-(\overline{z})$ with $U(1)$ charges ± 1. The energy momentum tensor T, the supercurrent $T_F^\pm$ together with Y form a global $N=2$ superconformal algebra

$$\begin{aligned} T(\overline{z})T(0) &\sim \frac{\frac{3}{4}\hat{c}}{\overline{z}^4} + \frac{2T(0)}{\overline{z}^2} + \frac{\partial T(0)}{\overline{z}} + \cdots \ , \\ T(\overline{z})T_F^\pm(0) &\sim \frac{\frac{3}{2}T_F^\pm(0)}{\overline{z}^2} + \frac{\partial T_F^\pm(0)}{\overline{z}} + \cdots \ , \end{aligned}$$

$$
\begin{aligned}
T_F^+(\overline{z})T_F^-(0) &\sim \frac{\frac{1}{8}\hat{c}}{\overline{z}^3} + \frac{\frac{1}{4}Y(0)}{\overline{z}^2} + \frac{\frac{1}{4}T(0)}{\overline{z}} + \frac{\frac{1}{8}\partial Y(0)}{\overline{z}} + \cdots \ , \\
Y(\overline{z})T_F^{\pm}(0) &\sim \frac{\pm T_F^{\pm}(0)}{\overline{z}} + \cdots \ , \\
Y(\overline{z})Y(0) &\sim \frac{\frac{1}{2}\hat{c}}{\overline{z}^2} + \cdots \ ,
\end{aligned}
\tag{14}
$$

where only the singular terms are shown. In our case, $\hat{c} = \frac{2}{3}c = 8$. (The space-time part of the supercurrent carries $\hat{c} = 2$ and the internal part has $\hat{c} = 6$.)

Let us consider the cases where the Kac-Moody algebra $\mathcal{G}'_R$ corresponding to the right-moving part of the lattice is realized at level 1 and has central charge 6. Then there exist six right-moving real bosons ϕ^I such that $i\partial\phi^I$ are the vertex operators for the Cartan generators of $\mathcal{G}'_R$. As discussed in the previous section, in order to rewrite the twists in ∂X^a as shifts in ϕ^I, ∂X^a must take the following form:

$$
i\partial X^a = \sum_{Q^2=2} \xi^a(Q) J_Q \ , \quad a = 1, 2, 3 \ ,
\tag{15}
$$

where

$$
J_Q(\overline{z}) = \exp(iQ \cdot \phi(\overline{z}))\ c(Q) \ .
\tag{16}
$$

We have introduced six-dimensional real vectors $Q = (Q^1, ..., Q^6)$ which are root vectors of $\mathcal{G}_R$ with length squared 2. This ensures that $i\partial X^a$ has conformal dimension 1. The $c(Q)$ are cocycle operators necessary in the Kac-Moody algebra. Note that J_Q are Kac-Moody currents for root generators. The supercurrent is therefore a linear combination of terms with different H and Q charges.

Here $\xi^a(Q)$ are numerical coefficients constrained by the OPEs:

$$
\begin{aligned}
\partial X^a(\overline{z})\partial X^b(0) &\sim \text{regular} \ , \\
\partial X^a(\overline{z})\partial X^{b\dagger}(0) &\sim -\overline{z}^{-2}\delta^{ab} + \text{regular} \ .
\end{aligned}
\tag{17}
$$

The monodromies (V_i^a, T_i^a) for ψ^a and ∂X^a can now be translated into monodromies (V_i^a, U_i^I) of ρ^a and ϕ^I:

$$
\begin{aligned}
\rho_a(\overline{z}e^{-2\pi i}) &= \rho_a(\overline{z}) - 2\pi V_i^a \ , \\
\phi^I(\overline{z}e^{-2\pi i}) &= \phi^I(\overline{z}) - 2\pi U_i^I \ .
\end{aligned}
\tag{18}
$$

In this basis, the *triplet* constraint on the supercurrent becomes

$$
\xi^a(Q) = 0 \text{ unless } V_i^a + U_i \cdot Q = s_i \ (\text{mod } 1) \ .
\tag{19}
$$

In terms of the chiral bosons ρ_a and ϕ^I, the energy momentum tensor T is given by:

$$
T(\overline{z}) = -\frac{1}{2}\sum_a (\partial\rho_a)^2 - \frac{1}{2}\sum_I (\partial\phi^I)^2 - \frac{1}{2}\sum_\mu (\partial X_R^\mu)^2 \ .
\tag{20}
$$

The above form of T together with the bosonic supercurrent $T_F^{\pm}$ and Y satisfy the $N = 2$ SCFT (14).

Having discussed the right-moving degrees of freedom, let us turn to the left-movers. Since $\mathcal{G}'_L$ is realized at level 1 and has central charge 22, we have the following monodromy conditions on the fields φ^A:

$$\varphi^A(ze^{2\pi i}) = \varphi^A(z) + 2\pi U_i^A \ . \tag{21}$$

The monodromies V_i^a, U_i^I, U_I^A can be conveniently combined into $(10, 22)$ dimensional Lorentzian vectors (with metric $((-)^{10}, (+)^{22})$):

$$V_i = (V_i^a | U_i^I || U_I^A) \ . \tag{22}$$

The monodromies V_i can be viewed as fields Φ (where Φ is a collective notation for ρ_a, ϕ^I and φ^A) being periodic $\Phi(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = \Phi(z, \overline{z})$ up to the identification $\Phi \sim g(V_i)\Phi g^{-1}(V_i)$, where $g(V_i)$ is an element of the orbifold group G. For G to be a finite discrete group, the element $g(V_i)$ must have a finite order $m_i \in \mathbf{N}$, *i.e.*, $g^{m_i}(V_i) = 1$. This implies that the vector V_i must be a rational multiple of a vector in $\Delta^4 \otimes \Gamma^{6,22}$, that is, $m_i V_i \in \Delta^4 \otimes \Gamma^{6,22}$. Here Δ^4 is the odd self-dual Euclidean lattice spanned by the four-dimensional vectors p that correspond to the vector $\mathbf{v}$ and spinor $\mathbf{s}$ irreps of $SO(8)_1$. Thus, this lattice can be described as being spanned by vectors of the following form: $p = (p^0, p^1, p^2, p^3)$, where either $p^a \in \mathbf{Z}$ and $\sum_a p^a \in 2\mathbf{Z} + 1$ (the momenta corresponding to the $\mathbf{v}$ irrep), or $p^a \in \mathbf{Z} + \frac{1}{2}$ and $\sum_a p^a \in 2\mathbf{Z}$ (the momenta corresponding to the $\mathbf{s}$ irrep). Here we note that we could have chosen Δ^4 to be spanned by the the four-dimensional vectors p that correspond to the vector $\mathbf{v}$ and conjugate $\mathbf{c}$ irreps of $SO(8)_1$. This freedom in choosing the lattice Δ^4 is immaterial, and is related to the choice of the structure constant k_{00} to be introduced below.

To describe all the elements of the group G, it is convenient to introduce the set of generating vectors $\{V_i\}$ such that $\alpha V = \mathbf{0}$ if and only if $\alpha_i \equiv 0$. Here $\mathbf{0}$ is the null vector, *i.e.*, the vector of the form (22) with all its entries being null:

$$\mathbf{0} = (0^4 | 0^6 || 0^{22}) \ . \tag{23}$$

Also, $\alpha V \equiv \sum_i \alpha_i V_i$ (the summation is defined as, say, $(V_i + V_j)^a = V_i^a + V_j^a$), α_i being integers that take values from 0 to $m_i - 1$. The elements of the group G are then in one-to-one correspondence with the vectors αV and will be denoted by $g(\alpha V)$. It is precisely the Abelian nature of G that allows this correspondence (by simply taking all the possible linear combinations of the generating vectors V_i).

Now we can identify the sectors of the model. They are labeled by the vectors αV, and in a given sector αV the monodromies of the string degrees of freedom are given by $\Phi(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = g(\alpha V)\Phi(z, \overline{z})g^{-1}(\alpha V)$. It is clear that the sectors with $\alpha s \in \mathbf{Z}$ give rise to the space-time bosons, whereas those with $\alpha s \in \mathbf{Z} + 1/2$ give rise to the space-time fermions.

Note that a sector described by the vector

$$V_0 = ((-\frac{1}{2})^4 | 0^6 || 0^{22}) \tag{24}$$

is always present in any orbifold model. This is related to $N = 4$ SUSY of the original Narain model [9]. In other words, the V_0 sector is the Ramond sector of the Narain model, whereas $\mathbf{0}$ sector is the Neveu-Schwarz sector. It is convenient to include this sector V_0 into the set of generating vectors $\{V_i\}$ (then $i = 0, 1, 2, \ldots$). Since we have included V_0 into the set $\{V_i\}$, without loss of generality we can set $s_i = 0$ for $i \neq 0$. Then the space-time bosons come from the sectors αV with $\alpha_0 = 0$, and the space-time fermions come from the sectors αV with $\alpha_0 = 1$. (Note that $m_0 = 2$.) With this definition, we can relax the constraint on $m_i V_i$, and require that $m_i V_i \in \mathbf{Z}^4 \otimes \Gamma^{6,22}$. The odd self-dual lattice Δ^4 for the Narain model will then emerge as a consequence of the spectrum generating formula (see below).

We use the light-cone gauge in deriving the rules for computing the spectrum of a string model, since it is manifestly ghost-free and so all the states constructed are physical. However, it is more convenient to discuss scattering in the covariant gauge. Some of the issues such as picture changing will be more clear in the covariant approach. The translation from the light-cone gauge to the covariant gauge is straightforward. Given a vertex operator in the light-cone gauge, we can construct a vertex operator in the covariant gauge by simply covariantizing the light-cone coordinates and putting in the ghosts.

In the light-cone gauge of 4-dimensional heterotic string with $N = 1$ space-time supersymmetry (SUSY), massless physical states are created by vertex operators of the form $V(z,\overline{z}) = V(z)\overline{V}(\overline{z})$, where $V(z)$ is a left-moving vertex operator of conformal dimension 1 (the vacuum energy in the left-moving sector is -1 in the untwisted sector), and $\overline{V}(\overline{z})$ is a right-moving vertex operator of conformal dimension 1/2 (the vacuum energy in the right-moving sector, which has $N = 1$ local world-sheet supersymmetry and $N = 2$ global SUSY, is $-1/2$ for bosons in the untwisted sector).

In the covariant gauge, we have in addition to the light-cone degrees of freedom: two longitudinal space-time coordinates, two longitudinal components of the right-moving fermions, reparametrization ghosts b and c, and superconformal ghosts β and γ [10]. It is most convenient to bosonize the β, γ ghosts:

$$\beta = \partial\xi e^{-\phi}, \quad \gamma = \eta e^{\phi} \ , \tag{25}$$

where ξ and η are auxiliary fermions and ϕ is a bosonic ghost field obeying the OPE $\phi(\overline{z})\phi(\overline{w}) \sim \log(\overline{z} - \overline{w})$. The conformal dimension of $e^{q\phi}$ is $-\frac{1}{2}q(q+2)$. In covariant gauge, vertex operators are of the form $V(z,\overline{z}) = V(z)\overline{V}(\overline{z})$ where $V(z)$ and $V(\overline{z})$ are both dimension 1 operators constructed from the conformal fields. These include the longitudinal components as well as the ghosts. The vertex operators for space-time bosons carry integral ghost charges ($q \in \mathbf{Z}$) whereas for space-time fermions the ghost charges are half-integral ($q \in \mathbf{Z} + \frac{1}{2}$). Here, q specifies the picture. The canonical choice is $q = -1$ for space-time bosons and $q = -\frac{1}{2}$ for space-time fermions. We will denote the corresponding vertex operators by $V_{-1}(z,\overline{z})$ and $V_{-\frac{1}{2}}(z,\overline{z})$ respectively. Vertex operators in the $q = 0$ picture (with zero ghost charge) is given by *picture-changing* :

$$V_0(z,\overline{z}) = \lim_{\overline{w}\to\overline{z}} e^{\phi} T_F(\overline{z}) V_{-1}(z,\overline{w}) \ . \tag{26}$$

Because the supercurrents in the 3-family grand unified models have many terms, V_0 can be somewhat involved.

Having constructed the vertex operators for the massless states, one can in principle compute the scattering amplitudes, or the corresponding couplings in the superpotential. The coupling of M chiral superfields in the superpotential is given by the scattering amplitude of the component fields in the limit when all the external momenta are zero. Due to holomorphicity, one needs to consider only the scatterings of left-handed space-time fermions, with vertices $V_{-1/2}(z,\overline{z})$, and their space-time superpartners. Since the total ϕ ghost charge in any tree-level correlation function is -2, it is convenient to choose two of the vertex operators in the $-1/2$-picture, one in the -1-picture, and the rest in the 0-picture. Using the $SL(2,\mathbf{C})$ invariance, the scattering amplitude is therefore

$$\mathcal{A}_M = g_{\rm st}^{M-2} \int dz_4 d\overline{z}_4 \cdots dz_M d\overline{z}_M \langle V_{-\frac{1}{2}}(0,0) V_{-\frac{1}{2}}(1,1) V_{-1}(\infty,\infty) V_0(z_4,\overline{z}_4) \cdots V_0(z_M,\overline{z}_M)\rangle \ , \tag{27}$$

where we have normalized the c ghost part of the correlation function $\langle c(0,0)c(1,1)c(\infty,\infty)\rangle$ to 1.

B. Orbifold Rules for Level-1 Models

We are now ready to give the rules for constructing consistent orbifold models with multiple twists using the bosonic supercurrent formalism. We will not give the derivation of these rules as they can be deduced from Ref [3] which contains more generic (and, therefore, more complicated) rules. Instead, we just list all the requirements that a consistent model must satisfy. In this subsection we will concentrate on models that have no non-trivial left-moving coset $\mathcal{C}_L$. Models with left-moving twists (such as outer automorphisms) will be considered in the subsequent sections.

For a string model to be consistent, it must satisfy the following constraints which we impose:
(1) *Modular invariance.* One-loop partition function must be invariant under S and T modular transformations.
(2) *Physically sensible projection.* The physical states should appear in the partition function with proper weights and space-time statistics. The space-time bosons contribute $+1$ to the one loop vacuum amplitude while space-time fermions contribute -1.
(3) *Worldsheet supersymmetry.* This is essential for space-time Lorentz invariance. In order for the total supercurrent to have well defined boundary condition, the space-time supercurrent and the internal supercurrent defined above should have the same monodromies. This is guaranteed by the triplet constraint (19). Furthermore, the OPEs in (17) impose extra conditions on the coefficients $\xi^a(Q)$ in T_F. This ensures the $N=2$ superconformal algebra is satisfied.

We start from a Narain model with the momenta of the internal bosons spanning an even self-dual Lorentzian lattice $\Gamma^{6,22}$. We introduce a set of generating vectors $\{V_i\}$ that includes the vector V_0. Next, we find the structure constants k_{ij} that satisfy the following constraints:

$$k_{ij} + k_{ji} = V_i \cdot V_j \ (\text{mod } 1) \ , \tag{28}$$

$$k_{ii} + k_{i0} + s_i - \frac{1}{2} V_i \cdot V_i = 0 \ (\text{mod } 1) \ , \tag{29}$$

$$k_{ij} m_j = 0 \ (\text{mod } 1) \ . \tag{30}$$

Note that there is no summation over the repeated indices here. The dot product of two vectors is defined with Lorentzian signature $((-)^{10},(+)^{22})$, *i.e.*,

$$
\begin{aligned}
V_i \cdot V_j &= -\sum_a V_i^a V_j^a + \vec{U}_i \cdot \vec{U}_j \\
&= -\sum_a V_i^a V_j^a - \sum_I U_i^I \cdot U_j^I + \sum_A U_i^A \cdot U_j^A \ .
\end{aligned} \tag{31}
$$

Here we have combined the components U_i^I and U_i^A into a single $(6,22)$ dimensional vector $\vec{U}_i$. Note that this vector is a rational multiple of a vector in $\Gamma^{6,22}$, and, in particular, $m_i\vec{U}_i \in \Gamma^{6,22}$. The dot product of two vectors $\vec{U}_i$ and $\vec{U}_j$ is then defined in the same way as for two vectors in $\Gamma^{6,22}$.

The above rules are not sufficient for the model to be consistent. There must exist a supercurrent which satisfies the *triplet* constraint (19) for all generating vectors V_i. Furthermore, the OPEs in (17) impose extra conditions on the coefficients $\xi^a(Q)$ in T_F. Thus, the rules that constrain the set $\{V_i, k_{ij}\}$ together with the existence of the required supercurrent give the necessary and sufficient conditions for building a consistent string model.

The sectors of the model are αV. In a given sector αV the states are nothing but the momentum states of $\mathcal{X}^\mu$, ρ_a, ϕ^I and φ^A. This means that the vertex operator for a given state has the form (in the covariant formalism where $\mathcal{X}^\mu$, $\mu = 0,1,2,3$, are four space-time coordinates)

$$
V(z,\overline{z}) = \tilde{V}(z,\overline{z}) \exp(i\sum_a H_a \rho_a(\overline{z}) + i\sum_I Q^I \phi^I(\overline{z}) + i\sum_A Q^A \varphi^A(z)) \exp(ik_\mu \mathcal{X}^\mu(z,\overline{z})) \ , \tag{32}
$$

where $\tilde{V}(z,\overline{z})$ is a combination of ghost fields, derivatives of $\mathcal{X}^\mu$, ρ_a, ϕ^I and φ^A (*i.e.*, this is the corresponding oscillator excitation contribution), and an appropriate cocycle operator. The normal ordering is implicit here. Let us combine the H-charges H_a, Q-charges Q^I, and the gauge charges Q^A into a $(10,22)$ dimensional momentum vector $P_{\alpha V}$. Then the physical states are those that satisfy the following spectrum generating formula:

$$
V_i \cdot P_{\alpha V} = s_i + \sum_j k_{ij}\alpha_j \ (\mathrm{mod}\ 1) \ , \tag{33}
$$

and the momenta $P_{\alpha V} \in \mathbf{Z}^4 \otimes \Gamma^{6,22} + \alpha V$. Thus, for example, $H_a \in \mathbf{Z} + (\alpha V)^a$, and $\vec{\mathcal{Q}} \in \Gamma^{6,22} + \alpha\vec{U}$, where we have combined the Q-charges Q^I and the gauge charges Q^A into a single $(6,22)$ dimensional vector $\vec{\mathcal{Q}}$.

The above spectrum generating formula gives both on- and off-shell states. The on-shell states must satisfy the additional constraint that the left- and right-moving energies be equal. In the αV sector they are given by

$$
E^L_{\alpha V} = -1 + \sum_{q=1}^{\infty} q(\sum_\mu m_q^\mu + \sum_A n_q^A) + \frac{1}{2}(P^L_{\alpha V})^2 \ , \tag{34}
$$

$$
E^R_{\alpha V} = -\frac{1}{2} + \sum_{q=1}^{\infty} q(\sum_\mu \tilde{m}_q^\mu + \sum_I n_q^I + \sum_a k_q^a) + \frac{1}{2}(P^R_{\alpha V})^2 \ , \tag{35}
$$

where m_q^μ, $\tilde{m}_q^\mu$, n_q^A, n_q^I and k_q^a are the oscillator occupation numbers for the real bosons $\mathcal{X}_L^\mu$, $\mathcal{X}_R^\mu$, φ^A, ϕ^I and ρ_a, respectively. Also, we note that $(P^L_{\alpha V})^2 = (\vec{\mathcal{Q}}^L)^2$, and $(P^R_{\alpha V})^2 = (\vec{\mathcal{Q}}^R)^2 + \sum_a H_a^2$, where $\vec{\mathcal{Q}}^L$ and $\vec{\mathcal{Q}}^R$ are the left- and right-moving parts of the vector $\vec{\mathcal{Q}}$. (Here we note that for massless states that do not belong to the $N=1$ supergravity multiplet all the occupation numbers are zero and $\tilde{V}(z,\overline{z})=1$ up to a factor that involves ghosts and a cocycle.)

As a simple illustration of these rules let us consider the model generated by a single vector V_0. This is nothing but a Narain model. There is only one structure constant, namely, k_{00}. For $k_{00}=0$ we find that Δ^4 is spanned by $\mathbf{v}$ and $\mathbf{c}$ momenta of $SO(8)_1$, whereas for the other choice $k_{00}=1/2$ we find that Δ^4 is spanned by $\mathbf{v}$ and $\mathbf{s}$ momenta of $SO(8)_1$.

The above generating formula along with the constraints on the set $\{V_i, k_{ij}\}$ is all we need to construct the spectrum of any given model. For supersymmetric models, obtaining the spectrum is simplified further by space-time supersymmetry. For definiteness, let us concentrate on $N=1$ SUSY models. (Our discussion easily generalizes to models with larger SUSY.) In this case we have two SUSY generators $Q_L(\overline{z})$ and $Q_R(\overline{z})$, where the subscript indicates the space-time helicity. The vertex operators for these generators have the the same form as in (32), but with $\tilde{V}(z,\overline{z})=1$ (up to ghosts and a cocycle) and $Q^I=Q^A=k_\mu=0$, *i.e.*, only the H-charges are non-zero. Let us combine all of these charges into $(10,22)$ dimensional momenta $\mathcal{P}$. Then these momenta for the SUSY generators are determined by solving the following constraint:

$$V_i \cdot \mathcal{P} = -\sum_a V_i^a H_a(\mathcal{P}) = k_{i0} \ (\mathrm{mod}\ 1) \ , \tag{36}$$

where $H_a(\mathcal{P})=\pm\frac{1}{2}$ are the corresponding H-charges for the SUSY generators. We will use the convention that the massless fermion states with $H_0=-1/2$ are left-handed, whereas those with $H_0=+1/2$ are right-handed. Then the solution of Eq. (36) with $H_0(\mathcal{P})=-1/2$, which we will refer to as $\mathcal{P}_L$, corresponds to the left-handed SUSY generator $Q_L(\overline{z})$, whereas the solution with $H_0(\mathcal{P})=+1/2$, which we will refer to as $\mathcal{P}_R$, corresponds to the right-handed SUSY generator $Q_R(\overline{z})$. (Note that $\mathcal{P}_L=-\mathcal{P}_R$.) These generators are useful in this case in the following way. Instead of working out the entire spectrum, we can confine our attention to left-moving fermion states in the $-1/2$-picture. These have $H_0=-1/2$ according to the above convention, and let the corresponding momenta be $P_{\alpha V}$. (Their CPT-conjugate states are right-handed with $H_0=+1/2$). Their superpartners, which are boson states in the -1-picture, can be obtained by noting that their corresponding momenta would simply be $P_{\alpha' V}=\mathcal{P}_R+P_{\alpha V}$. Similarly, $P_{\alpha V}=\mathcal{P}_L+P_{\alpha' V}$. Note that $\alpha'_i=\alpha_i$ for $i\neq 0$, and $\alpha_0=1$, whereas $\alpha'_0=0$. Thus, all we need to work out in this case is the spectrum for the left-moving fermion states. So by a vertex operator in the $-1/2$-picture we will always mean vertex operators for such states, whereas by those in the -1-picture we will mean vertex operators of the corresponding bosonic superpartners. Here we comment that the number of supersymmetries in a general case is given by a half of the number of solutions of Eq. (36).

IV. ORBIFOLD RULES FOR HIGHER LEVEL MODELS

In this section we generalize the rules for the orbifold construction presented in section III. This generalization will allow us to construct models with reduced rank, in particular, models that contain gauge groups realized via higher level Kac-Moody algebras. The necessity of such a generalization can be seen from the fact that gauge symmetry in $N=1$ heterotic string models arises from the left-moving sector of the theory. On the other hand, the monodromies (21) for the left-moving bosons φ^A are such that they cannot project out Cartan generators of the original Kac-Moody algebra $\mathcal{G}_L$ of the Narain model that we orbifold. Thus, the final gauge group $\mathcal{G}'_L$ is always realized via a level-1 Kac-Moody algebra and has rank 22. Thus, to obtain models with reduced rank, we must project out some of the original Cartan generators, that is, we have to twist some of the 22 real left-moving bosons φ^A. Here we note that such twisting does not guarantee rank reduction. Sometimes it can happen that the model with such twists still has $\mathcal{G}'_L$ with rank 22. In this case it is always possible to rewrite the model so that all the left-moving bosons are shifted but not twisted.

A. Framework

In this subsection we will set up the framework for the remainder of this section. We will borrow the notation from section III. The right-moving degrees of freedom are the same. The left-movers come in two varieties. There are $22-2d$ real bosons φ^A, $A=1,...,22-2d$, and also d complex bosons Φ^r, $r=1,...,d$. (These can be viewed as complexifications of the original $2d$ real bosons $\varphi^{23-2d},...,\varphi^{22}$ via $\Phi^r=(\varphi^{21-2d+2r}+i\varphi^{22-2d+2r})/\sqrt{2}$.) The string sectors are labeled by the monodromies of the string degrees of freedom:

$$\begin{aligned}
\rho_a(\overline{z}e^{-2\pi i}) &= \rho_a(\overline{z}) - 2\pi V_i^a \ , \\
\phi^I(\overline{z}e^{-2\pi i}) &= \phi^I(\overline{z}) - 2\pi U_i^I \ , \\
\varphi^A(ze^{2\pi i}) &= \varphi^A(z) + 2\pi U_i^A \ , \\
\partial\Phi^r(ze^{2\pi i}) &= \exp(-2\pi i T_i^r)\partial\Phi^r(z) \ .
\end{aligned} \tag{37}$$

These monodromies can be combined into a single vector

$$V_i = (V_i^a|U_i^I||U_i^A|T_i^r) \ . \tag{38}$$

Without loss of generality we can restrict the values of T_i^r as follows: $0 \leq T_i^r < 1$. This restriction is actually necessary for correctly identifying the sectors of the orbifold model in what follows.

The monodromies V_i can be viewed as fields Φ (where Φ is a collective notation for ρ_a, ϕ^I, φ^A and Φ^r) being periodic $\Phi(ze^{2\pi i},\overline{z}e^{-2\pi i})=\Phi(z,\overline{z})$ up to the identification $\Phi\sim g(V_i)\Phi g^{-1}(V_i)$, where $g(V_i)$ is an element of the orbifold group G. (Since we are considering Abelian orbifolds, we have excluded shifts from the monodromies of the bosons Φ^r.) For G to be a finite discrete group, the element $g(V_i)$ must have a finite order $m_i\in\mathbf{N}$, *i.e.*, $g^{m_i}(V_i)=1$. This implies that the vector V_i must be a rational multiple of a vector in $\mathbf{Z}^4\otimes\Gamma^{6,22}\otimes\mathbf{N}^d$, that is, $m_iV_i\in\mathbf{Z}^4\otimes\Gamma^{6,22}\otimes\mathbf{N}^d$. In the component form we have: $m_iV_i^a\in\mathbf{Z}$, $m_i\vec{U}_i\in\Gamma^{6,22}$, and $m_iT_i^r\in\mathbf{N}$. Here we have combined the components U_i^I, $I=1,...,6$, and U_i^A, $A=1,...,22-2d$, along with $2d$ null entries into a single $(6,22)$ dimensional vector $\vec{U}_i=(U_i^I||U_i^A|0^{2d})$. Later we will use the dot product of such vectors defined in the same way as for two vectors in $\Gamma^{6,22}$.

To describe all the elements of the orbifold group G, it is convenient to introduce the set of generating vectors$\{V_i\}$ such that $\overline{\alpha V}=\mathbf{0}$ if and only if $\alpha_i\equiv 0$. Here $\mathbf{0}$ is the null vector

$$\mathbf{0}=(0^4|0^6||0^{22-2d}|0^d) \ . \tag{39}$$

Also, $\alpha V \equiv \sum_i \alpha_i V_i$ (the summation is defined as, say, $(V_i + V_j)^r = T_i^r + T_j^r$), α_i being integers that take values from 0 to $m_i - 1$. The overbar notation is defined as follows: $(\overline{\alpha V})^r = (\alpha V)^r \pmod 1$ and $0 \leq (\overline{\alpha V})^r < 1$.

Now we can identify the sectors of the model. They are labeled by the vectors $\overline{\alpha V}$, and in a given sector $\overline{\alpha V}$ the monodromies of the string degrees of freedom are given by $\Phi(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = g(\overline{\alpha V})\Phi(z,\overline{z})g^{-1}(\overline{\alpha V})$. It is clear from the supercurrent constraint (11), (19) that the sectors with $\alpha s \in \mathbf{Z}$ give rise to the space-time bosons, whereas those with $\alpha s \in \mathbf{Z} + 1/2$ give rise to the space-time fermions.

Note that a sector described by the vector

$$V_0 = ((-\frac{1}{2})^4 | 0^6 || 0^{22-2d} | 0^d) \tag{40}$$

is always present in any orbifold model. As before, the V_0 sector is the Ramond sector of the Narain model, whereas $\mathbf{0}$ sector is the Neveu-Schwarz sector. Without loss of generality we can set $s_i = 0$ for $i \neq 0$ (Recall that $s_i \equiv V_i^0$.) Then the space-time bosons come from the sectors $\overline{\alpha V}$ with $\alpha_0 = 0$, and the space-time fermions come from the sectors $\overline{\alpha V}$ with $\alpha_0 = 1$.

We finish this subsection by noting that we have not modified the action of the orbifold on the right-moving degrees of freedom, so that all the rules concerning the supercurrent construction, in particular, the constraints (17) and (19) remain unchanged and must be satisfied by a consistent orbifold model with left-moving twists within this framework just as in the case of models with no left-moving twists. The spectrum generating formula and the left-moving energy are different in the case of left-moving twists, and we turn to these issues next.

B. Orbifold Rules

We will not attempt to give the most general rules in this subsection as the rules for constructing a rather large class of orbifolds can be found in Ref [3]. Rather, we will confine our attention to the case where we have only one generating vector, which we choose to be V_1, with $T_1^r \neq 0$. For all the other vectors V_i, $i \neq 1$, we require that $T_i^r \equiv 0$. (Thus, without loss of generality, we can assume that $T_1^r \neq 0$ for all values of r.) Moreover, we only consider the cases where m_1 is a prime number.

Next, we start from a Narain model with the momenta of the internal bosons spanning an even self-dual Lorentzian lattice $\Gamma^{6,22}$. We introduce a set of generating vectors $\{V_i\}$ with the above properties. This set includes the V_0 vector, and also the V_1 vector with a left-moving twist. Here we assume that the lattice $\Gamma^{6,22}$ possesses $\mathbf{Z}_{m_1}$ symmetry generated by the twist part of the orbifold group element $g(V_1)$. Next, we find the structure constants k_{ij} that satisfy the following constraints

$$k_{ij} + k_{ji} = V_i \cdot V_j \pmod 1 ~, \quad i \neq j ~, \tag{41}$$

$$k_{ii} + k_{i0} + s_i - t_i - \frac{1}{2} V_i \cdot V_i = 0 \pmod 1 ~, \tag{42}$$

$$k_{ij} m_j = 0 \pmod 1 ~. \tag{43}$$

Note that there is no summation over the repeated indices here. The dot product of two vectors is defined as

$$\begin{aligned} V_i \cdot V_j &= -\sum_a V_i^a V_j^a + \vec{U}_i \cdot \vec{U}_j \\ &= -\sum_a V_i^a V_j^a - \sum_I U_i^I \cdot U_j^I + \sum_A U_i^A \cdot U_j^A \ . \end{aligned} \tag{44}$$

Also, we have introduced the following notation:

$$t_i \equiv \frac{1}{2} \sum_r T_i^r (1 - T_i^r) \ . \tag{45}$$

Note that $t_i = 0$ for $i \neq 1$.

Let $I(V_1)$ be the sublattice of $\Gamma^{6,22}$ invariant under the twist part of the orbifold group element $g(V_1)$, and let $\tilde{I}(V_1)$ be the lattice dual to $I(V_1)$. Then for the model to be consistent (in particular, for level-matching dictated by modular invariance) we must have

$$m_1 \vec{P}^2 \in \mathbf{Z} \text{ for all } \vec{P} \in \tilde{I}(V_1) \ . \tag{46}$$

Furthermore, for the sake of simplicity we will confine our attention to models with only one fixed point, so that we will require the following constraint to be satisfied:

$$\prod_r [2 \sin(\pi T_1^r)] = \sqrt{\mathrm{Vol}(I(V_1))} \ . \tag{47}$$

Here $\mathrm{Vol}(I(V_1))$ is the volume (or, equivalently, the determinant of the metric) of the lattice $I(V_1)$. (Note that Eq. (47) can be relaxed so that the r.h.s. also contains a factor which is some integer power of m_1. Then this factor is nothing but the number of fixed points for the twist given by (T_1^r). Here we only consider the models with one fixed point as we already mentioned.)

Now we turn to describing the sectors of the theory. They are labeled by $\overline{\alpha V}$. Let us start with the sectors with $\alpha_1 = 0$. To describe the vertex operators of states in these sectors, we will need to distinguish two different cases. They arise as follows. Note that there are two types of momenta $P \in \Gamma^{6,22}$: those that belong to the invariant sublattice $I(V_1)$, and those that do not. Let us consider the latter type first. Thus consider a set of momentum vectors $N(V_1) \subset \Gamma^{6,22}$ such that if $P \in N(V_1)$, then $P \notin I(V_1)$. Let $|P\rangle$ be the corresponding momentum states. It is clear that we can always decompose P into $P = P^\perp + P^\parallel$, where $P^\parallel \in I(V_1)$, and $P^\perp \cdot P' = 0$ for all $P' \in I(V_1)$. Then $|P\rangle = |P^\parallel\rangle \otimes |P^\perp\rangle$. Let $N^*(V_1) \subset N(V_1)$ be a set of momenta spanned by all $P^\perp$, and let $\mathcal{H}$ be the corresponding Hilbert space, *i.e.*, $\mathcal{H}$ is spanned by states $|P^\perp\rangle$, $P^\perp \in N^*(V_1)$. This space can be represented as $\mathcal{H} = \otimes_{\ell=0}^{m_1-1} \mathcal{H}_\ell$, where $\mathcal{H}_\ell$ is spanned by the states of the form

$$|P^\perp; \ell\rangle = \frac{1}{\sqrt{m_1}} \sum_{k=0}^{m_1-1} \exp(-2\pi i k \ell / m_1) g^k |P^\perp\rangle \ , \tag{48}$$

where $g = \exp(-2\pi i \sum_r T_1^r J^r)$, and J^r is the angular momentum operator (that acts on the momenta) for the complex boson Φ^r (or, equivalently, for the real bosons $\phi^{21-2d+2r}$ and $\phi^{22-2d+2r}$, and in this language J^r is the generator of $SO(2)$ rotations in the plane of these bosons). Note that $g|P^\perp;\ell\rangle = \exp(2\pi i\ell/m_1)|P^\perp;\ell\rangle$. Later, we will identify ℓ as part of a discrete charge (D-charge).

In the sectors with $\alpha_1 = 0$ there are two kinds of vertex operators. The first kind are momentum states (in the covariant formalism):

$$V(z,\overline{z}) = \tilde{V}(z,\overline{z}) \exp(i\sum_a H_a\rho_a(\overline{z}) + i\sum_I Q^I\phi^I(\overline{z}) + i\sum_A Q^A\varphi^A(z)) \exp(ik_\mu\mathcal{X}^\mu(z,\overline{z})) \ , \quad (49)$$

with $\vec{\mathcal{Q}} = (Q^I||Q^A) \in I(V_1)$. (Note that $\vec{\mathcal{Q}}$ is a $(6, 22-2d)$ dimensional vector.) Here $\tilde{V}(z,\overline{z})$ is a combination of derivatives of $\mathcal{X}^\mu$, ρ_a, ϕ^I, φ^A and Φ^r (*i.e.*, this is the corresponding oscillator excitation contribution), ghosts, certain cocycles, and the normal ordering is implicit here. Let us combine the H-charges H_a, Q-charges Q^I, and the gauge charges Q^A into a $(10, 22-2d)$ dimensional momentum vector $P_{\overline{\alpha V}}$. Then the physical states are those that satisfy the following spectrum generating formula:

$$V_i \cdot P_{\overline{\alpha V}} + \frac{\delta_{i1}}{m_1} D \equiv V_i \cdot P_{\overline{\alpha V}} + \sum_r T_i^r N^r = s_i + \sum_j k_{ij}\alpha_j \ (\text{mod } 1) \ , \quad (50)$$

and the momenta $P_{\overline{\alpha V}} \in \mathbf{Z}^4 \otimes I(V_1) + \alpha V$. Thus, for example, $H_a \in \mathbf{Z} + (\overline{\alpha V})^a$, and $\vec{\mathcal{Q}} \in I(V_1) + \alpha U$. Here we have introduced the boson number operators N^r for the bosons Φ^r. It can be expressed in terms of the occupation number operators s^r and $\tilde{s}^r$ for these bosons: $N^r_{\overline{\alpha V}} = \sum_q^\infty (s^r_q - \tilde{s}^r_q)$. The discrete D-charge is defined by the above equation. The origin of this D-charge (defined modulus m_1) is the $\mathbf{Z}_{m_1}$ twist acting on the d complex bosons $\partial\Phi^r$. This, however, is not the most general form of D as it has contributions only from the oscillators. We will give the general form of D in a moment.

The vertex operators (in the covariant formalism) for the second kind of states have the same form as (49), but now, in addition to the derivatives of $\mathcal{X}^\mu$, ρ_a, ϕ^I, φ^A and Φ^r, ghosts and cocycles, $\tilde{V}(z,\overline{z})$ contains also the vertex operator for a state $|P^\perp_{\overline{\alpha V}};\ell\rangle$. The latter can be written as

$$\frac{1}{\sqrt{m_1}} \sum_{k=0}^{m_1-1} \exp(-2\pi ik\ell/m_1) g^k \exp(i\sum_r [q^r(\Phi^r)^\dagger(z) + (q^r)^*\Phi^r(z)]) \ , \quad (51)$$

where q^r is the (complex) momentum of the Φ^r boson (and the $2d$ real components of all d momenta q^r give the momentum vector $P^\perp_{\overline{\alpha V}}$). The physical states are those that satisfy the following spectrum generating formula:

$$V_i \cdot P^{||}_{\overline{\alpha V}} + \frac{\delta_{i1}}{m_1} D \equiv V_i \cdot P^{||}_{\overline{\alpha V}} + \sum_r T_i^r N^r_{\overline{\alpha V}} + \frac{\delta_{i1}}{m_1}\ell = s_i + \sum_j k_{ij}\alpha_j \ (\text{mod } 1) \ . \quad (52)$$

Here we have combined the charges H_a, Q^I and Q^A into a single vector $P^{||}_{\overline{\alpha V}}$. The lattice momenta in this sector are given by $P_{\overline{\alpha V}} = P^{||}_{\overline{\alpha V}} + P^\perp_{\overline{\alpha V}}$.

The general form of the D-charge is given by:

$$D \equiv \ell + m_1 \sum_r T_1^r N^r_{\overline{\alpha V}} \quad (\text{mod } m_1) \quad (53)$$

which has contributions from both the lattice momentum and the oscillators. We recover the previous case if we set $\ell = 0$ (*i.e.* $P_{\overline{\alpha V}} \in I(V_1)$).

Finally, the vertex operators for states in the sectors $\overline{\alpha V}$ with $\alpha_1 \neq 0$ read:

$$
\begin{aligned}
V(z,\overline{z}) = & \tilde{V}(z,\overline{z})\sigma_{\overline{\alpha V}}(z) \times \\
& \exp(i\sum_a H_a \rho_a(\overline{z}) + i\sum_I Q^I \phi^I(\overline{z}) + i\sum_A Q^A \varphi^A(z)) \exp(ik_\mu \mathcal{X}^\mu(z,\overline{z})) \ ,
\end{aligned} \tag{54}
$$

where $\tilde{V}(z,\overline{z})$ is as defined in (49) and $\sigma_{\overline{\alpha V}}(z)$ is a vertex operator for the twisted ground state with conformal dimension $\sum_r \frac{1}{2}(\overline{\alpha V})^r(1-(\overline{\alpha V})^r)$. The physical states are those that satisfy the following spectrum generating formula:

$$
V_i \cdot P_{\overline{\alpha V}} = s_i + \sum_j k_{ij}\alpha_j \ (\mathrm{mod}\ 1) \ , \quad i \neq 1 \ . \tag{55}
$$

Here $P_{\overline{\alpha V}} \in \mathbf{Z}^4 \otimes \tilde{I}(V_1) + \overline{\alpha V}$. (Here we just take the momentum part of $\overline{\alpha V}$.) Note that here we are not imposing the constraint with respect to the vector V_1 as the latter is automatically satisfied in these sectors.

The above spectrum generating formulas give both on- and off-shell states. The on-shell states must satisfy the additional constraint that the left- and right-moving energies be equal. In the $\overline{\alpha V}$ sector they are given by

$$
\begin{aligned}
E^L_{\overline{\alpha V}} = & -1 + \sum_{q=1}^{\infty}(q\sum_\mu m^\mu_q + q\sum_A n^A_q \\
& + \sum_r [(q + (\overline{\alpha V})^r - 1)s_q + (q - (\overline{\alpha V})^r)\tilde{s}_q]) + \frac{1}{2}(P^L_{\overline{\alpha V}})^2 \ ,
\end{aligned} \tag{56}
$$

$$
E^R_{\overline{\alpha V}} = -\frac{1}{2} + \sum_{q=1}^{\infty} q(\sum_\mu \tilde{m}^\mu_q + \sum_I n^I_q + \sum_a k^a_q) + \frac{1}{2}(P^R_{\overline{\alpha V}})^2 \ , \tag{57}
$$

where m^μ_q, $\tilde{m}^\mu_q$, n^A_q, n^I_q and k^a_q are the oscillator occupation numbers for the real bosons $\mathcal{X}^\mu_L$, $\mathcal{X}^\mu_R$, φ^A, ϕ^I and ρ_a, respectively. Also, we note that $(P^L_{\overline{\alpha V}})^2 = (\vec{\mathcal{Q}}^L)^2$, and $(P^R_{\overline{\alpha V}})^2 = (\vec{\mathcal{Q}}^R)^2 + \sum_a H_a^2$, where $\vec{\mathcal{Q}}^L$ and $\vec{\mathcal{Q}}^R$ are the left- and right-moving parts of the vector $\vec{\mathcal{Q}}$.

C. Scattering Amplitudes

From the scattering amplitudes $\mathcal{A}_M$ (27), where the external space-time momenta are set to zero, we can read off the terms in the superpotential. For a non-zero coupling, the corresponding scattering amplitude must be present. Having taken care of the ghost factors, this means that all the gauge and discrete symmetries must be satisfied. In particular, a necessary condition is that the sum of all the lattice momenta must be zero in $\mathcal{A}_M$ (27). These selection rules impose very tight constraints on the possible terms that can appear in the superpotential of any model.
(1) *Gauge Invariance.* This local symmetry must be conserved. In the 4-dimensional $N = 1$ heterotic string models, the gauge symmetries come from left-movers only. We will refer to these as G-charges. Note that picture-changing does not touch the G-charges so each state carries well-defined gauge quantum numbers.
(2) *H- and Q-Charge Conservation.* They must be conserved in the scattering amplitude. Note that the supercurrent carries terms with different H- and Q-charges. Because of picture

changing, H- and Q- charges are not global charges even though they must be conserved exactly in $\mathcal{A}_M$. This is consistent with the fact that string theory has no global continuous symmetries [11]. Point group and space group selection rules follow from these conservation laws.

(3) *Invariance under Discrete Symmetries.* In higher level models, there is a discrete gauge charge (or quantum number) associated with the twist field. We shall call this a D-charge. As we shall see, in the models we are interested in, the selection rule coming from this discrete symmetry is subsumed in the other selection rules.

In principle, we can calculate the couplings in the superpotential explicitly, since we know all the vertex operators. However, in this paper, we shall consider only the selection rules coming from the conservation of the above G-, Q-, H- and D-charges. Note that space-time superpartners have identical G-, Q- and D-charges, but different H-charges.

V. SIMPLE LEVEL-1 EXAMPLES

In this section we use some simple examples to illustrate our rules for constructing level-1 models and calculating scattering amplitudes. After a detailed discussion of the familiar asymmetric $\mathbf{Z}_3$ orbifold, we will briefly discuss the original symmetric Z-orbifold.

Consider the Narain model with $\Gamma^{6,22} = \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^8 \otimes \Gamma^8$, where Γ^8 is the E_8 root lattice, whereas $\Gamma^{2,2} = \{(p_R||p_L)\}$ with $p_L, p_R \in \tilde{\Gamma}^2$ ($SU(3)$ weight lattice) and $p_L - p_R \in \Gamma^2$ ($SU(3)$ root lattice). This model has $N = 4$ SUSY and $SU(3) \otimes SU(3) \otimes SU(3) \otimes E_8 \otimes E_8$ gauge symmetry (counting only the gauge bosons coming from the left-moving sector of the string; the right-moving sector contributes 6 $U(1)$ vector bosons that are part of the $N = 4$ supergravity multiplet). Let us write the corresponding V_0 vector as

$$V_0 = ((-\frac{1}{2})^4|0^3||0^3|0^8|0^8) \ . \tag{58}$$

Here the first four entries stand for the right-moving complex world-sheet fermions ψ^a, $a = 0, 1, 2, 3$, next three stand for 3 right-moving complex bosons X^a, $a = 1, 2, 3$ (each corresponding to a factor $\Gamma^{2,2}$). Double vertical line separates the right-movers from the left-movers. The first three left-moving entries correspond to the left-moving counterparts of the X^a bosons. The next $8 + 8$ entries correspond to the $E_8 \otimes E_8$ lattice which we will describe using 16 real bosons, and we will use the $\mathrm{Spin}(16)/\mathbf{Z}_2$ basis for each of the E_8 factors (*i.e.*, the E_8 roots are described as those of $SO(16)$ plus 128 additional roots in the corresponding irrep of $SO(16)$; thus, for example, $(+1, 0, -1, 0, 0, 0, 0, 0)$ is a root of $SO(16)$, and the roots in the 128 irrep are those with all eight entries equal $+1/2$ or $-1/2$ and total number of positive signs being even).

(1) An asymmetric Z-orbifold model.

Consider the following asymmetric $\mathbf{Z}_3$ orbifold of the $SU(3)^3 \otimes (E_8)^2$ Narain model described above:

$$V_1 = (0(-\frac{1}{3})^3|(e_1/3)^3||0^3|\frac{1}{3}\frac{1}{3}\frac{2}{3}0^5|0^8) \ . \tag{59}$$

This is the original asymmetric Z-orbifold model of Ref [2]. It has $N = 1$ SUSY and $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$ gauge symmetry. Here e_1 is a simple root of $SU(3)$. We denote the other simple root by e_2, and define $e_3 \equiv -e_1 - e_2$. Note that $e_\alpha \cdot e_\beta = 2$ for $\alpha = \beta$ and -1 for $\alpha \neq \beta$ ($\alpha, \beta = 1, 2, 3$). Let ϕ^a, $a = 1, 2, 3$, be the two-component real bosonic fields corresponding to each of the three $SU(3)$ lattices in the model with $e_1/3$ shifts. The $i\partial X^a$ in the supercurrent T_F is given by

$$i\partial X^a = \frac{1}{\sqrt{3}} \sum_\alpha e^{-ie_\alpha \phi^a} c(-e_\alpha) \ . \tag{60}$$

It is easy to see that the *triplet* constraint is satisfied:

$$V_1^a + U_1 \cdot Q = -\frac{1}{3} + \frac{e_1}{3} \cdot (-e_\alpha) = 0 \pmod 1 \tag{61}$$

We can now compute the massless spectrum of the model. First, let us choose $k_{00} = 0$ (note that there is a freedom in choosing k_{00} to be 0 or 1/2 which only reflects in flipping the chirality of the states). The other structure constants are fixed: $k_{10} = 1/2$, $k_{01} = 0$ and $k_{11} = 1/3$. It is convenient to define

$$P_{\alpha V} = (H_0, \cdots, H_3, \mathbf{Q}_1^R, \mathbf{Q}_2^R, \mathbf{Q}_3^R | \mathbf{Q}_1^L, \mathbf{Q}_2^L, \mathbf{Q}_3^L, Q_4, \cdots, Q_{19}) \tag{62}$$

where $\mathbf{Q}^{L,R}$ are charges under $U(1)^2$ of $SU(3)$ and all other Q's are $U(1)$ charges.

First, consider the untwisted sector, the spectrum generating formulae read:

$$\frac{1}{2}(H_0 + H_1 + H_2 + H_3) = \frac{1}{2} \pmod 1 \tag{63}$$

$$\frac{1}{3}(Q_4 + Q_5 + 2Q_6) + \frac{1}{3}(H_1 + H_2 + H_3) - \frac{e_1}{3} \cdot (\mathbf{Q}_1^R + \mathbf{Q}_2^R + \mathbf{Q}_3^R) = 0 \pmod 1 \tag{64}$$

The first spectrum generating formula requires that one of the H-charges must be equal to 1. It then follows from the massless condition that $\mathbf{Q}^R = 0$ which implies that $\mathbf{Q}^L \in \Gamma^2$ ($SU(3)$ root lattice). The choice $a = 0$ gives rise to gauge bosons of $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$. For $a = 1, 2,$ or 3, the second spectrum generating formula gives $Q_4 = -1$ or $Q_5 = -1$ or $Q_6 = 1$. Therefore, $\mathbf{Q}^L = 0$ or else $(\mathbf{Q}^L)^2 \geq 2$ and the state is massive. Note that even though the orbifold group does not act on $\mathbf{Q}^L$, constraints such as $\mathbf{Q}^L - \mathbf{Q}^R \in \Gamma^2$ and massless condition restrict the possible values of $\mathbf{Q}^L$. The fields that survive the projection are χ_a which transform in the $(\mathbf{3}, \mathbf{27}, \mathbf{1})$ irrep of $SU(3) \otimes E_6 \otimes E_8$, and are neutral under the other $SU(3)^3$. The index a labels the choice of H-charges: $H = (1, 0, 0)$ for $a = 1$, $H = (0, 1, 0)$ for $a = 2$ and $H = (0, 0, 1)$ for $a = 3$.

We now turn to the twisted sectors. Note that in the -1 picture, $H_a \in \mathbf{Z} - \frac{1}{3}$ in V_1 sector while $H_a \in \mathbf{Z} + \frac{1}{3}$ in $2V_1$ sector. The possible choices of H_a which do not give rise to massive states are $H_a = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$ in V_1 sector and $H_a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in $2V_1$ sector. The left-handed chiral supermultiplets all come from $2V_1$ and $V_0 + 2V_1$ sectors. In $2V_1$ sector, the spectrum generating formulae give:

$$\frac{1}{2}(H_0 + H_1 + H_2 + H_3) = \frac{1}{2} \pmod 1 \tag{65}$$

$$\frac{1}{3}(Q_4 + Q_5 + 2Q_6) + \frac{1}{3}(H_1 + H_2 + H_3) - \frac{e_1}{3} \cdot (\mathbf{Q}_1^R + \mathbf{Q}_2^R + \mathbf{Q}_3^R) = \frac{2}{3} \pmod 1 \tag{66}$$

where $Q_4, Q_5 \in \mathbf{Z} - \frac{1}{3}$, $Q_6 \in \mathbf{Z} + \frac{1}{3}$ and $\mathbf{Q}^R \in \tilde{\Gamma}^2 - e_1/3$. For massless states, $\sum_j (\mathbf{Q}^R_j)^2 = 2/3$ and therefore, there are 27 choices of $\mathbf{Q}^R$:

$$\mathbf{Q}^R = \left(-\frac{e_\alpha}{3}, -\frac{e_\beta}{3}, -\frac{e_\gamma}{3}\right) \quad \text{for } \alpha, \beta, \gamma = 1, 2, 3 \tag{67}$$

The Q charges $\mathbf{Q}^L$ and $\mathbf{Q}^R$ are correlated by $p_L - p_R \in \Gamma^2$. For simplicity, let us consider for the moment only the first $SU(3)$. We have

$$\begin{array}{lll} & & \underline{\text{irrep of } SU(3)} \\ \alpha = 1 & \mathbf{Q}^L_1 = 0 & \mathbf{1} \\ \alpha = 2 & \mathbf{Q}^L_1 = \tilde{e}_2, -\tilde{e}_1, \tilde{e}_1 - \tilde{e}_2 & \mathbf{3} \\ \alpha = 3 & \mathbf{Q}^L_1 = -\tilde{e}_2, \tilde{e}_1, \tilde{e}_2 - \tilde{e}_1 & \overline{\mathbf{3}} \end{array} \tag{68}$$

where $\tilde{e}_1 = \frac{2}{3}e_1 + \frac{1}{3}e_2$ and $\tilde{e}_2 = \frac{1}{3}e_1 + \frac{2}{3}e_2$ are $SU(3)$ weights such that $\tilde{e}_i \cdot e_j = \delta_{ij}$. The conformal dimension of $\mathbf{3}$ (and $\overline{\mathbf{3}}$) of $SU(3)$ is $\frac{1}{2}(\mathbf{Q}^L_1)^2 = \frac{1}{3}$ as expected.

Massless states are created by conformal fields with total left-moving conformal dimension 1. Therefore, they must transform non-trivially under $SU(3)$ as well as other gauge group such as E_6. For instance, a field that transforms as $\mathbf{3}$ of $SU(3)$ and $\overline{\mathbf{27}}$ of E_6 (which has conformal dimension 2/3) has the right conformal dimension. It remains to check if the spectrum generating formulae are satisfied. To summarize, we have the following left-handed chiral supermultiplets from the twisted sectors:

- The twisted sector field $\overline{\chi}$.

This field transforms in the $(\mathbf{3}, \overline{\mathbf{27}}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ irrep of $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$. The Q-charges for this field read $(-e_1/3, -e_1/3, -e_1/3)$.

- The twisted sector fields $\chi_{A\pm}$.

The field χ_{A+} transforms in the $(\mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ irrep of $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$ with $\mathbf{x} = \mathbf{3}$, $\mathbf{y} = \mathbf{z} = \mathbf{1}$ for $A = 1$, $\mathbf{y} = \mathbf{3}$, $\mathbf{x} = \mathbf{z} = \mathbf{1}$ for $A = 2$, and $\mathbf{z} = \mathbf{3}$, $\mathbf{x} = \mathbf{y} = \mathbf{1}$ for $A = 3$. Similarly, the field χ_{A-} transforms in the $(\mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ irrep of $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$ with $\mathbf{x} = \overline{\mathbf{3}}$, $\mathbf{y} = \mathbf{z} = \mathbf{1}$ for $A = 1$, $\mathbf{y} = \overline{\mathbf{3}}$, $\mathbf{x} = \mathbf{z} = \mathbf{1}$ for $A = 2$, and $\mathbf{z} = \overline{\mathbf{3}}$, $\mathbf{x} = \mathbf{y} = \mathbf{1}$ for $A = 3$. The Q-charges for the fields χ_{A+} are given by $(-e_2/3, -e_1/3, -e_1/3)$ for $A = 1$, $(-e_1/3, -e_2/3, -e_1/3)$ for $A = 2$, and $(-e_1/3, -e_1/3, -e_2/3)$ for $A = 3$. Similarly, the Q-charges for the fields χ_{A-} are given by $(-e_3/3, -e_1/3, -e_1/3)$ for $A = 1$, $(-e_1/3, -e_3/3, -e_1/3)$ for $A = 2$, and $(-e_1/3, -e_1/3, -e_3/3)$ for $A = 3$.

- The twisted sector fields $T_{\mathbf{xyz}}$.

The field $T_{\mathbf{xyz}}$ transforms in the $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ irrep of $SU(3) \otimes E_6 \otimes E_8 \otimes SU(3)^3$ where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are irreps of $SU(3)$ such that only one of them is a singlet and the others can be either $\mathbf{3}$ or $\overline{\mathbf{3}}$. The Q-charges are correlated with the $SU(3)$ irrep as described above. For example, if $\mathbf{x} = \mathbf{1}, \mathbf{y} = \mathbf{z} = 3$, the Q-charges are given by $(-e_1/3, -e_2/3, -e_2/3)$.

The gauge quantum numbers as well as the Q- and H-charges of the massless spectrum are summarized in Table I.

Having described the bosonic supercurrent and the massless spectrum of the model, we are ready to calculate scattering amplitudes. Let us start with the three-point Yukawa interactions of the chiral families in $\mathbf{27}$ of E_6. The three-point Yukawa coupling $\chi_a \chi_b \chi_c$ is non-zero only for $a \neq b \neq c \neq a$ as only in this case are the H-charges conserved. For illustrative purposes let us consider this case in more detail. If $a = 1$, $b = 2$ and $c = 3$, then in the -1-picture the H-charges are $(+1, 0, 0)$, $(0, +1, 0)$ and $(0, 0, +1)$, respectively.

Two of the fields must be in the $-1/2$-picture, and the third one must be in the -1-picture, however. Let the χ_a and χ_b fields in the $-1/2$-picture. Their H-charges then are given by $(+1/2, -1/2, -1/2)$ and $(-1/2, +1/2, -1/2)$. Then we see that the total H-charge is $(+1/2, -1/2, -1/2) + (-1/2, +1/2, -1/2) + (0, 0, +1) = (0, 0, 0)$, and the process is allowed by the H-charge conservation. All the other quantum numbers are also conserved in this case. It is easy to see that in all the other cases $\chi_a \chi_b \chi_c$ Yukawa coupling vanishes.

Now consider the three-point couplings of the twisted sector fields. Naively, just from the conservation of gauge charges, one might expect that, say, the coupling $\chi_{A+}\chi_{B+}\chi_{C+}$ is non-zero for $A = B = C$. This is, however, not the case. Indeed, to have a singlet of the corresponding $SU(3)$ (note that each of these three fields transform in the irrep **3** of the corresponding $SU(3)$), we must completely antisymmetrize these fields. This means that we must take the completely antisymmetric combination of the three **27**s (note that each of these three fields transform in the irrep **27** of E_6). The latter antisymmetric product does not contain a singlet of E_6, so that the trace of this product vanishes. The same conclusion can be drawn from the Q-charge non-conservation. Note that the Q-charges in this case are not conserved. (Recall that for the three-point couplings there is no picture changing insertions of the supercurrent.) In fact, it is easy to see that for the same reason all three-point couplings involving at least one twisted sector field vanish. Some higher point couplings, however, are non-zero. For example, consider the following six-point couplings: $\overline{\chi\chi\chi}\chi_a\chi_b\chi_c$. It can be easily checked that H-, Q- and G-charge conservations in this case imply that this coupling is non-zero if and only if $a \neq b \neq c \neq a$. This is a typical situation for asymmetric orbifolds: lower point couplings typically vanish because of the Q-charge non-conservation, whereas there are (usually) somewhat higher point couplings that are non-zero. This observation will play an important role for the superpotentials of the three-family grand unified string theories which we will discuss in the subsequent sections.

For illustrative purpose, we give the lowest order non-vanishing terms in the superpotential of this asymmetric $\mathbf{Z}_3$ orbifold:

$$W = (\lambda_1 + \lambda_2 \overline{\chi}^3) \sum_{a\neq b\neq c\neq a} \chi_a \chi_b \chi_c + \dots \ . \tag{69}$$

In passing, we remark that the above model was originally constructed in the twist basis:

$$V_1 = (0(-\frac{1}{3})^3|\theta^3||0^3|\frac{1}{3}\frac{1}{3}\frac{2}{3}0^5|0^8) \ . \tag{70}$$

Here θ denotes a $2\pi/3$ rotation in the corresponding $SU(3)$ lattice. The above generating vector defines the same asymmetric Z_3 orbifold model. The relation between the complex bosons $i\partial X^a$ in the twist formalism and the real bosons ϕ^a in the shift formalism is given in Eq.(60).

(2) A symmetric Z-orbifold model.

Consider the following $\mathbf{Z}_3$ orbifold of the above Narain model:

$$V_1 = (0(-\frac{1}{3})^3|(e_1/3)^3||(e_1/3)^3|\frac{1}{3}\frac{1}{3}\frac{2}{3}0^5|0^8) \ . \tag{71}$$

This model has $N = 1$ SUSY, $U(1)^6 \otimes SU(3) \otimes E_6 \otimes E_8$ gauge group, and 36 chiral families of fermions in the **27** of E_6. Nine of these come from the untwisted sector, and the other 27 from the twisted sector. This orbifold model is nothing but the original Z-orbifold model of Ref [1] at the special values of the moduli of the compactification torus. This is precisely the reason for the enhanced $U(1)^6$ gauge symmetry. Since the group actions are identical on the right-movers for these symmetric and asymmetric orbifolds, we can use the same supercurrent T_F.

Let us fix the structure constants of the model. They are the same as in the asymmetric case except for $k_{11} = -1/3$. We have the following left-handed chiral supermultiplets:

• The untwisted sector fields $S_{aA\alpha}$.

The fields $S_{aA\alpha}$ are $SU(3) \otimes E_6 \otimes E_8$ singlets that are charged under $U(1)^6$. Thus, for $A = 1$ the $U(1)$ charges are given by $(e_\alpha, 0, 0)$, for $A = 2$ the $U(1)$ charges are given by $(0, e_\alpha, 0)$, and for $A = 3$ the $U(1)$ charges are given by $(0, 0, e_\alpha)$. There is no correlation between the gauge quantum numbers and the index a. The latter is related to the H-charges (here we only give $H_{1,2,3}$ charges in the -1 picture; the corresponding H-charges for Q_L left-handed SUSY generator are given by $(-1/2, -1/2, -1/2)$). For $a = 1$ they are $(+1, 0, 0)$, for $a = 2$ they are $(0, +1, 0)$, and for $a = 3$ they are $(0, 0, +1)$.

• The untwisted sector fields χ_a.

These are the same as in the asymmetric case.

• The twisted sector fields $\chi_{\alpha\beta\gamma}$.

The 27 fields $\chi_{\alpha\beta\gamma}$ transform in the irrep $(\mathbf{1}, \mathbf{27}, \mathbf{1})$ of $SU(3) \otimes E_6 \otimes E_8$. Their $U(1)^6$ charges are given by $(-e_\alpha/3, -e_\beta/3, -e_\gamma/3)$. Their Q-charges are the same as their $U(1)^6$ charges in the spectrum picture. For illustrative purposes we give the part of the vertex operator for the field $\chi_{\alpha\beta\gamma}$ that corresponds to the right-moving internal bosons:

$$\exp(-i[e_\alpha \cdot \phi^1(\overline{z}) + e_\beta \cdot \phi^2(\overline{z}) + e_\gamma \cdot \phi^3(\overline{z})]/3)\ c((e_\alpha, e_\beta, e_\gamma)) \ . \tag{72}$$

• The twisted sector fields $T_{A\alpha\beta\gamma}$.

The 81 fields $T_{A\alpha\beta\gamma}$ transform in the $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1})$ irrep of $SU(3) \otimes E_6 \otimes E_8$. Their Q-charges are given by $(-e_\alpha/3, -e_\beta/3, -e_\gamma/3)$. Their $U(1)^6$ charges are given by $(+2e_\alpha/3, -e_\beta/3, -e_\gamma/3)$ for $A = 1$, $(-e_\alpha/3, +2e_\beta/3, -e_\gamma/3)$ for $A = 2$, and $(-e_\alpha/3, -e_\beta/3, +2e_\gamma/3)$ for $A = 3$. Note that in the twisted sector all the left-handed fermions have H-charges $(-1/6, -1/6, -1/6)$, whereas their bosonic superpartners have H-charges $(+1/3, +1/3, +1/3)$.

We are now ready to calculate scattering amplitudes in this model. Let us start with the three-point Yukawa interactions of the chiral families in $\mathbf{27}$ of E_6. Note that terms like $\chi_a \chi_{\alpha\beta\gamma} \chi_{\alpha'\beta'\gamma'}$ and $\chi_a \chi_b \chi_{\alpha\beta\gamma}$ are not allowed by H-charge and also G-charge conservation. The three-point Yukawa coupling $\chi_a \chi_b \chi_c$ is the same as in the asymmetric case and is non-zero only for $a \neq b \neq c \neq a$.

Next, we turn to the scattering $\chi_{\alpha\beta\gamma} \chi_{\alpha'\beta'\gamma'} \chi_{\alpha''\beta''\gamma''}$. The G-charge and Q-charge conservation (which in this case give the same selection rules as the Q-charges for these fields are the same as the $U(1)^6$ G-charges) give the following selection rules for the scattering: $\alpha \neq \alpha' \neq \alpha'' \neq \alpha$, and similarly for the β and γ indices.

We will not give all the couplings for this model. We will finish our discussion here by considering the couplings of the type $S_{aA\delta} \chi_{\alpha\beta\gamma} \chi_{\alpha'\beta'\gamma'} \chi_{\alpha''\beta''\gamma''}$. Since all the other couplings are similar, for definiteness let us consider the case $a = 1$ and $A = 1$. Let us take $S_{aA\delta}$ to be in the 0-picture, two other fields in the $-1/2$-picture, and the last one in the -1-picture. Note that for the H-charges to be conserved, we must have $H_a = 0$ for the field $S_{aA\delta}$ in the 0-picture. This means that the term in the OPE $T_F S_{aA\delta}$ that can possibly contribute into this scattering must be of the form $\sim \exp(-i\rho_1)\sum_\alpha \exp(ie_\alpha \cdot \phi^1)$. Then the Q-charge and G-charge conservation together tell us that the above four-point coupling is non-zero if and only if $\alpha=\alpha'=\alpha''=\delta$, $\beta\neq\beta'\neq\beta''\neq\beta$, and $\gamma\neq\gamma'\neq\gamma''\neq\gamma$.

Let us compare our results with those of Ref [4]. To do so let us first note that this Z-orbifold is a factorized orbifold, and it is convenient to carry out the discussion on the example of a $\mathbf{Z}_3$ orbifold of one complex boson. Then 27 families of chiral fermions in the twisted sector come from $3\times3\times3$ fixed points. Thus, within the first set of 3, fixed points in our notation are labeled by α. So we will discuss the couplings only for one index α, and just point out that the other two indices β and γ obey similar selection rules. We see that the Yukawa couplings in our case are non-zero only for $\alpha\neq\alpha'\neq\alpha''\neq\alpha$. All the other Yukawa couplings are zero. In Ref [4] there are also additional non-zero Yukawa couplings with $\alpha=\alpha'=\alpha''$ (although all the others, with say, $\alpha=\alpha'\neq\alpha''$, are zero). The reason why we do not have the latter couplings is because we are considering a special point in the moduli space with enhanced gauge symmetry, and the G-charge conservation forbids these couplings. Note that if we move away from this point of enhanced gauge symmetry, and consider generic points as in Ref [4], we will account for additional non-vanishing couplings. This can be seen from the four-point couplings $S_{aA\delta}\chi_{\alpha\beta\gamma}\chi_{\alpha'\beta'\gamma'}\chi_{\alpha''\beta''\gamma''}$. As we move away from the special point in the moduli space we break the $U(1)^2$ (for one complex boson) gauge symmetry. In terms of effective field theory this corresponds to giving vevs to the fields $S_{aA\delta}$. Then effectively we generate three-point Yukawa couplings for $\alpha=\alpha'=\alpha''$, but the couplings with, say, $\alpha=\alpha'\neq\alpha''$, remain zero, because there were no corresponding higher point couplings in the superpotential to begin with. The latter was due to the Q-charge non-conservation. On the other hand, note that the absence of the couplings of the $\alpha=\alpha'\neq\alpha''$ type in the orbifold language is due to the orbifold space-group selection rules. Thus, the Q-charge conservation has this space-group discrete symmetry encoded in it, just as H-charge conservation guarantees that the orbifold point-group selection rules are satisfied.

Again, the above model can be written in the twist basis. The following generating vector:

$$V_1=(0(-\frac{1}{3})^3|\theta^3||\theta^3|\frac{1}{3}\frac{1}{3}\frac{2}{3}0^5|0^8)\ . \tag{73}$$

produces the same symmetric $\mathbf{Z}_3$ orbifold as above.

VI. THREE-FAMILY E_6 MODEL

In this and the following sections, we describe the construction of some of the three-family grand unified string theories previously presented in Refs [3]. In particular, we will rewrite these models in the bases where the supercurrent is in the bosonized form. As explained previously, scattering amplitudes are most easily calculated in such a representation. This section describes the E_6 model. It is organized as follows. In subsection A we set up the notation and describe the construction of the Narain model to be orbifolded. In subsection B we construct the unique three-family E_6 model by $\mathbf{Z}_6$ orbifolding this Narain model. This E_6 model can be realized as two different $\mathbf{Z}_6$ orbifolds which we refer to as $E1$ and $E2$ models. In subsection C we present the shift construction for $E1$ model and compute the superpotential using the bosonic supercurrent. The shift construction and the superpotential for $E2$ model will be given in subsection D. In section VII, we will discuss a three-family $SO(10)$ model. We use this example to illustrate how one can calculate the superpotential for models that do not admit a bosonic supercurrent. In section VIII, we will discuss two three-family $SU(6)$ models which are obtained by adding a $\mathbf{Z}_3$ Wilson line to the E_6 model. Finally, we will briefly comment on other three-family grand unified string models.

A. Narain Model

Consider the Narain model, which we will refer to as $N(1,1)$, with $\Gamma^{6,22} = \Gamma^{6,6} \otimes \Gamma^{16}$, where Γ^{16} is the Spin$(32)/\mathbf{Z}_2$ lattice, and $\Gamma^{6,6}$ is the lattice spanned by the vectors $(p_R||p_L)$, where $p_L, p_R \in \Gamma^6$ (E_6 weight lattice), and $p_L - p_R \in \Gamma^6$ (E_6 root lattice). Recall that, under $E_6 \supset SU(3)^3$,

$$\mathbf{27} = (\mathbf{3},\mathbf{3},\mathbf{1}) + (\overline{\mathbf{3}},\mathbf{1},\mathbf{3}) + (\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}}) \ , \tag{74}$$

$$\mathbf{78} = (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) + (\mathbf{3},\overline{\mathbf{3}},\mathbf{3}) + (\overline{\mathbf{3}},\mathbf{3},\overline{\mathbf{3}}) \ , \tag{75}$$

so we can write $\mathbf{p} \in \Gamma^6$ as

$$\mathbf{p} = (\mathbf{q}_1 + \lambda \mathbf{w}_1, \mathbf{q}_2 + \lambda \overline{\mathbf{w}}_2, \mathbf{q}_3 + \lambda \mathbf{w}_3) \ , \tag{76}$$

where $\mathbf{q}_i \in \Gamma^2$ ($SU(3)$ root lattice), $\mathbf{w}_i$ ($\overline{\mathbf{w}}_i$) is in the $\mathbf{3}$ ($\overline{\mathbf{3}}$) weight of $SU(3)$, and $\lambda = 0,\ 1,\ 2$.

Next, consider the model, which we will refer to as $N1(1,1)$, generated from the $N(1,1)$ model by adding the following Wilson lines:

$$\begin{aligned} V_1 &= (0^4|0^3||e_1/2\ 0\ 0|\mathbf{s}\ \mathbf{0}\ \mathbf{0}|\overline{S}) \ , \\ V_2 &= (0^4|0^3||e_2/2\ 0\ 0|\mathbf{0}\ \mathbf{s}\ \mathbf{0}|\overline{S}) \ . \end{aligned} \tag{77}$$

Here the first four entries correspond to the right-moving world-sheet fermions, the next three right-moving entries stand for the three right-moving complex bosons X^a, $a = 1,2,3$ (each corresponding to one of the three $SU(3)$s). The double vertical line separates the right-movers from the left-movers. The first three left-moving entries correspond to the left-moving counterparts of the X^a bosons. The remaining 16 left-moving world-sheet bosons generate the Spin$(32)/\mathbf{Z}_2$ lattice. The $SO(32)$ shifts are given in the $SO(10)^3 \otimes SO(2)$ basis. In this basis, $\mathbf{0}$ stands for the null vector, $\mathbf{v}(V)$ is the vector weight, whereas $\mathbf{s}(S)$ and $\overline{\mathbf{s}}(\overline{S})$ are the spinor and anti-spinor weights of $SO(10)(SO(2))$. (For $SO(2)$, $V = 1$, $S = 1/2$ and $\overline{S} = -1/2$). The untwisted sector provides gauge bosons of $SU(3)^2 \otimes U(1)^2 \otimes SO(10)^3 \otimes SO(2)$. There are additional gauge bosons from the new sectors. Recall that under $E_6 \supset SO(10) \otimes U(1)$,

$$\mathbf{78} = \mathbf{1}(0) + \mathbf{45}(0) + \mathbf{16}(3) + \overline{\mathbf{16}}(-3) \ . \tag{78}$$

It is easy to see that the V_1, V_2 and $V_1 + V_2$ sectors provide the necessary $\mathbf{16}(3)$ and $\overline{\mathbf{16}}(-3)$ gauge bosons to the three $SO(10)$'s respectively. The resulting Narain $N1(1,1)$ model has $N = 4$ SUSY and gauge group $SU(3)^2 \otimes (E_6)^3$ provided that we set $k_{10} = k_{20} = 0$. The permutation symmetry of the three E_6 factors should be clear from the above construction. Since there is only one $N = 4$ SUSY $SU(3)^2 \otimes (E_6)^3$ model in 4-dimensional heterotic string theory, this permutation symmetry may also be explicitly checked by looking at its one-loop modular invariant partition function.

B. E_6 Model: Twist Construction

Before we describe the $\mathbf{Z}_6$ asymmetric orbifold that leads to the three-family E_6 model, we will introduce some notation. By θ we will denote a $2\pi/3$ rotation of the corresponding complex (or, equivalently, two real) chiral world-sheet boson(s). Thus, θ is a $\mathbf{Z}_3$ twist. Similarly, by σ we will denote a π rotation of the corresponding complex chiral world-sheet boson. Thus, σ is a $\mathbf{Z}_2$ twist. By $\mathcal{P}$ we will denote the outer-automorphism of the three $SO(10)$s that arise in the breaking $SO(32) \supset SO(10)^3 \otimes SO(2)$. Note that $\mathcal{P}$ is a $\mathbf{Z}_3$ twist. Finally, by (p_1, p_2) we will denote the outer-automorphism of the corresponding two complex chiral world-sheet bosons. Note that (p_1, p_2) is a $\mathbf{Z}_2$ twist.

The E_6 model can be constructed by performing the following asymmetric $\mathbf{Z}_6$ orbifolds on the $N=4$ SUSY $SU(3)^2 \otimes (E_6)^3$ model (*i.e.*, $N1(1,1)$ model):

• The $E1$ model. Start from the $N1(1,1)$ model and perform the following twists:

$$\begin{aligned} T_3 &= (0(-1/3)^3|\theta,\theta,\theta||\theta,e_1/3,0|\mathcal{P}|2/3) \ , \\ T_2 &= (0\ (-1/2)^2\ 0|\sigma,p_1,p_2||0,e_1/2,e_1/2|0^{15}|0) \ . \end{aligned} \tag{79}$$

This model has $SU(2)_1 \otimes (E_6)_3 \otimes U(1)^3$ gauge symmetry. The massless spectrum of the $E1$ model is given in Table II. They are grouped according to where they come from, namely, the untwisted sector U, the $\mathbf{Z}_3$ twisted (*i.e.*, T_3 and $2T_3$) sector T3, the $\mathbf{Z}_6$ twisted (*i.e.*, T_3+T_2 and $2T_3+T_2$) sector T6, and $\mathbf{Z}_2$ twisted (*i.e.*, T_2) sector T2. Note that all particles have integer $U(1)$ charges. The normalization, or compactification radius r, of each left-moving world-sheet boson is given at the bottom of the table. The $U(1)$ charge of a particle with charge n contributes $n^2r^2/2$ to its conformal highest weight. That is, the corresponding part of the vertex operator has momentum nr.

• The $E2$ model. Start from the $N1(1,1)$ model and perform the following twists:

$$\begin{aligned} T_3 &= (0\ 0\ (-1/3)^2|0,\theta,\theta||\theta,e_1/3,0|\mathcal{P}|2/3) \ , \\ T_2 &= (0\ (-1/2)^2\ 0|\sigma,p_1,p_2||0,e_1/2,e_1/2|0^{15}|0) \ . \end{aligned} \tag{80}$$

This model has $SU(2)_1 \otimes (E_6)_3 \otimes U(1)^3$ gauge symmetry. The massless spectrum of the $E2$ model is given in Table II.

Note that the spin structures of the world-sheet fermions in the right-moving sector are fixed by the world-sheet supersymmetry consistency. The string consistency conditions impose tight constraints on the allowed twists. Using the approach given in Ref [3] one can check that both sets of twists presented above are consistent provided that the appropriate choices of the structure constants k_{ij} are made. It is then straightforward, but somewhat tedious, to work out the massless spectrum of the model. (Again, more details can be found in Ref [3]). We will give an alternative way of constructing this model in the following subsections, and working out the massless spectrum there is somewhat easier.

The models $E1$ and $E2$ have the same tree-level massless spectra. We will show that interactions on the two orbifolds are the same even though naively the fields in the two models seem to have different origins (in particular, the same states come from different twisted sectors of the two orbifolds). They are possibly a T-dual pair.

In the following subsections, we will give the shift representation for the twists given in models $E1$ and $E2$. To be able to do so it is necessary and sufficient, as we already discussed in the previous sections, that the corresponding Kac-Moody algebras $\mathcal{G}'_R$ are realized at level 1 and have central charge 6. This is indeed the case for the $E1$ and $E2$ models. Let us first consider the $E1$ model. The original Kac-Moody algebra $\mathcal{G}_R$ before orbifolding was $(E_6)_1$. The $\mathbf{Z}_3$ twist reduces it to $(SU(3)_1)^3$, whereas the $\mathbf{Z}_2$ twist breaks it to $SU(6)_1 \otimes SU(2)_1$. The combined action of the $\mathbf{Z}_3$ and $\mathbf{Z}_2$ twists, *i.e.*, the corresponding $\mathbf{Z}_6$ twist breaks $(E_6)_1$ to $\mathcal{G}'_R = [SU(2)_1 \otimes U(1)]^3$, which is realized at level one. Similarly, in the $E2$ model we have $\mathcal{G}'_R = (SU(2)_1)^4 \otimes U(1)^2$. Even though the right moving Kac-Moody algebra is not $SU(3)^3$ for both cases, we can always write the Kac-Moody charges in this basis. The translation of the charges in different bases are given in Table VII.

C. $E1$ Model: Shift Construction

Let us present the generating vectors in the shift formalism which produce the $E1$ model. Here V_1 and V_2 correspond to the T_3 and T_2 twists acting on the $N1(1,1)$ model:

$$\begin{aligned} V_1 &= (0(-1/3)^3|e_1/3, e_1/3, e_1/3||\theta, e_1/3, 0|\mathcal{P}|2/3)\ , \\ V_2 &= (0\ (-1/2)^2\ 0|e_1/2, 0, 0||0, e_1/2, e_1/2|0^{15}|0)\ . \end{aligned} \tag{81}$$

We may rewrite the vector V_1 in the following way. Consider the branching of $SO(32)$ to $SO(10)^3 \otimes SO(2)$. The three $SO(10)$s are permuted by the action of the $\mathbf{Z}_3$ outer-automorphism twist $\mathcal{P}$: $\phi^I_1 \to \phi^I_2 \to \phi^I_3 \to \phi^I_1$, where the real bosons ϕ^I_p, $I = 1, ..., 5$, correspond to the $p^{\rm th}$ $SO(10)$ subgroup, $p = 1, 2, 3$. We can define new bosons $\varphi^I \equiv \frac{1}{\sqrt{3}}(\phi^I_1 + \phi^I_2 + \phi^I_3)$; the other ten real bosons are complexified via the linear combinations $\Phi^I \equiv \frac{1}{\sqrt{3}}(\phi^I_1 + \omega\phi^I_2 + \omega^2\phi^I_3)$ and $(\Phi^I)^\dagger \equiv \frac{1}{\sqrt{3}}(\phi^I_1 + \omega^2\phi^I_2 + \omega\phi^I_3)$, where $\omega = \exp(2\pi i/3)$. Under $\mathcal{P}$, φ^I is invariant, while Φ^I ($(\Phi^I)^\dagger$) are eigenstates with eigenvalue ω^2 (ω), i.e., $\mathcal{P}$ is equivalent to a $\mathbf{Z}_3$ twist θ on each Φ^I. Finally, string consistency requires the inclusion of the 2/3 shift in the $SO(2)$ lattice of the boson η. This simply changes the radius of this left-moving world-sheet boson. So, in the Φ^I, φ^I and η basis, V_1 becomes

$$V_1 = (0(-1/3)^3|e_1/3, e_1/3, e_1/3||\theta, e_1/3, 0|\theta^5 0^5_r|(2/3)_r) \tag{82}$$

where the subscript r indicates real bosons.

Including the V_0 vector, the generating vectors for $E1$ model can then be written as:

$$\begin{aligned} V_0 &= ((-1/2)^4|0^6||0^{22}) \\ V_1 &= (0(-1/3)^3|e_1/3, e_1/3, e_1/3||(1/3)_t, e_1/3, 0|(1/3)^5_t 0^5_r|(2/3)_r)\ , \\ V_2 &= (0\ (-1/2)^2\ 0|e_1/2, 0, 0||0, e_1/2, e_1/2|0^{15}|0)\ . \end{aligned} \tag{83}$$

where the subscript t indicates that the corresponding complex boson is twisted.

The dot products $V_i \cdot V_j$ and the choice of the structure constants k_{ij} are given by:

$$V_i \cdot V_j = \begin{pmatrix} -1 & -1/2 & -1/2 \\ -1/2 & -1/3 & -1/3 \\ -1/2 & -1/3 & 0 \end{pmatrix}\ , \qquad k_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 2/3 & 1/2 \end{pmatrix}\ . \tag{84}$$

This defines a consistent string model, provided that a supercurrent satisfying (17) and (19) can be found.

1. Bosonic Supercurrent

Before we compute the spectrum and calculate the superpotential, it is important to construct the supercurrent. First of all, the supercurrent satisfying the constraints (17), (19) must exist in order for the model to be consistent. Furthermore, in calculating higher point couplings, picture-changing requires insertions of the supercurrent. In the following, we will construct the supercurrent for the $E1$ model in detail.

The currents ∂X^a can be written as linear combinations of vertex operators for the root generators of the original $\mathcal{G}_R = (E_6)_1$ Kac-Moody algebra. It is convenient to express the roots of E_6 in $SU(3)^3$ basis (Eq.(76)). As before, we define $\pm e_1, \pm e_2$ and $\pm e_3 = \mp(e_1 + e_2)$ as the roots of $SU(3)$. The weight vectors $\tilde{e}^1, \tilde{e}^2$ are defined by $\tilde{e}^i e_j = \delta^i_j$ and $\tilde{e}^0 = \tilde{e}^2 - \tilde{e}^1$. Therefore, the weight $\mathbf{3}$ of $SU(3)$ is represented by $\tilde{e}^2, -\tilde{e}^1, -\tilde{e}^0$. The triplet constraints (Eq.(19)) from the V_1 and V_2 vectors restrict the possible terms in ∂X^a. The currents ∂X^a and $\partial X^{b\dagger}$ must also obey the OPEs (Eq.(17)). A solution satisfying all the constraints is given by:

$$
\begin{aligned}
i\partial X^1 &= \frac{1}{\sqrt{12}} \left(J_1 + \sqrt{2} J_2 + \sqrt{3} J_3 + \sqrt{2} J_4 + J_5 + \sqrt{2} J_6 + J_7 \right) \ , \\
i\partial X^2 &= \frac{1}{\sqrt{12}} \left(L_1 - \sqrt{2} L_2 + \sqrt{3} L_3 - \sqrt{2} L_4 - L_5 - \sqrt{2} L_6 + L_7 \right) \ , \\
i\partial X^3 &= \frac{1}{\sqrt{6}} \left(K_1 - K_2 - K_3 + K_1^{'} - K_2^{'} + K_3^{'} \right) \ ,
\end{aligned} \tag{85}
$$

where $J_i = \exp(-ia_i\phi)c(-a_i)$, $L_i = \exp(-il_i\phi)c(-l_i)$, $K_i = \exp(-ik_i\phi)c(-k_i)$ and $K_i^{'} = \exp(-ik_i^{'}\phi)c(-k_i^{'})$. Here a_i, l_i, k_i and $k_i^{'}$ are roots of E_6 defined in Table IV. We will define the cocycle operators $c(K)$ in a moment.

Notice that out of the 72 roots of E_6, only 7 of them contribute to ∂X^1. It is easy to see that $a_1, \cdots, a_6$ form a set of simple roots of E_6 and $-a_7 = a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6$ is the highest root. (Note that the coefficients are simply the co-marks of E_6 simple roots). Similarly, $l_1, \cdots, l_6$ which contribute to ∂X^2 form another set of simple roots of E_6 with $-l_7$ being the highest root. Since the $\mathbf{Z}_2$ action (V_2 vector) does not act on ∂X^3 and ψ^3, ∂X^3 can be expressed in terms of the root generators of $SU(3)^2$. We see that k_1 and k_2 form a set of simple roots of $SU(3)$ and $-k_3$ is the highest root. Similarly for the $k_i^{'}$. The co-marks of the simple roots of E_6 and $SU(3)$ are also listed in Table IV. Other choices of the supercurrent are equivalent to this one (involving only a change of basis). In this sense, the supercurrent consistent with the twists-shifts of this model is unique.

Any weight vector K of E_6 can then be expressed in terms of the basis vectors $a_1, \cdots, a_6$, *i.e.*, $K = \sum_j n_j a_j$. We can define an ordered product of weight vectors K and $K^{'}$,

$$
K * K^{'} = \sum_{i>j} n_i n_j^{'} a_i \cdot a_j \tag{86}
$$

The cocycle operator and the cocycle structure constant can then be given by [12]:

$$
c(K) = (-1)^{\mathbf{p} * K} \tag{87}
$$

$$
\epsilon(K, K^{'}) = (-1)^{K * K^{'}} \tag{88}
$$

where $\mathbf{p}$ is the momentum operator, i.e., $\mathbf{p}|P\rangle = P|P\rangle$. The cocycles defined above depend crucially on our choice of the basis vectors.

The coefficient of each term in ∂X^a (*i.e.* $\sqrt{m_i/h}$ where m_i is the co-mark and $h = 1+\sum_i m_i^2$ is the Coxeter number) are determined up to a phase by the OPE of ∂X^1 with $\partial X^{1\dagger}$. The phases we have chosen in (85) ensures that the OPEs of ∂X^a and ∂X^b for $a \neq b$ are non-singular.

The supercurrent T_F (the internal part) is therefore a linear combination of 40 terms, each with different right-moving quantum numbers (given in Table IV):

$$
\begin{aligned}
T_F &= \frac{i}{2}\sum_{a=1}^{3}\psi^a\partial X^a + \text{h.c.} \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{12}}e^{iH_1\rho_1}\sum_{i=1}^{7}\sqrt{m_i}e^{-ia_i\phi}c(-a_i) + \frac{1}{\sqrt{12}}e^{iH_2\rho_2}\sum_{i=1}^{7}\sqrt{m_i}e^{-il_i\phi}c(-l_i)\right. \\
&\quad \left.+\frac{1}{\sqrt{6}}e^{iH_3\rho_3}\sum_{i=1}^{3}(e^{-ik_i\phi}c(-k_i)+e^{-ik_i'\phi}c(-k_i'))\right) + \text{h.c.}\ .
\end{aligned}
\tag{89}
$$

where m_i are the co-marks of the E_6 simple roots and $h = 12$ for E_6.

2. Spectrum

We are now ready to compute the massless spectrum. Let us recall what the left- and the right-moving quantum numbers are. For the left-movers, we have the H-charges $H_0, \cdots, H_3$, and the Q^R charges $(\mathbf{Q}_1^R, \mathbf{Q}_2^R, \mathbf{Q}_3^R)$ in the $SU(3)^3$ basis. The left-moving charges are: $SU(3)$ charges $\mathbf{Q}_2^L, \mathbf{Q}_3^L$, a $SO(10)$ charge $\mathbf{q}$, a $U(1)$ charge Q and a discrete charge D coming from the $\mathbf{Z}_3$ twist. Here, the charge Q is the $U(1)$ charge of the real boson which is shifted by $2/3$.

Let us first consider the untwisted sector. In the $\mathbf{0}$ sector, the spectrum generating formulae (50),(52) read:

$$
V_0\cdot N = \frac{1}{2}(H_0+H_1+H_2+H_3) = \frac{1}{2} \pmod 1 \tag{90}
$$

$$
\begin{aligned}
V_1\cdot N &= \frac{1}{3}(H_1+H_2+H_3) - \frac{e_1}{3}\cdot(\mathbf{Q}_1^R+\mathbf{Q}_2^R+\mathbf{Q}_3^R) \\
&\quad +\frac{e_1}{3}\cdot\mathbf{Q}_2^L + \frac{1}{3}D + \frac{2}{3}Q = 0 \pmod 1
\end{aligned}
\tag{91}
$$

$$
V_2\cdot N = \frac{1}{2}(H_1+H_2) - \frac{e_1}{2}\cdot\mathbf{Q}_1^R + \frac{e_1}{2}\cdot(\mathbf{Q}_2^L+\mathbf{Q}_3^L) = 0 \pmod 1 \tag{92}
$$

Here, the quantum number D is the eigenvalue of the twist operator on the 6 twisted complex bosons. Since the twist is $\mathbf{Z}_3$, $D = 0,1,2 \pmod 3$. For gauge bosons, $H_0 = 1$. The right-moving energy given by

$$
E_R = -\frac{1}{2} + \frac{1}{2}\sum_{j=0}^{3}H_j^2 + \frac{1}{2}\sum_{j=1}^{3}(\mathbf{Q}_j^R)^2 + \text{oscillators} \tag{93}
$$

implies that $\mathbf{Q}^R=(0,0,0)$. The original Narain model has gauge symmetry $SU(3)^2\otimes(E_6)^3$. It is easy to see from the last spectrum generating formula that all the $SU(3)^2$ roots are projected out except $\mathbf{Q}^L_3=\pm e_1$. As a result, $SU(3)^2$ is broken to $SU(2)\otimes U(1)^3$. On the other hand, $(E_6)^3$ is broken to the diagonal $(E_6)_3$ by the $\mathbf{Z}_3$ outer-automorphism. The resulting gauge group is therefore $SU(2)_1\otimes(E_6)_3\otimes U(1)^3$.

There are other massless states in the $\mathbf{0}$ sector with their fermionic partners in V_0 sector. To determine the chiralities of the fields, let us consider the V_0 sector. We choose the convention that $H_0=-1/2$ for left handed fermions. The spectrum generating formulae (50),(52) read:

$$V_0\cdot N=\frac{1}{2}(H_0+H_1+H_2+H_3)=\frac{1}{2}\pmod 1 \tag{94}$$

$$\begin{aligned}V_1\cdot N=&\frac{1}{3}(H_1+H_2+H_3)-\frac{e_1}{3}\cdot(\mathbf{Q}^R_1+\mathbf{Q}^R_2+\mathbf{Q}^R_3)\\&+\frac{e_1}{3}\cdot\mathbf{Q}^L_2+\frac{1}{3}D+\frac{2}{3}Q=\frac{1}{2}\pmod 1\end{aligned} \tag{95}$$

$$V_2\cdot N=\frac{1}{2}(H_1+H_2)-\frac{e_1}{2}\cdot\mathbf{Q}^R_1+\frac{e_1}{2}\cdot(\mathbf{Q}^L_2+\mathbf{Q}^L_3)=\frac{1}{2}\pmod 1 \tag{96}$$

Again, massless states have $\mathbf{Q}^R=(0,0,0)$. First, consider states that are charged only under $\mathbf{Q}^L_2$ and $\mathbf{Q}^L_3$ and hence do not carry D and Q quantum number. The left-moving energy is zero only if $(\mathbf{Q}^L_2)^2+(\mathbf{Q}^L_3)^2=2$. The spectrum generating formulae further restrict the choices to $(\mathbf{Q}^L_2,\mathbf{Q}^L_3)=(e_1,0)$ for the field U_0, $(e_2,0)$ for $U_{\pm+}$, and $(e_3,0)$ for $U_{\pm-}$. The definition of the fields as well as their H-charges are given in Table III.

Now we turn to states that has non-trivial D quantum numbers. The adjoint scalar Φ has $\mathbf{Q}^L_2=\mathbf{Q}^L_3=0$. The spectrum generating formulae give $(H_1,H_2,H_3)=(+1/2,+1/2,+1/2)$ and

$$D=2(1-Q)\pmod 3 \tag{97}$$

Upon decomposition of $\mathbf{78}$ of E_6 into $SO(10)\otimes U(1)$ representations:

$$\mathbf{78}=\mathbf{1}(0)+\mathbf{45}(0)+\mathbf{16}(+3)+\overline{\mathbf{16}}(-3)\ , \tag{98}$$

we notice that the component fields of $\mathbf{78}$ carry different $U(1)$ charges. ($Q=0,0,1/2,-1/2$ respectively). It then follows from (97) that the component fields of a E_6 multiplet can carry different discrete $\mathbf{Z}_3$ charges. We will return to this point when we discuss the superpotential.

Now, let us turn to the twisted sectors. In $\overline{V_0+V_1}$ sector,

$$\overline{V_0+V_1}=(-1/2\ (1/6)^3|(e_1/3)^3||(1/3)_t(e_1/3)0|(1/3)^5 0^5_r|(2/3)_r)\ , \tag{99}$$

the spectrum generating formulae (55) give:

$$V_0\cdot N=\frac{1}{2}(H_0+H_1+H_2+H_3)=\frac{1}{2}\pmod 1 \tag{100}$$

$$\begin{aligned}V_1\cdot N=&\frac{1}{3}(H_1+H_2+H_3)-\frac{e_1}{3}\cdot(\mathbf{Q}^R_1+\mathbf{Q}^R_2+\mathbf{Q}^R_3)\\&+\frac{e_1}{3}\cdot\mathbf{Q}^L_2+\frac{1}{3}D+\frac{2}{3}Q=\frac{1}{2}\pmod 1\end{aligned} \tag{101}$$

$$V_2\cdot N=\frac{1}{2}(H_1+H_2)-\frac{e_1}{2}\cdot\mathbf{Q}^R_1+\frac{e_1}{2}\cdot(\mathbf{Q}^L_2+\mathbf{Q}^L_3)=+\frac{1}{6}\pmod 1 \tag{102}$$

Since $H_a \in \mathbf{Z}+\frac{1}{6}$ for $a=1,2,3$, for massless states, $(H_1,H_2,H_3)=(1/6,1/6,1/6)$ and hence $H_0=1/2$. Therefore, this sector provides antiparticle states whereas $\overline{V_0+2V_1}$ provides the corresponding particles. The H-charges (and also Q-charges) of particle and antiparticle states have opposite sign and so it suffices to consider only $\overline{V_0+V_1}$ sector. In this sector, $\mathbf{Q}^R_j$ for $j=1,2,3$ and $\mathbf{Q}^L_2$ are shifted by $e_1/3$. The simplest choice $\mathbf{Q}^R=(e_1/3,e_1/3,e_1/3)$ and $(\mathbf{Q}^L_2,\mathbf{Q}_3)=(e_1/3,0)$ has $E_R=0$ and

$$\begin{aligned} E_L &= -1+\frac{1}{2}\sum_{r=1}^{6}T^r(1-T^r)+\frac{1}{2}\left((\mathbf{Q}^L_2)^2+(\mathbf{Q}^L_3)^2\right)+\frac{1}{2}(\frac{\mathbf{q}}{\sqrt{3}})^2+\frac{1}{2}(\frac{2}{3}+Q)^2+\ldots \\ &= -\frac{2}{9}+\frac{1}{6}\mathbf{q}^2+\frac{1}{2}(\frac{2}{3}+Q)^2+\ldots \end{aligned} \tag{103}$$

The only possible massless states come from $(\mathbf{q},Q)=(\mathbf{0},0)$, $(\mathbf{v},-1)$ and $(\mathbf{c},-1/2)$. Here $\mathbf{0}$, $\mathbf{v}$ and $\mathbf{c}$ are the singlet, vector and spinor representation of $SO(10)$ respectively, and together they form $\overline{\mathbf{27}}$ of E_6. It is easy to see that the spectrum generating formulae are satisfied.

This, however, is not the only choice of $(\mathbf{Q}^R,\mathbf{Q}^L)$ which gives rise to massless states. The appearance of the other massless states requires a careful explanation. Recall that the outer-automorphism $\theta-\mathcal{P}$ in (81) does not commute with the Wilson lines (77) and the combined action can be viewed as a non-Abelian orbifold corresponding to modding out the original Spin(32)/$\mathbf{Z}_2$ Narain model (which we refer to as $N(1,1)$ model) by the tetrahedral group $\mathcal{T}$ [13]. This group is generated by three elements Θ,R_1,R_2, where $\Theta^3=1$, $(R_1)^2=(R_2)^2=1$, and $R_2=\Theta R_1$. In our case, the Wilson lines V_1 and V_2 in (77) correspond to the elements R_1 and R_2, respectively, whereas the $\mathbf{Z}_3$ twist in (81) corresponds to the element Θ. The $\mathbf{Z}_2$ twist in (81) commutes with all the elements Θ,R_1,R_2 and will not be important for this discussion. The resulting $E1$ model is therefore a $\mathcal{T}\times\mathbf{Z}_2$ orbifold. The spectrum generating formulae are written in the basis in which the Wilson lines (77) are diagonal. Therefore, the $\mathbf{Z}_3$ action θ in (81) is represented as a twist but not a shift. However, in determining the $\mathbf{Q}^L$ charges, we can always go to the basis where the θ twist is replaced by a shift $\mathbf{Q}^L_1\to\mathbf{Q}^L_1+e_1/3$. (The Wilson lines in this basis are not diagonal). Notice that the conformal dimension of the momentum state $h=\frac{1}{2}(e_1/3)^2=1/9$ is the same as that of a $\mathbf{Z}_3$ twist field. We therefore have $\mathbf{Q}^R=p_R+(e_1/3,e_1/3,e_1/3)$ and $\mathbf{Q}^L=p_L+(e_1/3,e_1/3,0)$ where $p_R,p_L\in\tilde{\Gamma}^6$ (E_6 weight lattice), and $p_L-p_R\in\Gamma^6$ (E_6 root lattice). We can add weight vectors (p_R,p_L) to the above $(\mathbf{Q}^R,\mathbf{Q}^L)$ provided that the lengths $(\mathbf{Q}^R)^2$ and $(\mathbf{Q}^L_2)^2+(\mathbf{Q}^L_3)^2$ are preserved (hence $E_R=E_L=0$) and the spectrum generating formulae are satisified. It turns out there are 4 choices:

$$\begin{array}{llll} \chi_{++}: & \mathbf{Q}^R = (e_3/3,e_3/3,e_1/3) & \mathbf{Q}^L = & (e_3/3,e_3/3,0) \\ \chi_{-+}: & \phantom{\mathbf{Q}^R = }(e_2/3,e_1/3,e_3/3) & & (e_3/3,e_3/3,0) \\ \chi_{+-}: & \phantom{\mathbf{Q}^R = }(e_3/3,e_1/3,e_2/3) & & (e_2/3,e_2/3,0) \\ \chi_{--}: & \phantom{\mathbf{Q}^R = }(e_2/3,e_2/3,e_1/3) & & (e_2/3,e_2/3,0) \end{array} \tag{104}$$

For example, the $(\mathbf{Q}^R,\mathbf{Q}^L)$ charges of χ_{++} are obtained by taking $p_R=p_L=(-\tilde{e}^1,-\tilde{e}^1,0)$.

The other twisted sector fields coming from $T6$ and $T2$ sectors can be worked out in a similar way. The Q- and H- charges for the $E1$ model are summarized in Table III. The discrete D-charges are given in Table VIII.

3. Superpotential

Now we are ready to calculate scattering amplitudes and deduce the non-vanishing terms in the superpotential for $E1$ model. Here we are only interested in whether a given term is vanishing or not, not in the actual numerical values of these couplings. That is, we are only concerned with the selection rules for the scattering. Since the coefficients $\xi^a(Q)$ in the supercurrent are completely determined, calculating the actual numerical values of the coupling is rather straightforward, although certain couplings might be tedious to work out. We will return to such a calculation in future publications if there is a necessity for it to be done for phenomenological purposes.

Let us recall our selection rules. First, all the terms in the superpotential must be gauge singlets. This is dictated by gauge invariance (G-charge conservation). Next, the H-charges must be conserved. Here one should be careful as the H-charges are altered by picture changing. Similarly, the Q-charges also must be conserved. These are also altered by picture changing, and Table IV provides the quantum numbers of terms that are relevant when doing the picture changing by inserting the supercurrent T_F. Finally, the discrete charges (D-charges) coming from the left-moving coset $\mathcal{C}_L$ (recall that this is a level-3 model with reduced rank) must also be conserved. As we will see, D-charge conservation is guaranteed by G-, H-, and Q-charge conservation. We will see this explicitly in an $SO(10)$ model in the next section.

Let us first focus on gauge symmetries which are common for the $E1$ and $E2$ models. Gauge invariance implies that some of the fields cannot enter into the superpotential by themselves. For instance, D_+ must couple with D_- due to the $SU(2)$ symmetry. Similarly, the only fields that are charged under the first $U(1)$ are $\tilde{U}_\pm$ and $\tilde{\chi}_\pm$. Therefore, they must enter into the superpotential in the invariant combinations $\tilde{U}_+\tilde{U}_-$, $\tilde{\chi}_+\tilde{\chi}_-$ and $\tilde{U}_\pm\tilde{\chi}^3_\mp$. In the same fashion, $U_{i\pm}$ and $\chi_{i\pm}$, $i=+,-$, are bound to form the invariant combinations $U_{i+}U_{j-}$, $\chi_{i+}\chi_{j-}$ and $U_{i\pm}\chi_{j_1\mp}\chi_{j_2\mp}\chi_{j_3\mp}$ as a consequence of invariance under the first $U(1)$.

Naively, the three-point couplings $\chi_0\chi_{i\pm}\chi_{j\mp}$ are allowed by G-charge conservation. However, the Q-charges are not conserved. Therefore, there is no three-point coupling in the superpotential. Nevertheless, conservation of Q- and H-charges allows a limited set of four-point couplings: $\chi_0\chi_{++}\chi_{--}\Phi$ and $\chi_0\chi_{+-}\chi_{-+}\Phi$. To see this, let us consider the H- and Q-charges of χ_0, χ_{++} and χ_{--} in the -1, $-1/2$ and $-1/2$ picture respectively. The total H-charge is then $(1/3,1/3,1/3)+(-1/6,-1/6,-1/6)+(-1/6,-1/6,-1/6)=(0,0,0)$, whereas the total Q-charge is $(-e_1/3,-e_1/3,-e_1/3)+(-e_3/3,-e_3/3,-e_1/3)+(-e_2/3,-e_2/3,-e_1/3)=(0,0,-e_1)$. The adjoint Φ has $H=(0,0,1)$ and $Q=(0,0,0)$ in the -1 picture. Using the supercurrent we have constructed in the previous section, it is easy to see that Φ in the 0-picture contains a term with $H=(0,0,0)$ and $Q=(0,0,e_1)$. Hence, the four-point coupling $\chi_0\chi_{++}\chi_{--}\Phi$ is allowed. Similarly, one can show that $\chi_0\chi_{+-}\chi_{-+}\Phi$ is allowed but other four-point couplings such as $\chi_0\chi_{++}\chi_{+-}\Phi$ and $\chi_0\chi_{-+}\chi_{--}$ are forbidden. This analysis has been performed in the $SU(3)^3$ basis, but we could have equally successfully used, say, the $SU(2)^3\otimes U(1)^3$ basis for the $E1$ model, and obtained the same result. Notice that Φ carries H-charges in the -1 picture and so Φ^n by itself for any integer n is not allowed in the superpotential due to H-charge conservation. Hence, the adjoint Φ is a moduli. However, in the 0-picture, Φ contains terms with $H=(0,0,0)$ and $Q=(0,e_\alpha,0)$ or $(0,0,e_\alpha)$. Therefore, Φ^3 contains terms with no H- and Q-charges when all the Φ are in the

0-picture. An immediate consequence is that if w is an allowable N-point coupling ($N \geq 3$) in the superpotential, then $w\Phi^{3n}$ is also allowable.

Let us now turn to the D-charge. As mentioned before, in the branching $E_6 \supset SO(10) \otimes U(1)$, the component fields (*i.e.*, different representations of $SO(10)$) of a E_6 multiplet carry different D-charges. They are summarized in Table VIII. This D-charge is a $\mathbf{Z}_3$ symmetry charge and so it must be conserved modulus 3. The origin of this D-charge is the S_3 permutation symmetry of the three $(E_6)^3 \supset SO(10)^3 \otimes U(1)^3$. A priori, D-charge conservation imposes an extra constraint on the superpotential. However, it turns out that this D-charge conservation is subsumed in the other selection rules. To see this, consider the example of $\chi_0\chi_{++}\chi_{--}\Phi$ coupling. In terms of $SO(10) \otimes U(1)$ representations,

$$\begin{aligned}\chi_0\chi_{++}\chi_{--}\Phi = \quad & [Q_0Q_{++}H_{--} + Q_0Q_{--}H_{++} + Q_{++}Q_{--}H_0 \\ & + H_0H_{++}S_{--} + H_0H_{--}S_{++} + H_{++}H_{--}S_0](\Phi + \phi) \\ & + [Q_0H_{++}S_{--} + Q_0H_{--}S_{++} + Q_{++}H_0S_{--} + Q_{++}H_{--}S_0 \\ & + Q_{--}H_0S_{++} + Q_{--}H_{++}S_0 + Q_0Q_{++}Q_{--}]Q \\ & + [Q_0H_{++}H_{--} + Q_{++}H_0H_{--} + Q_{--}H_0H_{++}]\overline{Q}\end{aligned} \tag{105}$$

From Table VIII, one can easily check that the D-charge is conserved for every term on the right-handed side. The same is true for other terms in the superpotential.

It is, therefore, useful to organize the fields and the invariant combinations according to their $U(1)^3$ G-charges, Q-charges and H-charges when deducing the selection rules. Instead of trying to write down the most general form of the superpotential, for illustrative purposes we will give the lowest order non-vanishing couplings here:

$$\begin{aligned}W = \quad & \lambda_1(\Phi^3, D_+D_-)\chi_0(\chi_{++}\chi_{--} + \chi_{+-}\chi_{-+})\Phi \\ & + \lambda_2(\Phi^3, D_+D_-)\tilde{\chi}_+\tilde{\chi}_-(\chi_{++}\chi_{--} + \chi_{+-}\chi_{-+}) \\ & + \lambda_3(\Phi^3, D_+D_-)U_0(U_{++}U_{--} + U_{+-}U_{-+}) \\ & + \lambda_4(\Phi^3, D_+D_-)[\chi^3_{++}U_{--} + \chi^3_{+-}U_{-+} + \chi^3_{-+}U_{+-} + \chi^3_{--}U_{++}]\Phi^2 \\ & + \lambda_5(\Phi^3, D_+D_-)(\chi_0)^3U_0D_+D_-\Phi^2 \\ & + \lambda_6(\Phi^3, D_+D_-)[(\tilde{\chi}_+)^3\tilde{U}_- + \tilde{\chi}^3_-\tilde{U}_+]D_+D_-\Phi + \ldots\ ,\end{aligned} \tag{106}$$

where traces over the irreps of the gauge group are implicit. Here, λ_k are certain polynomials of their respective arguments such that $\lambda_k(0) \neq 0$, *i.e.*,

$$\lambda_k(\Phi^3, D_+D_-) = \sum_{m,n} \lambda_{kmn}\Phi^{3m}(D_+D_-)^n \tag{107}$$

It is clear that all discrete (local) symmetries impose stringent constraints on the couplings. This is clearly an important property of string theory.

D. $E2$ Model: Shift Construction

After a detail discussion of the $E1$ model, our discussion of the $E2$ model will be brief. The generating vectors in the shift formalism which produce the $E2$ model are:

$$V_1 = (0\ 0\ (-1/3)^2|0, e_1/3, e_1/3||\theta, e_1/3, 0|\mathcal{P}|2/3)\ ,$$
$$V_2 = (0\ (-1/2)^2\ 0|e_1/2, 0, 0||0, e_1/2, e_1/2|0^{15}|0)\ . \tag{108}$$

As in $E1$ model, we can rewrite the generating vectors as follows:

$$V_1 = (0\ 0\ (-1/3)^2|0, e_1/3, e_1/3||(1/3)_t, e_1/3, 0|(1/3)_t^5 0_r^5|(2/3)_r)\ ,$$
$$V_2 = (0\ (-1/2)^2\ 0|e_1/2, 0, 0||0, e_1/2, e_1/2|0^{15}|0)\ . \tag{109}$$

The structure constants can then be determined: $k_{00} = k_{10} = k_{02} = 0$, $k_{12} = k_{20} = k_{22} = 1/2$, and $k_{01} = k_{11} = k_{21} = 2/3$.

The currents $i\partial X^a$ that satisfy the triplet constraint (19) and the OPEs (17) are given by

$$\begin{aligned} i\partial X^1 &= \frac{1}{2}(J_1 + J_2 + J_3 - J_4)\ , \\ i\partial X^2 &= \frac{1}{\sqrt{6}}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6)\ , \\ i\partial X^3 &= \frac{1}{\sqrt{6}}(K_1 + K_2 + K_3 - K'_1 + K'_2 + K'_3)\ , \end{aligned} \tag{110}$$

where $J_i = \exp(-i\alpha_i\phi)c(-\alpha_i)$, $L_i = \exp(-i\beta_i\phi)c(-\beta_i)$, $K_i = \exp(-ik_i\phi)c(-k_i)$ and $K'_i = \exp(-ik'_i\phi)c(-k'_i)$. Here, α_i, β_i, k_i and k'_i are the roots of E_6 defined in Table VI. It is easy to see that $\alpha_1, \ldots, \alpha_4$ are simple roots of $SU(2)^4$ whereas $\beta_1, \ldots, \beta_5$ and $-\beta_6$ form the simple roots and the highest weight of $SU(6)$. The roots of E_6 that enter into $i\partial X^3$ are the same as that in $E1$ model because again the $\mathbf{Z}_2$ action does not act on ∂X^3 and ψ^3 and so $i\partial X^3$ can be expressed in terms of the root generators of $SU(3)^2$.

The massless spectrum of the $E2$ model can be obtained in a similar way. The G-, Q- and H-charges for the $E2$ model are given in Table V. Notice that $E1$ and $E2$ models have the same massless spectrum with some of the twisted and untwisted sector fields interchanged. By examining some of the lowest order non-vaishing couplings, one sees that the $E1$ and $E2$ models have the same tree level superpotential.

VII. $SO(10)$ MODEL

We have seen in the previous section that for models which admit a bosonic supercurrent, scattering amplitudes become straightforward to compute. However, there are other three-family grand unified models classified in [3] which do not admit a basis where the supercurrent can be bosonized. It is, nonetheless, possible to deduce the superpotential in an indirect way. We will use the $T1(1,1)$ and $T2(1,1)$ models of Ref [3] to illustrate how this can be done.

Let us begin with the construction of the models. Start from $N(1,1)$, *i.e.*, the $N = 4$ SUSY Spin$(32)/\mathbf{Z}_2$ model, and add the following Wilson lines:

$$V_1 = (0^4|0\ e_1/2\ e_1/2||e_1/2\ 0\ 0|\mathbf{s}\ \mathbf{0}\ \mathbf{0}|\overline{S})\ , \tag{111}$$
$$V_2 = (0^4|0\ e_2/2\ e_2/2||e_2/2\ 0\ 0|\mathbf{0}\ \mathbf{s}\ \mathbf{0}|\overline{S})\ . \tag{112}$$

The resulting model, which we will refer to as $N2(1,1)$, is a $N=4$ SUSY model with gauge symmetry $SU(3)^2 \otimes SO(10)^3 \otimes U(1)^3$ provided that we set $k_{10}=k_{20}=0$.

• The $T1(1,1)$ model. Start from the $N2(1,1)$ model and perform the same twists T_3 and T_2 as in the $E1$ model (79). This model has $SU(2)_1 \otimes SO(10)_3 \otimes U(1)^4$ gauge symmetry. The massless spectrum of the $T1(1,1)$ model is given in Table IX.

• The $T2(1,1)$ model. Start from the $N2(1,1)$ Narain model and perform the same twists T_3 and T_2 as in the $E2$ model (80). This model has $SU(2)_1 \otimes SO(10)_3 \otimes U(1)^4$ gauge symmetry. The massless spectrum of the $T2(1,1)$ model is given in Table IX.

The models $T1(1,1)$ and $T2(1,1)$ are possibly a T-dual pair just as $E1$ and $E2$ are. We will compute the superpotentials for these models, and show that they are the same.

Let us briefly discuss the underlying conformal field theory, or the Kac-Moody algebra $\mathcal{G}'_R$, for the $T1(1,1)$ and $T2(1,1)$ models. Without going into details, we simply state the results: $\mathcal{G}'_R = SU(2)_3 \otimes U(1)$ for the $T1(1,1)$ model, and $\mathcal{G}'_R = SU(2)_3 \otimes SU(2)_1$ for the $T2(1,1)$ model. Thus, what happens is that the first three $SU(2)_1$s in both models (meaning starting from $E1$ and $E2$ and arriving at $T1(1,1)$ and $T2(1,1)$, respectively) are broken down to their diagonal subgroup $SU(2)_3$. The last two $U(1)s$ are completely broken. The first $U(1)$ in the $E1$ model, and the last $SU(2)_1$ in the $E2$ model are untouched. In the process of this breaking one encounters additional cosets $\mathcal{C}_R$, which give rise to certain discrete symmetries.

Therefore, the $SO(10)$ models $T1(1,1)$ and $T2(1,1)$ do not admit bosonization of the supercurrent because the corresponding right-moving Kac-Moody algebras $\mathcal{G}'_R$ have reduced rank which is less than 6. There is, however, a way to deduce their superpotentials which is due to the fact that they are connected by (classically) flat directions to the E_6 model.

Comparing the massless spectrum of the $SO(10)$ model to that of the E_6 model, given in Table II, we see that the two spectra are very similar. In particular, the spectrum of the $SO(10)$ model is the same as that of the E_6 model with a non-zero vev of the adjoint Φ of $(E_6)_3$ such that the $(E_6)_3$ is broken down to $SO(10)_3 \otimes U(1)$ (the last $U(1)$ in the first and second columns of Table IX). Therefore, the $\mathbf{27}$ of $(E_6)_3$ branches into $\mathbf{16}(-1)+\mathbf{10}(+2)+\mathbf{1}(-4)$ of $SO(10)_3 \otimes U(1)$. The adjoint $\mathbf{78}$ branches into $\mathbf{45}(0)+\mathbf{1}(0)+\mathbf{16}(+3)+\overline{\mathbf{16}}(-3)$. Note that the $\mathbf{45}(0)$ and $\mathbf{1}(0)$ are present in the $SO(10)$ model, the $\mathbf{16}(+3)$ and $\overline{\mathbf{16}}(-3)$ are missing, however. This is due to the fact that the latter have been eaten by the corresponding gauge bosons of $(E_6)_3$ in the super-Higgs mechanism. Thus, in the effective field theory language, the $SO(10)$ model is the same as the E_6 model with the adjoint vev turned on. There is a subtlety here. The two models are equivalent only if we also turn on the vev of the singlet ϕ in the $SO(10)$ model correspondingly. This can be seen from the fact that once the adjoint of $(E_6)_3$ acquires a vev, there is an effective three-point Yukawa coupling in the $SO(10)$ model which has the form $H_0 Q_+ Q_-$ (here we define $Q_\pm$ and $Q'_\pm$ in the same way as $\chi_\pm$ and $\chi'_\pm$). In the $SO(10)$ model without the vev of ϕ turned on there are only four-point couplings $H_0 Q_+ Q_- \phi$ and $H_0 Q_+ Q_- \Phi$. Thus, to get the three-point coupling, we have to turn on the vev of ϕ (whereas turning on the vev of Φ in the $SO(10)$ model would break the $SO(10)$ symmetry further, say, to $SU(5) \otimes U(1)$). From this discussion it is clear how to get the superpotential for the $SO(10)$ model from that of the E_6 model. One simply starts from the latter and turns on the vev of the adjoint of $(E_6)_3$. In practice, just to get the selection rules for the $SO(10)$ model, simply replace χs by $(Q+H+S)$s (and similarly for $\tilde{\chi}s$), and Φ by $\Phi+\phi$. Let us write down the first few lowest order terms in the superpotential:

$$
\begin{aligned}
W = \quad & \lambda_1((\Phi+\phi')^3, D_+D_-)\,[Q_0(Q_{++}H_{--}+Q_{+-}H_{-+}+Q_{-+}H_{+-}+Q_{--}H_{++}) \\
&+ H_0(Q_{++}Q_{--}+Q_{+-}Q_{-+}) + H_0(H_{++}S_{--}+H_{+-}S_{-+}+H_{-+}S_{+-}+H_{--}S_{++}) \\
&+ S_0(H_{++}H_{--}+H_{+-}H_{-+})]\,(\Phi+\phi') \\
&+ \lambda_2((\Phi+\phi')^3, D_+D_-)\,\Big[\tilde{Q}_+\tilde{Q}_-(Q_{++}Q_{--}+Q_{+-}Q_{-+}) \\
&+ \tilde{Q}_+\tilde{Q}_-(H_{++}S_{--}+H_{+-}S_{-+}+H_{-+}S_{+-}+H_{--}S_{++}) \\
&+ (\tilde{Q}_+\tilde{S}_-+\tilde{Q}_-\tilde{S}_+)(Q_{++}S_{--}+Q_{+-}S_{-+}+Q_{-+}S_{+-}+Q_{--}S_{++}) \\
&+ (\tilde{H}_+\tilde{S}_-+\tilde{H}_-\tilde{S}_+)(H_{++}S_{--}+H_{+-}S_{-+}+H_{-+}S_{+-}+H_{--}S_{++}) \\
&+ (\tilde{H}_+\tilde{S}_-+\tilde{H}_-\tilde{S}_+)(Q_{++}Q_{--}+Q_{+-}Q_{-+}) \\
&+ (\tilde{Q}_+\tilde{H}_-+\tilde{Q}_-\tilde{H}_+)(Q_{++}H_{--}+Q_{+-}H_{-+}+Q_{-+}H_{+-}+Q_{--}H_{++}) \\
&+ \tilde{H}_+\tilde{H}_-(H_{++}H_{--}+H_{+-}H_{-+}) + \tilde{S}_+\tilde{S}_-(S_{++}S_{--}+S_{+-}S_{-+})\Big] \\
&+ \lambda_3((\Phi+\phi')^3, D_+D_-)U_0(U_{++}U_{--}+U_{+-}U_{-+}) \\
&+ \lambda_4((\Phi+\phi')^3, D_+D_-)\,\Big[(Q^2_{++}H_{++}+H^2_{++}S_{++})U_{--}+(Q^2_{+-}H_{+-}+H^2_{+-}S_{+-})U_{-+} \\
&+ (Q^2_{-+}H_{-+}+H^2_{-+}S_{-+})U_{+-}+(Q^2_{--}H_{--}+H^2_{--}S_{--})U_{++}\Big]\,(\Phi+\phi')^2+\ldots\,, \qquad (113)
\end{aligned}
$$

where traces over the irreps of the gauge group are implicit and the Clebsch-Gordan coefficients are of order 1. The field ϕ' is defined to be

$$
\phi' = \phi + \langle\phi\rangle \qquad (114)
$$

Here, λ_k are certain polynomials of their respective arguments such that $\lambda_k(0)\neq 0$, *i.e.*,

$$
\lambda_k((\Phi+\phi')^3, D_+D_-) = \sum_{m,n}\lambda_{kmn}(\Phi+\phi')^{3m}(D_+D_-)^n \qquad (115)
$$

VIII. $SU(6)$ MODELS

In this section we give the construction of two three-family $SU(6)$ models. Both of them can be obtained from the E_6 models discussed in section VI by adding the following Wilson line:

$$
V_3 = (0^4|0^3||0,\tilde{e}^2,\tilde{e}^2|(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3})^5|0)\ . \qquad (116)
$$

Since the Wilson line V_3 does not act on the right-movers, the supercurrent is the same as that of the original E_6 model.

• The $S1$ model. Add the V_3 Wilson line to the set (81) and set $k_{13}=0$ or $k_{13}=1/3$ (both choices give the same model). This model has $SU(3)_1\otimes SU(6)_3\otimes U(1)^3$ gauge symmetry. The Q- and H-charges of the massless spectrum are given in Table X.

• The $S2$ model. Add the V_3 Wilson line to the set (81) and set $k_{13}=2/3$. This model has $SU(2)_1\otimes SU(2)_1\otimes SU(6)_3\otimes U(1)^3$ gauge symmetry. The Q- and H-charges of the massless spectrum are given in Table XI.

• The $S3$ model. Add the V_3 Wilson line to the set (108) and set $k_{13}=0$ or $k_{13}=1/3$ (both choices give the same model). This model has $SU(3)_1\otimes SU(6)_3\otimes U(1)^3$ gauge symmetry.

The Q- and H-charges of the massless spectrum are given in Table XII.

• The $S4$ model. Add the V_3 Wilson line to the set (108) and set $k_{13} = 2/3$. This model has $SU(2)_1 \otimes SU(2)_1 \otimes SU(6)_3 \otimes U(1)^3$ gauge symmetry. The Q- and H-charges of the massless spectrum are given in Table XIII.

The models $S1$ and $S3$ have the same tree-level massless spectra. We will show that interactions are also the same. These two models are possibly T-dual to each other. Similarly, $S2$ and $S4$ are a possible T-dual pair.

A. Superpotential For $SU(6)$ Model $S1 = S3$

.

Next we give the superpotential for the $S1 = S3$ model. We will be brief here as the techniques involved in deducing the selection rules should be clear by now after we have given the example of the E_6 model.

The superpotential for the $SU(3)_1 \otimes SU(6)_3 \otimes U(1)^3$ model ($S1 = S3$) reads:

$$
\begin{aligned}
W = \quad & \lambda_1(\Phi,\phi)U_0(U_{++}U_{--} + U_{+-}U_{-+}) + \lambda_2(\Phi,\phi)T_0(\tilde{T}_+U_{-+} + \tilde{T}_-U_{++}) \\
& + \lambda_3(\Phi,\phi)\tilde{T}_0T_-U_0 + \lambda_4(\Phi,\phi)\tilde{T}_0T_0\tilde{U}_- \\
& + \lambda_5(\Phi,\phi)\sum_{A,B,C} y_{ABC}[U_{++}(F^A_+F^B_+F^C_- + \frac{1}{3}F^A_-F^B_-F^C_-) + U_{-+}(F^A_+F^B_-F^C_- + \frac{1}{3}F^A_+F^B_+F^C_+)] \\
& + [\lambda_6(\Phi,\phi)\Phi^2 + \lambda_7(\Phi,\phi)\Phi\phi + \lambda_8(\Phi,\phi)\phi^2] \\
& \qquad \times \sum_A \{U_{++}[(F^A_-)^3 + F^A_-(F^A_+)^2] + U_{-+}[(F^A_+)^3 + F^A_+(F^A_-)^2]\} \\
& + \lambda_9(\Phi,\phi)\sum_A [F^A_+\tilde{S}^A_- + F^A_-\tilde{S}^A_+]\tilde{S}^A_0 \\
& + [\lambda_{10}(\Phi,\phi)\Phi + \lambda_{11}(\Phi,\phi)\phi]\sum_{A,B,C} y_{ABC}(F^A_+\tilde{S}^B_- + F^A_-\tilde{S}^B_+)\tilde{S}^C_0 \\
& + \lambda_{12}(\Phi,\phi)\tilde{U}_+T_+\tilde{T}_0\sum_{A,B,C} y_{ABC}\tilde{F}^A\tilde{F}^B\tilde{F}^C \\
& + [\lambda_{13}(\Phi,\phi)\Phi + \lambda_{14}(\Phi,\phi)\phi]\tilde{U}_+T_+\tilde{T}_0\sum_A (\tilde{F}^A)^3 \\
& + \lambda_{15}(\Phi,\phi)\sum_{A,B} z_{AB}\tilde{F}^AS^B(F^A_+\tilde{S}^B_- + F^A_-\tilde{S}^B_+ + F^B_+\tilde{S}^A_- + F^B_-\tilde{S}^A_+) \\
& + [\lambda_{16}(\Phi,\phi)\Phi + \lambda_{17}(\Phi,\phi)\phi]\sum_{A,B,C} y_{ABC}\tilde{F}^AS^A(F^B_+\tilde{S}^C_- + F^B_-\tilde{S}^C_+) \\
& + [\lambda_{18}(\Phi,\phi)\Phi^2 + \lambda_{19}(\Phi,\phi)\Phi\phi + \lambda_{20}(\Phi,\phi)\phi^2]\sum_{A,B,C} y_{ABC}[F^A_+\tilde{S}^A_- + F^A_-\tilde{S}^A_+]\tilde{F}^BS^C \\
& + [\lambda_{21}(\Phi,\phi)\Phi^3 + \lambda_{22}(\Phi,\phi)\Phi^2\phi + \lambda_{23}(\Phi,\phi)\Phi\phi^2 + \lambda_{24}(\Phi,\phi)\phi^3] \\
& \qquad \times \sum_A [F^A_+\tilde{S}^A_- + F^A_-\tilde{S}^A_+]\tilde{F}^AS^A \\
& + \lambda_{25}(\Phi,\phi)\sum_{A,B,C} y_{ABC}[(F^A_+)^2 + (F^A_-)^2]S^BS^C(U_{++}U_{+-} + U_{-+}U_{--}) \\
& + \lambda_{26}(\Phi,\phi)\sum_{A,B,C} y_{ABC}F^A_+F^A_-S^BS^C(U_{++}U_{--} + U_{+-}U_{-+})
\end{aligned}
$$

$$
\begin{aligned}
&+[\lambda_{27}(\Phi,\phi)\Phi+\lambda_{28}(\Phi,\phi)\phi]\sum_{A,B} z_{AB}F_+^AF_+^BS^AS^B(U_{++}U_{+-}+U_{-+}U_{--})\\
&+[\lambda_{29}(\Phi,\phi)\Phi^2+\lambda_{30}(\Phi,\phi)\Phi\phi+\lambda_{31}(\Phi,\phi)\phi^2]\\
&\qquad\times\sum_{A,B,C} y_{ABC}F_+^AF_+^B(S^C)^2(U_{++}U_{+-}+U_{-+}U_{--})\\
&+[\lambda_{32}(\Phi,\phi)\Phi+\lambda_{33}(\Phi,\phi)\phi]\sum_{A,B} z_{AB}F_+^AF_-^BS^AS^B(U_{++}U_{--}+U_{+-}U_{-+})\\
&+[\lambda_{34}(\Phi,\phi)\Phi^2+\lambda_{35}(\Phi,\phi)\Phi\phi+\lambda_{36}(\Phi,\phi)\phi^2]\\
&\qquad\times\sum_{A,B,C} y_{ABC}F_+^AF_-^B(S^C)^2(U_{++}U_{--}+U_{+-}U_{-+})\\
&+\lambda_{37}(\Phi,\phi)\sum_{A,B,C} y_{ABC}\tilde S_+^A\tilde S_-^B(\tilde F^C)^2U_0\\
&+[\lambda_{38}(\Phi,\phi)\Phi+\lambda_{39}(\Phi,\phi)\phi]\sum_{A,B,C} y_{ABC}\tilde S_+^A\tilde S_-^A\tilde F^B\tilde F^CU_0\\
&+[\lambda_{40}(\Phi,\phi)\Phi^2+\lambda_{41}(\Phi,\phi)\Phi\phi+\lambda_{42}(\Phi,\phi)\phi^2]\sum_{A}\tilde S_+^A\tilde S_-^A(\tilde F^A)^2U_0\\
&+[\lambda_{43}(\Phi,\phi)\Phi^2+\lambda_{44}(\Phi,\phi)\Phi\phi+\lambda_{45}(\Phi,\phi)\phi^2]\sum_{A,B} z_{AB}\tilde S_+^A\tilde S_-^B\tilde F^A\tilde F^BU_0+\ldots\ , \qquad (117)
\end{aligned}
$$

where λ_k, $k=1,...,45$, are certain polynomials of their respective arguments, which combine into the terms of the form $\Phi^{3n-m}\phi^m$, $n,m\in\mathbf{N}$, such that $\lambda_k(0,\phi)\not\equiv 0$, and $\lambda_k(\Phi,0)\not\equiv 0$, and traces over the irreps of the gauge group are implicit here. The coefficients y_{ABC} and z_{AB} are defined as follows: $y_{ABC}=\epsilon_{ABC}$, and $z_{AB}=1-\delta_{AB}$.

B. Superpotential For $SU(6)$ Model $S2=S4$

.

The superpotential for the $SU(2)_1^2\otimes SU(6)_3\otimes U(1)^3$ model ($S2=S4$) reads:

$$
\begin{aligned}
W=\quad&\lambda_1(\Phi,\phi)U_0(d_{++}d_{--}+d_{+-}d_{-+})+\lambda_2(\Phi,\phi)(D_+\tilde d_-\Delta_-+D_-\tilde d_+\Delta_+)\\
&+\lambda_3(\Phi,\phi)U_0\Delta_+\Delta_-\sum_{A,B,C} y_{ABC}F^AF^BF^C\\
&+[\lambda_4(\Phi,\phi)\Phi^2+\lambda_5(\Phi,\phi)\Phi\phi+\lambda_6(\Phi,\phi)\phi^2]U_0\Delta_+\Delta_-\sum_A(F^A)^3\\
&+\lambda_7(\Phi,\phi)\sum_A F^A[\tilde S_{++}^A\tilde S_{--}^A+\tilde S_{+-}^A\tilde S_{-+}^A]\\
&+[\lambda_8(\Phi,\phi)\Phi+\lambda_9(\Phi,\phi)\phi]\sum_{A,B,C} y_{ABC}F^A(\tilde S_{++}^B\tilde S_{--}^C+\tilde S_{-+}^B\tilde S_{+-}^C)\\
&+\lambda_{10}(\Phi,\phi)\sum_{A,B} z_{AB}S_+^AS_-^B(\tilde S_{++}^A\tilde S_{--}^B+\tilde S_{-+}^A\tilde S_{+-}^B+\tilde S_{+-}^A\tilde S_{-+}^B+\tilde S_{--}^A\tilde S_{++}^B)\\
&+[\lambda_{11}(\Phi,\phi)\Phi^3+\lambda_{12}(\Phi,\phi)\Phi^2\phi+\lambda_{13}(\Phi,\phi)\Phi\phi^2+\lambda_{14}(\Phi,\phi)\phi^3]\\
&\qquad\times\sum_A S_+^AS_-^A(\tilde S_{++}^A\tilde S_{--}^A+\tilde S_{+-}^A\tilde S_{-+}^A)\\
&+[\lambda_{15}(\Phi,\phi)\Phi+\lambda_{16}(\Phi,\phi)\phi]\sum_{A,B,C} y_{ABC}S_+^AS_-^A(\tilde S_{++}^B\tilde S_{--}^C+\tilde S_{-+}^B\tilde S_{+-}^C)
\end{aligned}
$$

$$
\begin{aligned}
&+ [\lambda_{17}(\Phi,\phi)\Phi^2 + \lambda_{18}(\Phi,\phi)\Phi\phi + \lambda_{19}(\Phi,\phi)\phi^2] \\
&\qquad \times \sum_{A,B,C} y_{ABC} S_+^B S_-^C (\tilde{S}_{++}^A \tilde{S}_{--}^A + \tilde{S}_{+-}^A \tilde{S}_{-+}^A) + \dots \,.
\end{aligned} \tag{118}
$$

The couplings λ_k and the coefficients y_{ABC} and z_{AB} are defined in the same way as in the $S1 = S3$ model.

IX. REMARKS

Using the bosonic supercurrent formalism, we give a prescription how to calculate the correlation functions in the 3-family grand unification models. We use this approach to determine the quantum numbers of the massless spectra of some of these models. This gives us selection rules for the allowed couplings in the respective superpotentials. Many couplings that are allowed by gauge symmetries are forbidden by stringy discrete symmetries. The explicit values of the couplings are not hard to determine; however we will leave them for the future. Even without their explicit determination, there are still plenty of phenomenological issues one can address. This will also be discussed elsewhere. One question that might need clarification is whether there are additional discrete quantum numbers associated with the coset ($\mathcal{C}_L$) coming from the rank reduction.

In this paper, we discuss explicitly some 3-family grand unified string models: the unique E_6 model, an $SO(10)$ model and two $SU(6)$ models. There are other $SO(10)$, $SU(5)$ and $SU(6)$ three-family grand unified string models classified in Refs [3]. Most of these models are connected to the unique E_6 model by classically flat moduli. Some $SU(5)$ models are connected to the two $SU(6)$ models. Finally, there are a few $SU(5)$ models and one unique $SO(10)$ model that are isolated from the E_6 and two $SU(6)$ models we have described here. The superpotentials of the models that are connected to the above models via flat moduli can be easily obtained simply by giving the flat moduli appropriate vevs. These vevs can be of the order of string scale.

As we have already mentioned, some of these models do not admit bosonization of the supercurrent because their corresponding right-moving Kac-Moody algebras $\mathcal{G}'_R$ have reduced rank which is less than 6. Thus, deducing the discrete symmetries is complicated by the presence of the right-moving cosets $\mathcal{C}_R$. There is, however, a way to deduce their superpotentials when they are connected by (classically) flat directions to the above three models. We have illustrated how this can be done with the $SO(10)$ model $T1(1,1) = T2(1,1)$.

We note that all the couplings (in string units) λ_k are of order one for vanishing values of the Φ and ϕ vevs. The latter fields are (classically) flat moduli in these models. In the E_6 and the two $SU(6)$ models we have studied in this paper, there are no other completely flat moduli of a geometric origin, but such flat directions are present in other three-family grand unified string models classified in Refs [3]. Their stabilizations can only be achieved via non-perturbative dynamics.

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	Field	$SU(3)\otimes E_6\otimes E_8\otimes SU(3)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
U	χ_1	$(\mathbf{3},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,0)$	$(+1,0,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	χ_2	$(\mathbf{3},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,0)$	$(0,+1,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	χ_3	$(\mathbf{3},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
$T3$	$\tilde{\chi}$	$(\mathbf{3},\overline{\mathbf{27}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-e_1/3,-e_1/3,-e_1/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{1+}	$(\mathbf{1},\mathbf{27},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	$(-e_2/3,-e_1/3,-e_1/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{1-}	$(\mathbf{1},\mathbf{27},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	$(-e_3/3,-e_1/3,-e_1/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{2+}	$(\mathbf{1},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})$	$(-e_1/3,-e_2/3,-e_1/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{2-}	$(\mathbf{1},\mathbf{27},\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1})$	$(-e_1/3,-e_3/3,-e_1/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{3+}	$(\mathbf{1},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(-e_1/3,-e_1/3,-e_2/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{3-}	$(\mathbf{1},\mathbf{27},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(-e_1/3,-e_1/3,-e_3/3)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$T_{\mathbf{xyz}}$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z})$	$(Q(\mathbf{x}),Q(\mathbf{y}),Q(\mathbf{z}))$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$

TABLE I. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) of the massless fields in the asymmetric $\mathbf{Z}_3$ orbifold model. Here, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are irreps of $SU(3)$ such that only one of them is a singlet and the others can be either $\mathbf{3}$ or $\overline{\mathbf{3}}$. The charges Q as a function of irrep is given by $Q(\mathbf{1})=-e_1/3$, $Q(\mathbf{3})=-e_2/3$ and $Q(\overline{\mathbf{3}})=-e_3/3$.

M	$E1$ $SU(2)\otimes E_6\otimes U(1)^3$	Field	$E2$ $SU(2)\otimes E_6\otimes U(1)^3$	Field
U	$(\mathbf{1},\mathbf{78})(0,0,0)_L$ $(\mathbf{1},\mathbf{1})(0,+6,0)_L$ $2(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	Φ U_0 $U_{+\pm},U_{-\pm}$	$(\mathbf{1},\mathbf{78})(0,0,0)_L$ $(\mathbf{1},\mathbf{1})(0,+6,0)_L$ $(\mathbf{2},\mathbf{1})(0,0,\pm3)_L$ $(\mathbf{1},\mathbf{1})(\pm3,+3,0)_L$	Φ U_0 $D_\pm$ $\tilde{U}_\pm$
$T3$	$(\mathbf{1},\mathbf{27})(0,-2,0)_L$ $2(\mathbf{1},\mathbf{27})(0,+1,\pm1)_L$	χ_0 $\chi_{+\pm},\chi_{-\pm}$	$(\mathbf{1},\mathbf{27})(0,-2,0)_L$ $(\mathbf{1},\overline{\mathbf{27}})(\pm1,-1,0)_L$	χ_0 $\tilde{\chi}_\pm$
T6	$(\mathbf{1},\overline{\mathbf{27}})(\pm1,-1,0)_L$	$\tilde{\chi}_\pm$	$2(\mathbf{1},\mathbf{27})(0,+1,\pm1)_L$	$\chi_{+\pm},\chi_{-\pm}$
$T2$	$(\mathbf{2},\mathbf{1})(0,0,\pm3)_L$ $(\mathbf{1},\mathbf{1})(\pm3,+3,0)_L$	$D_\pm$ $\tilde{U}_\pm$	$2(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	$U_{+\pm},U_{-\pm}$
$U(1)$	$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6})$		$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6})$	

TABLE II. The massless spectra of the T-dual pair of E_6 models $E1$ and $E2$ both with gauge symmetry $SU(2)_1\otimes(E_6)_3\otimes U(1)^3$. The $U(1)$ normalization radii are given at the bottom of the Table. The gravity, dilaton and gauge supermultiplets are not shown.

$E1$	Field	$SU(2)\otimes E_6\otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
U	Φ	$(\mathbf{1},\mathbf{78})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1})(0,+6,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	$U_{+\pm}$	$(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	$(0,0,0)$	$(+1,0,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$U_{-\pm}$	$(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	$(0,0,0)$	$(0,+1,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
$T3$	χ_0	$(\mathbf{1},\mathbf{27})(0,-2,0)_L$	$-(e_1/3,e_1/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{++}	$(\mathbf{1},\mathbf{27})(0,+1,+1)_L$	$-(e_3/3,e_3/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{-+}	$(\mathbf{1},\mathbf{27})(0,+1,+1)_L$	$-(e_2/3,e_1/3,e_3/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{+-}	$(\mathbf{1},\mathbf{27})(0,+1,-1)_L$	$-(e_3/3,e_1/3,e_2/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	χ_{--}	$(\mathbf{1},\mathbf{27})(0,+1,-1)_L$	$-(e_2/3,e_2/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
$T6$	$\tilde{\chi}_+$	$(\mathbf{1},\overline{\mathbf{27}})(+1,-1,0)_L$	$(-e_1/6,e_3/3,e_3/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	$\tilde{\chi}_-$	$(\mathbf{1},\overline{\mathbf{27}})(-1,-1,0)_L$	$(-e_1/6,e_2/3,e_2/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
$T2$	$D_\pm$	$(\mathbf{2},\mathbf{1})(0,0,\pm3)_L$	$(e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$\tilde{U}_\pm$	$(\mathbf{1},\mathbf{1})(\pm3,+3,0)_L$	$-(e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6})$		$(1,1,1)$	$(1,1,1)$

TABLE III. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) of the massless fields in the $E1$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table. The Q^R-charges (in the -1 picture which are the same as in the $-1/2$ picture) are written in $SU(3)^3$ basis.

$E1$	Q^R-charges in $SU(3)^3$ basis	co-marks	(H_1,H_2,H_3)
$i\partial X^1$	$a_1=(-\tilde{e}^1,\tilde{e}^0,-\tilde{e}^0)$	1	$(-1,0,0)$
	$a_2=(-\tilde{e}^0,\tilde{e}^1,\tilde{e}^2)$	2	
	$a_3=(e_2,0,0)$	3	
	$a_4=(-\tilde{e}^0,\tilde{e}^0,-\tilde{e}^1)$	2	
	$a_5=(-\tilde{e}^1,-\tilde{e}^2,\tilde{e}^2)$	1	
	$a_6=(-\tilde{e}^0,-\tilde{e}^2,-\tilde{e}^0)$	2	
	$a_7=(-\tilde{e}^1,\tilde{e}^1,-\tilde{e}^1)$		
	$=-a_1-2a_2-3a_3-2a_4-a_5-2a_6$	1	
$i\partial X^2$	$l_1=(\tilde{e}^0,-\tilde{e}^0,\tilde{e}^0)=-a_1-a_2-a_3-a_4-a_6$	1	$(0,-1,0)$
	$l_2=(\tilde{e}^1,\tilde{e}^2,\tilde{e}^1)=a_1+2a_2+2a_3+a_4+a_6$	2	
	$l_3=(e_3,0,0)=-a_2-2a_3-a_4-a_6$	3	
	$l_4=(\tilde{e}^1,-\tilde{e}^1,\tilde{e}^0)=a_2+2a_3+2a_4+a_5+a_6$	2	
	$l_5=(\tilde{e}^0,\tilde{e}^2,-\tilde{e}^2)=-a_2-a_3-a_4-a_5-a_6$	1	
	$l_6=(\tilde{e}^1,-\tilde{e}^0,-\tilde{e}^2)=-a_1-a_2-a_3-a_4-a_5$	2	
	$l_7=(\tilde{e}^0,-\tilde{e}^1,\tilde{e}^1)=-l_1-2l_2-3l_3-2l_4-l_5-2l_6$		
	$=a_1+a_2+2a_3+a_4+a_5+a_6$	1	
$i\partial X^3$	$k_1=(0,e_1,0)=-a_1-a_2-2a_3-2a_4-a_5-a_6$	1	$(0,0,-1)$
	$k_2=(0,e_2,0)=a_1+a_2+a_3+a_4$	1	
	$k_3=(0,e_3,0)=a_3+a_4+a_5+a_6$	1	
	$k_1'=(0,0,e_1)=a_1+a_2+a_3+a_6$	1	
	$k_2'=(0,0,e_2)=a_2+a_3+a_4+a_5$	1	
	$k_3'=(0,0,e_3)=-a_1-2a_2-2a_3-a_4-a_5-a_6$	1	

TABLE IV. The terms that enter the expressions for the currents $i\partial X^a$ for $E1$ model. In the first column they are given in terms of their quantum numbers under the Kac-Moody algebra $\mathcal{G}_R=E_6$ in the $SU(3)^3$ basis. The weight vectors $\tilde{e}^1$, $\tilde{e}^2$ are defined by $\tilde{e}^i e_j=\delta^i_j$, and $\tilde{e}^0=\tilde{e}^2-\tilde{e}^1$. The co-marks of the roots are given in the second column. The corresponding H-charges carried by the supercurrent are given in the last column.

$E2$	Field	$SU(2)\otimes E_6\otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
U	Φ	$(\mathbf{1},\mathbf{78})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1})(0,+6,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	$D_\pm$	$(\mathbf{2},\mathbf{1})(0,0,\pm3)_L$	$(0,0,0)$	$(-1,0,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	$\tilde{U}_\pm$	$(\mathbf{1},\mathbf{1})(\pm3,+3,0)_L$	$(0,0,0)$	$(0,-1,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
$T3$	χ_0	$(\mathbf{1},\mathbf{27})(0,-2,0)_L$	$(0,-e_1/3,-e_1/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{\chi}_+$	$(\mathbf{1},\overline{\mathbf{27}})(+1,-1,0)_L$	$(0,e_3/3,e_3/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$\tilde{\chi}_-$	$(\mathbf{1},\overline{\mathbf{27}})(-1,-1,0)_L$	$(0,e_2/3,e_2/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
$T6$	χ_{++}	$(\mathbf{1},\mathbf{27})(0,+1,+1)_L$	$(\tilde{e}^2/2,-e_3/3,-e_1/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	χ_{-+}	$(\mathbf{1},\mathbf{27})(0,+1,+1)_L$	$(-\tilde{e}^2/2,-e_1/3,-e_3/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	χ_{+-}	$(\mathbf{1},\mathbf{27})(0,+1,-1)_L$	$(\tilde{e}^2/2,-e_1/3,-e_2/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	χ_{--}	$(\mathbf{1},\mathbf{27})(0,+1,-1)_L$	$(-\tilde{e}^2/2,-e_2/3,-e_1/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
$T2$	$U_{+\pm}$	$(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	$(e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$U_{-\pm}$	$(\mathbf{1},\mathbf{1})(0,-3,\pm3)_L$	$(-e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6})$		$(1,1,1)$	$(1,1,1)$

TABLE V. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) of the massless fields in the $E2$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table. The Q^R-charges (in the -1 picture which are the same as in the $-1/2$ picture) are written in $SU(3)^3$ basis.

$E2$	Q^R-charges in $SU(3)^3$ basis	co-marks	(H_1, H_2, H_3)
$i\partial X^1$	$\alpha_1 = -(e_2, 0, 0) = -a_3$ $\alpha_2 = (-\tilde{e}^1, \tilde{e}^0, -\tilde{e}^0) = a_1$ $\alpha_3 = (-\tilde{e}^1, -\tilde{e}^2, \tilde{e}^2) = a_5$ $\alpha_4 = (-\tilde{e}^1, \tilde{e}^1, -\tilde{e}^1) = -a_1 - 2a_2 - 3a_3 - 2a_4 - a_5 - 2a_6$	1 1 1 1	$(-1, 0, 0)$
$i\partial X^2$	$\beta_1 = (\tilde{e}^0, -\tilde{e}^1, -\tilde{e}^2) = -a_2$ $\beta_2 = (-\tilde{e}^0, \tilde{e}^0, \tilde{e}^2) = a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 + a_6$ $\beta_3 = (\tilde{e}^0, -\tilde{e}^0, \tilde{e}^1) = -a_4$ $\beta_4 = (-\tilde{e}^0, -\tilde{e}^2, -\tilde{e}^1) = -a_1 - a_2 - a_3$ $\beta_5 = (\tilde{e}^0, \tilde{e}^2, \tilde{e}^0) = -a_6$ $\beta_6 = (-\tilde{e}^0, \tilde{e}^1, -\tilde{e}^0) = -a_3 - a_4 - a_5$	1 1 1 1 1 1	$(0, -1, 0)$
$i\partial X^3$	$k_1 = (0, e_1, 0) = -a_1 - a_2 - 2a_3 - 2a_4 - a_5 - a_6$ $k_2 = (0, e_2, 0) = a_1 + a_2 + a_3 + a_4$ $k_3 = (0, e_3, 0) = a_3 + a_4 + a_5 + a_6$ $k_1^{'} = (0, 0, e_1) = a_1 + a_2 + a_3 + a_6$ $k_2^{'} = (0, 0, e_2) = a_2 + a_3 + a_4 + a_5$ $k_3^{'} = (0, 0, e_3) = -a_1 - 2a_2 - 2a_3 - a_4 - a_5 - a_6$	1 1 1 1 1 1	$(0, 0, -1)$

TABLE VI. The terms that enter the expressions for the currents $i\partial X^a$ for $E2$ model. In the first column they are given in terms of their quantum numbers under the Kac-Moody algebra $\mathcal{G}_R = E_6$ in the $SU(3)^3$ basis. The weight vectors $\tilde{e}^1$, $\tilde{e}^2$ are defined by $\tilde{e}^i e_j = \delta^i_j$, and $\tilde{e}^0 = \tilde{e}^2 - \tilde{e}^1$. The co-marks of the roots are given in the second column. The corresponding H-charges carried by the supercurrent are given in the last column.

$\mathcal{G}'_R$	$SU(2)^3 \otimes U(1)^3$	$SU(2)^4 \otimes U(1)^2$	$SU(3)^3$
	$(\mathbf{1},\mathbf{1},\mathbf{1})(+3,0,0)$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}_+)(0,0,0)$	$(e_1/2,0,0)$
	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,+12,0)$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(+12,0)$	$(0,e_1,e_1)$
	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,0,+4)$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(0,+4)$	$(0,-\tilde{e}^2,-\tilde{e}^2)$
	$(\mathbf{1},\mathbf{1},\mathbf{2}_+)(0,0,0)$	$(\mathbf{1},\mathbf{1},\mathbf{2}_+,\mathbf{1})(0,0,0)$	$(\tilde{e}^2/2,\tilde{e}^0/2,-\tilde{e}^0/2)$
	$(\mathbf{1},\mathbf{2}_+,\mathbf{1})(0,0,0)$	$(\mathbf{1},\mathbf{2}_+,\mathbf{1},\mathbf{1})(0,0,0)$	$(\tilde{e}^2/2,\tilde{e}^1/2,-\tilde{e}^1/2)$
	$(\mathbf{2}_+,\mathbf{1},\mathbf{1})(0,0,0)$	$(\mathbf{2}_+,\mathbf{1},\mathbf{1},\mathbf{1})(0,0,0)$	$(\tilde{e}^2/2,-\tilde{e}^2/2,\tilde{e}^2/2)$
$U(1)$	$(1/3\sqrt{2},1/6,1/2\sqrt{3})$	$(1/6,1/2\sqrt{3})$	

TABLE VII. The quantum numbers under the Kac-Moody algebra $\mathcal{G}'_R$ in the $SU(3)^3$ basis. For $E1$ model, $\mathcal{G}'_R = SU(2)^3 \otimes U(1)^3$. For $E2$ model, $\mathcal{G}'_R = SU(2)^4 \otimes U(1)^2$. The $U(1)$ normalization radii are given at the bottom of the Table. Here $\mathbf{2}_+$ and $\mathbf{2}_-$ stand for the upper and lower components of an $SU(2)$ doublet. In the $SU(2) \supset U(1)$ basis we have $\mathbf{2}_\pm = (\pm 1)$.

Field	$SU(2)\otimes E_6\otimes U(1)^3$	Field	$SU(2)\otimes SO(10)\otimes U(1)^4$	D-charge
Φ	$(\mathbf{1},\mathbf{78})(0,0,0)_L$	Φ	$(\mathbf{1},\mathbf{45})(0,0,0,0)_L$	2
		ϕ	$(\mathbf{1},\mathbf{1})(0,0,0,0)_L$	2
		Q	$(\mathbf{1},\mathbf{16})(0,0,0,3)_L$	1
		$\overline{Q}$	$(\mathbf{1},\overline{\mathbf{16}})(0,0,0,-3)_L$	0
χ_0	$(\mathbf{1},\mathbf{27})(0,-2,0)_L$	Q_0	$(\mathbf{1},\mathbf{16})(0,-2,0,-1)_L$	1
		H_0	$(\mathbf{1},\mathbf{10})(0,-2,0,+2)_L$	0
		S_0	$(\mathbf{1},\mathbf{1})(0,-2,0,-4)_L$	2
$\chi_{+\pm}$	$(\mathbf{1},\mathbf{27})(0,+1,\pm1)_L$	$Q_{+\pm}$	$(\mathbf{1},\mathbf{16})(0,+1,\pm1,-1)_L$	2
		$H_{+\pm}$	$(\mathbf{1},\mathbf{10})(0,+1,\pm1,+2)_L$	1
		$S_{+\pm}$	$(\mathbf{1},\mathbf{1})(0,+1,\pm1,-4)_L$	0
$\chi_{-\pm}$	$(\mathbf{1},\mathbf{27})(0,+1,\pm1)_L$	$Q_{-\pm}$	$(\mathbf{1},\mathbf{16})(0,+1,\pm1,-1)_L$	2
		$H_{-\pm}$	$(\mathbf{1},\mathbf{10})(0,+1,\pm1,+2)_L$	1
		$S_{-\pm}$	$(\mathbf{1},\mathbf{1})(0,+1,\pm1,-4)_L$	0
$\tilde{\chi}_\pm$	$(\mathbf{1},\overline{\mathbf{27}})(\pm1,-1,0)_L$	$\tilde{Q}_\pm$	$(\mathbf{1},\overline{\mathbf{16}})(\pm1,-1,0,+1)_L$	1
		$\tilde{H}_\pm$	$(\mathbf{1},\mathbf{10})(\pm1,-1,0,-2)_L$	2
		$\tilde{S}_\pm$	$(\mathbf{1},\mathbf{1})(\pm1,-1,0,+4)_L$	0

TABLE VIII. The discrete D-charges for $E1$ model. The second column gives the gauge quantum numbers of the fields in $E1$ model. The fourth column gives the gauge quantum numbers of the fields in the branching $E_6\supset SO(10)\otimes U(1)$. The last column gives the discrete D-charges. This D-charge comes from a $\mathbf{Z}_3$ symmetry and must be conserved modulus 3 in the scattering amplitude. Fields in the $E1$ model that have $D=0$ for the entire E_6 multiplet are not shown.

M	$T1(1,1)$ $SU(2)\otimes SO(10)\otimes U(1)^4$	Field	$T2(1,1)$ $SU(2)\otimes SO(10)\otimes U(1)^4$	Field
U	$(\mathbf{1},\mathbf{45})(0,0,0,0)_L$ $(\mathbf{1},\mathbf{1})(0,0,0,0)_L$ $2(\mathbf{1},\mathbf{1})(0,-3,\pm3,0)_L$ $(\mathbf{1},\mathbf{1})(0,+6,0,0)_L$	Φ ϕ $U_{+\pm},U_{-\pm}$ U_0	$(\mathbf{1},\mathbf{45})(0,0,0,0)_L$ $(\mathbf{1},\mathbf{1})(0,0,0,0)_L$ $(\mathbf{1},\mathbf{1})(\pm3,+3,0,0)_L$ $(\mathbf{1},\mathbf{1})(0,+6,0,0)_L$ $(\mathbf{2},\mathbf{1})(0,0,\pm3,0)_L$	Φ ϕ $\tilde{U}_\pm$ U_0 $D_\pm$
$T3$	$(\mathbf{1},\mathbf{16})(0,-2,0,-1)_L$ $(\mathbf{1},\mathbf{10})(0,-2,0,+2)_L$ $(\mathbf{1},\mathbf{1})(0,-2,0,-4)_L$ $2(\mathbf{1},\mathbf{16})(0,+1,\pm1,-1)_L$ $2(\mathbf{1},\mathbf{10})(0,+1,\pm1,+2)_L$ $2(\mathbf{1},\mathbf{1})(0,+1,\pm1,-4)_L$	Q_0 H_0 S_0 $Q_{+\pm},Q_{-\pm}$ $H_{+\pm},H_{-\pm}$ $S_{+\pm},S_{-\pm}$	$(\mathbf{1},\mathbf{16})(0,-2,0,-1)_L$ $(\mathbf{1},\mathbf{10})(0,-2,0,+2)_L$ $(\mathbf{1},\mathbf{1})(0,-2,0,-4)_L$ $(\mathbf{1},\overline{\mathbf{16}})(\pm1,-1,0,+1)_L$ $(\mathbf{1},\mathbf{10})(\pm1,-1,0,-2)_L$ $(\mathbf{1},\mathbf{1})(\pm1,-1,0,+4)_L$	Q_0 H_0 S_0 $\tilde{Q}_\pm$ $\tilde{H}_\pm$ $\tilde{S}_\pm$
$T6$	$(\mathbf{1},\overline{\mathbf{16}})(\pm1,-1,0,+1)_L$ $(\mathbf{1},\mathbf{10})(\pm1,-1,0,-2)_L$ $(\mathbf{1},\mathbf{1})(\pm1,-1,0,+4)_L$	$\tilde{Q}_\pm$ $\tilde{H}_\pm$ $\tilde{S}_\pm$	$2(\mathbf{1},\ \mathbf{16})(0,+1,\pm1,-1)_L$ $2(\mathbf{1},\ \mathbf{10})(0,+1,\pm1,+2)_L$ $2(\mathbf{1},\mathbf{1})(0,+1,\pm1,-4)_L$	$Q_{+\pm},Q_{-\pm}$ $H_{+\pm},H_{-\pm}$ $S_{+\pm},S_{-\pm}$
$T2$	$(\mathbf{2},\mathbf{1})(0,0,\pm3,0)_L$ $(\mathbf{1},\mathbf{1})(\pm3,+3,0,0)_L$	$D_\pm$ $\tilde{U}_\pm$	$2(\mathbf{1},\mathbf{1})(0,-3,\pm3,0)_L$	$U_{+\pm},U_{-\pm}$
$U(1)$	$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6},\ 1/6)$		$(1/\sqrt{6},\ 1/3\sqrt{2},\ 1/\sqrt{6},\ 1/6)$	

TABLE IX. The massless spectra of the two $SO(10)$ models $T1(1,1)$ and $T2(1,1)$ both with gauge symmetry $SU(2)_1\otimes SO(10)_3\otimes U(1)^4$. The $U(1)$ normalization radii are given at the bottom of the Table. The gravity, dilaton and gauge supermultiplets are not shown.

$S1$	Field	$SU(3)\otimes SU(6)\otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
	Φ	$(\mathbf{1},\mathbf{35})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	ϕ	$(\mathbf{1},\mathbf{1})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1})(+6,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
U	T_0	$(\mathbf{3},\mathbf{1})(0,0,-4)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	$U_{+\pm}$	$(\mathbf{1},\mathbf{1})(-3,\pm3,\pm3)_L$	$(0,0,0)$	$(+1,0,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$\tilde{T}_+$	$(\overline{\mathbf{3}},\mathbf{1})(+3,-3,+1)_L$	$(0,0,0)$	$(+1,0,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$U_{-\pm}$	$(\mathbf{1},\mathbf{1})(-3,\pm3,\pm3)_L$	$(0,0,0)$	$(0,+1,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	$\tilde{T}_-$	$(\overline{\mathbf{3}},\mathbf{1})(+3,-3,+1)_L$	$(0,0,0)$	$(0,+1,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	$\tilde{S}^1_0$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$-\frac{1}{3}(e_1,e_1,e_1)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^2_0$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$-\frac{1}{3}(e_1,e_2,e_2)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^3_0$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$-\frac{1}{3}(e_1,e_3,e_3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$F^1_\pm$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$-\frac{1}{3}(e_3,e_2,e_3),-\frac{1}{3}(e_2,e_3,e_2)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
$T3$	$\tilde{S}^1_\pm$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$-\frac{1}{3}(e_3,e_2,e_3),-\frac{1}{3}(e_2,e_3,e_2)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$F^2_\pm$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$-\frac{1}{3}(e_3,e_3,e_1),-\frac{1}{3}(e_2,e_1,e_3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^2_\pm$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$-\frac{1}{3}(e_3,e_3,e_1),-\frac{1}{3}(e_2,e_1,e_3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$F^3_\pm$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$-\frac{1}{3}(e_3,e_1,e_2),-\frac{1}{3}(e_2,e_2,e_1)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^3_\pm$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$-\frac{1}{3}(e_3,e_1,e_2),-\frac{1}{3}(e_2,e_2,e_1)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	S^1	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(-e_1/6,e_1/3,e_1/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	$\tilde{F}^1$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(-e_1/6,e_1/3,e_1/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	S^2	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(-e_1/6,e_2/3,e_2/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
$T6$	$\tilde{F}^2$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(-e_1/6,e_2/3,e_2/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	S^3	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(-e_1/6,e_3/3,e_3/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	$\tilde{F}^3$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(-e_1/6,e_3/3,e_3/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	T_+	$(\mathbf{3},\mathbf{1})(+3,-3,-1)_L$	$(e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$T2$	$\tilde{T}_0$	$(\overline{\mathbf{3}},\mathbf{1})(-3,+3,+1)_L$	$(e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	T_-	$(\mathbf{3},\mathbf{1})(-3,-3,-1)_L$	$(-e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$\tilde{U}_\pm$	$(\mathbf{1},\mathbf{1})(+3,\pm3,\mp3)_L$	$(-e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(\frac{1}{3\sqrt{2}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}})$		$(1,1,1)$	$(1,1,1)$

TABLE X. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) for the massless fields of the $S1$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table.

$S2$	Field	$SU(2)^2\otimes SU(6)\otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
U	Φ	$(\mathbf{1},\mathbf{1},\mathbf{35})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	ϕ	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,0,-6)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	$D_\pm$	$(\mathbf{2},\mathbf{1},\mathbf{1})(\pm2,0,+3)_L$	$(0,0,0)$	$(0,0,+1)$	$(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$
	$d_{+\pm}$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm1,\mp3,+3)_L$	$(0,0,0)$	$(+1,0,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$d_{-\pm}$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm1,\mp3,+3)_L$	$(0,0,0)$	$(0,+1,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
$T3$	F^1	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$-(e_1/3,e_1/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	F^2	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$-(e_1/3,e_2/3,e_2/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	F^3	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$-(e_1/3,e_3/3,e_3/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^1_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_3/3,e_2/3,e_3/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^1_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_2/3,e_3/3,e_2/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^2_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_3/3,e_3/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^2_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_2/3,e_1/3,e_3/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^3_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_3/3,e_1/3,e_2/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}^3_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm1,\mp1,-1)_L$	$-(e_2/3,e_2/3,e_1/3)$	$(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3})$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$
$T6$	$S^1_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm1,\pm1,+1)_L$	$(-e_1/6,e_1/3,e_1/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	$S^2_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm1,\pm1,+1)_L$	$(-e_1/6,e_2/3,e_2/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
	$S^3_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm1,\pm1,+1)_L$	$(-e_1/6,e_3/3,e_3/3)$	$(+\frac{1}{6},+\frac{1}{6},+\frac{2}{3})$	$(-\frac{1}{3},-\frac{1}{3},+\frac{1}{6})$
$T2$	$\Delta_\pm$	$(\mathbf{2},\mathbf{2},\mathbf{1})(\pm1,\mp3,0)_L$	$(e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$\tilde{d}_\pm$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm1,\pm3,-3)_L$	$(-e_1/2,0,0)$	$(+\frac{1}{2},+\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(\frac{1}{2},\ \frac{1}{2\sqrt{3}},\ \frac{1}{3\sqrt{2}})$		$(1,1,1)$	$(1,1,1)$

TABLE XI. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) for the massless fields of the $S2$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table.

$S3$	Field	$SU(3)\otimes SU(6)\otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1,H_2,H_3)_{-1}$	$(H_1,H_2,H_3)_{-1/2}$
	Φ	$(\mathbf{1},\mathbf{35})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	ϕ	$(\mathbf{1},\mathbf{1})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1})(+6,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
U	T_0	$(\mathbf{3},\mathbf{1})(0,0,-4)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	T_+	$(\mathbf{3},\mathbf{1})(+3,-3,-1)_L$	$(0,0,0)$	$(-1,0,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	$\tilde{T}_0$	$(\overline{\mathbf{3}},\mathbf{1})(-3,+3,+1)_L$	$(0,0,0)$	$(-1,0,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	T_-	$(\mathbf{3},\mathbf{1})(-3,-3,-1)_L$	$(0,0,0)$	$(0,-1,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$\tilde{U}_\pm$	$(\mathbf{1},\mathbf{1})(+3,\pm 3,\mp 3)_L$	$(0,0,0)$	$(0,-1,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	$\tilde{S}_0^1$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$(0,-e_1/3,-e_1/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}_0^2$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$(0,-e_2/3,-e_2/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{S}_0^3$	$(\mathbf{1},\overline{\mathbf{6}})(+1,0,+2)_L$	$(0,-e_3/3,-e_3/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	$\tilde{F}^1$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(0,e_1/3,e_1/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
$T3$	S^1	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(0,e_1/3,e_1/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$\tilde{F}^2$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(0,e_2/3,e_2/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	S^2	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(0,e_2/3,e_2/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$\tilde{F}^3$	$(\mathbf{1},\overline{\mathbf{15}})(-1,-1,+1)_L$	$(0,e_3/3,e_3/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	S^3	$(\mathbf{1},\mathbf{6})(+2,+1,+1)_L$	$(0,e_3/3,e_3/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$\tilde{S}_\pm^1$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_2}{3},-\frac{e_3}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_3}{3},\frac{e_2}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$F_\pm^1$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_2}{3},-\frac{e_3}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_3}{3},\frac{e_2}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
$T6$	$\tilde{S}_\pm^2$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_3}{3},-\frac{e_1}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_1}{3},\frac{e_3}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$F_\pm^2$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_3}{3},-\frac{e_1}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_1}{3},\frac{e_3}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$\tilde{S}_\pm^3$	$(\mathbf{1},\overline{\mathbf{6}})(-2,+1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_1}{3},-\frac{e_2}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_2}{3},\frac{e_1}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$F_\pm^3$	$(\mathbf{1},\mathbf{15})(+1,-1,-1)_L$	$(\frac{\tilde{e}^2}{2},-\frac{e_1}{3},-\frac{e_2}{3}),-(\frac{\tilde{e}^2}{2},\frac{e_2}{3},\frac{e_1}{3})$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$U_{\pm+}$	$(\mathbf{1},\mathbf{1})(-3,+3,+3)_L$	$(e_1/2,0,0),(-e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$T2$	$U_{\pm-}$	$(\mathbf{1},\mathbf{1})(-3,-3,-3)_L$	$(e_1/2,0,0),(-e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$\tilde{T}_\pm$	$(\overline{\mathbf{3}},\mathbf{1})(+3,-3,+1)_L$	$(e_1/2,0,0),(-e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(\frac{1}{3\sqrt{2}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}})$		$(1,1,1)$	$(1,1,1)$

TABLE XII. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) for the massless fields of the $S3$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table.

$S4$	Field	$SU(2)^2 \otimes SU(6) \otimes U(1)^3$	Q^R-charges in $SU(3)^3$	$(H_1, H_2, H_3)_{-1}$	$(H_1, H_2, H_3)_{-1/2}$
	Φ	$(\mathbf{1},\mathbf{1},\mathbf{35})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	ϕ	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,0,0)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	U_0	$(\mathbf{1},\mathbf{1},\mathbf{1})(0,0,-6)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
U	$D_\pm$	$(\mathbf{2},\mathbf{1},\mathbf{1})(\pm 2,0,+3)_L$	$(0,0,0)$	$(0,0,+1)$	$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$
	$\Delta_\pm$	$(\mathbf{2},\mathbf{2},\mathbf{1})(\pm 1,\mp 3,0)_L$	$(0,0,0)$	$(-1,0,0)$	$(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$
	$\tilde{d}_\pm$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm 1,\pm 3,-3)_L$	$(0,0,0)$	$(0,-1,0)$	$(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
	F^1	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$(0,-e_1/3,-e_1/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	F^2	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$(0,-e_2/3,-e_2/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
$T3$	F^3	$(\mathbf{1},\mathbf{1},\mathbf{15})(0,0,+2)_L$	$(0,-e_3/3,-e_3/3)$	$(0,-\frac{2}{3},+\frac{1}{3})$	$(+\frac{1}{2},-\frac{1}{6},-\frac{1}{6})$
	$S^1_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm 1,\pm 1,+1)_L$	$(0,e_1/3,e_1/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$S^2_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm 1,\pm 1,+1)_L$	$(0,e_2/3,e_2/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$S^3_\pm$	$(\mathbf{1},\mathbf{1},\mathbf{6})(\pm 1,\pm 1,+1)_L$	$(0,e_3/3,e_3/3)$	$(0,-\frac{1}{3},+\frac{2}{3})$	$(+\frac{1}{2},+\frac{1}{6},+\frac{1}{6})$
	$\tilde{S}^1_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(\tilde{e}^2/2,-e_2/3,-e_3/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$\tilde{S}^1_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(-\tilde{e}^2/2,-e_3/3,-e_2/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
$T6$	$\tilde{S}^2_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(\tilde{e}^2/2,-e_3/3,-e_1/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$\tilde{S}^1_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(-\tilde{e}^2/2,-e_1/3,-e_3/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$\tilde{S}^3_{+\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(\tilde{e}^2/2,-e_1/3,-e_2/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
	$\tilde{S}^1_{-\pm}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{6}})(\pm 1,\mp 1,-1)_L$	$(-\tilde{e}^2/2,-e_2/3,-e_1/3)$	$(-\frac{1}{2},-\frac{1}{6},+\frac{1}{3})$	$(0,+\frac{1}{3},-\frac{1}{6})$
$T2$	$d_{+\pm}$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm 1,\mp 3,+3)_L$	$(e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
	$d_{-\pm}$	$(\mathbf{1},\mathbf{2},\mathbf{1})(\pm 1,\mp 3,+3)_L$	$(-e_1/2,0,0)$	$(-\frac{1}{2},-\frac{1}{2},0)$	$(0,0,-\frac{1}{2})$
$U(1)$		$(\frac{1}{2},\ \frac{1}{2\sqrt{3}},\ \frac{1}{3\sqrt{2}})$		$(1,1,1)$	$(1,1,1)$

TABLE XIII. The G-, Q^R- and H-charges (in the -1 and $-1/2$ pictures) for the massless fields of the $S4$ model. The $U(1)$ normalization radii for the G- and H-charges are given at the bottom of the Table.

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