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ON THE PHYSICAL LIMITATIONS TO THE LOWEST EMITTANCE (TOWARD COLLIDING ELECTRON-POSITRON CRYSTALLINE BEAMS)[1]

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In this paper we explore the conditions under which a relativistic beam of electrons or positrons, treated as a Fermi gas, becomes degenerative. The quantum effects give an absolute limit to the emittance of the beam. We discuss here some methods to prepare this ultra-cold electron-positron gas in a framework of colliding beams at high energy.

1. INTRODUCTION

It is well known, that the electron gas has a tendency to generate with lowering it's temperature. In the fully degenerated state, each particle occupied the volume in a phase space of the order $\Delta p \Delta x \approx \hbar$. Typical requirement for degeneration is that the temperature of the electron gas must be small compared with Fermi energy ε_F. The Fermi energy is [1]

$$\varepsilon_F \cong \left(3\pi^2\right)^{2/3} \frac{\hbar^2}{2m} \rho^{2/3} \tag{1}$$

or

$$\varepsilon_F = (3\pi^2)^{1/3} \hbar c \rho^{1/3},$$

if the particles in the rest frame are relativistic. Here $\rho \approx \left(\frac{N}{V}\right)$ --is a density, N -- is the number of states in volume V, m -- is a rest mass of electron, $\hbar$ -- is Plank constant. The condition for degeneration becomes

$$k_B T \le \frac{\hbar^2}{m} \rho^{2/3} = mc^2 \lambda\!\!\!^{-2}_C \rho^{2/3},$$

where $k_B \cong 1.38 \cdot 10^{-23}\ J/^{\circ}K$ is Boltzmann's constant, $\lambda\!\!\!^{-}_C = \hbar / mc$. One can see, that the gas becomes more degenerative with increasing the density ρ, keeping the temperature constant. Typical density in metals is $\rho \approx 5 \cdot 10^{22} 1/cm^3$ what gives a tendency for electron gas to degenerate at the temperature about $1eV$.

Normally in metals the charge of electrons gas is neutralized by the charge of nuclei-- eZ. To satisfy the condition, that the Coulomb interaction energy of the electrons and nuclei is small compared with Fermi energy, it needs to be

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$\rho \geq \left(e^2 m / \hbar^2\right)^3 Z^2$ [1]. Again, with increasing the density, this condition is better satisfy.

Ultrarelativistic electron (positron) beam is an unique object of statistical physics due to cancellation of forces between charged particles $\approx 1/\gamma^2$, where $\gamma = E/mc^2$, E -- is the energy of the electron (or positron). The electron gas becomes more ideal with increasing its energy. For intermediate energy what is, say, 10 GeV, the factor of cancellation is of the order $1/\gamma^2 \approx 2.5 \cdot 10^{-9}$. In principal, some amount of particles with opposite charge can be added to the beam, which is moving along straight line.

2. CONDITION FOR DEGENERATION

For an electron gas in a volume V, the number of states with absolute magnitude of momentum in the interval from $p_{x,y,s}$ to $p_{x,y,s} + dp_{x,y,s}$ is

$$dn \cong 2\frac{dp_x dp_y dp_s \cdot V}{(2\pi\hbar)^3},$$

where factor 2 reflects two possibilities for spin statement. The total number of electrons in these states can be estimated for uniform distribution as

$$N = \int dn \cong 2\frac{p_x p_y \Delta p_{\parallel} S_{\perp} l_b \gamma}{(2\pi\hbar)^3} \cong 2\frac{\gamma\varepsilon_x \gamma\varepsilon_y \gamma l_b (\Delta p / p_0)}{(2\pi\lambda_C)^3}, \qquad (2)$$

where $\gamma l_b (\Delta p / p_0)$ -- is an invariant longitudinal emittance, l_b is the bunch length, $\gamma\varepsilon_x$ and $\gamma\varepsilon_y$ are the transverse horizontal and vertical emittances. Again, if N is close to the number of the particles in the bunch, then the particles in the bunch close to generation condition. We can also say, that the beam with the number of the particles N, can not have emittances, lower, than defined by (2). The physical sense is clear: all lower states are occupied.

3. THERMALIZATION

The electron gas temperature T in a moving frame can be represented as the following (see for example [2])

$$\frac{3}{2}k_B T \cong \frac{1}{2}mc^2\gamma\left[\frac{\gamma\varepsilon_x}{\beta_x} + \frac{\gamma\varepsilon_y}{\beta_y} + \frac{1}{\gamma}\left(\frac{\Delta p}{p_0}\right)^2\right], \qquad (3)$$

where $(\Delta p / p_0)$ is the energy spread in the beam, $\beta_{x,y}$ is the envelope function in the damping ring.. For damping rings, developed as injectors for future Linear Colliders, the typical values are $\gamma\varepsilon_x \cong 3 \cdot 10^{-4} cm \cdot rad$, $\gamma\varepsilon_y \cong 3 \cdot 10^{-6} cm \cdot rad$, $\beta_{x,y} \cong 10 m$, $\Delta p / p \cong 10^{-3}$, $\gamma \cong 6 \cdot 10^3$ (3 GeV). This gives for the temperature in the beam $\frac{3}{2}k_B T \cong \frac{1}{2}mc^2\gamma\left[3 \cdot 10^{-7} + 3 \cdot 10^{-9} + 1.5 \cdot 10^{-10}\right]$. One can see, that the longitudinal temperature is the lowest one. This yields the possibility for redistribution the temperatures from transverse to longitudinal one due to intra-beam

scattering for example, or with help of mechanism, described in [10]. We will discuss this later. Let's investigate the condition for degeneration from the temperature consideration. Compare formula (3) with (1) one can obtain

$$\frac{1}{2}mc^2\gamma\left[\frac{\gamma\varepsilon_x}{\beta_x}+\frac{\gamma\varepsilon_y}{\beta_y}+\frac{1}{\gamma}\left(\frac{\Delta p}{p_0}\right)^2\right]\le(3\pi^2)^{1/3}\hbar c\tilde{\rho}^{1/3},$$

where $\tilde{\rho}=\rho/\gamma$ is the density in the moving frame. Neglecting the longitudinal temperature and supposing that $\gamma\varepsilon_x\approx\gamma\varepsilon_y$ one can obtain the condition $(\gamma\varepsilon_x)^4\le\frac{\beta_x^2}{l_b}\lambda\!\!\!^{-3}_C N$. Supposing for estimation $\beta_x\approx l_b\approx 1cm$, $N\approx 10^{10}$, one can obtain $\gamma\varepsilon_x\le(5.5\cdot 10^{-22})^{1/4}\cong 4.8\cdot 10^{-6}cm\cdot rad$. This is an estimation for the maximal possible transverse emittance required for degeneration. One can see, that radial emittance need to lowered about two orders of magnitude to satisfy the condition for degeneration.

From the expression for temperature (3) yields that transverse velocity of oscillations is

$$v_\perp^2\cong c^2\gamma\cdot\left[\frac{\gamma\varepsilon_x}{\beta_x}+\frac{\gamma\varepsilon_y}{\beta_y}+\frac{1}{\gamma}\left(\frac{\Delta p}{p_0}\right)^2\right].$$

The frequency of collisions $f_{x,y}$ is $\frac{1}{Nf_{x,y}}\cong\frac{a_{x,y}}{v_{x,y}}\cong\sqrt{\frac{\gamma\varepsilon_{x,y}\beta_{x,y}}{\gamma}}\Big/c\sqrt{\gamma(\frac{\gamma\varepsilon_{x,y}}{\beta_{x,y}})}=\frac{\beta_{x,y}}{c\gamma}$ and is not dependent of emittance. For longitudinal motion one can estimate $f_s^{-1}\cong\gamma l_b/v_{\parallel}\cong\frac{\gamma l_b}{c(\Delta p/p)}$. Longitudinal mass is $\mu^{-1}=\frac{1}{m\gamma}\left(\frac{1}{\gamma^2}-\alpha\right)=\frac{1}{m\gamma}\left(\frac{1}{\gamma^2}-\frac{1}{\gamma_{tr}^2}\right)$. It becomes negative above the transition energy, when $\gamma>\gamma_{tr}\cong\alpha^{-1/2}$. Here $\alpha=\frac{p}{C}\frac{\partial C}{\partial p}$, *C*--is the circumference -- is a momentum compaction factor .

Considerations [6,7] shows that thermalization due to intrabeam scattering (not only direct Coulomb interaction between particles, but through the third agent, such as a resistive wall or a parasitic cavity) can happen only *below* the critical energy (the longitudinal mass is *positive* here). So, for thermalization, the damping ring must have *zero* or *negative* α. The kayak-paddle type cooler and LDS system, considered below, satisfy this condition. Some investigations are made recently to verify the conditions for operation with positive longitudinal mass [9]. This circumstance is out of designer's attention, who are working with damping rings for Linear Colliders, however.

The thermalization can be also used in OSC method for cooling *all degrees of freedom* of the beam, although the cooling is applied only to *one* degree of freedom. Here thermalization will equalize the temperature over all degrees of freedom. This can reduce requirements for emittance arising from simultaneous cooling in transverse and longitudinal direction with Hereward cooling method [8]. One method, also considered recently [10], gives a possibility to redistribute effectively

the modes of damping in focusing-dominated system, what may be a quadrupole wiggler, for example.

Particles spacing.-- The distance between particles δ in the laboratory frame can be estimated as

$$\delta^3 \cong \frac{V}{N} \cong \frac{\sqrt{\varepsilon_x \beta_x \varepsilon_y \beta_y} \cdot l_b}{N}.$$

Substitute here expressions for emittances from previous relation and estimate $\beta_{x,y} \cong l_b$, one can obtain, that in the case of degeneration

$$\delta^6 \cong (2\pi\lambda\!\!\!^{-}_C)^3 \frac{l_b^3}{2\gamma^3 N(\Delta p / p_0)}.$$

Let us estimate possible values for emittances what is possible to obtain in term of satisfaction the condition of degeneration (1). We shall consider the ordinary synchrotron damping with the minimal possible value of invariant, obtained in Linear Damping System -- in LDS [3] type damping ring -- kayak-paddle cooler [4] and with help of OSC method [5].

4. THE COOLERS

In previous chapter we considered the conditions, what can be obtained with existing damping rings. Now let us consider some possible improvements.

4.1. Linear damping system (LDS)--[3] is a sequence of wigglers and accelerating structures, installed along a straight line, basically in the beginning of the Linear Collider main accelerating structure. The losses in the wigglers are the same as the gain of the energy given by the accelerating structures, so the particles are moving with practically constant energy. The formulas are the same as for kayak-paddle ring, so we will consider them in the following section.

4.2. Kayak-paddle cooler (KPC) [4] is a damping ring, which made as a sequence of wigglers and accelerating structures, installed along a straight line and having the bends at the end. The bends can be made to give a small input into cooling dynamics, so we will neglect it's influence. In that sense it is similar to LDS. But here there is no requirement that the linear rate of losses is the same as the energy gain per unit length. In LDS, like in any damping ring, particle needs to re-radiate its full energy a few times. The length, what characterized the speed of damping is $l_s = -\frac{\gamma}{d\gamma / ds}$, $d\gamma / ds \cong -\frac{2}{3} r_0 \frac{K^2}{\lambda\!\!\!^{-2}} \gamma^2$, where $K = eH_\perp \lambda\!\!\!^{-} / mc^2$, $H_\perp$ is a magnetic field value in the wiggler, $2\pi\lambda\!\!\!^{-}$ is the wiggler period, $r_0 = e^2 / mc^2$. Notice here, that $d\gamma / ds$ *is not* a function of the wiggler period. Substitute here for estimation $\gamma \cong 8 \cdot 10^3 (4 \quad GeV)$, $K \approx 5 \quad (H_\perp \cong 0.36T, \quad 2\pi\lambda\!\!\!^{-} \cong 30\,cm)$, one can obtain $\frac{d\gamma}{ds} \cong -\frac{2}{3} 2.8 \cdot 10^{-13} \frac{25}{25} 6.4 \cdot 10^7 \cong 1.2 \cdot 10^{-5}$ $[1/cm]$ or $0.0012[1/m]$. For the characteristic damping length one can obtain $l_s = -\frac{\gamma}{d\gamma / ds} \cong \frac{8 \cdot 10^3}{1.3 \cdot 10^{-3}} \cong 6 \cdot 10^6\, m$.

The fraction of the wigglers is about 1 due to relatively slow damping (in contrast to LDS, where the fraction is about 1/2 --half the space is occupied by the accelerating structures). If we suppose, that the circumference is about $3km$, the number of

revolutions will be 2000, what will take about 20*msec*. Emittance dynamics defined by equations

$$\frac{d\varepsilon_x}{ds} = \left\langle \left(H_x + \frac{\beta_x}{\gamma^2} \right) \frac{d(\Delta E / E)^2_{tot}}{ds} \right\rangle - 2\alpha_x \varepsilon_x ,$$

with similar equation for vertical motion, where

$$H_{x,y} = \frac{1}{\beta_{x,y}} \left(\eta^2_{x,y} + (\beta_{x,y}\eta'_{x,y} - \frac{1}{2}\beta'_{x,y}\eta_{x,y})^2 \right),$$

$\eta_{x,y}$ are the dispersion functions. Derivatives are taken over longitudinal direction, which is *s*. Decrements $\alpha_{x,y,s}$ are defined by usual way $\alpha_i = \frac{J_i}{2\, l_s}$, where $J_x \cong 1, J_y = 1, J_s \cong 2$. Notice here that the decrement for the energy spread is the same as for emittance decrease.

$$\frac{d(\Delta E / E)^2_{tot}}{ds} = \frac{d(\Delta E / E)^2_{IBS}}{ds} + \frac{d(\Delta E / E)^2_{QE}}{ds} - \alpha_s \left(\frac{\Delta E}{E} \right)^2 ,$$

where $\frac{d(\Delta E / E)^2_{QE}}{ds} \cong \frac{55}{48\sqrt{3}} \frac{r_0^2 \cdot \gamma^5}{\alpha |\rho|^3}$, $\frac{d(\Delta E / E)^2_{IBS}}{ds} \cong \frac{N\, r_0^2 \ln_C}{\gamma^3 \varepsilon_x \sqrt{\varepsilon_y \beta_y} l_b}$, and ln_C is Coulomb's logarithm. For a dipole wiggler, the periodic solution for η_x can be expressed as

$$\eta_x = \frac{K\lambda\!\!\!^{-}}{\gamma} Sin(s / \lambda\!\!\!^{-}) = \frac{\lambda\!\!\!^{-2}}{\rho} Sin(s / \lambda\!\!\!^{-}) ,$$

where $\rho = \lambda\!\!\!^{-}\gamma / K$ is the bending radius in magnetic field of the wiggle. For the function H_x we can estimate $H_x \cong \beta_x \eta'^2_x$ and for $\eta'_x \cong K / \gamma$. For vertical emittance the term β_y / γ^2 dominates. This gives the equilibrium emittances under condition $d\varepsilon_{x,y} / ds = 0$

$$(\gamma\varepsilon_x) \cong \frac{1}{2} \lambda\!\!\!^{-}_C \overline{\beta}_x K^2 \gamma / \rho \cong \frac{1}{2} \lambda\!\!\!^{-}_C \overline{\beta}_x K^3 / \lambda\!\!\!^{-}$$

$$(\gamma\varepsilon_y) \cong \frac{1}{4} \lambda\!\!\!^{-}_C \overline{\beta}_y \gamma / \rho \cong \frac{1}{2} \lambda\!\!\!^{-}_C \overline{\beta}_y K / \lambda\!\!\!^{-} ,$$

where $\overline{\beta}_{x,y}$ is averaged values of the envelope functions in the wiggle. Notice here that quantum equilibrium vertical emittance does not dependent of the wiggler period and both equilibrium invariant emittances does not depend on energy. Substitute here for estimation $\overline{\beta}_{x,y} \approx 1m$, $\lambda\!\!\!^{-} \cong 5\, cm$, $K \cong 5$,

$$(\gamma\varepsilon_x) \cong \frac{1}{2} 3.86 \cdot 10^{-11} \cdot 100 \cdot 125 / 5 \cong 5 \cdot 10^{-8}\ cm \cdot rad$$

$$(\gamma\varepsilon_x) \cong \frac{1}{2} 3.86 \cdot 10^{-11} \cdot 100 \cdot 5 / 5 \cong 2 \cdot 10^{-9}\ cm \cdot rad .$$

According the formula (2), number of states will be for these emittances $N \cong 2\frac{5\cdot 10^{-8}\cdot 2\cdot 10^{-9}\cdot 8\cdot 10^{3}\cdot 10^{-1}\cdot 10^{-4}}{(2\pi\cdot 3.86\cdot 10^{-11})^{3}} \cong 1.1\cdot 10^{12}$, what indicates, that these emittances still too big to raise a degeneration. Here we substitute, optimistically, $l_b \cong 0.1cm$, $(\Delta p / p_0) \cong 10^{-4}$ at $\gamma \cong 8\cdot 10^{3}$ $(4\ GeV)$.

4.3. OSC method --An appropriate method for obtaining the beam with extremely small emittance is the Optical Stochastic Cooling (OSC) method [4]. The amplification coefficient κ of optical amplifier can be expressed as following

$$\kappa \cong \frac{\varepsilon_{\parallel}}{r_0}\frac{1}{N}\frac{\Delta f}{f},$$

where $\gamma\varepsilon_{\parallel} = \gamma l_b \Delta E / E$ is an invariant longitudinal emittance $\Delta f / f$ is a relative bandwidth, N is the number of the particles. One can see the only phase density in longitudinal direction $\rho_{\varepsilon} = \gamma\varepsilon_{\parallel} / N$ is important for the level of amplification. The decrease of emittance is equal to

$$\frac{\varepsilon_f}{\varepsilon_0} \cong \frac{1}{\alpha N_u},$$

here $\alpha = e^2/\hbar c \cong 1/137$, N_u is the number of the particles in the bandwidth, $N_u \cong \frac{f}{\Delta f}N\frac{\lambda_u}{l_b} \cong \frac{f}{\Delta f}N\frac{L}{l_b\gamma^2}$, where $L = 2\pi\lambda$ is the wiggler period, l_b is the bunch length, λ_u is the central wavelength of radiation amplified. For $N \cong 10^{10}$, $\lambda_u \cong 10^{-4}\ cm\ (1\ \mu m)$, $l_b \cong 1cm$, $\Delta f / f \cong 20\%$ one can expect $N_u \cong 5\cdot 10^{10}\cdot 10^{-4} \cong 5\cdot 10^{6}$ and $\varepsilon_f / \varepsilon_0 \cong 2.7\cdot 10^{-5}$. This is of cause the maximal possible value of cooling. Real value will be somewhere within $\approx 10^{-3}$. This procedure can be applied after LDS or KPC, so one can expect the invariant emittance $\gamma\varepsilon_y \le 2\cdot 10^{-9}\ cm\ rad$. According to formula (2), number of states for these emittances will be $N \cong 5.6\cdot 10^{11}$, where it was substituted $(\Delta p / p_0) \cong 10^{-3}$. We supposed also that the cooling is going with the beam energy about $1GeV$. As we have the number of the particles in the beam $N \cong 10^{10}$ only, this method also does not give the yield for full degeneration. Nevertheless, the figures are not drastically out of range and some optimization is possible. But this method need to be tested experimentally, however.

5. CONCLUSION

These considerations indicate, that for present day coolers the condition for degeneration couldn't be satisfied. In the same time, the *optimistic* estimations indicate, that the real condition for degeneration is not out of scale the near day coolers. One can try to investigate experimentally the condition for degeneration in routinely working machines meanwhile.

In the same time the physics of this quantum Crystal state in a cooled relativistic beam, may be interesting in general. Behavior of the beam with degeneration may

modify the characteristics of Synchrotron radiation as well as modify the scattering process in IP of electron-positron Collider.

For protons situation is not so optimistic. Despite to smaller emittance of the proton machines, the Compton wavelength of the proton is about 1836 times smaller, thereby excluding satisfaction of equation (1).

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