The Maxwell-Bloch Theory in Quantum Optics and the Kondo Model

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In this letter, the problem of radiation in a fiber geometry interacting with a two level atom is mapped onto the anisotropic Kondo model. Thermodynamical and dynamical properties are then computed exploiting the integrability of this latter system. We compute some correlation functions, decay rates and Lamb shifts. In turn this leads to an analysis of the classical limit of the anisotropic Kondo model.

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I. INTRODUCTION

The interaction of radiation with two-level atoms is a fundamental problem with many important applications. The model goes under the name of the Maxwell-Bloch theory in quantum optics. The problem has been extensively studied theoretically in a framework where the radiation is essentially classical and the so-called slowly varying envelope and rotating wave approximations are commonly made, which are valid near resonance [1]. In one dimension, fully quantum integrable versions of this system, refered to as the reduced Maxwell-Bloch theory, have been solved exactly [2]. In some applications, off-resonance effects are likely to be significant; for example in doped photonic band gap materials. In this letter, we show that the single atom case is exactly solvable without making any near-resonance approximations, by mapping the problem onto the anisotropic Kondo model. Furthermore, we are able to compute important physical quantities that have hitherto been inaccessible in previous treatments of even the approximate reduced Maxwell-Bloch theory. In particular, we compute thermodynamic properties in the form of impurity corrections to the Stephan-Boltzman law, and some electric field correlators. From the latter we obtain the atomic spontaneous emission decay rate and the Lamb shifted energy splitting.

II. THE MODELS

The interaction of radiation with a two-level atomic impurity at x_0 is described by the total hamiltonian

$$H = H^{atom} + H^{field} + H^{int} \tag{1}$$

with

$$H^{field} = \frac{1}{2}\int_{-\infty}^{\infty} dx[(\partial_t\phi)^2 + (\partial_x\phi)^2], \tag{2}$$

$$H^{atom} = \frac{\omega_0}{2}\sigma_3, \qquad H^{int} = \frac{\beta}{2}\partial_t\phi(x_0)\,(\sigma_+ + \sigma_-)\,,$$

and the conventions $[\sigma_3,\sigma_\pm] = \pm 2\sigma_\pm$, $[\sigma_+,\sigma_-] = \sigma_3$. We have made a reduction from the Maxwell theory to a one-dimensional fiber geometry, following the conventions in [3]. The dimensionless coupling constant β is determined by the strength d of the dipole transition and the effective cross-sectional area of the fiber, $\beta = \sqrt{\frac{16\pi}{\mathcal{A}_{eff}}}d$. The coupling β is also related to the lifetime τ of the excited state of the atom; to lowest order, $\gamma = 1/\tau = \beta^2\omega_0/4$.

The hamiltonian (3) is known in the condensed matter literature as the spin boson hamiltonian [4], [5]. It can be mapped onto the anisotropic Kondo hamiltonian H_K: defining the unitary operator

$$U = \frac{1}{\sqrt{2}}\exp\left(i\frac{\beta}{2}\sigma_3\phi(x_0)\right)(\sigma_3 + \sigma_+ + \sigma_-), \tag{3}$$

then $H = U^\dagger H_K U$, where

$$H_K = H^{field} + \frac{\omega_0}{2}(\sigma_+ e^{i\beta\phi(x_0)} + \sigma_- e^{-i\beta\phi(x_0)}). \tag{4}$$

This allows us to formulate the quantum mechanics in either the "optical picture" based on H or the "Kondo picture" based on H_K.

In order to complete the map to the conventional Kondo model on the half-line $x<0$, we fold the system. Let $\phi_L(x+t), \phi_R(x-t)$ be the left and right-moving components in the bulk, with $\phi = \phi_L + \phi_R$. Let us set $x_0 = 0$, and define even and odd fields in the region $x<0$

$$\begin{aligned}
\varphi_L^e(x,t) &= \beta\,(\phi_L(x,t) + \phi_R(-x,t))\\
\varphi_L^o(x,t) &= \beta\,(\phi_L(x,t) - \phi_R(-x,t))\\
\varphi_R^e(x,t) &= \varphi_L^e(-x,t), \qquad \varphi_R^o(x,t) = -\varphi_L^o(-x,t).
\end{aligned} \tag{5}$$

Introduce new fields $\phi^{e,o} = \varphi_L^{e,o} + \varphi_R^{e,o}$. The odd and even fields decouple; the odd field just becomes a free field, while for the even field the hamiltonian reads

$$\begin{aligned}
H_K^e = \frac{1}{2}\int_{-\infty}^{0} dx\ &[8\pi g\Pi^{2,e} + \frac{1}{8\pi g}(\partial_x\phi^e)^2]\\
&+\frac{\omega_0}{2}(\sigma_+ e^{i\phi(0)^e/2} + \sigma_- e^{-i\phi^e(0)/2}),
\end{aligned} \tag{6}$$

where $g = \beta^2/4\pi$.

Recent progress in the understanding of the anisotropic Kondo problem will now allow us to compute exactly the quantities of interest in the optics problem.

III. THERMODYNAMICS

We start with thermodynamics. Consider the system at finite temperature T. The partition function of the optical system is identical to the Kondo one: $Z = Tre^{-H/T} = Tre^{-H_K/T}$. Without the atom, the photon energy spectrum is given by the usual Plank distribution. The total partition function is

$$\log Z|_{g=0} = -\frac{L}{2\pi}\int_{-\infty}^{\infty} dk \log(1-e^{-|k|/T}) + \log Z_{\rm atom} = \frac{\pi}{6}LT + \log\left(2\cosh\left(\frac{\omega_0}{2T}\right)\right), \qquad (7)$$

where L is the length of the system. Of particular interest is the mean energy density

$$U(T) = \frac{T^2}{L}\partial_T \log Z = \frac{\pi}{6}T^2 - \frac{\omega_0}{2L}\tanh\left(\frac{\omega_0}{2T}\right). \qquad (8)$$

The first term in (8) is the 1d analog of the Stephan-Boltzman law.

In the interacting theory, we work with equation (8). The partition function factorizes into a bulk term and an impurity contribution $Z = Z_{\rm bulk}\cdot Z_{\rm imp}$. The bulk contribution, including both $\phi^{e,o}$, is the same as (7): $\log Z_{\rm bulk} = \frac{\pi}{6}LT$.

The impurity contribution in the context of the Kondo problem has recently been studied using novel techniques in [6] [7]. In particular, the partition function $Z_{\rm imp}$ is the trace of the monodromy matrix studied in detail in [6] [1].

It is convenient to express physical quantities in terms of a "Kondo temperature" T_K, defined such that the leading low temperature behaviour is $\log Z_{\rm imp} \sim T_K/T$. The Kondo temperature is then a fixed function of the energy splitting and g

$$T_K = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1-2g}{2-2g}\right)}{\Gamma\left(\frac{2-3g}{2-2g}\right)}\left(\frac{\omega_0}{2}\Gamma(1-g)\right)^{\frac{1}{1-g}}. \qquad (9)$$

The leading high and low temperature behaviors for arbitrary g are then as follows. At low temperatures,

$$\log Z_{\rm imp} \asymp \frac{T_K}{T} + \sum_{n=0}^{\infty} a_n(g)\left(\frac{T}{T_K}\right)^{2n+1} \qquad (10)$$

where $a_0 = \frac{\pi}{6}\tan\frac{\pi g}{2-2g}$,

$$a_1 = \frac{\pi^2(1-4g)(4-g)}{360}\frac{\Gamma^3\left(\frac{1-2g}{2-2g}\right)\Gamma\left(\frac{3}{2-2g}\right)}{\Gamma^3\left(\frac{2-3g}{2-2g}\right)\Gamma\left(\frac{3g}{2-2g}\right)}. \qquad (11)$$

Higher $a_n(g)$, up to $n=7$, can be deduced from results in [6]. At high temperatures, one finds

$$Z_{\rm imp} = 2 + \widetilde{a}(g)\left(\frac{T_K}{T}\right)^{2-2g} + \mathcal{O}\left(\frac{T_K}{T}\right)^{4-4g} \qquad (12)$$

where

$$\widetilde{a}(g) = 4\pi^2\frac{\Gamma(1-2g)}{\Gamma^4(1-g)}\left(\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{2-3g}{2-2g}\right)}{\Gamma\left(\frac{1-2g}{2-2g}\right)}\right)^{2-2g} \qquad (13)$$

The mean energy density $U(T)$ has the form $U = U_{\rm bulk} + U_{\rm imp}$ where $U_{\rm bulk} = \pi T^2/6$. At low and high temperatures for arbitrary g, one finds

$$U_{\rm imp} \approx \frac{1}{L}\left(-T_K + a_0\frac{T^2}{T_K}\right), \qquad T \ll T_K,$$
$$\approx -\widetilde{a}(1-g)\frac{T}{L}\left(\frac{T_K}{T}\right)^{2-2g}, \qquad T \gg T_K. \qquad (14)$$

In the quantum optical context, g is generally very small, thus it is more relevant to expand $Z_{\rm imp}$ in powers of g. For small g, $T_K \approx \omega_0/2$. In [6], the low temperature expansion of $Z_{\rm imp}$ is expressed in terms of the so-called local integrals of motion I_{2n-1}. With the conventions of [6], one can show that

$$g^{n-1}I_{2n-1}(g)|_{g=0} = -\frac{1}{2^{n+1}}\frac{(n-1)!}{(2n-3)!!}|B_{2n}|, \qquad (15)$$

where B_{2n} are the Bernoulli numbers. Using this, one can show that to order g the partition function at arbitrary temperature is given by

$$Z_{\rm imp} = 2\cosh\left(\frac{T_K}{T}\right) + 2g\frac{T_K}{T}\sinh\left(\frac{T_K}{T}\right)\times Re\left[\Psi\left(1+\frac{iT_K}{\pi T}\right) - \log\left(\frac{T_K}{\pi T}\right)\right], \qquad (16)$$

where $\Psi(x) = \partial_x \log\Gamma(x)$. One can easily check that (16) correctly reproduces the high and low temperature expansions in (10) (12) at small g.

The classical limit of the sine-Gordon model thermodynamics had been partially studied in [8]. We see here that there are, in fact, physical probes exploring the different Bethe ansatz excitations: these are the spin j impurities, whose thermal properties depend on the pseudoenergies ϵ_{2j} of the thermodynamic bethe ansatz. For instance, when $g=0$, $Z_{\rm imp}$ corresponds to the partition function of a decoupled two-level system with energies $\pm\omega_0/2$: $Z_{\rm imp} = 2\cosh(\omega_0/2T)$, and this can be recovered using the expressions for ϵ_1 in [8].

The mean energy density can be easily obtained for all T from (8) and (16).

[1] For the conventions in [6] , the 'spectral parameter' argument λ of the monodromy matrix $T(\lambda)$ is a function of ω_0 and T, namely $\lambda = \omega_0(2\pi T)^{g-1}/2$. The parameter ω_0 has anomalous scaling dimension $-g$, which is consistent with the the renormalization group equation derived in the optical picture in [3].

IV. PHOTON CORRELATION FUNCTIONS

Let E denote the relevant component of the electric field perpendicular to the fiber. Given some initial state $|i\rangle$, or some mixed state with density matrix $\rho = \sum_i P_i |i\rangle\langle i|$, one is interested in the intensity of the radiation in the electric field

$$I(x) = \frac{1}{4\pi} \langle E^{(-)}(x,t) E^{(+)}(x,t) \rangle_\rho, \tag{17}$$

where $\langle \mathcal{O} \rangle_\rho = Tr \rho \mathcal{O}$, and $E^{(-)}$ ($E^{(+)}$) is the creation (annihilation) part of the field, $E = E^{(+)} + E^{(-)}$. In the quantum measurement theory of Glauber, $I(x)$ is proportional to the number of photons detected at x given the initial state specified by ρ. One is also interested in the power spectrum $S(\omega)$ of detected photons satisfying $\int_0^\infty d\omega S(\omega) = I(x)$. One can express the above quantities in terms of the usual electric field correlators. Taking as the definition of $E^{(\pm)}$:

$$E^{(\pm)}(x,t) = \pm \frac{i}{2\pi} \int_{-\infty}^{\infty} dt' \frac{1}{t' - t \pm i\epsilon} E(x,t'), \tag{18}$$

one can show that if $|\psi\rangle$ is an eigenstate of the hamiltonian with energy E_ψ, then $\langle \psi' | E^{(+)} | \psi \rangle = \langle \psi' | E | \psi \rangle$ if $E_{\psi'} \leq E_\psi$ and zero otherwise. Likewise, $\langle \psi' | E^{(-)} | \psi \rangle = \langle \psi' | E | \psi \rangle$ if $E_{\psi'} \geq E_\psi$ and zero otherwise. Inserting complete sets of states in (17), one finds

$$S(\omega) = \theta(\omega) \frac{1}{8\pi^2} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle E(x,\tau) E(x,0) \rangle_\rho. \tag{19}$$

The power spectrum $S(\omega)$ is non-zero only for density matrices ρ containing excited states, e.g. a thermal distribution, since the integral in (19) is proportional to $\theta(-\omega)$ for $\rho = |0\rangle\langle 0|$. For example, such correlators were computed for the harmonic oscillator version of the present problem in [11]. In this paper, in order to illustrate our approach to correlation functions, we deal with the simpler quantity $\langle 0 | E(x,t) E(x,0) | 0 \rangle$, which probes how easy it is to *add* a photon to (rather than *detect* a photon in) the vacuum. In the reduction to one dimension, $E = \sqrt{\frac{4\pi}{\mathcal{A}_{eff}}} \partial_t \phi$. Thus we define

$$i(x) = \langle 0 | : \partial_t \phi(x,t) \partial_t \phi(x,t) : | 0 \rangle = \int_0^\infty d\omega \ s(\omega). \tag{20}$$

We define the normal ordering in (20) such that the free photon vacuum energy $|\omega|/4\pi$ is subtracted from $s(\omega)$. To lowest order in perturbation theory in the optical picture, one has

$$s(\omega) = i \frac{g\omega_0}{2} \frac{\omega^2}{\omega^2 - \omega_0^2 + i\epsilon} e^{2i|\omega x|}. \tag{21}$$

All the previous quantities were defined in the unrotated and unfolded theory. The rotation by the unitary operator U changes the operator $\partial_t \phi$ to

$$\partial_t \phi(x,t) \to \partial_t \phi(x,t) - \frac{\beta}{2} \sigma_3 \delta(x). \tag{22}$$

Since the detector is taken to be away from the impurity (and at $x < 0$ in our conventions), the second term of the previous expression can be dropped. Then under folding, we find

$$\partial_t \phi(x,t) = \frac{1}{2\beta} \left(\partial_t \phi^e(x,t) + \partial_t \phi^o(x,t) \right), \tag{23}$$

where now the odd and even field are defined on the negative axis. The odd field is free and has Dirichlet boundary conditions, $\phi^o(0) = 0$, whereas the even field has Neumann boundary conditions and interacts with the boundary. When computing $s(\omega)$ the odd field will not contribute because the Wick contraction is substracted. Thus

$$i(x) = \frac{1}{16\pi g} \langle 0 | : \partial_t \phi^e(x,t)^2 : | 0 \rangle. \tag{24}$$

In order to compute these correlation functions, we will make use of the techniques used for the anisotropic Kondo model in [9]. There, the "natural" basis of the excitations of the system, as described by the massless sine-Gordon model, is used to compute correlation functions. At first sight, the problem appears untractable since, as $g \to 0$, the spectrum of the sine-Gordon theory consists of a very large number of particles: soliton, anti-soliton and n bound states or breathers where $n \simeq 1/g$. The problem simplifies however when considering the different contributions of these particles to the correlators. A quick way to see this is to consider $< \partial_z \phi(z,\overline{z}) \partial_{z'} \phi(z',\overline{z}') >= 2g/(z-z')^2$, which is unaffected by the boundary interaction. (Above, $z = x + iy$, $\overline{z} = x - iy$, and $y = it$) This correlator can also be computed using form factors, and expands as a sum of positive contributions. This gives a good idea of the order of magnitude of neglected terms when only the most important excitations are taken into account: for instance the one breather is exact to order g and is an excellent approximation for values of g up to $g \sim 1/5$ (see below).

The general expression for the current-current correlator is

$$< \partial_{\overline{z}} \phi(z,\overline{z}) \partial_{z'} \phi(z',\overline{z}') >= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{\epsilon_i\}} \int \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} \tag{25}$$

$$R(\theta_i - \theta_B) \cdot |f^z(\theta_1, ..., \theta_n)|^2 \ e^{-(\overline{z}+z')(M_{\epsilon_1} e^{\theta_1} + ... + M_{\epsilon_n} e^{\theta_n})}$$

where f^z is the form factor of the right moving current operator and the dot product means that there is a contraction between the isotopic indices of the R matrices and that of the form factors.

The one breather has mass $M_1 = 2\sin(\frac{\pi g}{2-2g})$ and using the results of [9] we find the intensity to be

$$i(x) = -\int_0^\infty \frac{d\omega}{4\pi} \omega \left[K_1 (\log(\frac{\omega}{M_1 T_B})) - 1 \right] e^{-2\omega x} \tag{26}$$

where for convenience we have defined $T_B = T_K/\tan(\pi g/2(1-g))$, and K_1 is related to the reflection matrix for the breather by $K_1(\theta) = R_1(i\frac{\pi}{2} - \theta)$, where

$$R_1(\theta) = \frac{\tanh(\frac{\theta}{2} - i\frac{\pi}{4}(\frac{g}{(1-g)})}{\tanh(\frac{\theta}{2} + i\frac{\pi}{4}(\frac{g}{(1-g)})}. \tag{27}$$

The 'power spectrum' $s(\omega)$ thus takes a very simple form

$$s(\omega) = i\frac{\gamma}{2\pi}\left(\frac{\omega^2}{\omega^2 - \widehat{\omega}_0^2 + i\omega\gamma}\right)e^{-2i\omega x}, \tag{28}$$

where

$$\widehat{\omega}_0 = 2\cos\left(\frac{\pi g}{2-2g}\right)T_K, \qquad \gamma = 2\sin\left(\frac{\pi g}{1-g}\right)T_K. \tag{29}$$

This result reproduces the perturbation theory computation at small g (21). Near resonance $\omega \approx \widehat{\omega}_0$, $s(\omega)$ takes on the familiar Wigner-Weisskopf Lorentzian lineshape, with a Lamb shifted resonant frequency $\widehat{\omega}_0$ and a half-width $\gamma/2$:

$$s(\omega) = \frac{\gamma^2}{8\pi}\left(\frac{\widehat{\omega}_0}{(\omega-\widehat{\omega}_0)^2 + (\gamma/2)^2}\right)e^{-2i\omega x}. \tag{30}$$

We remark that since γ is twice the imaginary part of the pole of R_1 in the momentum variable $k = M_1 T_B e^{\theta}$, in a sense γ represents the exact decay rate into the first breather. In the small g limit, the first breather is identified with the photon and thus γ agrees with the lowest order perturbative result $\gamma = \pi g \omega_0$.

One can check that this one-breather approximation is better than the latter perturbative result. Actually, it turns out that it is an excellent approximation for the correlators for values of g at least up to $g = 1/5$! Consider for instance the case $g = 1/5$. Writing $s(\omega) = \frac{\omega}{4\pi}e^{-2i\omega x}F(\omega/T_B)$, where F is a universal function, let us compare the contributions to the imaginary part of F coming from other intermediate states. (The contributions for the real part have similar properties). This can be done by using explicit formulas for form factors deduced from [10]. We display in figure 1 the next dominant contributions, i.e. the one particle form factor of the breather $n = 3$ and the two particle soliton/anti-soliton form factor. It is clear from the figure that these contributions, which can barely be seen on the graph, are negligible when compared to the one breather contribution for most of the range. In the deep IR, when $\omega/T_B > 10$, then the soliton/anti-soliton form factor starts to contribute (indeed it is necessary to get the correct IR exponent) but the function F is very small in that limit and this is certainly not the dominant region. The perturbative expression is also given on the graph to give an idea of the differences. The g dependence of the correlator is then entirely included in the reflection matrix of the breather (27) and is very easy to extract from the previous formulae.

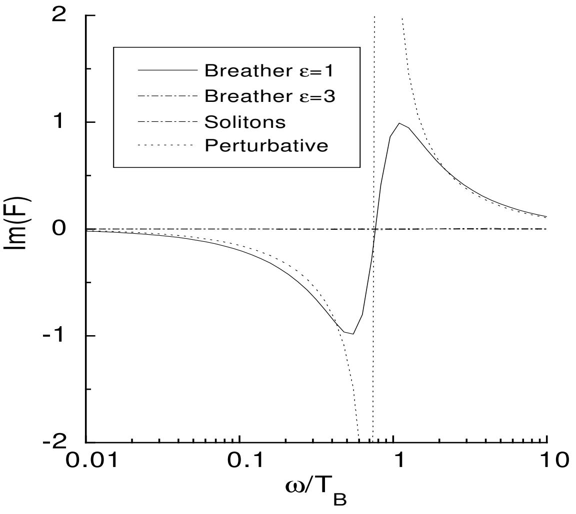

FIG. 1. Accuracy comparison for various form factor contributions for $g = 1/5$.

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