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Some Features of (0,2) Moduli Space

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We discuss some aspects of perturbative $(0,2)$ Calabi-Yau moduli space. In particular, we show how models with different $(0,2)$ data can meet along various sub-loci in their moduli space. In the simplest examples, the models differ by the choice of desingularization of a holomorphic V-bundle over the same resolved Calabi-Yau base while in more complicated examples, even the smooth Calabi-Yau base manifolds can be topologically distinct. These latter examples extend and clarify a previous observation which was limited to singular Calabi-Yau spaces and seem to indicate a multicritical structure in moduli space. This should have a natural F-theory counterpart in terms of the moduli space of Calabi-Yau four-folds.

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1. Introduction

Recent developments have yielded tremendous insights into numerous areas of string theory. The physics of left-right symmetric Calabi-Yau string compactifications is one such area in which our understanding has been sharply refined through the discovery of a variety of remarkable physical phenomenon. As has long been recognized, these $(2,2)$ string models are but a small subclass of the more general $(0,2)$ models initially introduced in [1]. After a number of years of limited progress, the work of [2] has given rise to renewed hope that our understanding of $(0,2)$ Calabi-Yau string models may one day be on par with that for $(2,2)$ models. This hope has received support from recent developments in string duality, most notably F-theory. In particular, F-theory on a special class of Calabi-Yau four-folds is dual to heterotic $(0,2)$ vacua on Calabi-Yau three-folds, with the left-right symmetric geometric moduli of the former capturing the left-right asymmetric moduli of the latter. Fully exploiting this tool, though requires a more precise dictionary between the two than has presently been achieved, so we shall perform our analysis directly in the $(0,2)$ heterotic models themselves. It would be of great interest to deepen our explicit understanding of this dictionary in order to be able to approach the kind of questions we study here in the context of four-folds [1].

The present paper is concerned with a few unusual features of the moduli space of $(0,2)$ Calabi-Yau string compactifications. Recall that a perturbative $(0,2)$ model is specified by a choice of a Calabi-Yau manifold M and a holomorphic vector bundle V satisfying

$$c_1(V) = 0 \tag{1.1}$$

and

$$c_2(V) = c_2(T) \tag{1.2}$$

[1] Very recently, a number of new results [3] have sharpened the F-theory/heterotic dictionary. We have not determined, as yet, how the work presented here relates to those results.

where T is the holomorphic tangent bundle of M [2]. One immediate solution to these conditions is $V = T$ and this yields the familiar $(2,2)$ case. Our interest is on solutions which are not of this special form. Progress in studying these more general solutions was hampered by the substantial complexity and relative paucity of known examples. Witten's linear sigma model provides a powerful tool for straightforwardly generating new solutions. As we will briefly review in section II, by working at the level of two-dimensional non-conformal quantum field theories, the linear sigma model dispenses with much of the complexity of the associated conformal fixed point theory, while retaining many of its essential features. Furthermore, models derived from the linear sigma model approach, as argued in [4] are particularly robust as they avoid instanton destabilization. For these reasons, all of our remarks will be confined to $(0,2)$ models with a linear sigma model realization.

In [2], the global discussion of both $(2,2)$ and $(0,2)$ models in an irreducible toric variety such as a hypersurface in a weighted projective space was limited, for the most part, to a one-dimensional slice through the Kähler moduli space. Typically, this slice probes a family of singular Calabi-Yau spaces. Through the link between the physics of linear sigma models and the mathematics of toric geometry, it was shown in [5,6] how to systematically include the other Kähler moduli in the context of $(2,2)$ models and thereby gain access to the full Kähler moduli space [3]. As shown through local analysis in [2] and global analysis in [7], inclusion of these other degrees of freedom in $(2,2)$ models yields important physical phenomenon such as topology changing flop transitions in left-right symmetric compactifications. With the importance of studying the properties of the full Kähler moduli space of $(2,2)$ models thereby established, a natural direction to pursue is a similar study of $(0,2)$ models. This is a more delicate question since the extra modes included must not only resolve

[2] Generally $c_1(V)$ is only required to vanish mod 2 to admit spinors. This additional freedom has not as yet been explored in any detail.

[3] We note that this approach actually gives access to the full toric part of the Kähler moduli space, which is sometimes a proper subspace of the full Kähler moduli space.

singularities in the Calabi-Yau base M, but they must also repair singularities in the vector bundle V. In [8], this problem was studied, and a fairly systematic solution was offered. The new features of bundle singularities was addressed in the context of the linear sigma model. Amongst the most striking new local features encountered are examples in which the nonlinear sigma model gauge data is naturally described in the language of sheaves, with corresponding "protuberance" structures in the associated linear sigma model. In the present paper our interest is in the more global aspect of these $(0,2)$ models.

In section II we review the method of [8] for resolving a class of the perturbative singularities which can arise in $(0,2)$ models. We frame this approach by discussing, in section III, the kinds of singularities — both perturbative and nonperturbative — which can arise in the $(0,2)$ setting, and contrast the situation with that relevant for $(2,2)$ models. In section IV we focus on a variety of examples which serve to illustrate some novel features of resolved $(0,2)$ moduli space. We discuss, for instance, examples in which there are multiple resolutions of given singular geometric data. Our prime focus, however, is on carrying forth an observation made in [9] regarding the possibility of a novel duality between various $(0,2)$ models. These original observations were left somewhat inconclusive as only models along singular loci in moduli space were discussed. Applying the desingularizing methodology of [8] to these examples proves to be a bit subtle; however we are able to construct new examples which avoid these complexities. In this way we are able to clarify and extend our understanding of these striking dual examples. In section V we briefly describe the constraints which stability seems to require, and finally we offer some conclusions in section VII.

2. Constructing Resolved $(0,2)$ Models

In this section we briefly review the algorithm put forth in [8] for desingularizing a particular class of $(0,2)$ models.

As mentioned above, the data for a $(0,2)$ model consists of a Calabi-Yau base M and a vector bundle V. The typical class of Calabi-Yau spaces dealt

with in the literature over the years are complete intersections in toric varieties. Of these, the most familiar are those in which the initial toric variety is a weighted projective space, or possibly a product of such spaces. These are the ones we shall consider. The essential point is that weighted projective spaces in which the individual weights are not all relatively prime have singularities from the projective identifications. The full parameter space of the corresponding physical model is found only by resolving these singularities. Bundles — or more precisely V-bundles — over the original singular space must correspondingly be augmented by data specifying their behaviour over regions of the base which are modified by the desingularizing procedure. All the while, attention must be paid to avoid spoiling any of the delicate geometric and topological consistency requirements to have a genuine solution.

To accomplish this task we use the formalism of Witten's linear sigma model augmented by methods from toric geometry. To keep the notation from getting out of hand, we consider a Calabi-Yau hypersurface M in a single weighted projective space, $\mathrm{WCP}^{n-1}_{q_1,q_2,\ldots,q_n}$. The procedure for defining a $(0,2)$ model on M is as follows:

One considers right moving $(0,2)$ chiral superfields P (with scalar and fermionic components p and π) and $\Phi_1,...,\Phi_n$ (respectively with scalar and fermionic components ϕ_i and ψ_i) with U(1) gauge charges $q_0,q_1,...,q_n$, as well as left moving $(0,2)$ fermi multiplets Γ (with first component γ) and $\Lambda_1,...,\Lambda_m$ (respectively with first component λ_i) with charges $\widetilde{q}_0,\widetilde{q}_1,...,\widetilde{q}_m$.

The superpotential then takes the following form:

$$\int d^2z d\theta(\Gamma G(\Phi_i)+\sum_a \Lambda^a P F_a(\Phi_i)) \tag{2.1}$$

where G and F_a $(a=1,...,m)$ are homogeneous polynomials. $U(1)$-invariance dictates that the degrees of G and F_a $(a=1,...,m)$ are respectively $-\widetilde{q}_0$, $-\widetilde{q}_1-q_0,...,-\widetilde{q}_m-q_0$.

2.1. Yukawa couplings and anomaly cancellation

The gauge charges of these fields are not fully arbitrary as they are subject to anomaly cancellation conditions. To understand the geometrical meaning of these conditions, we examine the various Yukawa-couplings in the action.

Let us first consider the right moving fermions. From (2.1) we obtain the coupling:

$$\sum_i \gamma\psi_i \frac{\partial G}{\partial\phi_i} \tag{2.2}$$

This gives mass to a linear combination of the ψ_i's, if the Calabi-Yau is transverse in the sense that $\frac{\partial(\gamma G)}{\partial\phi_i}d\phi_i$ [4] does not vanish on the manifold. Another coupling with the gaugino α of the bosonic gauge symmetry:

$$-\sum_i q_i\alpha\psi_i\overline{\phi_i} - q_0\alpha\pi\overline{p} \tag{2.3}$$

gives mass to a different linear combination of the ψ_i's, provided p vanishes (as in a typical Calabi-Yau phase). These two combine to imply that the massless right-moving fermions transform as sections of the cohomology of the following sequence ("monad"):

$$0 \longrightarrow \mathcal{O} \stackrel{\otimes q_i\phi_i}{\longrightarrow} \bigoplus_{i=1}^{n}\mathcal{O}(q_i) \stackrel{\otimes\frac{\partial G}{\partial\phi_i}}{\longrightarrow} \mathcal{O}(-\widetilde{q}_0) \longrightarrow 0 \tag{2.4}$$

In simple terms, this means that they transform as sections of the tangent bundle T. As the Calabi-Yau condition requires that $c_1(T)=0$, we obtain the following condition on the charges:

$$\sum_{i=1}^{n} q_i = -\widetilde{q}_0 \tag{2.5}$$

The analysis for the left-moving fermions follows a similar sequence of steps. From the superpotential we obtain the coupling

$$\sum_a \pi\lambda^a F_a \tag{2.6}$$

[4] This form easily generalizes to complete intersections.

This gives mass to a certain combination of λ^a's provided that a similar transversality condition holds, i.e. that the F_a's do not simultaneously vanish anywhere on the manifold. (2.6) is the analog of (2.2) in the left-moving case. To see a similar analog for (2.3), we recall that in a $(2,2)$ model, the gauge multiplet may be decomposed as a combination of a bosonic and fermionic $(0,2)$ gauge multiplet. Our interest being in $(0,2)$ models which are generally not holomorphic deformations of their $(2,2)$ cousins, we are led to a notion of *fermionic gauge* symmetries that can be incorporated independently of the bosonic symmetries. The introduction of l such symmetries [10] requires the existence of $l(m+1)$ homogeneous polynomials E^a_j , $a=0,...,m$, $j=1,...,l$ which form part of the action on the fermi multiplets:

$$\Gamma \to \Gamma + 2PE^0_j(\Phi)\Omega^j \qquad \Lambda^a \to \Lambda^a + 2E^a_j(\Phi)\Omega^j \tag{2.7}$$

where the Ω^j's are $U(1)$-neutral fermionic chiral multiplets. Often times, the fermionic symmetries are chosen in order to obtain a sensible Landau-Ginzburg interpretation, so that, for instance, there are no chargeless fermions in the Landau-Ginzburg phase. Invariance of (2.1) under (2.7) then implies

$$E^0_j G + \sum_{a=1}^{m} E^a_j F_a = 0 \tag{2.8}$$

Assuming we can find such E's, we still have to ensure the invariance of the left-moving fermion kinetic terms under (2.7). To do so, we introduce l additional *unconstrained* $U(1)$ neutral fermionic superfields Σ^j transforming as $\Sigma^j \to \Sigma^j + \Omega^j$ and couple them appropriately to the other fields in the action. For our purposes, it is essential to know only the two main consequences of doing this. First of all, a scalar potential of the following form arises:

$$\sum_j\sum_k(|p|^2\overline{E^0_j(\phi_i)\sigma_j}E^0_k(\phi_i)\sigma_k + \sum_{a=1}^{m}\overline{E^a_j(\phi_i)\sigma_j}E^a_k(\phi_i)\sigma_k) \tag{2.9}$$

where σ_j is related to the gauge invariant quantity $\overline{D}_+\Sigma^j$ by

$$\overline{D}_+\Sigma_j = \frac{1}{2}(\sigma_j + \theta^-\beta_j + \theta^-\theta^+\partial_{\overline{z}}\sigma_j) \tag{2.10}$$

Secondly, the Σ_j couplings also contain a Yukawa coupling of the form

$$\sum_j \overline{\beta}_j(\overline{pE_j^0(\phi_i)}\gamma + \sum_{a=1}^m \overline{E_j^a(\phi_i)}\lambda_a) \tag{2.11}$$

which, in a typical Calabi-Yau phase where p has no expectation value, gives mass to l combinations of the λ's, provided that $(E_j^1, ..., E_j^m)$ form l linearly-independent vectors at all points on the manifold. This gives us the desired analog of (2.3). Combined with (2.6), we therefore see that the massless left-moving fermions transform as sections of the bundle V over M defined by the cohomology of the following sequence:

$$0 \longrightarrow \oplus^l \mathcal{O} \stackrel{\otimes E_j^a}{\longrightarrow} \oplus_{a=1}^m \mathcal{O}(\widetilde{q}_a) \stackrel{\otimes F^a}{\longrightarrow} \mathcal{O}(-q_0) \longrightarrow 0 \tag{2.12}$$

Equations (1.1) and (1.2) then gives us the remaining conditions on the charges[5].

$$q_0 = -\sum_{a=1}^m \widetilde{q}_a \tag{2.13}$$

$$\sum_{i=1}^n q_i^2 = \sum_{a=1}^m \widetilde{q}_a^2 \tag{2.14}$$

From the point of view of field theory, (2.14) is the condition that $U(1)$ gauge symmetry is nonanomalous. (2.5) and (2.13) are the conditions that ensure the existence of a nonanomalous global $U(1) \times U(1)_R$ symmetry, which becomes the left- and right-moving $U(1)$s in the conformal limit. Before generalizing these conditions as appropriate for a *nonsingular* base, we briefly review the basic features of the 1-dimensional Kähler moduli space.

[5] The second condition is somewhat stronger than just requiring that the Chern classes be equal since it amounts to ignoring relations between the ambient projective cohomology classes restricted to M. It also however follows from cancelling the worldsheet gauge $U(1)$ anomaly.

2.2. The one-parameter moduli space

Here the typical complete $(0,2)$ bosonic potential is:

$$U = |G(\phi_i)|^2 + |p|^2 \sum_a |F_a|^2 + \frac{1}{2e^2} D^2 + |\sigma|^2 (|p|^2 |E^0(\phi_i)|^2 + \sum_a |E^a(\phi_i)|^2) \tag{2.15}$$

where

$$D = -e^2 (\sum_{i=1}^n q_i |\phi_i|^2 + q_0 |p|^2 - r) \tag{2.16}$$

Now the q_i's are positive since they are the weights of the the ambient projective space. Furthermore the $\widetilde{q}_a$'s for $a = 1, .., m$ are positive because otherwise the bundle over the (possibly singular) base as defined in (2.12) would not be stable. Thus from (2.13) we obtain that q_0 is negative. The semiclassical vacuum is found by setting $U = 0$ which implies that D and G vanish and that either p or all the F's must vanish. Let us analyze this in detail, first for the case when $r > 0$.

In this case, D vanishing means not all the ϕ's can vanish. Hence as G vanishes and one of our assumptions was that the F's do not simultaneously vanish at any point in the manifold, p must also vanish. Furthermore we also assumed that the E^a's for $a = 1, ..., m$ do not simultaneously vanish when G does, which means from (2.15) that the σ must vanish. Looking at the D-term again then gives $\sum_i q_i |\phi_i|^2 = r$, which, after $U(1)$ identifications, is just a copy of $\mathrm{WCP}^{n-1}_{q_1,...,q_n}$. The locus $G = 0$ then defines a Calabi-Yau in this weighted projective space. Actually, since p transforms nontrivially under the gauge group it is natural to consider the manifold as embedded in the total space of $\mathcal{O}(q_0)$ over $\mathrm{WCP}^{n-1}_{q_1,...,q_n}$. The latter space is nothing but the toric variety found by setting D to zero modulo gauge equivalences without any further constraints. The resulting model is therefore the Calabi-Yau sigma model on M with the left-moving fermions transforming as sections of V.

One apparent problem with the previous analysis is that, unlike in $(2,2)$ models, there is no relation between q_0 and $q_1, ..., q_n$. The charge relations (2.5) and (2.13) relate q's and $\widetilde{q}$'s instead. This would contribute to a 1-loop renormalization of r proportional to $q_{tot} = \sum_{i=0}^n q_i$ so that the theory would be strongly renormalized from our semiclassical treatment above. A means of avoiding this was described in [10] and amounts to introducing a pair of chiral superfields, termed "spectator" superfields, S and Ξ with charges $-q_{tot}$ and q_{tot} respectively. They contribute to the superpotential a term of the form ΞS, which means that they are massive and can be set to zero in determining the classical vacuum at any RG-invariant radius. However, their 1-loop contribution cancels the renormalization of r. Effectively, the linear sigma model trades the two parameters (the coefficient of ΞS in the superpotential and $r + i\theta$) for a single RG-invariant parameter, which we will call $r + i\theta$. In particular, our previous analysis remains valid with the addition of such spectators.

Having thus discussed the scenario for $r > 0$, we turn to the other interesting case of $r < 0$. Here, $D = 0$ implies that p cannot vanish, and hence that the F's must simultaneously vanish, which occurs, as G vanishes as well, only when all the ϕ_i's are zero. This means that $|p|$ is $\sqrt{r/q_0}$. Our vacuum is thus, after $U(1)$ gauge identifications, a single point with $(\phi_1, ..., \phi_n, p) = (0, ..., 0, \sqrt{r/q_0})$. If $E^0(0, ..., 0)$ does not vanish at this point, σ again drops out in the infrared. Actually [2] there is a residual $\mathbb{Z}_{-q_0}$ gauge symmetry in the vacuum, which is embedded naturally in $\mathbb{C}^n/\mathbb{Z}_{-q_0}$. We thus obtain an orbifold of an Landau-Ginzburg model defined over the vacuum.

The Calabi-Yau sigma model and Landau-Ginzburg orbifold regimes described above form only a one-dimensional slice of the full Kähler moduli space of the theory. The other phases are found by including $N = 2$ superconformal marginal operators that can be interpreted as modes that geometrically resolve singularities in the base manifold. Most of these modes are captured naturally in the formalism of toric geometry. A detailed introduction to toric geometry for physicists can be found in [7]; here we will freely make use of this formalism. Toric geometry allows our Calabi-Yau to be associated with a Gorenstein cone in $\mathbb{R}^d$ with k edges. This is reflected in a new linear sigma model with $s = h^{1,1} - 1 = k - d$ extra chiral superfields and $U(1)^{s+1}$ gauge symmetry where $h^{1,1}$ is the hodge number of the resolved space [6]. The charges of the $k = s + d$ chiral superfields are determined by the kernel of the $(k + 1) \times d$ point set matrix formed from the vertices and interior of the polytopic base of the cone [5]. Corresponding to each regular triangulation of the base is a phase of the linear σ-model defined by a region in the parameter space of the $s + 1$ Fayet-Iliopoulos coefficients r_i.

[6] More precisely $s = h^{1,1}_{toric} - 1$ where $h^{1,1}_{toric}$ is the number of divisors that arise from the toric ambient space.

2.3. *Desingularization of* $(0, 2)$ *models*

The $(2, 2)$ toric formalism thus naturally allows for desingularization of the base, and along with it the desingularization of the tangent bundle. In the $(0, 2)$ case however we must supply more information to pull the original V-bundle V back to the nonsingular base. In [8] a straightforward approach for doing this was proposed: if the resolved base has a $h^{1,1}$ dimensional Kähler moduli space, then line bundles over it — which can be classified by their first Chern class — are specified by the equivalent data of $h^{1,1}$ integers: their charges under each of the $h^{1,1}$ $U(1)$ gauge factors. These charges must be such as to meet conditions (1.1) and (1.2), which are physically equivalent to the cancellation of all anomalies in the linear sigma model. Thus, the desingularization of the original V-bundle is obtained by searching for $h^{1,1} - 1$ additional charges for each line bundle in the definition of the gauge bundle meeting the anomaly cancellation conditions. It would be nice to have a more intrinsic procedure, but this is our present formulation.

So to summarize, the steps needed in defining a $(0, 2)$ model on a hypersurface M in a weighted projective space are as follows:

- Construct the appropriate toric data for the base, giving us the the charges of the left-moving fermion Γ and n right-moving bosons ϕ_i under a $U(1)^s$ gauge group. The set of charges can be denoted by $\widetilde{q}_0^{(k)}$ and $q_i^{(k)}$ where i runs from 1 to n and k from 1 to s.
- Construct an appropriate set of charges $q_0^{(k)}$ and $\widetilde{q}_a^{(k)}$ where a runs from 1 to m satisfying the following anomaly cancellation conditions which are a generalization of (2.13) and (2.14):

$$\sum_{a=1}^{m} \widetilde{q}_a^{(k)} + q_0^{(k)} = 0 \tag{2.17}$$

$$\sum_{a=1}^{m} \widetilde{q}_a^{(j)} \widetilde{q}_a^{(k)} + \widetilde{q}_0^{(j)} \widetilde{q}_0^{(k)} = \sum_{i=1}^{n} q_i^{(j)} q_i^{(k)} + q_0^{(j)} q_0^{(k)} \tag{2.18}$$

where j and k run from 1 to s. The charges $q_0^{(k)}$ is to be interpreted as that of a right-moving bosonic superfield P and $\widetilde{q}_a^{(k)}$ are the charges of m left-moving fermionic superfields Λ^a.

• Choose a number l (which can be 0) of fermionic gauge symmetries, and determine suitable E's and F's satisfying (2.8). The massless left-moving fermions thus transform under a vector bundle V which is defined by the following two *exact* sequences (as a generalization of (2.12)):

$$0 \longrightarrow \oplus^l \mathcal{O} \stackrel{\otimes E_j^a}{\longrightarrow} \oplus_{a=1}^{m} \mathcal{O}(\widetilde{q}_a^{(1)}, \widetilde{q}_a^{(2)}, ..., \widetilde{q}_a^{(s)}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{2.19}$$

$$0 \longrightarrow V \longrightarrow \mathcal{E} \stackrel{\otimes F_a}{\longrightarrow} \mathcal{O}(-q_0^{(1)}, -q_0^{(2)}, ..., -q_0^{(s)}) \longrightarrow 0 \tag{2.20}$$

The maps above are as follows:

$$\otimes E_j^a : (c_1, ..., c_l) \to (\sum_{j=1}^{l} c_j E_j^1(\phi_i), ..., \sum_{j=1}^{l} c_j E_j^m(\phi_i)) \tag{2.21}$$

$$\otimes F_a : (\xi_1(\phi_i), ..., \xi_m(\phi_i)) \to \sum_{a=1}^{m} \xi_a(\phi_i) F_a(\phi_i) \tag{2.22}$$

3. Singularities in $(0,2)$ Models

Singularities in a linear σ-model are generally detected by searching for noncompact directions in the classical vacuum. The rationale for this follows, roughly speaking, from the possibility of bad behaviour in the path integrand used in calculating correlation functions at large values of the relevant *noncompact* fields, i.e. fields that have a noncompact range in the σ-model vacuum. A better indication of a singularity is a non-compact direction in the full quantum effective potential [2]. For the most part we shall ignore these complexities and base our analysis on the classical potential; these aspects deserve further study.

In $(2,2)$ models, perturbative singularities are known to arise, for a one-parameter case, when the complex Fayet-Iliopoulos parameter vanishes, classically, (more precisely, when it is equal to a particular nonzero value, taking one-loop linear sigma model corrections into account) [2] or when the manifold becomes singular. In the former case, the noncompact field is the gauge field σ while in the latter it is a right-moving boson p. They correspond respectively to Kähler and complex structure degenerations.

For $(0,2)$ models, the story has some similarities as well as some differences. In particular, the presence or absence of fermionic gauge symmetries is a crucial factor. In the $(2,2)$ case, if the bosonic gauge fields were absent in the scalar potential [2], there would be no singularities in the Kähler moduli space of a smooth Calabi-Yau M. Similarly, in the $(0,2)$ case, if there are no fermionic gauge symmetries, singularities can only possibly occur when the bundle is *nontransverse*, i.e. when the F's simultaneously vanish at a point on the "manifold". This means that, for instance, there are a priori no physical singularities associated with a base space having bad *complex structure*, so long as none of the bosonic fields acquire a noncompact range. The lesson we learn from this is that the natural arena of $(0,2)$ models is geometrically realized in terms of coherent sheaves over base spaces that can be relatively badly behaved.

When fermionic gauge symmetries are present, a new potential difficulty arises. Given a model defined as a monad over the singular manifold, when we resolve the manifold, we may find that there is no choice of E's on the *resolved* manifold, M, for which the first map in (2.19) is injective. In that case, the resulting (0,2) theory is, classically, *generically* singular, for there is a noncompact branch to the vacuum, corresponding to some σ_j having a flat potential. It might be that these models require non-perturbative effects to be taken into account everywhere in their moduli space, or it might be that they are only physically sensible before resolving the geometrical singularities. Although each possibility is quite interesting, in this paper, we will skirt the issue by eschewing such models.

When we have a Landau-Ginzburg phase, one can sometimes show that fermionic gauge symmetries are required for consistency. In the absence of a

Landau-Ginzburg phase (or a detailed check of the stability of the bundle), we need to make some assumptions about what fermionic gauge symmetries are present. We will assume that the only ones which are present are those which lead *classically* to a generically-nonsingular (in the sense that there are, generically, no noncompact directions in the vacuum) model on the resolved manifold. We should point out that we presently lack a first principles argument for which fermionic gauge symmetries a particular model requires; the assumption we make here is consistent with our present level of understanding but deserves further investigation.

Having thus summarized what is to come, we begin with a detailed discussion by examining the bosonic potential for a $(0,2)$ model based on a Calabi-Yau hypersurface M in $\mathrm{WCP}^{n-1}_{q_1,\ldots,q_n}$. Similar conclusions can be obtained for other models based on more general complete intersections in toric varieties.

3.1. Compactness of p

Let us first determine when, in the notation of the previous section, any of the ϕ's can become noncompact. This is for our purposes actually equivalent to determining when p can become noncompact, because in all the examples described in this paper, there is always a special basis of the gauge group $U(1)^s$ under which all the ϕ's have nonnegative multi-charges, while p and γ have negative multi-charges. This is certainly true for most $(2,2)$ models, for instance. Equation (2.15) then clearly shows that if some ϕ_i were to grow arbitrarily while maintaining $U=0$ at a fixed point in the Kähler moduli space, p must also grow with ϕ_i to maintain the vanishing of the D-terms. Thus we need only consider when p or the σ_j's could become noncompact.

We examine first the case with p. To be concrete, consider a *typical* smooth Calabi-Yau phase. From (2.15) we see that p can grow arbitrarily large only where G as well as all the F's simultaneously vanish in the ambient toric variety. Here the ambient toric variety is the total space of a negative line bundle (which p transforms under) over a compact toric variety C in which the Calabi-Yau M is embedded. If the F's simultaneously vanish at some locus S in C, then p would become noncompact over $S\cap M$. If this happens, we then obtain

a singularity in the gauge bundle which is the $(0,2)$ analog of being on the discriminant locus of a $(2,2)$ model. Indeed, in [4] it was shown that for $(0,2)$ deformations of the quintic, the correlation function of 3 **27**'s develop a simple pole at such a singularity. In the case where M is a $K3$ and $S\cap M$ consists of a set of points, this has been understood from the point of view of nonperturbative gauge symmetries in 6 dimensions [11].

While the discussion just given holds for a typical Calabi-Yau, one may wonder if it leads to similar conclusions for all $(0,2)$ vacua with an obvious geometrical interpretation. The answer is no, and the reason is that the discussion just given relied on p being unconstrained by the D-terms at all points of C. In some cases, however, as is familiar from studies of certain $(2,2)$ models, the range of p can be forced to be compact over a toric subvariety Z in C. If the F's vanish only on Z, then there would be no singularity in the underlying linear σ-model as p is then compact over M. In the $(2,2)$ case, the relevant phases where this occurs are related to ordinary Calabi-Yau phases in the same moduli space by a flop in a *noncompact* direction [12]. In the $(0,2)$ case the situation is similar, an example being given by page 19 of [8]. In that example, the degrees of the F^a's were *not generic* enough to allow for avoidance of gauge bundle singularities on the base. Nonetheless, the D-terms constrain p to be compact over the common vanishing locus of the F's. We may thus view the compactness of p as the linear σ-model's way of coping with what would have been an "unavoidable" singularity. V, as it turns out, is then no longer a vector bundle but rather a coherent sheaf. We will see later that sheaves commonly occur as perturbative vacua in $(0,2)$ models. For at least some classes of examples in which p actually is non-compact, [13] have shown that there is a non-perturbative mechanism leading to a sensible physical theory. We will not consider such examples in this paper.

3.2. Compactness of σ_j

From (2.9), we see that if the matrix M whose elements are given by $M_{jk}=|p|^2\overline{E^0_j}E^0_k+\sum_a\overline{E^a_j}E^a_k$ is not positive definite, there will be a noncompact direction in the space of σ_j's. This yields a situation analogous to a Kähler degeneration in a $(2,2)$ model, which occurs at non-generic points on the walls of a Kähler cone in the enlarged Kähler moduli space. In the $(0,2)$ case, though, there are some differences. The essential difference for our present discussion is that Kähler degenerations in $(2,2)$ models occur at complex codimension one in the enlarged Kähler moduli space, at those loci where the (2,2) linear sigma model gets an unbroken $U(1)$ gauge symmetry. More generally, in complex codimension n, the linear sigma model has an unbroken $U(1)^n$, and there are n noncompact directions in the vacuum.

In $(0,2)$ models, on the other hand, the singularity loci in the moduli space, associated to zero eigenvalues of the matrix M, do not directly track the unbroken gauge symmetry points (as they do in the $(2,2)$ case). This means that the number of zero eigenvalues (the number of noncompact σ-directions) is largely decoupled from the rank of the unbroken gauge symmetry and can only be determined by direct calculation.

This is important. While we will continue to speak of the "phases" of the (0,2) model, and the walls of the Kähler cone, at which some $U(1)$ gauge symmetry is restored, these are *not* necessarily the locations of singularities of the (0,2) model. *Those* are determined by the fermionic gauge symmetries and the associated coupling to σ. In the simplest examples – those with no fermionic gauge symmetries – there are no singularities in the Kähler moduli space.

One of the tricky matters in formulating a (0,2) linear sigma model is deciding when, and how many, fermionic gauge symmetries ought to be present. Sometimes the answer is forced upon you by consistency considerations. Clearly, if the bundle V defined by (2.19) and (2.20) is stable for l fermionic gauge symmetries, it will not be stable if we tried to formulate it with $l-1$ fermionic gauge symmetries. Since stability may be hard to check directly, the inconsistency is most often uncovered by examining the spectrum of the putative model at Landau-Ginzburg [14]. Models whose Landau-Ginzburg spectra lead to an anomaly in the discrete R-symmetry in spacetime can often be "cured" by imposing additional fermionic gauge symmetries.

Conversely, one might easily formulate a model with "too many" fermionic gauge symmetries. In that case, one might find that, at a generic point in the

Kähler moduli space, it is impossible to find a choice of E's so that the matrix M is everywhere nondegenerate on the space of zeroes of the scalar potential. In that case, one finds that some σ's become noncompact in codimension *zero*. That is, the *generic* point in the toric moduli space gives rise to a singular (0,2) model.

This singularity is often easiest to see in a phase in which p is nonzero. Then, G and all the F's must vanish in the vacuum as well. We would now like to consider if all the E's for *any* fermionic gauge symmetry might also vanish. If this happens, then all the σ's would become noncompact, and we would have a singularity in the linear σ-model. The main strategy to establish that some examples exist, therefore, is to show that the degrees of F's, G's and the E's corresponding to an arbitrary fermionic gauge symmetry are not generic enough to prevent them from vanishing *in* the ambient toric variety. We illustrate this in a specific example:

Let M be the complete intersection of a quartic and a quintic in $\mathrm{WCP}^5_{1,1,1,2,2,2}$. The data for a $(0,2)$ model based on M is as follows:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	χ	p
2	2	2	1	1	1	0	-8
1	1	1	0	0	0	1	-3

Λ_1	Λ_2	Λ_3	Λ_4	Γ_1	Γ_2
0	1	1	6	-4	-5
1	0	0	2	-2	-2

E'_1	E'_2	E_1	E_2	E_3	E_4	F_1	F_2	F_3	F_4	G_1	G_2
4	3	0	1	1	6	8	7	7	2	4	5
1	1	1	0	0	2	2	3	3	1	2	2

For convenience we have indicated the degrees of the various polynomials required. pE'_1 and pE'_2 correspond to the functions used in fermionic gauge transformations of Γ_1 and Γ_2 respectively. There are 2 hybrid phases in the $(0,2)$ phase diagram of this model, corresponding respectively to $\phi_1, \phi_2, \phi_3, \chi$ or ϕ_4, ϕ_5, ϕ_6 not simultaneously vanishing in the toric variety. Our aim is to show that in each phase one cannot choose polynomials specified in the bottom table that simultaneously vanish only outside the ambient space. Let us first consider the phase where ϕ_1, ϕ_2, ϕ_3 and χ cannot be all zero. Here, because of transversality in the ordinary Calabi-Yau phase ($r_1 > 0$, $r_1 - 2r_2 > 0$), $\phi_4 = \phi_5 = \phi_6 = 0$. It can be seen from the D-terms that the remaining bosonic fields form a $\mathrm{WCP}^3_{1,1,1,4}$. This means that, among the E's, F's and W's, the only ones that are nonvanishing are those that can be written only in terms of ϕ_1, ϕ_2, ϕ_3 and χ. It is clear from the tables above that such a polynomial must have multi-degree (m, n) where $m \leq 2n$. This leaves only G_1, F_4 and E_1, which are sufficient only to cut $\mathrm{WCP}^3_{1,1,1,4}$ to a single point at best. The proof for the other phase is even simpler. Here similarly we have a $\mathbb{P}^2$ of points where ϕ_1, ϕ_2, ϕ_3 and χ simultaneously vanish. This means that functions which are expressible in terms of only ϕ_4, ϕ_5 and ϕ_6 will not vanish, which leaves only E_2 and E_3, which again at best cuts $\mathbb{P}^2$ down to a point. Thus we see in this case that even if we can actually construct E's and F's which satisfy (2.8), they will simultaneously vanish, and hence lead to noncompact σ's, at some loci in the hybrid vacuum.

It appears that this is a model which is generically singular at the level of perturbative string theory. Perhaps some nonperturbative string effects might cure the disease, but in this paper, we will be content with understanding models that are closely related to these, but avoid the subtlety of possibly having an important dependence on higher order quantum effects. The example just discussed serves to illustrate our approach. The point is that, in this example, *no choices* of G's or F's will give an actual Landau-Ginzburg phase anywhere in the Kähler moduli space. This again follows from a similar degree-matching argument as above, except now only with the F's and G's. In the phase where ϕ_1, ϕ_2, ϕ_3 and χ cannot simultaneously vanish, we can at most cut the $\mathrm{WCP}^3_{1,1,1,4}$ with G_1 and F_4 down to a curve. Similarly for the other phase, where all of the G's and F's must vanish on the $\mathbb{P}^2$ given above.

This suggests that whereas one naively would have assumed a fermionic gauge symmetry was necessary due to Λ_1 being uncharged under the first $U(1)$, it isn't actually the case. Nowhere in the phase diagram is there a Landau-Ginzburg phase where the gauge group is actually broken to leave a discrete

subgroup of the first $U(1)$, as would be typical in $(2,2)$ models. What one finds instead are two hybrid phases, the transition from one to the other involving a *gauged* Landau-Ginzburg theory. In such a theory one has an unbroken $U(1)$ even though the vacuum is given by the typical Landau-Ginzburg form of fixed p and vanishing ϕ's and χ. Further studies of the conformal field theory of hybrids or gauged Landau-Ginzburg vacua may confirm that the model as defined by the data above gives rise to a sensible spectrum. Here we will assume that is the case for some of the examples to be discussed later.

In summary, we consider only those models for which the fermionic gauge symmetries are such as to allow the model to be generically nonsingular on the full toric moduli space.

3.3. Coherent Sheaves and $(0,2)$ *models*

In section 1 we saw that, if a model is sufficiently well behaved, the massless left moving fermions would transform as sections of the bundle defined by (2.19), (2.20) over a Calabi-Yau base. If the first map in (2.19) is not one-to-one, then we need to modify it by the kernel as follows:

$$0 \longrightarrow \oplus^l \mathcal{O}_K \longrightarrow \oplus^l \mathcal{O} \xrightarrow{\otimes E^a_j} \oplus_{a=1}^m \mathcal{O}(\tilde{q}_a^{(1)}, \tilde{q}_a^{(2)}, ..., \tilde{q}_a^{(s)}) \longrightarrow \varepsilon \longrightarrow 0 \tag{3.1}$$

where K is the locus on which the map given by the E's fails to be one-to-one. This could occur, for instance, if the E^a_j's for fixed j simultaneously vanish at some loci in the vacuum. In an ordinary Calabi-Yau phase where p vanishes, this would imply that the corresponding σ_j becomes noncompact, a signal of a singularity in the infrared.

Similarly, if the F's simultaneously vanish at some loci S in the vacuum, then (2.20) is no longer exact as the last map is no longer surjective. An exact sequence can be obtained by adding the cokernel of the last map as follows:

$$0 \longrightarrow V \longrightarrow \mathcal{E} \xrightarrow{\otimes F_a} \mathcal{O}(-q_0^{(1)}, -q_0^{(2)}, ..., -q_0^{(s)}) \longrightarrow \mathcal{O}_S(-q_0^{(1)}, -q_0^{(2)}, ..., -q_0^{(s)}) \longrightarrow 0 \tag{3.2}$$

where $\mathcal{O}_S(-q_0^{(1)}, ..., -q_0^{(s)})$ is restricted to S. V is now generally not a vector bundle but rather a coherent sheaf. As discussed previously, in a Calabi-Yau phase, if p nonetheless remains compact — spanning suitable *protuberances* over M — string theory is expected to be well behaved.

We should hence think of $(0,2)$ models as sheaves rather than V-bundles defined over Calabi-Yau spaces. In the next section we show how this is reflected in the complex structure variations of some models.

3.4. *Complex structure variations in* $(0,2)$ *models*

Consider what happens in a $(0,2)$ model over M defined by $G = 0$ as we vary the complex structure of G. Unlike in the $(2,2)$ case, the bosonic potential (2.15) does not generally depend on the derivatives of G. In $(2,2)$ models, these derivatives are crucial in establishing that p is compact in a typical Calabi-Yau phase. A singular G in a $(2,2)$ model would therefore not make perturbative sense. Here, however, the F^a's replace the derivatives of G. It is hence possible that the linear σ-model is largely insensitive to changes in M's complex structure. We now pursue this issue in a clear-cut example.

Consider the example studied in page 19 of [8], which we summarize by the data in the following table:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	χ	p
2	2	2	1	1	0	-9
1	1	1	0	0	1	-5

Λ_1	Λ_2	Λ_3	Λ_4	Γ
1	1	2	5	-8
0	0	2	3	-4

Because no fermionic gauge symmetries are necessary here, we do not need to worry about E's in this model. We now make the following choices for the polynomials defining the manifold and the gauge bundle:

$$\begin{aligned} G &= H(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\chi) \\ F_1 &= \phi_1^4\chi \\ F_2 &= \phi_2^4\chi \\ F_3 &= \phi_4^7\chi^3 \\ F_4 &= \phi_3^2 \end{aligned} \tag{3.3}$$

In the above H is an arbitrary polynomial of multidegree $(8,4)$ which contains a nonzero term in $\phi_5^8\chi^4$. These choices ensure that the bundle is transverse in all phases except one. In the latter phase, as alluded to previously, p nonetheless remains compact over the base. By varying H arbitrarily over the relevant space of polynomials, one encounters various singularities in the base manifold which are *irrelevant* to the linear σ-model. The left-moving fermions here transform as sections of a coherent sheaf V while the right-moving fermions transform as sections of the appropriate tangent sheaf T.

4. Aspects of $(0,2)$ Moduli Space

An important aspect of $(2,2)$ moduli space is that sometimes there isn't a unique desingularization of a Calabi-Yau base. In the $(2,2)$ toric formalism this arises by different triangulations of the toric point set data; correspondingly, in the linear sigma model it arises by different classes of choices for the Fayet-Iliopoulos coefficients. Physically, this gave rise to the first kind of smooth topology changing transitions in string theory. These $(2,2)$ transitions have an immediate $(0,2)$ extension since the bundle part of our desingularization formalism is insensitive to distinct triangulations of the toric base. So, for instance, the $(2,2)$ model with $h^{1,1} = 2$ given by a Calabi-Yau hypersurface in $\mathrm{WCP}^4_{3,2,2,1,1}$ has the phase diagram given in Figure 1.

In figure 1 region II corresponds to an ordinary Calabi-Yau phase. Region I corresponds to a Calabi-Yau phase related to II by a flop. Region III corresponds to a Calabi-Yau orbifold with $\mathbb{Z}_2$ quotient singularities. Region IV is a Landau-Ginzburg phase and region V a hybrid phase. In the geometrical phases the left and right moving fermions both transform as sections of the tangent bundle.

We can now build a $(0,2)$ model on this base whose initial bundle data over the singular Calabi-Yau hypersurface is given by

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	p
3	2	2	1	1	-9

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Γ
1	1	2	2	3	-9

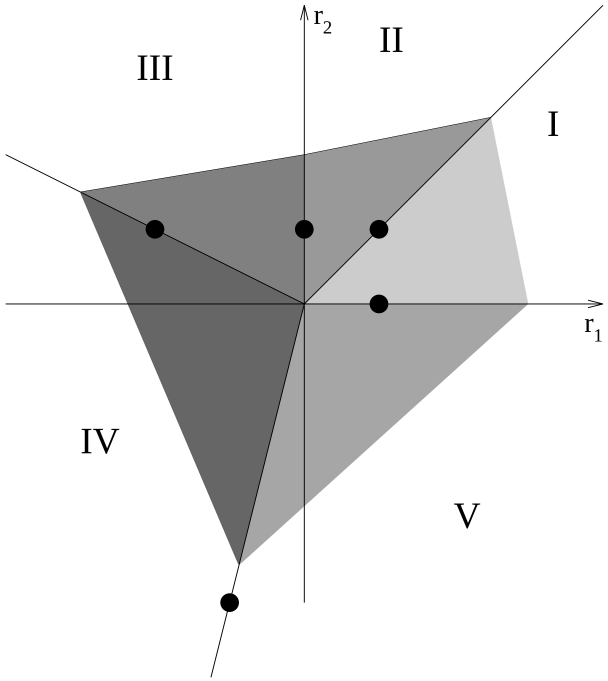

Figure 1. Phase diagram for a Calabi-Yau in $\mathrm{WCP}^4_{3,2,2,1,1}$

This can be desingularized by

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	χ	p
1	0	0	1	1	-2	-1
1	1	1	0	0	1	-4

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Γ
-1	-1	2	0	1	-1
1	1	0	1	1	-4

In the above, we have changed basis so that the original first $U(1)$ charge is given by $Q_1 + 2Q_2$ where the Q_i's refer to the i^{th} current $U(1)$ charges. An

appropriate set of equations for the manifold and gauge bundle are as follows:

$$\begin{aligned} G &= \phi_1(\phi_2^3 + \phi_3^3) + (\phi_4^9 + \phi_5^9)\chi^4 \\ F_1 &= \phi_2^3\phi_4^2 \\ F_2 &= \phi_3^3\phi_5^2 \\ F_3 &= (\phi_4^7 + \phi_5^7)\chi^4 \\ F_4 &= \phi_3^3\phi_4 + \phi_2^3\phi_5 + \phi_1\phi_2\phi_3 + \phi_4^4\phi_5^3\chi^3 \\ F_5 &= \phi_1^2\chi \end{aligned} \tag{4.1}$$

The above equations ensure bundle transversality in the ordinary Calabi-Yau phase. If this theory had a Landau-Ginzburg phase, then the fact that Γ has the same charge as p would certainly require a fermionic gauge symmetry and hence specification of the relevant E's. In this case, however, the charges are such that there is no ordinary Landau-Ginzburg phase in the Kähler moduli space, for any choice of manifold and bundle data. Hence, adopting the philosophy of section 3.2, we assume that the above model is defined *without* any fermionic gauge symmetries.

The phase diagram is again the same as Figure 1, although the geometric interpretation of each phase, besides involving the gauge bundle is also somewhat different in the $(0, 2)$ case. Region II corresponds as expected to a vector bundle over a smooth Calabi-Yau. Regions I and III, however, are now coherent sheaves as the polynomials above do vanish simultaneously in the ambient toric variety. Regions IV and V are their appropriate desingularizations, which turn out, in this case, to be "extended hybrids". This relatively unfamiliar form of hybridization has the expectation value of p being compact but not *fixed* in the semiclassical vacuum. Nevertheless, some of the bosons transform as a point in a V-bundle over the compact space formed by the rest, a distinct feature of typical hybrid vacua.

Notice that in going from region II to region I we alter the topology of the Calabi-Yau by flopping a curve. The bundle data and the physical model make perfectly good sense on the flopped model, and hence we directly see the immediate extension of flop transitions to this more general setting.

4.1. Multiple (0,2) Resolutions

The additional freedom in $(0,2)$ models, though, gives rise to new features not present in the $(2,2)$ case. One of these is related to the above, so we describe it first. In each (geometric) phase of a $(2,2)$ model, the structure of the tangent bundle is uniquely inherited from the structure of the base. The latter, in the toric formalism, is determined by the charges of the chiral fields. In a given geometrical phase of the base for a $(0,2)$ model, however, there can be *different sets of charges* which desingularize the initial V-bundle. In general these different charges correspond to topologically distinct total spaces of the pull-back of the bundle over the desingularized manifold. By passing through the geometrically singular locus of the original unresolved model, physics can apparently pass from one such $(0,2)$ model to another. We note that the number of generations will not change in such a transition, and therefore these transitions are somewhat akin to flop transitions. Here, though, the charges of the linear sigma model fields — not just the value of Fayet-Iliopoulos parameters — actually change, all in a perturbatively smooth manner.

As an example of the multiple resolutions, the following provides a valid data for a gauge bundle over a singular Calabi-Yau hypersurface M in $\mathrm{WCP}^4_{7,2,2,2,1}$:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	p
7	2	2	2	1	-13

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Γ
4	3	2	2	1	1	-14

On resolving the base manifold, both of the following are valid solutions to the anomaly cancellation condition for the gauge bundle:

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6
4	3	2	2	1	1
2	2	2	1	0	0
4	3	2	2	1	1
3	1	1	1	1	0

5. $(0,2)$ Duality

Part of the motivation for the present work is the observation of a novel kind of duality found in [9]. In that paper it was found that apparently distinct geometrical $(0,2)$ models can become isomorphic in their Landau-Ginzburg phases. This opens up the remarkable possibility of perturbative transitions from a $(0,2)$ model on (M_1,V_1) to one on (M_2,V_2), with subscripts denoting topologically distinct data. As emphasized in [9], though, their analysis was severely hampered by only working on one-dimensional singular loci in the respective moduli spaces. This prevented direct comparison between topological properties in Landau-Ginzburg and geometrical phases, as the latter were not desingularized. It also left open the possibility that subtle differences in the respective desingularizations of the two geometrical models might lead to previously undetected differences in their Landau-Ginzburg realizations, thus spoiling the identification. Finally, working only on the singular locus made it impossible to extract any global information regarding how the relevant moduli spaces might attach. The desingularizing work reviewed above allows us to at least partially address some of these issues, as we now do.

5.1. Duality for complete intersections

Let us first recall the set up in [9]. We consider a complete-intersection Calabi-Yau M_1 in a product of weighted projective spaces. A $(0,2)$ model can be built on the base as a coherent sheaf V_1 over M_1. We now consider a phase in the Kähler moduli space in which $|p|$ is forced to be nonzero in the semiclassical vacuum. This occurs for instance in a typical hybrid phase. By making an appropriate choice of F's, one may ensure the nonzero value of p is constant and proportional to a certain linear combination of the Fayet-Iliopoulos parameters. Taking that combination to infinity so as to fully freeze any point-like anti-instantons contributed by p in the σ-model, we may then regard p as a constant $<p>$. The resulting superpotential has the form

$$\int d^2zd\theta(\Gamma^b G_b(\Phi_i) + \Lambda^a <p> F_a(\Phi_i)) \tag{5.1}$$

We now consider a change of variables defined by tilded quantities as follows:

$$
\begin{array}{llll}
\widetilde{\Gamma}_1 = <p> \Lambda_1 & \widetilde{\Gamma}_2 = <p> \Lambda_2 & \widetilde{\Lambda}_1 = \dfrac{\Gamma_1}{<p>} & \widetilde{\Lambda}_2 = \dfrac{\Gamma_2}{<p>} \\
\widetilde{G}_1 = F_1 & \widetilde{G}_2 = F_2 & \widetilde{F}_1 = G_1 & \widetilde{F}_2 = G_2 \\
\widetilde{E}_1^{'(j)} = E_1^{(j)} & \widetilde{E}_2^{'(j)} = E_2^{(j)} & \widetilde{E}_1^{(j)} = E_1^{'(j)} & \widetilde{E}_2^{(j)} = E_2^{'(j)}
\end{array}
\tag{5.2}
$$

The $(0,2)$ action has the same form in terms of the new quantities as it had in terms of the old ones, i.e. (5.2) is an automorphism of the theory. Now, by taking the large radius limit again would result, if the exchange was done in a way consistent with anomaly cancellation, in a $(0,2)$ model over a distinct complete-intersection Calabi-Yau! This is clear from (5.2), where the new G's are related to the old F's. Similarly, the new gauge sheaf is defined by G_1 and G_2, instead of F_1 and F_2, and the remaining F's. We can thus view this effectively as an exchange of complex structure and gauge bundle moduli. Notice that the bosonic phase diagram remains identical, because the charges of the right-moving bosons remain the same, but that, a priori, the new and the old Calabi-Yaus are not related.

The duality noted essentially involves a permutation of the left-moving fermionic superfields when certain bosonic fields are frozen in moduli space. After desingularization, we can imagine a generalization of the previous discussion whereby, at some point in the moduli space, in addition to p, some of the extra chiral superfields introduced by the resolution of the base manifold are also frozen to be constants. For simplicity we will consider here only the case for which one additional field χ is frozen. In this case, the G's and F's in (5.1) will be functions of the ϕ's and χ, and generally one can perform exchanges of the Γ's and Λ's treating p and χ as nonzero constants. A simple attempt to define a χ-duality therefore goes as follows:

$$
\begin{array}{llll}
\widetilde{\Gamma}_1 = \dfrac{<p> \Lambda_1}{<\chi>} & \widetilde{\Gamma}_2 = <p> \Lambda_2 & \widetilde{\Lambda}_1 = \dfrac{\Gamma_1}{<p>} & \widetilde{\Lambda}_2 = \dfrac{\Gamma_2}{<p>} \\
\widetilde{G}_1 = \chi F_1 & \widetilde{G}_2 = F_2 & \widetilde{F}_1 = G_1 & \widetilde{F}_2 = G_2 \\
\widetilde{E}_1^{'(j)} = \dfrac{E_1^{(j)}}{\chi} & \widetilde{E}_2^{'(j)} = E_2^{(j)} & \widetilde{E}_1^{(j)} = E_1^{'(j)} & \widetilde{E}_2^{(j)} = E_2^{'(j)}
\end{array}
\tag{5.3}
$$

where $E_1^{(j)}$ is divisible by χ. The $(0,2)$ action is invariant under (5.3), however the resulting model is never anomaly-free. The simplest way to obtain an anomaly free model is to, in addition to redefining various quantities as in (5.3), also add another left-moving fermion $\widetilde{\Lambda}$ whose charge is precisely that of χ. One way of understanding how this can be done is as follows.

We can consider the original model on M_1 but with an extra gauge-neutral fermion Λ. In the superpotential, Λ multiplies $PF(\phi_i, \chi)$ where F is a polynomial of degree $-P$. We then extend (5.3) by the following exchange:

$$
\widetilde{\Lambda} = \chi \Lambda \qquad \widetilde{E}^{(j)} = \chi E^{(j)} \qquad \widetilde{F} = \frac{F}{\chi} \tag{5.4}
$$

The gauge-neutral fermion becomes a charged fermion after the exchange. This allows us to understand one way in which gauge-neutral fermions could come about in the first place. They are simply the relics of performing χ-duality in the opposite direction.

Before proceeding with details of some concrete examples, we briefly examine the consequences for singularities after performing the transformations discussed so far.

5.2. Singularities and Duality Transformations

As far as singularities in σ_j's are concerned, they need to be dealt with appropriately in each case. The singularities in p can however be discussed more generally. As we learned from section 3, p becomes noncompact when all the G's and F's simultaneously vanish in the relevant toric ambient space. For ordinary dual models, the loci where p becomes noncompact is clearly isomorphic to that of the original model, as we have merely interchanged the G's and F's in going from one model to the other. Hence, beginning with a model with no gauge bundle singularities on M_1, we obtain one on M_2 by duality, and vice versa.

We now proceed to show some examples of such dualities. One might first try to resolve the examples discussed in [9] but it turns out that none of them are fully amenable to toric methods. We thus concentrate on new examples which are.

5.3. An ordinary duality for complete intersections

We begin with an example in a product of ordinary projective spaces to avoid having any singularities at all in the base space. Consider then

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	p
1	1	1	1	0	0	0	-3
0	0	0	0	1	1	1	-2

Λ_1	Λ_2	Λ_3	Λ_4	Γ_1	Γ_2
0	0	1	2	-2	-2
1	1	0	0	-2	-1

which is a complete intersection of polynomials of bidegree $(2,2)$ and bidegree $(2,1)$ in $\mathrm{CP}^3\times\mathrm{CP}^2$. The phase diagram consists of the expected smooth Calabi-Yau phase as well as two hybrid phases. One cannot find, for any choice of manifold or bundle data, an ordinary Landau-Ginzburg phase in the moduli space, so we may as before construct the model without any fermionic gauge symmetries. The following set of polynomials then ensure that the bundle is transverse in the smooth Calabi-Yau phase:

$$\begin{aligned}
G_1 &= (\phi_1^2+\phi_2^2+\phi_3^2)\phi_7^2+(\phi_1^2+2\phi_3^2+\phi_4^2)\phi_6^2+(\phi_1^2+3\phi_2^2+2\phi_4^2)\phi_5^2\\
G_2 &= (\phi_1^2+\phi_4^2)\phi_7+(\phi_1^2+\phi_2^2)\phi_6+(\phi_1^2+\phi_3^2)\phi_5\\
F_1 &= \phi_4^3\phi_6+\phi_2^3\phi_5+\phi_3^3\phi_7+\phi_1^3(\phi_5+\phi_7)\\
F_2 &= \phi_1^3\phi_5\\
F_3 &= \phi_2^2\phi_6^2\\
F_4 &= \phi_1\phi_7^2+\phi_2\phi_6^2+\phi_3\phi_5^2
\end{aligned}\tag{5.5}$$

We now perform a duality on the model given by the following exchange:

$$\widetilde{G}_1=F_1\qquad \widetilde{G}_2=F_4\qquad \widetilde{F}_1=G_1\qquad \widetilde{F}_4=G_2$$

This results in the following data:

$\widetilde{\Lambda}_1$	$\widetilde{\Lambda}_2$	$\widetilde{\Lambda}_3$	$\widetilde{\Lambda}_4$	$\widetilde{\Gamma}_1$	$\widetilde{\Gamma}_2$
1	0	1	1	-3	-1
0	1	0	1	-1	-2

which represents, given the polynomials above, a smooth complete intersection of two polynomials of bidegree $(3,1)$ and $(1,2)$ in $\mathrm{CP}^3\times\mathrm{CP}^2$. In both of these models the third Chern number of the respective gauge bundles are the same as expected, and have the value -168. Note that the Euler number and Hodge numbers of the two Calabi-Yau's involved are *not* the same. Rather, the Hodge numbers of the bidegree $(2,2),(2,1)$ model are given by $h^{1,1}=2, h^{2,1}=62$ while those of the $(3,1),(1,2)$ model are given by $h^{1,1}=2$ and $h^{2,1}=59$.

Interesting examples abound when we start introducing weighted projective spaces as ambient varieties. To see how these are constructed, we briefly discuss the relevance of these dualities to $(0,2)$ models on hypersurfaces.

5.4. Complete intersection dualities on hypersurfaces

Consider a degree 9 hypersurface M in $\mathrm{WCP}^4_{3,3,1,1,1}$. The data for a $(0,2)$ model on M is as follows:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	χ	p
3	3	1	1	1	0	-9
1	1	0	0	0	1	-3

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Γ
3	3	1	1	1	0	-9
1	1	0	0	0	1	-3

This $(0,2)$ model is simply a deformation of the (2,2) model, in which the G and F's are, respectively, a transverse polynomial and its derivatives with respect to all the bosonic fields.

We now modify the above model in a way that again satisfies the anomaly cancellation conditions. Namely we add a right moving boson ϕ_6 with charges $(3,1)$ and a left-moving fermion Γ_2 with charges $(-3,-1)$ to the theory. Γ_2 here is a fermion that multiples a polynomial G_2 of multi-degree $(3,1)$ in the superpotential. We may therefore simply choose that polynomial to be ϕ_6. This effectively realizes M in a trivial way as a complete intersection in $\mathrm{WCP}^5_{3,3,1,1,1,3}$. To satisfy (2.8), we need to choose polynomials E^1 and E^2 of multi-degree $(6,2)$ for Γ_2 under the two fermionic gauge symmetries. One way to do so is to modify G by adding a monomial involving ϕ_6^2: $G_{new}=G+\phi_6^2H$. This does not alter the vacuum structure because $G_2=\phi_6$ always vanishes in the vacuum. We can then solve for E^1 and E^2 from (2.8) as the terms not involving ϕ_6 cancel out by construction.

Embedding the model in a weighted projective space with one higher dimension via a hyperplane section has not changed either the manifold or the bundle at all. In the linear sigma model language, we have added a new bosonic superfield Φ_6 and a fermionic superfield Γ_2, with a mass term which pairs them up. The infrared physics of the linear sigma model is completely unchanged.

Thus we have a good $(0,2)$ model on M realized as a complete intersection, with toric data as follows (after renaming ϕ_6 as ϕ_1):

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	χ	p
3	3	3	1	1	1	0	-9
1	1	1	0	0	0	1	-3

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Γ_1	Γ_2
3	3	1	1	1	0	-9	-3
1	1	0	0	0	1	-3	-1

In the Landau-Ginzburg phase ($r_2 < 0$, $r_1 - 3r_2 < 0$), Λ_6 and Γ_1 can be gauged away using the fermionic gauge symmetries.

At the Landau-Ginzburg point, we can perform an ordinary duality:

$$\widetilde{G}_1 = F_1 \qquad \widetilde{G}_2 = F_2 \qquad \widetilde{F}_1 = G_1 \qquad \widetilde{F}_2 = G_2$$

with the corresponding Es and E's also exchanged. The resulting manifold is a complete intersection of two sextics in $\mathrm{WCP}^5_{3,3,3,1,1,1}$. One of the fermions, $\widetilde{\Lambda}_1$, is neutral under both $U(1)$'s. Hence it can be gauged away using one of the fermionic gauges symmetries in *any* of the phases of the model. This complete intersection model could thus have been formulated *without* Λ_1 and with only *one* fermionic gauge symmetry.

One can check that the new model has no singularities in σ's in the various phases. From our earlier discussion, there are no gauge bundle singularities either.

To see this, we consider the E's as listed below:

E_1	E_2	E_3	E_4	E_5	E_6	E'_1	E'_2
$3\phi_2$	$3\phi_3$	ϕ_4	ϕ_5	ϕ_6	0	-9	$9\phi_1^2$
ϕ_2	ϕ_3	0	0	0	χ	-3	$3\phi_1^2$

The general solution to (2.8)is given by

$$\begin{aligned} G_1 &= \phi_1^2 f_1 + \phi_2 f_2 + \phi_3 f_3 + \chi(\phi_4 f_4 + \phi_5 f_5 + \phi_6 f_6) \\ G_2 &= f_1 \\ F_1 &= 3 f_2 \\ F_2 &= 3 f_3 \\ F_3 &= 9\chi f_4 \\ F_4 &= 9\chi f_5 \\ F_5 &= 9\chi f_6 \\ F_6 &= 3(\phi_4 f_4 + \phi_5 f_5 + \phi_6 f_6) \end{aligned} \tag{5.6}$$

with appropriately chosen polynomials $f_{1,\ldots 6}$. One choice, which leads to a bundle on a smooth Calabi-Yau, for both the original and dual theory is

$$\begin{aligned} f_1 &= \phi_1 \\ f_2 &= \phi_2^2 + \frac{\phi_3^2}{2} + \frac{\chi^2}{6}(\phi_4^6 + \frac{\phi_5^6}{3} + \frac{\phi_6^6}{4}) \\ f_3 &= \phi_2^2 + \phi_1^2 + \frac{\chi^2}{6}(\phi_4^6 + \phi_5^6 + \phi_6^6) \\ f_4 &= \phi_4^6 \chi^2 \\ f_5 &= \phi_5^6 \chi^2 \\ f_6 &= \phi_6^6 \chi^2 \end{aligned} \tag{5.7}$$

We have thus shown a perturbatively valid transition between a Calabi-Yau hypersurface in $\mathrm{WCP}^4_{3,3,1,1,1}$, which had Hodge numbers $h^{1,1} = 4, h^{2,1} = 112$ and a complete intersection in $\mathrm{WCP}^5_{3,3,3,1,1,1}$, which has Hodge numbers $h^{1,1} = 5, h^{2,1} = 77$. In the first of these, the bundle is a deformation of the tangent bundle; in the second, it is not. Both of these models have third Chern number of their respective gauge bundles being -216 with, in fact, $h^1(V) = 4$, and $h^2(V) = 112$. Similar methods go through for many other hypersurfaces.

5.5. Chain of dualities

Consider the degree 12 hypersurface M in $\mathrm{WCP}^4_{4,4,2,1,1}$. We construct as before an embedding of M as a complete intersection of a quartic and a degree 12 polynomial in $\mathrm{WCP}^5_{4,4,4,2,1,1}$.

The toric data for the appropriate $(0,2)$ model is given by:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	χ_1	χ_2	p
4	4	4	2	1	1	0	0	−12
2	2	2	1	0	0	1	0	−6
1	1	1	0	0	0	0	1	−3

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Γ_1	Γ_2
4	4	2	1	1	0	0	−4	−12
2	2	1	0	0	1	0	−2	−6
1	1	0	0	0	0	1	−1	−3

The following is an ordinary duality of the model:

$$\widetilde{G}_1 = F_1 \qquad \widetilde{G}_2 = F_2 \qquad \widetilde{F}_1 = G_1 \qquad \widetilde{F}_2 = G_2$$

Performing the duality results in a complete intersection of two octics in $\mathrm{WCP}^5_{4,4,4,2,1,1}$. The interesting point here is that M is actually a $K3$ fibration. The relevant $K3$ can be found by essentially slicing off the first rows in the above tables.

That is, the following is a $(0,2)$ model on a $K3$ essentially derived from the above model on the Calabi-Yau:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	χ	p
2	2	2	1	1	0	−6
1	1	1	0	0	1	−3

Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Γ_1	Γ_2
2	2	1	1	0	−2	−6
1	1	0	0	1	−1	−3

The duality now *extends* to this fiber, giving a transition of a vector bundle over a $K3$ to a different vector bundle over another $K3$. The second Chern number of each of these bundles over $K3$ is 24 , as they must be for anomaly

cancellation. Thus, each of these is dual to F-theory on the degree-84 hypersurface in $\mathrm{WCP}^4_{42,28,12,1,1}$.

It turns out that $K3$ is also elliptically fibered, with a typical fiber being that given by slicing off the first rows in the above:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	p
1	1	1	1	−3

Λ_1	Λ_2	Λ_3	Γ_1	Γ_2
1	1	1	−1	−3

Again the duality extends to the duality over the toroidal fibers.

We can also work backwards from a duality between sheaves over $K3$ to a duality over $K3$-fibered Calabi-Yaus and so on.

6. Moduli Space Structure

An important question that arises in all of these examples is whether the attachment loci are in fact multicritical points, or whether the physical model forces the respective moduli spaces to either be isomorphic or possibly to join together to fill out to an ‘enlarged’ manifold moduli space. The latter would be analogous to the ‘enlarged’ Kähler moduli space of $(2,2)$ models while the former would be closer to $(2,2)$ conifold-like attachments. Of course, all of our present discussion is perturbative. This is not a question which we have been able to fully settle, although preliminary evidence seems to indicate that the full moduli space has a multicritical structure such as that given seems in Figure 2, where the loci of intersection denotes the points in the phase diagram where the exchange takes place.

One way to settle this question, at least in specific examples, is to calculate the bundle moduli from $H^1(EndV)$ in dual examples. If the moduli spaces are to be identified or glued together in an enlarged structure, we expect a one-to-one correspondence between the full set of gauge singlet moduli (complex structure, Kähler structure, holomorphic bundle structure) in each of the dual pairs. We have made some progress in calculating $h^1(EndV)$ for some of the models discussed here, but the calculations are relatively challenging. We will not present the details here [15].

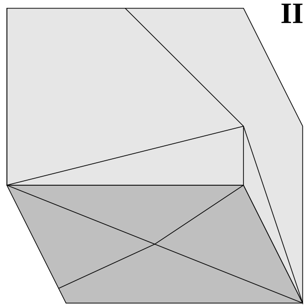

Figure 2. Two moduli spaces meeting at a loci

We do note that in the example of section *5.4* the moduli corresponding to deformations of the holomorphic structure of V (in both cases) do not all have polynomial representatives. That is, there are deformations of V which are not captured by the linear sigma model. (From what we have said above, it is already evident that not all of the deformations of the Kähler structure are captured by the linear sigma model.) So, though we can clearly line up the polynomial deformations in the two theories (there are 380 of them, corresponding in each case to some mixture of deformations of complex structure and holomorphic deformations of V), as well as the two Kähler moduli which are captured by the linear sigma model, we do not know how to make a correspondence between the other moduli which are not representable in the linear sigma model. Indeed, as the Landau-Ginzburg theory has more singlets (431) than are expected as moduli for either the M or the $\widetilde{M}$ theory, it appears likely that the Landau-Ginzburg locus is a multicritical point, with the M and $\widetilde{M}$ theories as different branches of the moduli space which meet at Landau-Ginzburg. Fully establishing this requires completion of the geometric calculation of moduli, as we will present elsewhere.

If the dual models we have found are in fact multicritical, many distinct $(0,2)$ models would be joined by passing through suitable points in their enlarged Kähler moduli space — points at which certain bosonic fields are frozen

to constant values. This would be a perturbative cousin of the nonperturbative $(2,2)$ multicritical conifold transitions.

7. Remarks on Toric Resolutions of $(0,2)$ Models

Before concluding, we wanted to point out one feature of $(0,2)$ resolutions which has not been clearly delineated in either [8] or in the above. For a given singular model, the number of solutions to (1.1) and (1.2) generally greatly exceeds the number of physically sensible resolutions. At first sight, our discussion has identified solutions of these equations with physical resolutions, but in reality, equations (1.1) and (1.2) are necessary but not sufficient conditions. The condition that we have not addressed is that of stability. We recall from [2] that the bundle V over a smooth base M must be a stable bundle in order to be able to solve the vanishing beta function equations for the gauge fields. This has always been a difficult condition to systematically incorporate into physical models, and at present it is still unclear how to generally do so at the level of the linear sigma model. What has been noticed [9], for instance, is that certain naively sensible choices of V can sometimes lead to pathological behaviour. The natural guess is that this is due to failing to meet the stability condition.

In the treatment of [8] there are some additional necessary conditions which must be satisfied beyond equations (1.1) and (1.2) in order for there to be a chance that the resulting solution is stable. For instance, if $h^3(V)$ is nonzero, then V is not stable as the dual bundle V^* has a global section.

These conditions turn out to be rather efficient at eliminating solutions to equations (1.1) and (1.2). For example, the following are all valid solutions to the anomaly cancellation conditions for a gauge bundle over a smooth Calabi-

Yau in $WCP^4_{3,2,2,1,1}$:

	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	p
1.				2	3	5	-10
				0	2	3	-5
2.			0	1	1	15	-17
			-1	1	1	7	-8
3.			0	1	1	15	-17
			1	0	0	6	-7
4.		1	1	2	2	3	-9
		1	1	0	1	1	-4
5.	0	1	1	2	2	3	-9
	1	0	0	1	1	1	-4

We can see however that solution 2 is problematic whether or not fermionic gauge symmetries are considered. The reason is that the relevant E's for Λ_3 have to be either 0 or a suitable rational function. The latter not being well-defined over the ordinary Calabi-Yau phase, we must set the E's for Λ_3 to be 0. Having done so, all the E's,F's and G's will simultaneously vanish in the hybrid phases, resulting in a model that appears to be generically singular. On the other hand, if we consider instead a model defined without fermionic gauge symmetries, we find from the exactness of (2.19) and (2.20) that $h^0(V^*) = 1$. As alluded to above, this corresponds to the kernel bundle being unstable as a perturbative solution to the string equations of motion.

Ideally, a more robust formalism can be found in which the necessary and sufficient conditions for a bona fide string solution are directly applied and manifestly satisfied by candidate solutions. As the fundamental ingredients in the approach taken here involves line bundles over toric varieties, it would seem possible that an extended toric formalism should exist for this purpose. Finding such a formalism is worthy of concerted effort. Presumably it would also be the appropriate framework for discussing $(0,2)$ mirror symmetry, a topic on which there has been some interesting recent progress [16].

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