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Fine Structure Constant obtained from an Improved Calculation of the Electron $\mathbf{g-2}$*

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Abstract

A preliminary result is presented for an accurate determination of the fine structure constant α based on an improved evaluation of the sixth- and eighth-order radiative corrections to the electron anomalous magnetic moment a_e. In combination with a very precise measurement of a_e, this leads to the most accurate value of α obtained thus far, with an uncertainty of less than 0.004 parts per million.

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Introduction

Currently three of the best measurements of the fine structure constant α, with precision exceeding 0.1 ppm (parts per million), are those based on the quantum Hall effect [1], the ac Josephson effect [2], and the de Broglie wave length of a neutron beam [3]:

$$\begin{aligned} \alpha^{-1}(\mathrm{q.Hall}) &= 137.035\ 997\ 9(32) \quad (0.024\ \mathrm{ppm}), \\ \alpha^{-1}(\mathrm{acJ}) &= 137.035\ 977\ 0(77) \quad (0.056\ \mathrm{ppm}), \\ \alpha^{-1}(\mathrm{h/m_n}) &= 137.036\ 010\ 82\ (524) \quad (0.039\ \mathrm{ppm}). \end{aligned} \tag{1}$$

The most precise value of α, however, is obtained by combining the theoretical calculation of the electron anomalous magnetic moment a_e with its best measurements [4]:

$$\begin{aligned} a_{e^-}(\mathrm{exp}) &= 1\ 159\ 652\ 188.4\ (4.3) \times 10^{-12}, \\ a_{e^+}(\mathrm{exp}) &= 1\ 159\ 652\ 187.9\ (4.3) \times 10^{-12}, \end{aligned} \tag{2}$$

where e^- and e^+ refer to the electron and positron, respectively. This method utilizes the fact that the theoretical value of a_e is determined to a very high degree by α only. The value of α thus obtained and reported in [5] is

$$\alpha^{-1}(\mathrm{a_e}) = 137.035\ 991\ 85(42)(24)(51) \qquad (0.005\ \mathrm{ppm}), \tag{3}$$

where the numerals enclosed in parentheses are, respectively, errors in the numerical integration of the sixth-order and eighth-order terms and the experimental uncertainty in the measurement of a_{e^-} given in (2).

From the perspective of metrology as well as fundamental physics there is a strong interest in seeing how far the precision of $\alpha(a_e)$ can be pushed. This paper reports the progress made since then on the evaluation of $\alpha(a_e)$.

The QED contribution to a_e can be written as

$$\begin{aligned} a_e^{\mathrm{QED}} &= A_1 + A_2(m_e/m_\mu) \\ &\quad + A_2(m_e/m_\tau) \ + \ A_3(m_e/m_\mu, m_e/m_\tau), \end{aligned}$$

where m_e, m_μ, and m_τ are the masses of the electron, muon, and tau, respectively. A_2 and A_3 are very small as shown later. The first four coefficients in the perturbation expansion of the mass-independent A_1 term

$$A_1 = A_1^{(2)}\,(\alpha/\pi) + A_1^{(4)}\,(\alpha/\pi)^2 + \; A_1^{(6)}\,(\alpha/\pi)^3 + A_1^{(8)}\,(\alpha/\pi)^4 + \;\ldots, \tag{4}$$

which represent 1, 7, 72, and 891 Feynman diagrams, respectively, have been evaluated. $A_1^{(2)}$ and $A_1^{(4)}$ are known analytically:

$$\begin{aligned} A_1^{(2)} &= 0.5, \\ A_1^{(4)} &= -0.328\ 478\ 965\ldots. \end{aligned} \tag{5}$$

The α given in (3) was calculated using the preliminary results [5]

$$\begin{aligned} A_1^{(6)} &= 1.176\ 15\ (28), \\ A_1^{(8)} &= -1.557\ (70). \end{aligned} \tag{6}$$

The value of $A_1^{(6)}$ above is a "hybrid" obtained by combining the analytic results for 57 diagrams and the best numerical values of 15 diagrams obtained by the adaptive-iterative Monte-Carlo integration routine VEGAS [6] for which analytic results were not available. $A_1^{(8)}$ was obtained by purely numerical means.

New results

The most significant progress is the complete elimination of the uncertainty in $A_1^{(6)}$ by two independent works. One is a substantial improvement of $A_1^{(6)}$ by numerical integration made possible by advances in massively-parallel computing [7]. This work has led to a 7-fold improvement in $A_1^{(6)}$:

$$A_1^{(6)} = 1.181\ 259\ (40). \tag{7}$$

The other is the recent completion of analytic evaluation [8,9], which gives an exact result

$$A_1^{(6)} = 1.181\ 241\ 456 \ldots . \tag{8}$$

Note that $A_1^{(6)}$ of (6) is smaller than (7) or (8) by 0.43 % (or $\sim$ 18 standard deviations). This was discovered when a previously undetectable mismatch in combining analytic and best numerical results became noticeable as a consequence of improved numerical precision. This was traced to an error in the analytic evaluation of a certain two-loop integral [7]. It is another warning that analytic results by themselves cannot be fully trusted until they are confirmed independently. Meanwhile, the excellent agreement of (7) and (8) obtained by radically different methods assures that $A_1^{(6)}$ is now known exactly.

The numerical and analytic results of the most difficult integrals contributing to $A_1^{(6)}$ are summarized in Table I. A detailed comparison of these results shows that the numerical approach based on VEGAS can be trusted if it is handled with some precaution. This is very important because calculation of the eighth-order term $A_1^{(8)}$ must rely mostly on the numerical method for the foreseeable future.

The work on $A_1^{(8)}$ is now moving from a 'qualitative' stage to a 'quantitative' stage, made possible by the availability of a massively-parallel computer. The new evaluation of $A_1^{(8)}$ by VEGAS, carried out with substantially larger statistics than before, has led to

$$A_1^{(8)} = -1.409\ 2\ (384). \tag{9}$$

This is still being improved. This is consistent with the earlier results, if one takes into account that previous works suffer from severe shortages of integrand sampling and must also be corrected for the dependence of some integrals on the infrared cutoff (discussed in the next section). Details will be published elsewhere.

To obtain the theoretical value of a_e, one must include A_2, A_3, as well as the contributions of hadronic and electroweak interactions:[1]

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[1]The values of a_e(hadron) and a_e(electro$-$weak) are calculated based on the information provided by F. Jegerlehner, B. Krause, and W. J. Marciano.

$$
\begin{aligned}
(\alpha/\pi)^2 A_2^{(4)}(m_e/m_\mu) &= 2.804 \times 10^{-12}, && [10] \\
(\alpha/\pi)^2 A_2^{(4)}(m_e/m_\tau) &= 0.992 \times 10^{-14}, && [10] \\
(\alpha/\pi)^3 A_2^{(6)}(m_e/m_\mu) &= -0.924 \times 10^{-13}, && [11, 12] \\
(\alpha/\pi)^3 A_2^{(6)}(m_e/m_\tau) &= -0.825 \times 10^{-15}, && [11, 12] \\
(\alpha/\pi)^3 A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) &\sim 2.4 \times 10^{-21}, \\
a_e(\mathrm{hadron}) &= 1.648 \times 10^{-12}, \\
a_e(\mathrm{electro-weak}) &= 0.030 \times 10^{-12}. && (10)
\end{aligned}
$$

¿From (8), (9), and (10), one obtains

$$
a_e(\mathrm{theory}) = 1\ 159\ 652\ 205.4\ (1.2)\ (27.1) \times 10^{-12}, \qquad (11)
$$

if α(q.Hall) listed in (1) is chosen as the value of α. The first error is theoretical and the second is from the measurement of α(q.Hall).

Note that the overall uncertainty in (11) is dominated by that of α(q.Hall), the intrinsic theoretical uncertainty being much smaller. Note also that the error given in (2) for a_e(exp) is quite small. This means that a more precise value of α can be obtained by comparison of theory and measurement of a_e. Combining the measurements of a_{e^-} and a_{e^+} statistically assuming the validity of the CPT theorem, one obtains

$$
\begin{aligned}
\alpha^{-1}(\mathrm{a_e}) &= 137.035\ 999\ 93\ (14)\ (50) \\
&= 137.035\ 999\ 93\ (52) \qquad (0.003\ 8\ \mathrm{ppm}), \qquad (12)
\end{aligned}
$$

where the errors on the first line are from (9) and (2), respectively. This is the most accurate determination of α available at present.

Outline of the $A_1^{(8)}$ calculation

Now that the uncertainty in $A_1^{(6)}$ is eliminated completely, $A_1^{(8)}$ has become the dominant source of theoretical uncertainty. $A_1^{(8)}$ consists of contributions from 891 Feynman diagrams. They are represented by 86 integrals with the help of the Ward-Takahashi identity. A typical integrand consists of up to 20,000 terms, each a product of about 10 rational functions of Feynman parameters, and occupies up to 300 kilobytes of FORTRAN source code.

The integral is in general divergent at both large momentum (UV) and small momentum (IR) regions. The former can be dealt with by renormalization of mass and charge. The latter must be separated out before integration. Both are achieved by point-by-point cancellation of singularities by carefully tailored counterterms, which renders the integrand integrable everywhere in the domain. (The whole procedure will be referred to as "renormalization" for simplicity.) The IR subtraction terms are combined into $\sim$ 30 convergent integrals. Renormalized integrals are evaluated by VEGAS.

Adoption of the method of point-by-point singularity cancellation is crucial for enabling us to evaluate $A_1^{(8)}$ without going through an analytic construction of renormalized integrals of the conventional renormalization scheme, a forbiddingly complex and difficult procedure. The price to pay for our 'easy' method is that it is also the main source of numerical uncertainty which is caused by round-off errors inherent in the numerical integration method, such as VEGAS, that relies on integrand sampling at randomly chosen points.

Of course, if each step of computation were carried out with infinite precision, the scheme of pointwise cancellation would work as designed. In reality we calculate in finite precision, typically in double precision. The intended cancellation of divergences may thus fail because cancelling terms have no more than 12 or 13 significant digits (in double precision) and their difference will be dominated by round-off noises around a singularity. The difficulty is compounded further by register overflow. In many cases this may not cause a serious error since it is proportional to the fractional volume of the integration domain near the singularity. In cases where this causes a problem, however, it is found necessary to control it by resorting to quadruple precision arithmetic.

There are further problems associated with VEGAS. One is that the error estimate generated internally by VEGAS, which will behave as $1/\sqrt{N}$ for a sufficiently large number of total samplings N, may not be reliable until N becomes very large. Unfortunately, whether N is large enough or not depends on the individual integral and we do not know whether the

$1/\sqrt{N}$ behavior has been established or not until a calculation with larger N is carried out. Another complication arises from the fact that our integrand has peaks at the boundary of the domain. Samplings of such an integrand may generate a skewed distribution, rather than the normal distribution assumed by VEGAS. In fact, one standard deviation calculated internally by VEGAS appears to be too optimistic for our integrals. To compensate for a possible non-normal distribution to some extent, I found it desirable to enlarge the error generated by VEGAS by a factor of 1.6 or more, utilizing the χ^2 information generated by VEGAS.

All these problems have been anticipated from the beginning and measures have been taken to deal with them. It is the sheer size of the integrands contributing to $A_1^{(8)}$, however, that made it very difficult to accumulate an adequate number of integrand samplings necessary to resolve these problems.

In particular, in the past it has not been practical to rely on an extensive use of quadruple precision arithmetic since it slows down the computation by a factor of ~ 30. Thus, in the early runs, I was forced to control the bad fluctuations caused by round-off errors by means of an IR cutoff in some cases and exclusion of dangerous regions of the integration domain in others. Of course, this procedure introduces an error of its own. Since a reliable estimate of such an error requires an extensive computation, it was often difficult to obtain the necessary information. It is only recently that a satisfactory solution has become more accessible. The problem associated with the IR cutoff is now being eliminated by resorting to quadruple precision arithmetic whenever it is needed.

Table I shows that two of the diagrams ($6A$ and $6G$) required quadruple precision in some part of the integration domain. Double precision was sufficient for other diagrams. Comparison of numerical results and analytic results from Table I shows that N of order 10^{11} is large enough to be regarded as asymptotic and that the enlarged error estimate is more or less adequate, justifying the *ad hoc* measure discussed above. These observations are reinforced by numerous experiences with other sixth-order and lower-order integrals.

Over the years various preliminary values of $A_1^{(8)}$ have been reported. In hindsight, it is clear that they suffer from problems associated with woefully insufficient integrand samplings. As was noted above, for some integrals, it was necessary to introduce a small but finite photon mass to avoid wild fluctuations caused by round-off errors. A typical chosen photon mass λ (measured in units of the electron mass) was 10^{-5}. This may introduce an error of order $\lambda(\ln\lambda)^n$ in the integral, where n is some integer. Unfortunately, the precise value of n may not be determined without elaborate analytic work. Alternatively, its magnitude may be estimated by numerical integration. This requires, however, an extensive numerical work for several chosen values of λ. Recent evaluations indicate that n might be as large as 4 for some integrals. The correction may then amount to 10 % for $\lambda \simeq 10^{-5}$, well beyond the one standard deviation bound for these integrals.

At present the number N of sampling points used for evaluation of the integrals contributing to $A_1^{(8)}$ ranges from 10^{10} to 10^{11}. It appears that some integrals are safely in the asymptotic region, while others need further work. This is why the result (9) must be regarded as preliminary.

Concluding remarks

Currently the uncertainty in $\alpha(a_e)$ comes mostly from the measurement of a_e, even if allowance is made for the preliminary nature of the estimated error of $A_1^{(8)}$ in (9). Efforts are under way to improve the precision of measured a_e by more than an order of magnitude, to 0.000 4 ppm (= 0.4 ppb) or better [13,14]. On the theoretical side, I should like to emphasize that the successful conclusion of the $A_1^{(6)}$ calculation has not only reduced the theoretical uncertainty substantially but also enhanced the credibility of the numerical evaluation of $A_1^{(8)}$ significantly. Further work will certainly reduce the uncertainty of $A_1^{(8)}$ to at least 0.02, which corresponds to an uncertainty in a_e of about 0.6 ppb. Eventually, however, the theory will run into a brick wall due to the tenth-order term $A_1^{(10)}$, as is clear from

$$(\alpha/\pi)^5 \simeq 6.76 \times 10^{-14}. \tag{13}$$

Unless some way is found to estimate $A_1^{(10)}$, the determination of α by means of the electron anomalous magnetic moment will be unable to go beyond the precision of ~ 0.1 ppb.

At present two promising approaches are being pursued which may lead to measurement of α with a precision of 1 ppb or better [15,16]. It will be interesting to see whether some of these methods, or improvements to the methods employed in measuring α listed in (1), eventually surpasses the precision of $\alpha(a_e)$.

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TABLES

TABLE I. Comparison of numerical and analytic results for eight groups of sixth-order diagrams contributing to the electron anomalous magnetic moment. Infrared divergent terms are omitted for simplicity. All integrals are evaluated in double precision except in a small domain of $6A$, where the integral is evaluated in quadruple precision (using 10^7 function calls, 60 iterations) as is indicated by the second numbers in the row for $6A$; similarly for $6G$. $\Delta =$ (numerical result $-$ analytic result)/(uncertainty in numerical result).

Diagram	Function calls per iteration (in units of 10^8)	No. of iterations	Numerical result	Analytic result	Δ
$6A$	40,0.1	60, 60	$-0.529491(89)$	$-0.529448726\ldots$	-0.455
$6B$	40	60	2.545351(152)	$2.545526100\ldots$	-1.100
$6C$	40	60	$-1.416898(142)$	$-1.417104129\ldots$	1.396
$6D$	80	62	$-0.871075(138)$	$-0.871237459\ldots$	1.130
$6E$	20	60	$-1.242410(121)$	$-1.242417882\ldots$	0.065
$6F$	40	65	1.551989(107)	$1.552137597\ldots$	-1.341
$6G$	10, 1	60, 60	0.081997(143)	$0.082122612\ldots$	-0.851
$6H$	200	68	0.785419(39)	$0.785401156\ldots$	0.461