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Integrable Quantum Field Theories in Finite Volume: Excited State Energies.

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Abstract

We develop a method of computing the excited state energies in Integrable Quantum Field Theories (IQFT) in finite geometry, with spatial coordinate compactified on a circle of circumference R. The IQFT "commuting transfer-matrices" introduced in [1] for Conformal Field Theories (CFT) are generalized to non-conformal IQFT obtained by perturbing CFT with the operator $\Phi_{1,3}$. We study the models in which the fusion relations for these "transfer-matrices" truncate and provide closed integral equations which generalize the equations of Thermodynamic Bethe Ansatz to excited states. The explicit calculations are done for the first excited state in the "Scaling Lee-Yang Model".

1. Introduction

Much progress in understanding $2D$ Integrable Quantum Field Theories (IQFT) has been achieved during the last two decades (see e.g. [2] for a review). In particular, many models have been exactly solved "on-shell" i.e. their mass spectra and their factorizable S-matrices have been determined. Important progress was also made towards "off-shell" solution of IQFT. The approach based on so called Thermodynamic Bethe Ansatz (TBA) technique [3] has proven to be very successful [4]. This approach allows one to calculate the ground state energy $E_0(R)$ of IQFT in the finite-size geometry, where the spatial coordinate is compactified on the circle of circumference R, provided the "on-shell" solution is known. The problem being reduced to solving certain non-linear integral equation (TBA equation). The quantity $E_0(R)$ makes it possible to probe the scale dependence in the IQFT, in particular, to find in the limit $R \to 0$ the central charge c of the Conformal Field Theory (CFT) which governs the short distance behavior of the IQFT.

So far the power of TBA approach was limited to the ground state energy $E_0(R)$ (or the ground state energies in "twisted" sectors [5], [6]). At the same time it seems to be interesting to calculate the full energy spectrum $E_n(R)$ of the finite-size IQFT. These energies would provide interpolation between the CFT spectrum (determined by the CFT central charge c and the conformal dimensions Δ_a) at $R \to 0$ and massive spectrum (determined by the particle masses and the S-matrix amplitudes) at $R \to \infty$. However it is not clear how the ideas behind traditional derivation of TBA equation could be generalized to incorporate the excited states.

In our recent paper [1] we have shown how one can define the family of operators $\mathbf{T}_j(\lambda)$; $j = 1/2, 1, 3/2, \ldots$ — the "QFT transfer-matrices" [1] which act in the space of states of the finite size IQFT and commute for different values of the parameter λ

$$\big[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')\big] = 0 \ . \tag{1.1}$$

We have also shown that the "fusion relations" for the operators $\mathbf{T}_j(\lambda)$

$$\mathbf{T}_j(q^{\frac{1}{2}}\lambda)\mathbf{T}_j(q^{-\frac{1}{2}}\lambda) = \mathbf{I} + \mathbf{T}_{j-1/2}(\lambda)\mathbf{T}_{j+1/2}(\lambda) \ . \tag{1.2}$$

[1] These operators appear as the continuous QFT versions of Baxter's commuting transfer-matrices [7], [8] and therefore we maintain using this term although the original meaning of the term "transfer-matrix" here is apparently lost.

(here q is a parameter related to the Virasoro central charge c), give rise to the functional equations for the eigenvalues of these "transfer-matrices" and that for the case of the ground state eigenvalue these reduce to the TBA equations; at the same time the functional equations apply to the excited states as well and following the method developed by Klümper and Pearce [9] one can derive from them the nonlinear integral equations which determine in principle the excited states $E_n(R)$.

Our analysis in [1] explicitly concerns the case of CFT (more precisely, the $c < 1$ "minimal" CFT [10]), where the space of states is decomposed in the sum of the direct products of "right" and "left" irreducible highest-weight Virasoro module $\mathcal{V}_{\Delta_a}$ and the operators $\mathbf{T}(\lambda)$ act in each of these spaces $\mathcal{V}_{\Delta_a}$ separately. However, one can consider more general non-conformal IQFT which are obtained by perturbing the "minimal CFT" with the operator $\Phi_{1,3}$ [11]. In this paper we show that the commuting operators similar to $\mathbf{T}_j(\lambda)$ can be constructed in the perturbed theory as well (we denote these "perturbed" transfer matrices as $\mathbb{T}_j(\mu|\lambda)$, where μ is the parameter of the perturbation), and that the most important properties of the operators $\mathbf{T}_j(\lambda)$ obtained in [1], including the fusion relation (1.2), remain valid for the "perturbed" operators $\mathbb{T}_j(\mu|\lambda)$. Although the analytic properties of the operators $\mathbb{T}_j(\mu|\lambda)$ as the functions of λ turn out to be significantly different from those of the "conformal" operators $\mathbf{T}_j(\lambda)$, it is still possible to derive the nonlinear integral equations which determine the finite-size energy levels $E_n(R)$ in the perturbed theory.

To demonstrate the efficiency of our approach we calculate the energy $E_1(R)$ of the first excited state of the so called Scaling Lee-Yang Model (SLYM). This model is obtained by perturbing the $c = -22/5$ "minimal" CFT $\mathcal{M}_{2/5}$ with the operator $\Phi = \Phi_{1,3}$ of the dimension $\Delta = -1/5$, i.e. it is defined by the action

$$\mathcal{A}_{SLYM} = \mathcal{A}_{\mathcal{M}_{2/5}} + \hat{\mu}^2 \int \Phi(x)\, d^2x \ , \tag{1.3}$$

where $\mathcal{A}_{\mathcal{M}_{2/5}}$ is the action of the CFT $\mathcal{M}_{2/5}$ and $\hat{\mu}$ is the coupling parameter. The QFT defined by (1.3) is integrable [11] and its on-shell solution (i.e. the particle content and the factorizable S-matrix) was found in [12]. The theory contains one sort of scalar particles with the mass m and their two-particle S-matrix is

$$S(\theta) = \frac{\sinh(\theta) + i\sin(\pi/3)}{\sinh(\theta) - i\sin(\pi/3)} \ , \tag{1.4}$$

where $\theta = \theta_1 - \theta_2$; θ_1 and θ_2 are the rapidities of the scattering particles (i.e. their momenta are $p_1 = m\ \sinh(\theta_1),\ p_2 = m\ \sinh(\theta_2)$). The finite size ground state energy $E_0(R)$ of (1.3) was obtained in [4] using TBA technique. Also, few first excited levels $E_n(R)$ was studied on the basis of "Truncated Conformal Space method" in [13]. More recently the exact relation between the mass m and the coupling parameter $\hat{\mu}$ in (1.3) was found in [14]

$$\hat{\mu}^2 = i\ \frac{2^{1/5}\,5^{3/4}}{16\,\pi^{6/5}}\ \frac{\big(\Gamma(2/3)\Gamma(5/6)\big)^{12/5}}{\Gamma(3/5)\Gamma(4/5)}\ m^{12/5}\ . \tag{1.5}$$

We apply the approach based on the functional equations for the transfer-matrices $\mathbb{T}$ to calculate the finite-size energy of the first excited state, the state which is interpreted as the one-particle state (with the zero-momentum particle) at $R \to \infty$ and approaches the CFT state associated with the identity operator of the CFT as $R \to 0$.

The paper is organized as follows. In Sect.2 we summarize the properties of the operators $\mathbf{T}_j(\lambda)$ of the Conformal Theory defined in [1]. The "transfer-matrices" $\mathbb{T}_j(\mu|\lambda)$ of the perturbed theory are introduced in Sect.3 where also our basic conjectures about the analytic properties of these operators and their relation to the local Integrals of Motion (IM) are formulated. In Sect.4 we study the simplest eigenvalues of the operator $\mathbf{T}_{1/2}(\lambda)$ in the CFT $\mathcal{M}_{2/5}$. Similar analysis of the eigenvalues of $\mathbb{T}_{1/2}(\mu|\lambda)$ in SLYM is carried out in Sect.5 where we derive the excited state energy $E_1(R)$. In Sect.6 the relation to the sine-Gordon theory is discussed and further applications of our approach are outlined.

2. The operators $\mathbf{T}_j(\lambda)$ in CFT

In this section we briefly summarize the properties of the operators $\mathbf{T}_j(\lambda)$ in CFT as they are defined in [1].

The space of states in a CFT decomposes as

$$\mathcal{H}_{CFT} = \oplus_a \big(\mathcal{V}_{\Delta_a} \otimes \bar{\mathcal{V}}_{\Delta_a}\big)\ , \tag{2.1}$$

(for simplicity we consider here the "diagonal" CFT) where $\mathcal{V}_{\Delta_a}$ is the irreducible highest weight representation of the "left" Virasoro algebra Vir

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,(n^3 - n)\,\delta_{n+m,0}\ , \tag{2.2}$$

with the highest weight Δ_a. Correspondingly, $\bar{\mathcal{V}}_{\Delta_a}$ stands for the representation space of the "right" Virasoro algebra $\overline{Vir}$ identical to Vir. The number c in (2.2) is known as the "central charge". The "left" Virasoro generators L_n are expressed in terms of the "left" chiral component of the energy-momentum tensor $T(z)$ $(z = x + iy)$ as

$$L_n = \frac{c}{24}\,\delta_{n,0} + \frac{R}{2\pi}\int_0^R \frac{dx}{2\pi}\,T(x+iy)\;\;e^{i\frac{2\pi}{R}nx}\;. \tag{2.3}$$

Here we assume that the CFT is defined in the geometry of an Euclidean cylinder so that y is interpreted as the Euclidean time and the spatial coordinate x is compactified on a circle of the circumference R. The field $T(z)$ therefore satisfies the periodicity condition $T(z+R) = T(z)$. The "right" Virasoro generators $\bar{L}_n$ are defined in terms of the "right" chiral component $\bar{T}(\bar{z})$, $\bar{z} = x - iy$ by similar integrals.

Any CFT possesses infinitely many local Integrals of Motion (IM) $\mathbf{I}_{2k-1}$ [15], [16] which can be written as

$$\mathbf{I}_{2k-1} = \int_0^R \frac{dx}{2\pi}\,T_{2k}(x+iy)\;, \tag{2.4}$$

where $T_{2k}(z)$ are certain local fields, polynomials in $T(z)$ and its derivatives. For example

$$\begin{gathered} T_2(z) = T(z),\;\; T_4(z) =\,:T^2(z):,\;\; T_6(z) =\,:T^3(z):+\frac{c+2}{12}:\big(T'(z)\big)^2:,\;\ldots, \\ T_{2k}(z) =\,:T^k(z):\;+\text{ terms with the derivatives}\;, \end{gathered} \tag{2.5}$$

Here : : denote appropriately regularized operator products, see [1] for details. There exists infinitely many densities (2.5) (one for each integer k [17]) such that all the integrals (2.4) commute

$$[\,\mathbf{I}_{2k-1}, \mathbf{I}_{2l-1}] = 0\;. \tag{2.6}$$

The "transfer-matrices" $\mathbf{T}_j$ are most conveniently defined in terms of Feigin-Fuchs free field representation of the Virasoro algebra [18], [19]. Introduce the "left" chiral Bose field[2]

$$\phi(z) = Q + \frac{2\pi z}{R}\,P - i\sum_{n\neq 0}\frac{a_{-n}}{n}\,e^{i\frac{2\pi}{R}nz}\quad, \tag{2.7}$$

where the operators Q, P, a_n satisfy the canonical commutation relations

$$[Q,P] = \frac{i}{2}\beta^2, \qquad [a_n, a_m] = \frac{n}{2}\,\beta^2\,\delta_{n+m,0} \tag{2.8}$$

and β is a parameter which will be defined below. Also, let $\mathcal{F}_p$ be the Fock space, i.e the space generated by the action of a_n with $n < 0$ on the "vacuum" state $\mid p\rangle$ which satisfies the equations

$$P\;\mid p\rangle = p\;\mid p\rangle\;, \qquad a_n\mid p\rangle = 0 \quad\text{for}\quad n > 0\;. \tag{2.9}$$

[2] This field differs in normalization from the field φ which is used in [1], $\varphi(z) = i\,\phi(z)$.

Note that (2.7) implies that the field $\phi(z)$ is quasi-periodic, $\phi(z+R) = \phi(z) + 2\pi P$. As is well known the energy-momentum tensor

$$T(z) = \beta^{-2} \ : (\partial_z \phi)^2 : +i \ (1-\beta^{-2}) \ \partial_z^2 \phi - \frac{1}{24} \tag{2.10}$$

generates the Virasoro algebra (2.2) with the central charge

$$c = 13 - 6 \, (\beta^2 + \beta^{-2}) \ , \tag{2.11}$$

which therefore is represented in the Fock space $\mathcal{F}_p$. We will always assume that $c < 1$ and hence that β is real. For generic values of β and p the space $\mathcal{F}_p$ is isomorphic to the irreducible highest weight module $\mathcal{V}_\Delta$ with

$$\Delta = \Delta(p) \equiv \Big(\frac{p}{\beta}\Big)^2 + \frac{c-1}{24} \ , \tag{2.12}$$

while for particular values of these parameters, when the "null-vectors" appear (so called "degenerate representations"), the irreducible representation $\mathcal{V}_\Delta$ can be obtained from $\mathcal{F}_p$ by factoring out the corresponding invariant subspaces. In fact, in what follows we always will imply that all such an invariant subspaces in $\mathcal{F}_p$ are factored out and $\mathcal{V}_p$ will stand for the "Virasoro irreducible Fock space", $\mathcal{V}_p = \mathcal{F}_p \, / \, (\text{invariant subspaces})$. With this convention the space $\mathcal{V}_p$ is isomorphic to $\mathcal{V}_{\Delta(p)}$.

Let j be a non-negative integer or half-integer number. Consider the following $(2j+1) \times (2j+1)$ matrix [1]

$$\mathbf{L}_j(\lambda) = e^{i\pi P H_j} \ \mathcal{P} \exp\Big(\lambda \int_0^R dz \, \big(: e^{-2i\phi(z)} : q^{\frac{H_j}{2}} E_j + : e^{2i\phi(z)} : q^{-\frac{H_j}{2}} F_j \big)\Big) \ , \tag{2.13}$$

where $E_j, \ F_j$ and H_j stand for the $2j+1$ dimensional representation π_j of the quantum algebra $U_q\big(sl(2)\big)$ with

$$q = e^{i\pi\beta^2} \ , \tag{2.14}$$

and the symbol $\mathcal{P}$ in (2.13) denotes the "path ordering" of the both operator and matrix products. The matrix elements $\big[\mathbf{L}_j(\lambda)\big]_A^B$, $A, B = -j, -j+1, ..., j$ are operators which act in the space

$$\hat{\mathcal{F}}_p = \oplus_{n=-\infty}^{\infty} \mathcal{F}_{p+n\beta^2} \ . \tag{2.15}$$

Note that the exponential fields $: e^{\pm 2i\phi(z)} :$ in (2.13) have anomalous dimensions and therefore the parameter λ carries the dimension

$$\lambda \sim \big[\, length \,\big]^{\beta^2 - 1} \ . \tag{2.16}$$

As discussed in [20], the expression (2.13) applies directly only for $\beta^2 < 1/2$ (i.e. for $c < -2$); for $\beta^2 > 1/2$ the integral in (2.13) can be defined through appropriate analytic continuation in β^2.

The operator matrices $\mathbf{L}_j(\lambda)$ satisfy the Yang-Baxter relation

$$\begin{aligned}&\sum_{C=-j}^{j}\sum_{C'=-j'}^{j'} \big[R_{j\,j'}(\lambda/\lambda')\big]_{A\,A'}^{C\,C'} \,\big[\mathbf{L}_j(\lambda)\big]_{C}^{B}\,\big[\mathbf{L}_{j'}(\lambda')\big]_{C'}^{B'} = \\ &\sum_{C=-j}^{j}\sum_{C'=-j'}^{j'} \big[\mathbf{L}_{j'}(\lambda')\big]_{A'}^{C'}\,\big[\mathbf{L}_j(\lambda)\big]_{A}^{C}\,\big[R_{j\,j'}(\lambda/\lambda')\big]_{C\,C'}^{B\,B'}\ ,\end{aligned} \tag{2.17}$$

where $\big[R_{j\,j'}(\lambda)\big]_{A\,A'}^{B\,B'}$ are the elements of the R-matrix acting in the product of $2j+1$ and $2j'+1$ dimensional representations of $U_q\big(sl(2)\big)$. As a simple consequence of (2.17) the "transfer-matrices"

$$\mathbf{T}_j(\lambda) = tr_{\pi_j}\big[\,e^{i\pi PH_j}\ \mathbf{L}_j(\lambda)\,\big] \equiv \sum_{A=-j}^{j}\big[\,e^{i\pi PH_j}\ \mathbf{L}_j(\lambda)\,\big]_A^A \tag{2.18}$$

form a commuting family of operators, i.e. they satisfy (1.1). Note that although the off-diagonal elements of the matrices (2.13) act between different components of (2.15) the operators (2.18) invariantly act in the spaces $\mathcal{V}_p$,

$$\mathbf{T}_j(\lambda):\qquad \mathcal{V}_p \to \mathcal{V}_p\ . \tag{2.19}$$

It is possible to show using (2.13), (2.17) that the operators $\mathbf{T}_j(\lambda)$ satisfy the fusion relations (1.2) [1], [21] which allow one to express any operator $\mathbf{T}_j(\lambda)$ with $j > 1/2$ through the "fundamental" one $\mathbf{T}(\lambda) \equiv \mathbf{T}_{1/2}(\lambda)$.

The operator $\mathbf{T}(\lambda)$ is an entire function of λ^2 (the same is true for all the operators $\mathbf{T}_j(\lambda)$). Its power series expansion in the variable λ^2 follows directly from the definitions (2.13), (2.18)

$$\mathbf{T}(\lambda) = 2\ \cos(2\pi P) + \sum_{k=1}^{\infty}\lambda^{2k}\,\mathbf{G}_k\ , \tag{2.20}$$

where $\mathbf{G}_k$ are basic "nonlocal IM"

$$\begin{aligned}\mathbf{G}_k = q^k \int_{\mathcal{D}_R(x_1,\ldots,x_{2k})}\Big(\ & e^{2i\pi P}\ :e^{-2i\phi(x_1)}::e^{2i\phi(x_2)}::e^{-2i\phi(x_3)}:\ldots:e^{2i\phi(x_{2k})}: \\ &+ e^{-2i\pi P}\ :e^{2i\phi(x_1)}::e^{-2i\phi(x_2)}::e^{2i\phi(x_3)}:\ldots:e^{-2i\phi(x_{2k})}:\ \Big)\ .\end{aligned} \tag{2.21}$$

Here the integration is performed over the domain $D_R(x_1,...,x_{2k}) = \{R > x_1 > x_2 > ... > x_{2k} > 0\}$. In the opposite limit $\lambda^2 \to \infty$ the operator $\mathbf{T}(\lambda)$ admits asymptotic expansion [1], [20]

$$\log \mathbf{T}(\lambda) \ \simeq \ (\kappa\, R/2\pi)\ \lambda^{(1+\xi)}\ \mathbf{I} - \sum_{n=1}^{\infty}\ C_n\ \lambda^{(1-2n)(1+\xi)}\ \mathbf{I}_{2n-1}\ . \tag{2.22}$$

in powers of $\lambda^{-(1+\xi)}$ with the coefficients proportional to the local IM (2.4). The numerical factors κ and C_n are given by

$$\begin{gathered}\kappa = \frac{2\sqrt{\pi}\ \Gamma(\frac{1}{2}-\frac{\xi}{2})}{\Gamma(1-\frac{\xi}{2})}\ \Big(\Gamma\big(1-\beta^2\big)\Big)^{1+\xi}\ ,\\ C_n = \frac{\sqrt{\pi}(1+\xi)}{n!}\ \big(\beta^2\big)^n\ \frac{\Gamma\big((n-\frac{1}{2})(1+\xi)\big)}{\Gamma\big(1+(n-\frac{1}{2})\xi\big)}\ \Big(\Gamma(1-\beta^2)\Big)^{-(2n-1)(1+\xi)}\ .\end{gathered} \tag{2.23}$$

Here and below

$$\xi = \frac{\beta^2}{1-\beta^2}\ . \tag{2.24}$$

The above expansions suggest that the eigenvalues of the operator $\mathbf{T}(\lambda)$ unite all the information about the eigenvalues of both nonlocal IM (2.21) and local IM (2.4) in a very interesting manner. In what follows we consider almost exclusively the vacuum eigenvalues

$$\mathbf{T}(\lambda)\mid p\rangle = T(t,p)\mid p\rangle\ ; \qquad t \equiv \lambda^2\ \Big(\frac{R}{2\pi}\Big)^{2-2\beta^2}\ . \tag{2.25}$$

According to (2.20) and (2.22) this function expands both in convergent series around $t=0$ and in asymptotic series in the vicinity of $t=\infty$,

$$T(t,p) = 2\cos(2\pi p) + \sum_{k=1}^{\infty} t^k\ G_k^{vac}(p) \tag{2.26a}$$

$$\simeq \exp\Big(\kappa\ t^{\frac{1+\xi}{2}}\ - \sum_{n=1}^{\infty}\ C_n\ t^{(1/2-n)(1+\xi)}\ I^{vac}_{2n-1}\big(\Delta(p)\big)\Big)\ , \tag{2.26b}$$

where $G_k^{vac}(p)$ and $I^{vac}_{2n-1}(\Delta)$ are, respectively, the vacuum eigenvalues of the nonlocal IM (2.21) and the local IM (2.4) with $R=2\pi$. Note, that the expression in (2.26b) is, in fact, an asymptotic series with zero radius of convergence. The eigenvalues $G_k^{vac}(p)$ are given by the integrals

$$\begin{gathered}G_k^{vac}(p) = \int_0^{2\pi} du_1 \int_0^{u_1} dv_1 \int_0^{v_1} du_2 \int_0^{u_2} dv_2 ... \int_0^{v_{n-1}} du_n \int_0^{u_n} dv_n\\ \prod_{j>i}^n\Big[\Big(2\ \sin\big(\frac{u_i-u_j}{2}\big)\Big)^{2\beta^2}\ \Big(2\ \sin\big(\frac{v_i-v_j}{2}\big)\Big)^{2\beta^2}\Big]\prod_{j\geq i}^n\Big(2\ \sin\big(\frac{u_i-v_j}{2}\big)\Big)^{-2\beta^2}\times\\ \prod_{j>i}^n\Big(2\ \sin\big(\frac{v_i-u_j}{2}\big)\Big)^{-2\beta^2}\ 2\ \cos\Big(2p\ \big(\pi+\sum_{i=1}^n(v_i-u_i)\big)\Big)\ ,\end{gathered} \tag{2.27}$$

while the eigenvalues $I^{vac}_{2n-1}(\Delta)$ can be calculated (in principle) from the explicit formulas for the associated local densities $T_{2n}(z)$ in (2.4). Unfortunately for general k neither the integrals (2.27) can be calculated explicitly nor the densities $T_{2k}(z)$ are known in a closed form. Some partial results about the vacuum eigenvalues $G^{vac}_k(p)$ and $I^{vac}_{2n-1}(\Delta)$ are collected in Appendices A and B.

For $\beta^2 < 1/2$ the entire function $T(t,p)$ is completely determined by the positions t_k of its zeroes in the complex t-plane. The locations of these zeroes are known exactly in two particular cases. First, in the "classical limit" $\beta^2 \to 0$, (with p and λ kept fixed) the explicit formula for the vacuum eigenvalue $T(t,p)$ can be easily obtained directly from the definition (2.18)[3]

$$T(t,p) = 2\ \cos\big(2\pi\sqrt{p^2-t}\,\big); \qquad \beta^2 = 0\ . \tag{2.28}$$

In this case

$$t_k = p^2 - \frac{(2k+1)^2}{16}\ , \quad k = 0,1,2,.... \tag{2.29}$$

Another case where $T(p,t)$ is known explicitly is the "free-fermion point" $\beta^2 = 1/2$ [20],

$$T(t,p) = \frac{2\pi\ e^{2\mathcal{C}t}}{\Gamma(1/2+2p+\pi t)\,\Gamma(1/2-2p+\pi t)}, \qquad \beta^2 = 1/2\ , \tag{2.30}$$

where $\mathcal{C}$ is some constant. This function has two sequences of zeroes

$$t^{(\pm)}_k = \frac{1}{\pi}\ (\pm 2p - 1/2 - k)\ , \quad k = 0,1,2,...\ . \tag{2.31}$$

For generic $0 < \beta^2 < \frac{1}{2}$ the zeroes t_k do not have any simple form. Nonetheless a brief inspection of (2.26a) and (2.29) , (2.31) makes it natural to adopt the following

Conjecture 1. *For real p and $0 \le \beta^2 \le 1/2$ all zeroes t_k of $T(t,p)$ are real. There are infinitely many negative zeroes which accumulate towards $t = -\infty$; for given value p there are also exactly* $\mathrm{Int}(2|p| + 1/2)$ *positive zeroes (*$\mathrm{Int}(x)$ *denotes integer part of x). In particular, if $|p| < 1/4$ all zeroes t_k are negative.*

This assumption will be used in Sect.4 in deriving closed nonlinear integral equation for the vacuum eigenvalues in particular case $\beta^2 = 2/5$.

[3] To derive (2.28) one observes that the oscillator modes a_n can be ignored in the vertex operators $:e^{\pm 2i\phi}:$ as they do not contribute to the vacuum eigenvalues in this limit. Also, in this limit the zero mode operators P and Q commute and one obtains $T_j(t,p) = tr_j\Big(e^{2\pi\,(\sqrt{t}E_j+\sqrt{t}F_j+ipH_j)}\Big)$.

3. The Operators $\mathbb{T}_j(\mu|\lambda)$ in the $\Phi_{1,3}$ Perturbed Theory

Our previous discussion was concerned with the Conformal Field Theory. It is well known that one can obtain more general non- conformal Integrable Quantum Field Theory by perturbing certain CFT with appropriately chosen relevant local operator. The most known example is a "minimal CFT" perturbed with the degenerate field $\Phi_{1,3}$, i.e. the primary field with the conformal dimensions

$$\Delta_{1,3} = \bar{\Delta}_{1,3} = -1 + 2\beta^2 = 1 - \frac{2}{1+\xi} \ , \tag{3.1}$$

which satisfy the third-level null vector equations [10]

$$\begin{aligned} &\big(2\,(1+2\beta^2)\,L_{-3} - 4\,L_{-1}L_{-2} + \beta^{-2}\,L_{-1}^3\big)\ \Phi_{1,3}(z,\bar{z}) = 0\ , \\ &\big(2\,(1+2\beta^2)\,\bar{L}_{-3} - 4\,\bar{L}_{-1}\bar{L}_{-2} + \beta^{-2}\,\bar{L}_{-1}^3\big)\ \Phi_{1,3}(z,\bar{z}) = 0\ . \end{aligned} \tag{3.2}$$

We will assume the canonical normalization of this field

$$\langle \Phi_{1,3}(z,\bar{z})\,\Phi_{1,3}(z',\bar{z}')\rangle \to \big[(z-z')(\bar{z}-\bar{z}')\big]^{-2\Delta_{1,3}} \quad \text{as} \quad (z,\bar{z}) \to (z',\bar{z}')\ . \tag{3.3}$$

The perturbed theory is defined by the formal action

$$\mathcal{A}_{PCFT} = \mathcal{A}_{CFT} + \hat{\mu}^2 \int d^2x\ \Phi_{1,3}(z,\bar{z})\ , \tag{3.4}$$

where $\mathcal{A}_{CFT}$ denotes the "action" of CFT and $\hat{\mu}$ is the coupling parameter with the dimension

$$\hat{\mu} \sim \big[\,length\,\big]^{\Delta_{1,3}-1} = \big[\,length\,\big]^{2\beta^2-2}\ . \tag{3.5}$$

As in Sect.2 we assume that the QFT is defined in the geometry of a cylinder with the spatial coordinate $x = \Re e\, z$ compactified on a circle with the circumference R. Correspondingly, the field $\Phi_{1,3}$ in (3.4) is taken to be periodic function of x, i.e. $\Phi_{1,3}(z+R,\bar{z}+R) = \Phi_{1,3}(z,\bar{z})$. As was shown in [11] the QFT defined by (3.4) is integrable in the sense that it possesses infinitely many local IM which appear as certain deformations of the local IM (2.4) of the original CFT. Although for $\hat{\mu} \neq 0$ the fields T_{2k} (2.5) are no longer holomorphic fields (i.e. they depend both on z and $\bar{z}$) they satisfy the continuity condition

$$\partial_{\bar{z}} T_{2k}(z,\bar{z}) = \hat{\mu}^2\,\partial_z \Theta_{2k-2}(z,\bar{z})\ , \tag{3.6}$$

where $\Theta_{2k-2}(z,\bar{z})$ are certain local fields. The right-moving chiral sector of the original CFT contains also the antiholomorphic fields $\bar{T}_{2k}$; in the perturbed theory (3.4) they satisfy similar continuity equations

$$\partial_z \bar{T}_{2k}(z,\bar{z}) = \hat{\mu}^2\, \partial_{\bar{z}} \bar{\Theta}_{2k-2}(z,\bar{z}) \ . \tag{3.7}$$

It follows that the integrals

$$\begin{aligned} \mathbb{I}_{2k-1} &= \int_0^R \frac{dx}{2\pi} \left(T_{2k}(x+iy, x-iy) + \hat{\mu}^2\, \Theta_{2k-2}(x+iy, x-iy)\right) \ , \\ \bar{\mathbb{I}}_{2k-1} &= \int_0^R \frac{dx}{2\pi} \left(\bar{T}_{2k}(x+iy, x-iy) + \hat{\mu}^2\, \bar{\Theta}_{2k-2}(x+iy, x-iy)\right) \end{aligned} \tag{3.8}$$

in fact do not depend on the "Euclidean time" y, i.e. they are Integrals of Motion (IM). It is possible to show [11] that the operators (3.8) commute among themselves

$$[\mathbb{I}_{2k-1}, \mathbb{I}_{2l-1}] = [\mathbb{I}_{2k-1}, \bar{\mathbb{I}}_{2l-1}] = [\bar{\mathbb{I}}_{2k-1}, \bar{\mathbb{I}}_{2l-1}] = 0 \ . \tag{3.9}$$

Note that the field $\hat{\mu}^2\, \Theta_0 = \hat{\mu}^2\, \bar{\Theta}_0 = \pi\, \hat{\mu}^2\, (1-\Delta_{1,3})\, \Phi_{1,3}$ coincides with the trace $T^{\alpha}_{\alpha}/4$ of the energy-momentum tensor $T_{\alpha\beta}$ of the QFT (3.4), and the IM $\mathbb{I}_1$ and $\bar{\mathbb{I}}_1$ are related to the Hamiltonian $\mathbb{H}$ and the spatial momentum $\mathbb{P}$ of this QFT

$$\mathbb{H} = \mathbb{I}_1 + \bar{\mathbb{I}}_1, \qquad \mathbb{P} = \mathbb{I}_1 - \bar{\mathbb{I}}_1 \ . \tag{3.10}$$

As in Sect.2 it is convenient to use the Feigin-Fuchs free-field realization for the original CFT. The space of states (2.1) can be realized as

$$\mathcal{H}_{CFT} = \oplus_p\, \mathcal{H}_p; \qquad \mathcal{H}_p \equiv \mathcal{V}_p \otimes \bar{\mathcal{V}}_{-p} \ , \tag{3.11}$$

where $\mathcal{V}_p$ are the "irreducible Fock spaces" defined in Sect.2 and $\bar{\mathcal{V}}_{-p}$ are similar spaces generated by the "right-moving" chiral Bose field

$$\bar{\phi}(\bar{z}) = \bar{Q} - \frac{2\pi\,\bar{z}}{R}\bar{P} - i\sum_{n\neq 0} \frac{\bar{a}_{-n}}{n}\, e^{-i\frac{2\pi}{R}n\bar{z}} \ . \tag{3.12}$$

Each of the spaces $\mathcal{H}_p$ forms an irreducible representation of $Vir \oplus \overline{Vir}$ with the highest weight $\big(\Delta(p), \Delta(p)\big)$ and the highest weight vector

$$\mid \Delta(p)\rangle = \mid p\rangle \otimes \overline{\mid -p\rangle} \ , \tag{3.13}$$

where $| \, p\rangle$ and $\overline{| -p\rangle}$ are the vacuum vectors in $\mathcal{F}_p$ and $\bar{\mathcal{F}}_{-p}$, respectively. The sum in (3.11) is taken over certain set of admitted values of the parameter p which is specific for each CFT and determines its Virasoro representation content. In different theories this set can be discrete or continuous.

One remark is in order here. The above perturbed QFT (3.4) can be constructed only if the original CFT meets certain criteria. For instance, as in CFT the space of local fields is isomorphic to the space of states (3.11), the sum in (3.11) must contain the term with $2p = 3\beta^2 - 1$ corresponding to the field $\Phi_{1,3}$ (and its conformal descendants). Moreover, according to the fusion rules which follow from the null-vector equations (3.2) the operator $\Phi_{1,3}$ acts as

$$\Phi_{1,3} \; : \quad \mathcal{H}_p \to \mathcal{H}_{p-\beta^2} \, \oplus \, \mathcal{H}_p \, \oplus \, \mathcal{H}_{p+\beta^2} \ , \tag{3.14}$$

and therefore to admit the action of this operator the space (3.11) must contain the whole sum

$$\hat{\mathcal{H}}_p = \oplus_n \, \mathcal{H}_{p+n\beta^2} \tag{3.15}$$

together with each $\mathcal{H}_p$ included therein. For particular values of p (corresponding to so called "completely degenerate" representations of Vir) this sum truncates and contains in fact only finitely many terms (see [10]). In any case we will assume that the space (3.11) has the structure

$$\mathcal{H}_{CFT} = \oplus_a \, \hat{\mathcal{H}}_{p_a} \ . \tag{3.16}$$

Obvious example of CFT with this structure is "minimal CFT" $\mathcal{M}_{r/r'}$ with $\beta^2 = \frac{r}{r'}$ [10] in which case p in (3.11) run over finitely many values $2\, p_{l,k} = l - k\, \beta^2$ $(l = 1, 2, ..., r-1; \; k = 1, 2, ..., r' - 1)$. Another example is $c = 1$ CFT of uncompactified free Bose field where p in (3.11) takes continuous values.

The perturbation term in (3.4) is relevant (i.e. $\Delta_{1,3} < 1$) for $0 < \beta^2 < 1$ and so the perturbation theory in μ^2 is superrenormalizable (moreover, for $\beta^2 < 1/2$ this perturbation theory does not have any short-distance divergences at all). Therefore in the finite-size system with $R < \infty$ where possible infrared divergences are also well contained there are all reasons to assume that the space of states $\mathcal{H}_{PCFT}$ of the perturbed theory (3.4) coincides with the space of states (2.1) of the original CFT

$$\mathcal{H}_{PCFT} \simeq \mathcal{H}_{CFT} \ . \tag{3.17}$$

Of course the highest weight states $|\,\Delta(p)\rangle$ (3.13) in $\mathcal{H}_{CFT}$ are no longer stationary states of the perturbed Hamiltonian (3.10); finding the eigenstates and eigenvalues of $\mathbb{H}$ in (3.17) is exactly the problem we want to address here.

In Sect.2 we have defined the operator valued matrices $\mathbf{L}_j(\lambda)$ for the "left" chiral sector of the CFT; these matrices satisfy the Yang-Baxter relation (2.17). Similar way one can define the "right" chiral counterparts of these matrices,

$$\bar{\mathbf{L}}_j(\lambda) = \Big[\mathcal{P}\exp\Big(\lambda\int_0^R \big(:e^{-2i\bar{\phi}(\bar{z})}:\, q^{\frac{H_j}{2}}E_j + :e^{2i\bar{\phi}(\bar{z})}:\, q^{-\frac{H_j}{2}}F_j\big)\,d\bar{z}\Big)\Big]\, e^{-i\pi\bar{P}\,H_j}\ , \tag{3.18}$$

where $\bar{\phi}(\bar{z})$ is the right chiral Bose field (3.12), $E_j,\ F_j$ and H_j denote the same $(2j+1)\times(2j+1)$ matrices as in (2.13), and like in (2.13) $\mathcal{P}$ indicates the path ordering of both operators and matrices along the integration path. The elements of the matrices (3.18) act in the space $\hat{\bar{\mathcal{F}}}_p$ defined similarly to (2.15) in terms of the "right" Fock spaces $\bar{\mathcal{F}}_p$. It is possible to check that the operator matrices (3.18) also satisfy the Yang-Baxter relation in the form

$$\begin{gathered}\sum_{C=-j}^{j}\ \sum_{C'=-j'}^{j'}\ \big[\bar{\mathbf{L}}_j(\lambda)\big]_A^{\ C}\ \big[\bar{\mathbf{L}}_{j'}(\lambda')\big]_{A'}^{\ C'}\ \big[R_{j\,j'}(\lambda/\lambda')\big]_{C\,C'}^{B\,B'} = \\ \sum_{C=-j}^{j}\ \sum_{C'=-j'}^{j'}\ \big[R_{j\,j'}(\lambda/\lambda')\big]_{A\,A'}^{C\,C'}\ \big[\bar{\mathbf{L}}_{j'}(\lambda')\big]_{C'}^{\ B'}\ \big[\bar{\mathbf{L}}_j(\lambda)\big]_C^{\ B}\ ,\end{gathered} \tag{3.19}$$

where $R_{j\,j'}(\lambda)$ are exactly the same R-matrices as in (2.17). Note that the R-matrices enter (3.19) different way as compared to (2.17), the difference being traced down to the difference in the commutation relations

$$[\phi(x),\phi(x')] = -[\bar{\phi}(x),\bar{\phi}(x')] = \frac{\pi\beta^2}{2\,i}\ \varepsilon\Big(\frac{x-x'}{R}\Big)\ , \tag{3.20}$$

where $\varepsilon(x)$ is the quasi-periodic continuation of the sign function,

$$\epsilon(x) = 2n+1,\quad n<x<n+1\ ;\ n\in\mathbb{Z}\ . \tag{3.21}$$

Now we can introduce the following "full transfer-matrices"

$$\mathbb{T}_j(\mu|\lambda) = tr_{\pi_j}\Big[\,\mathbf{L}_j(\lambda)\,\bar{\mathbf{L}}_j(\mu/\lambda)\,\Big] = \sum_{A,B=-j}^{j}\ \big[\mathbf{L}_j(\lambda)\big]_A^{\ B}\ \big[\bar{\mathbf{L}}_j(\mu/\lambda)\big]_B^{\ A}\ , \tag{3.22}$$

where μ is a parameter. It is easy to check that (3.22) defines the operator which acts in the space (3.15), i.e.

$$\mathbb{T}_j(\mu|\lambda)\ :\qquad \hat{\mathcal{H}}_p\rightarrow\hat{\mathcal{H}}_p\ . \tag{3.23}$$

It follows from (2.17), (3.19) and the fact that the R-matrices satisfy the "unitarity conditions"

$$\sum_{C=-j}^{j}\sum_{C'=-j'}^{j'}\left[R_{j\,j'}(\lambda)\right]_{A\,A'}^{C\,C'}\,\left[R_{j\,j'}(\lambda^{-1})\right]_{C\,C'}^{B\,B'}=\delta_A^B\,\delta_{A'}^{B'}\ , \tag{3.24}$$

that these operators form the commuting family, i.e.

$$[\mathbb{T}_j(\mu|\lambda),\mathbb{T}_{j'}(\mu|\lambda')]=0 \tag{3.25}$$

for all values of j, j'. The following simple properties of the operators (3.22) are immediately established

1. For $\mu=0$ the operator $\mathbb{T}_j$ reduces to the operator $\mathbf{T}_j$ defined in Sect.2,

$$\mathbb{T}_j(0|\lambda)=\mathbf{T}_j(\lambda)\otimes\bar{\mathbf{I}}\ , \tag{3.26}$$

i.e. it acts as $\mathbf{T}_j(\lambda)$ in the "left" spaces $\mathcal{V}_p$ and as the identity operator $\bar{\mathbf{I}}$ in the "right" spaces $\bar{\mathcal{V}}_p$ in (3.11)[4]. Similarly,

$$\mathbb{T}_j(\mu|\mu/\lambda)\big|_{\mu=0}=\mathbf{I}\otimes\bar{\mathbf{T}}_j(\lambda)\ , \tag{3.27}$$

where $\mathbf{I}$ is the identity in the "left" spaces $\mathcal{V}_p$ and $\bar{\mathbf{T}}_j(\lambda)=tr_{\pi_j}\left[\,\bar{\mathbf{L}}_j(\lambda)\,e^{-i\pi\bar{P}H_j}\,\right]$ act in the spaces $\bar{\mathcal{V}}_p$ as the "right" counterparts of the operators (2.18).

2. Like $\mathbf{T}_j(\lambda)$ the operators $\mathbb{T}_j(\mu|\lambda)$ are the single-valued functions of λ^2.

3. As follows directly from the definition (3.22) the operators $\mathbb{T}_j(\mu|\lambda)$ inherit their large λ asymptotics from those of the operators $\mathbf{T}_j(\lambda)$,

$$\log\mathbb{T}_j(\mu|\lambda)\sim\kappa_j\,R/(2\pi)\ \lambda^{1+\xi},\qquad \lambda\to\infty\ , \tag{3.28}$$

where

$$\kappa_j=\frac{\sin(\pi j\xi)}{\sin(\pi\xi/2)}\ \kappa \tag{3.29}$$

and κ is given by (2.23). Similarly, the asymptotics of these operators at $\lambda\to 0$ are controlled by the second factor in (3.22) and so

$$\log\mathbb{T}_j(\mu|\lambda)\sim\kappa_j\,R/(2\pi)\ (\mu/\lambda)^{1+\xi},\qquad \lambda\to 0\ . \tag{3.30}$$

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[4] Obviously, the zero mode operator $\bar{P}$, when acting on the space (3.11), can be identified with $-P$.

In particular, it follows from (3.30) that $\mathbb{T}_j(\mu|\lambda)$ has an essential singularity at $\lambda^2 = 0$.

It is clear that for generic value of the parameter μ the operators $\mathbb{T}_j(\mu|\lambda)$ do not commute with the CFT local IM (2.4). Instead, it is natural to expect that the operators $\mathbb{T}_j(\mu|\lambda)$ play the same role in perturbed theory (3.4) as $\mathbf{T}_j(\lambda)$ did in the CFT, namely, the asymptotic expansions of $\mathbb{T}_j(\mu|\lambda)$ near the essential singularities at $\lambda^2 = 0$ and $\lambda^2 = \infty$ generate the local IM (3.8) of the perturbed theory (3.4). Note that the spaces (3.15) of the perturbed theory admit the action of the perturbed local IM (3.8). The parameter μ in (3.22) is expected to be related to the coupling parameter $\hat{\mu}$ in (3.4).

Concerning the operators $\mathbb{T}_j$ we can formulate now our basic

Conjecture 2

a) The operators $\mathbb{T}_j(\mu|\lambda)$ are single-valued functions of λ^2, regular everywhere in the complex λ^2 plane except at $\lambda^2 = 0$ and $\lambda^2 = \infty$ where they have essential singularities.

b) The asymptotic expansions of the operators $\mathbb{T}_j(\mu|\lambda)$ near $\lambda^2 = \infty$ and $\lambda^2 = 0$ are controlled by the perturbed local IM (3.8); in particular, for $\mathbb{T}(\mu|\lambda) \equiv \mathbb{T}_{1/2}(\mu|\lambda)$ one has

$$\begin{aligned}
\log \mathbb{T}(\mu|\lambda) &\simeq \kappa\, R/(2\pi)\ \lambda^{1+\xi} - \sum_{n=1}^{\infty} C_n\ \lambda^{(1-2n)(1+\xi)}\ \mathbb{I}_{2n-1}\ , \\
\log \mathbb{T}(\mu|\lambda) &\simeq \kappa\, R/(2\pi)\ (\mu/\lambda)^{1+\xi} - \sum_{n=1}^{\infty} C_n\ \big(\mu/\lambda\big)^{(1-2n)(1+\xi)}\ \bar{\mathbb{I}}_{2n-1}\ ,
\end{aligned} \tag{3.31}$$

where κ and C_n are exactly the same constants as in (2.23) and the parameter μ is related to $\hat{\mu}$ in (3.4) as

$$\hat{\mu}^2 = \frac{\Gamma^4\big(1-\beta^2\big)}{\pi\,(1-2\beta^2)(3\beta^2-1)}\ \left(\frac{\Gamma\big(3\beta^2\big)\,\Gamma\big(\beta^2\big)}{\Gamma\big(1-3\beta^2\big)\,\Gamma\big(1-\beta^2\big)}\right)^{\frac{1}{2}}\ \mu^2. \tag{3.32}$$

In particular, the "transfer-matrices" $\mathbb{T}_j$ commute with all local IM (3.8)

$$[\mathbb{T}_j(\mu|\lambda), \mathbb{I}_{2n-1}] = [\mathbb{T}_j(\mu|\lambda), \bar{\mathbb{I}}_{2n-1}] = 0\ . \tag{3.33}$$

The relation (3.32) appears very natural in view of nearly obvious relation of the operators $\mathbb{T}(\mu|\lambda)$ to the sine-Gordon model (which we nonetheless discuss briefly in Sect.6 below); we have arrived at (3.32) by combining (2.23), (3.28)-(3.30) with known relations (6.3), (6.4) and (6.6).

It is possible to show that the operators (3.22) satisfy exactly the same "fusion relations" as the operators $\mathbf{T}_j$, namely

$$\mathbb{T}_j(\mu|q^{\frac{1}{2}}\lambda)\mathbb{T}_j(\mu|q^{-\frac{1}{2}}\lambda) = \mathbb{I} + \mathbb{T}_{j+1/2}(\mu|\lambda)\mathbb{T}_{j-1/2}(\mu|\lambda) \ . \tag{3.34}$$

Also, if q is a root of unity the fusion relations (3.34) truncate exactly the same way as it happens in the unperturbed theory [1] (see Sect.4 below). This leads to closed functional equations for the eigenvalues of the operators $\mathbb{T}_j(\mu|\lambda)$ in the space $\mathcal{H}_{PCFT}$. Solving these equations one can, in principle, determine those eigenvalues and whence find the eigenvalues of all the local IM (3.8) in the perturbed theory (3.4), including the eigenvalues of the Hamiltonian (3.10). In Sect.5 we will show how this approach works in the SLYM (1.3).

4. Fussion Truncation and Klümper-Pearce Equations in CFT

The properties of the operators $\mathbf{T}_j$ described Sect.2. hold for arbitrary positive β^2. As was discussed in [1] considerable simplifications occur if β^2 is a rational number (in this case q defined in (2.14) is a root of unity and associated value of c corresponds to a "minimal CFT" [10]). In these cases the "fusion relation" (1.2) truncates and reduces to a functional relation involving finitely many operators $\mathbf{T}_j$. This "truncated" fusion relation leads then to the closed system of functional equations for the eigenvalues $T_j(\lambda)$ of these operators.

Let q be a complex number of the form

$$q^N = \pm 1 \,, \qquad \text{and} \qquad q^n \neq \pm 1 \;\; \text{for any integer } 0 < n < N \ , \tag{4.1}$$

where $N \geq 2$ some integer. For such values of q the $N+1$ dimensional representation of $U_q\big(sl(2)\big)$ with the spin $j = N/2$ becomes reducible, while all the finite dimensional representation with $j < N/2$ remain irreducible. More precisely, this $N+1$ dimensional representation reduces to a direct sum of the $N-1$ dimensional representation with $j = N/2 - 1$ and two one-dimensional representations [5]. These representations bring separate contributions to the trace (2.18) and therefore

$$\mathbf{T}_{N/2}(\lambda) = 2\cos(2\pi N\mathbf{P}) + \mathbf{T}_{N/2-1}(\lambda) \ , \tag{4.2}$$

[5] The latter are special one-dimensional representations $\rho_\pm$ of $U_q\big(sl(2)\big)$ with $q^{2N} = 1$ for which $\rho_\pm(E) = \rho_\pm(F) = 0$ and $\rho_\pm(H) = \pm N$.

where we used the notation $\mathbf{P}$ for P to emphasis its operator nature. Due to (4.2) the fusion relations (1.2) close within N operators $\mathbf{T}_j(\lambda)$ with $j = 0, 1/2, 1, ..., N/2 - 1/2$.

This truncated fusion relation leads to closed functional equations for the eigenvalues of the operators $\mathbf{T}_j(\lambda)$; $j = 0, 1/2, 1, ..., N/2 - 1/2$. Let $T_j(\lambda)$ be the eigenvalues of these operators associated with some common eigenstate in $\mathcal{V}_p$. The functional equations take most convenient form if one makes use of the rapidity variable

$$\theta = \log\big(\,\lambda^{1+\xi} R/(2\pi)\,\big)\ , \tag{4.3}$$

and introduces the functions [9]

$$\begin{gathered} Y_j(\theta) = T_{j-\frac{1}{2}}(\lambda)\, T_{j+\frac{1}{2}}(\lambda), \quad j = \tfrac{1}{2}, 1, \tfrac{3}{2}, ..., \tfrac{N}{2} - 1\,; \\ Y_0(\theta) \equiv 0, \qquad \overline{Y}(\theta) = T_{\frac{N}{2}-1}(\lambda)\ . \end{gathered} \tag{4.4}$$

Then the functional equations can be written as

$$\begin{aligned} & Y_j\Big(\theta + \frac{i\pi\xi}{2}\Big)\, Y_j\Big(\theta - \frac{i\pi\xi}{2}\Big) = \big(1 + Y_{j-\frac{1}{2}}(\theta)\big)\big(1 + Y_{j+\frac{1}{2}}(\theta)\big),\ j = \tfrac{1}{2}, 1, \ldots, \tfrac{N}{2} - \tfrac{3}{2}, \\ & Y_{\frac{N}{2}-1}\Big(\theta + \frac{i\pi\xi}{2}\Big)\, Y_{\frac{N}{2}-1}\Big(\theta - \frac{i\pi\xi}{2}\Big) = \\ & \qquad\qquad = \big(1 + Y_{\frac{N}{2}-\frac{3}{2}}(\theta)\big)\,\big(1 + e^{2\pi i p N}\, \overline{Y}(\theta)\big)\,\big(1 + e^{-2\pi i p N}\, \overline{Y}(\theta)\big)\,, \\ & \overline{Y}\Big(\theta + \frac{i\pi\xi}{2}\Big)\, \overline{Y}\Big(\theta - \frac{i\pi\xi}{2}\Big) = \big(1 + Y_{\frac{N}{2}-1}(\theta)\big)\ , \end{aligned} \tag{4.5}$$

where as before p denotes the eigenvalue of the operator $\mathbf{P}$. Note that the system (4.5) coincides with the functional form of the TBA equations of D_N type [22], [23].

Further simplifications occur when p takes special values. Suppose that the operators $\mathbf{T}_j$ act on the space $\mathcal{V}_p$ with

$$p = \frac{\ell + 1}{2\,N}\ , \tag{4.6}$$

where $\ell \geq 0$ is an integer such that $2p \neq n\beta^2 + m$ for any integers n and m. Then one can show [21] that in addition to (4.2) the operators $\mathbf{T}_j$ satisfy

$$\begin{gathered} \mathbf{T}_{\frac{N}{2}-j-1}(\lambda) = (-1)^{\ell}\ \mathbf{T}_j(q^{\frac{N}{2}}\lambda), \quad \text{for} \quad j = 0, \tfrac{1}{2}, 1, ..., \tfrac{N}{2} - 1; \\ \mathbf{T}_{\frac{N}{2}-\frac{1}{2}}(\lambda) = 0\ . \end{gathered} \tag{4.7}$$

This is exactly the case for the "minimal CFT" $\mathcal{M}_{2/2n+3}$ with $n = 1, 2, ...$, where

$$q = e^{\frac{2\pi i}{2n+3}}$$

and the space of states has the form

$$\mathcal{H}_{\mathcal{M}_{2/2n+3}} = \oplus_{k=0}^{n} \, \mathcal{H}_{p_k} \ , \tag{4.8}$$

with

$$p_k = \frac{2k+1}{2(2n+3)}, \qquad \Delta(p_k) = -\frac{(n+k+1)(n-k)}{2(2n+3)}; \quad k = 0, ..., n \ . \tag{4.9}$$

In this case the extra relations (4.7) allow one to bring (4.5) to yet simpler form

$$\begin{aligned} &Y_j(\theta + i\pi/(2n+1))\, Y_j(\theta - i\pi/(2n+1)) = \big(1 + Y_{j-\frac{1}{2}}(\theta)\big)\, \big(1 + Y_{j+\frac{1}{2}}(\theta)\big), \\ &Y_0(\theta) = Y_{n+\frac{1}{2}}(\theta) \equiv 0; \qquad Y_{n+\frac{1}{2}-j}(\theta) = Y_j(\theta), \ j = \frac{1}{2}, ..., n \ . \end{aligned} \tag{4.10}$$

Again, (4.10) coincides with the functional form of the TBA equations for the perturbed "minimal CFT" $\mathcal{M}_{2/2n+3}$ [24]. The above truncation was discussed in [1].

In this paper we concentrate attention on the simplest of the above "minimal CFT", the model $\mathcal{M}_{2/5}$. This CFT describes the criticality associated with the Lee-Yang edge singularity [25] and therefore it is often referred to as the Lee-Yang CFT. The central charge is

$$c(\mathcal{M}_{2/5}) = -\frac{22}{5} \tag{4.11}$$

and the space (4.8) contains only two components

$$\mathcal{H}_{\mathcal{M}_{2/5}} = \mathcal{H}_{p_0} \oplus \mathcal{H}_{p_1} \ , \tag{4.12}$$

where $p_0 = 1/10$ and $p_1 = 3/10$, the associated conformal dimensions are

$$\Delta(p_0) = -1/5, \qquad \Delta(p_1) = 0 \ . \tag{4.13}$$

For this model the functional equation (4.10) takes the form

$$Y(\theta + i\pi/3)\, Y(\theta - i\pi/3) = 1 + Y(\theta) \ , \tag{4.14}$$

where according to (4.4) the function $Y(\theta) \equiv Y_{\frac{1}{2}}(\theta) = Y_1(\theta)$ simply coincides with corresponding eigenvalue $T_{1/2}(\lambda)$ of the operator $\mathbf{T}(\lambda) = \mathbf{T}_{1/2}(\lambda)$ in the space (4.12),

$$Y(\theta) = T_{1/2}(\lambda) \ , \quad e^{\theta} = \lambda^{\frac{5}{3}} \ R/(2\pi) \ . \tag{4.15}$$

The analytic properties of the eigenvalues $T_{1/2}(\lambda)$ was discussed in Sect.2. They are entire functions of the variable

$$t \equiv \lambda^2 \left(\frac{R}{2\pi}\right)^{\frac{6}{5}}$$

with the asymptotic behavior

$$\log T_{1/2}(\lambda) \sim \kappa\, t^{\frac{5}{6}}, \quad \text{as} \quad t \to \infty\ , \tag{4.16}$$

where

$$\kappa = \frac{2\sqrt{\pi}\,\Gamma(1/6)}{\Gamma(2/3)}\ \Big(\Gamma(3/5)\Big)^{\frac{5}{3}} = 28.29877111...\ ,$$

as it follows from (2.23) with $\beta^2 = 2/5$. Correspondingly, we are interested here in the solutions $Y(\theta)$ of (4.14) which are regular everywhere in the complex θ-plane, satisfy the periodicity condition [6]

$$Y\big(\theta + \frac{5}{3}\, i\pi\big) = Y(\theta)\ , \tag{4.17}$$

and have the asymptotic

$$Y(\theta) \sim \exp(\kappa\, e^{\theta}), \quad \text{as} \quad \theta \to +\infty\ . \tag{4.18}$$

Note that since the eigenvalues $T_{1/2}(\lambda)$ are regular at $t = 0$ the associated functions $Y(\theta)$ are bounded at $\theta \to -\infty$; in fact in this limit they approach the constant values

$$Y(\theta) \to \begin{cases} 2\cos(2\pi p_0) = (1+\sqrt{5})/2 & \text{for the eigenvalues in} \quad \mathcal{H}_{p_0}\ ; \\ 2\cos(2\pi p_1) = (1-\sqrt{5})/2 & \text{for the eigenvalues in} \quad \mathcal{H}_{p_1}\ . \end{cases} \tag{4.19}$$

as it follows from (2.26a).

The functional equation identical to (4.14) arises in the solvable lattice hard hexagon model [26]. It was studied by Klumper and Pearce [9], who developed the method of transforming the functional equations to nonlinear integral equations, the knowledge about analytic properties of the solutions being key ingredient in this approach. Although in the field theory the analytic properties of the functions $Y(\theta)$ are somewhat different from those considered in [9], the method of Klümper and Pearce can be easily adopted to our case. The derivation in [9] is based on the following

Lemma. *If a function $f(\theta)$ is regular and bounded in the strip $\Im m\,\theta \in (-\pi/3, \pi/3)$ and satisfies the relation*

$$f(\theta + i\pi/3) + f(\theta - i\pi/3) - f(\theta) = g(\theta) \tag{4.20}$$

[6] In fact, this periodicity condition can be derived from (4.14), see [22].

with some $g(\theta)$ *then*

$$f(\theta) = \int_{-\infty}^{\infty} d\theta' \ \Phi(\theta - \theta')\, g(\theta') \ , \qquad (4.21)$$

where

$$\Phi(\theta) = \frac{\sqrt{3}}{\pi} \ \frac{\sinh(2\theta)}{\sinh(3\theta)} \ . \qquad (4.22)$$

Note that

$$\Phi(\theta) = \frac{i}{2\pi} \partial_\theta \log\, S(\theta) \ , \qquad (4.23)$$

where $S(\theta)$ is the S-matrix (1.4). If $Y(\theta)$ was free of zeroes in the strip $\Im m\, \theta \in (-\pi/3, \pi/3)$ then the function $f(\theta) = \log Y(\theta) - \kappa\, \exp(\theta)$ would satisfy the conditions of this lemma with $g(\theta) = \log\big(1 + Y^{-1}(\theta)\big)$, so the integral equation would follow immediately. However a typical eigenvalue $T(t,p)$ does have zeroes in the wedge $-2\pi/5 < \arg t < 2\pi/5$ (which corresponds to the above strip) and so we need to take care of them.

Let us assume that $Y(\theta)$ has $N + 2M$ zeroes in the strip $\Im m\, \theta \in (-\pi/3, \pi/3)$, N real zeroes α_a, $a = 1, 2, ..., N$ and M complex-conjugated pairs $(\beta_b + i\gamma_b, \beta_b - i\gamma_b)$, $b = 1, 2, ..., M$. We introduce the following functions

$$\begin{aligned} \sigma_0(\theta) &= \tanh\big(\frac{3}{4}\,\theta\big) \ , \\ \sigma_1(\theta, \eta) &= \frac{\cosh(\theta) - \cos(\eta)}{\cosh(\theta) + \cos(\eta)} \ \frac{\cosh(\theta) - \sin(\pi/6 - \eta)}{\cosh(\theta) + \sin(\pi/6 - \eta)} \ , \end{aligned} \qquad (4.24)$$

which satisfy the equations

$$\sigma_0(\theta + i\pi/3)\, \sigma_0(\theta - i\pi/3) = 1; \qquad \sigma_1(\theta + i\pi/3, \eta)\, \sigma_1(\theta - i\pi/3, \eta) = \sigma_1(\theta, \eta) \ , \qquad (4.25)$$

and write $Y(\theta)$ as

$$\begin{aligned} Y(\theta) &= \prod_{a=1}^{N} \sigma_0(\theta - \alpha_a) \ e^{\epsilon(\theta)} \\ &= \exp(\kappa\ e^{\theta}) \ \prod_{a=1}^{N} \sigma_0(\theta - \alpha_a) \prod_{b=1}^{M} \sigma_1(\theta - \beta_b, \gamma_b) \ X(\theta) \ . \end{aligned} \qquad (4.26)$$

In view of (4.25) the function $X(\theta)$ satisfies the equation

$$X(\theta + i\pi/3)\, X^{-1}(\theta)\, X(\theta - i\pi/3) = \prod_{a=1}^{N} \sigma_0(\theta - \alpha_a)\big(1 + Y^{-1}(\theta)\big) \ . \qquad (4.27)$$

Note that because the σ-factors in (4.26) absorb all zeroes of $Y(\theta)$ in the strip $\Im m\ \theta \in (-\pi/3, \pi/3)$ and the exponential prefactor saturates the asymptotic (4.18) the function $\log X(\theta)$ is regular and bounded in this strip. Applying the above lemma we obtain the integral equation

$$\epsilon(\theta) = \kappa\, e^{\theta} + \sum_{b=1}^{M} \log\big(\sigma_1(\theta - \beta_b, \gamma_b)\big) + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta - \theta')\ \log\Big(\prod_{a=1}^{N} \sigma_0(\theta' - \alpha_a) + e^{-\epsilon(\theta')}\Big)\ , \tag{4.28}$$

where $\epsilon(\theta)$ is defined in (4.26) [9].

Given a set of real parameters $\alpha_a, \beta_b, \gamma_b$ the equation (4.28) uniquely determines the function $\epsilon(\theta)$ and hence $Y(\theta)$. Of course these parameters can not be taken arbitrary. With generic values of these parameters the function $Y(\theta)$ defined through the solution to (4.28) will have the poles in the domain $\Im m\, \theta \in (\pi/3, 2\pi)$ mod $5\pi/3$. Indeed, if, for instance, $Y(\beta_b + i\gamma_b) = 0$ then according to (4.14) the function $Y(\theta)$ must exhibit a pole at $\theta = \beta_b + i\gamma_b - 2\pi i/3$, unless, of course, $Y(\beta_b + i\gamma_b - i\pi/3) = -1$. Therefore the condition that $Y(\theta)$ is entire function leads to the equations

$$\begin{aligned} &Y(\alpha_a + i\pi/3) = Y(\alpha_a - i\pi/3) = -1, \quad a = 1, ..., N\ ; \\ &Y(\beta_b + i\gamma_b - i\pi/3) = Y(\beta_b - i\gamma_b + i\pi/3) = -1, \quad b = 1, ..., M\ . \end{aligned} \tag{4.29}$$

which determine the parameters α_a, β_b and γ_b in (4.26). The integral equation (4.28) together with the transcendental equations (4.29) are QFT versions of the equations obtained in the lattice theory in [9] and therefore we will refer to them as Klümper-Pearce equations.

The Klümper-Pearce equations (4.28), (4.29) enables one to determine (in principle) any eigenvalue $Y(\theta)$ of the operator $\mathbf{T}(\lambda)$ in the space (4.12). In this paper we consider only two simplest eigenvalues - the vacuum eigenvalues $T(t, p_0)$ and $T(t, p_1)$ which correspond to the vacuum vectors $|p_0>$ and $|p_1>$ in the spaces $\mathcal{V}_{p_0}$ and $\mathcal{V}_{p_1}$ in (4.12), respectively. We will use the notations $Y_0(\theta) = T(t, p_0)$ and $Y_1(\theta) = T(t, p_1)$ (which are not to be confused with the Y_j used in (4.4) – (4.10)) with $t = \exp(6\,\theta/5)$.

I. $Y_0(\theta)$. This eigenvalue corresponds to the ground state in the space (4.12). As discussed in Sect.2 all zeroes of $T(t, 1/10)$ lie on the negative part of the real t-axis and so all zeroes of $Y_0(\theta)$ are located along the lines $\Im m\, \theta = 5\pi/6\,(\mathrm{mod}\ 5\pi/3)$. In particular, there are no zeroes in the strip $\Im m\, \theta \in (-\pi/3, \pi/3)$. Therefore in this case the equation (4.28) takes the form

$$\epsilon_0(\theta) = \kappa\ \exp(\theta) + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta - \theta')\ \log(1 + e^{-\epsilon_0(\theta')})\ , \tag{4.30}$$

where $\epsilon_0(\theta) = \log Y_0(\theta)$, and the equations (4.29) do not appear. The Eq.(4.30) is exactly the massless TBA equation associated with the Lee-Yang CFT [4]. The solution of this equation was referred to as the "kink solution" in [4] where it was studied numerically; the solution is plotted in Fig.1.

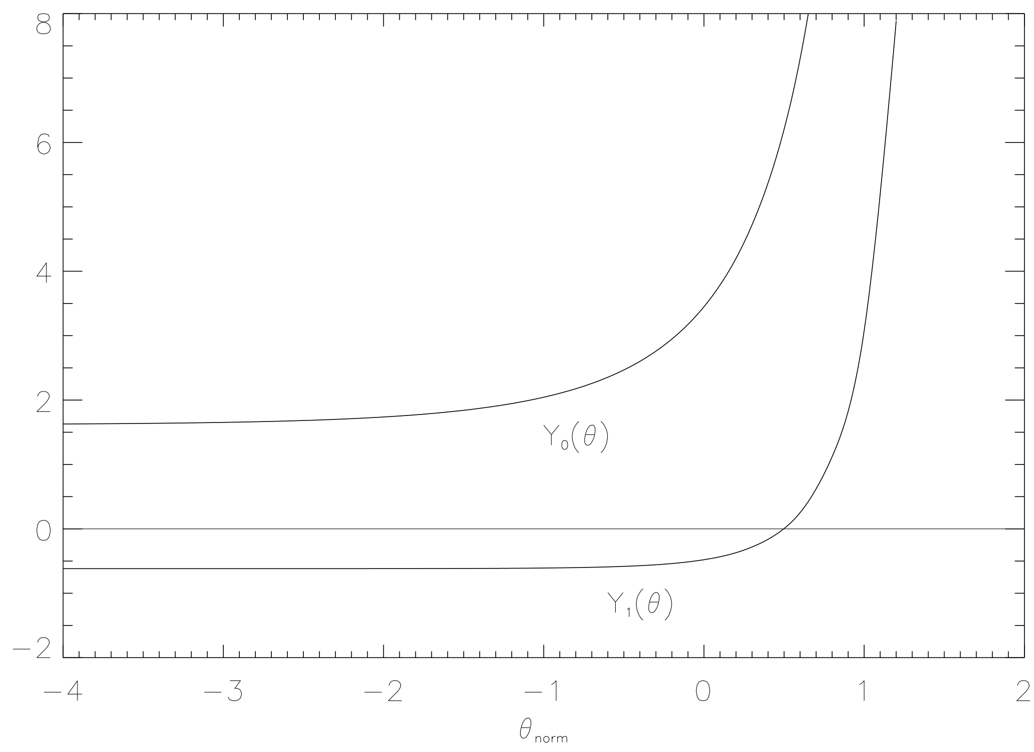

Fig.1. The functions $Y_0(\theta)$ and $Y_1(\theta)$ found from numerical solutions of the integral equations (4.30) and (4.37) plotted versus a "normalized" rapidity variable $\theta_{\text{norm}} = \theta + \log\kappa$.

Note that for $\theta \to -\infty$ the function $Y_0(\theta)$ approaches the value $2\cos(\pi p_0) = (\sqrt{5} + 1)/2 = 1.6180339\ldots$ in agreement with (4.19). We have used the numerical solution to the equation (4.30) to evaluate few first coefficients in the expansion (2.26a); the result is

Exact values	Numerical values
$G_0^{vac}(\frac{1}{10})= 1.6180339887\ldots$	$1.618033989\ldots$
$G_1^{vac}(\frac{1}{10}) = 7.01811993\ldots 10^1$	$7.0181198\ldots 10^1$
$G_2^{vac}(\frac{1}{10}) = 1.361347\ldots 10^3$	$1.36135\ldots 10^3$
$G_3^{vac}(\frac{1}{10}) = 1.6320\ldots 10^4$	$1.6317\ldots 10^4$
$G_4^{vac}(\frac{1}{10})$ =(unknown)	$1.41\ldots 10^5$

Table 1. The exact eigenvalues of a few first NIM for the $\Delta = -\frac{1}{5}$ vaccum state in the Lee-Yang CFT (as given by (A.2) and (A.9) with $a = 2/5$ and $p = 1/10$) and numerical values of the same eigenvalues obtained with a polynomial fit of $Y_0(\theta)$ determined from the numerical solution of the integral equation (4.30). The quantity $G_0^{vac}(p)$ denotes the constant term in the series expansion (2.26a)

given in Table 1 where the numerical results are also compared with the exact coefficients calculated as the integrals (2.27) (see Appendix A).

As is explained in Sect.2, the function $\epsilon_0(\theta)-\kappa\exp(\theta)$ can be expanded into asymptotic series in powers of $\exp(-\theta)$. The easiest way to obtain this expansion is to note that the kernel Φ in (4.30) expands for $\Re e\,\theta > 0$ as

$$\Phi(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\sin\frac{2\pi(n+1)}{3}\; e^{(1-2n)\theta}\;. \tag{4.31}$$

Substituting this into (4.30) one finds

$$\log Y_0(\theta) \simeq \kappa\, e^{\theta} - \frac{2}{\pi}\sum_{n=1}^{\infty}\sin\frac{2\pi(n+1)}{3}\;\chi_n\; e^{(1-2n)\theta}\;, \tag{4.32}$$

where the coefficients χ_n are calculated as

$$\chi_n = \int_{-\infty}^{\infty} d\theta\; e^{(2n-1)\theta}\;\log(1+e^{-\epsilon_0(\theta)})\;. \tag{4.33}$$

According to (2.26b) the coefficients in (4.32) are related to the vacuum eigenvalues of the local IM $\mathbf{I}_{2n-1}$,

$$I_{2n-1}^{vac}(-1/5) = \frac{2}{\pi C_n}\;\sin\frac{2\pi(n+1)}{3}\;\chi_n\;, \tag{4.34}$$

where

$$C_n = \frac{5}{3}\frac{\sqrt{\pi}}{n!}\,\Big(\frac{2}{5}\Big)^n\,\frac{\Gamma\big(\frac{10n-5}{6}\big)}{\Gamma\big(\frac{2n+2}{3}\big)}\,\Big(\Gamma(3/5)\Big)^{\frac{5-10n}{3}}\ . \tag{4.35}$$

The first of the integrals (4.33), can be evaluated exactly using the well known "dilogarithm trick" (see Appendix C)

$$\chi_1 = \frac{\pi^2}{15}\ \kappa^{-1}\ . \tag{4.36}$$

This agrees with the value $I_1^{vac}(-1/5) = -1/60$. We used the numerical solution to (4.30) to evaluate few further coefficients (4.33); these results are compared with the exact eigenvalues $I_{2n-1}^{vac}(-1/5)$ in Table 2.

Exact values	Numerical values
$I_1(-\frac{1}{5}) = -\frac{1}{60} = -1.6666666666666\ldots\ 10^{-2}$	$-1.6666666666665\ldots\ 10^{-2}$
$I_5(-\frac{1}{5}) = \frac{89}{756000} = 1.17724867724\ldots\ 10^{-4}$	$1.1772486773\ldots\ 10^{-4}$
$I_7(-\frac{1}{5}) = -\frac{211}{8100000} = -2.6049382716\ldots 10^{-5}$	$-2.604938279\ldots\ 10^{-5}$
$I_{11}(-\frac{1}{5}) = \frac{2160997}{464373000000} = 4.6535802\ldots\ 10^{-6}$	$4.653584\ldots\ 10^{-6}$
$I_{13}(-\frac{1}{5}) = -\frac{6283403}{1924560000000} = -3.264851\ldots\ 10^{-6}$	$-3.26488\ldots\ 10^{-6}$
$I_{17}(-\frac{1}{5}) =$ (unknown)	$3.5501\ldots\ 10^{-6}$

Table 2. The exact vacuum eigenvalues of a few first (non-vanishing) LIM given by (B.1)-(B.8) with $c = -22/5$ and $\Delta = -1/5$ and numerical values for the same LIM obtained from (4.34) with the numerical solution of the integral equation (4.30).

II. $Y_1(\theta)$. According to our discussion in Sect.2 all zeroes of the function $T(t, 3/10)$ are real, just one of them being positive. Correspondingly, the function $Y_1(\theta) = T(t, 3/10)$ has one real zero which we denote α and infinitely many zeroes located along the lines $\Im m\,\theta = 5\pi/6$ (mod $5\pi/3$). Therefore in this case the function $\epsilon_1(\theta) = \log Y_1(\theta) - \log\sigma_0(\theta-\alpha)$ solves the equation

$$\epsilon_1(\theta) = \kappa\ e^{\theta} + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta-\theta')\ \log\Big(\tanh\big(3(\theta'-\alpha)/4\big) + e^{-\epsilon_1(\theta')}\Big)\ , \tag{4.37}$$

and the parameter α satisfies

$$i\ e^{\epsilon_1(\alpha+i\pi/3)} = -i\ e^{\epsilon_1(\alpha-i\pi/3)} = -1\ . \tag{4.38}$$

The last relation can be written as

$$\sqrt{3}\,\kappa\; e^{\alpha}+\frac{3}{\pi}\,⨍_{-\infty}^{\infty}d\theta\;\frac{\cosh\big(2(\theta-\alpha)\big)}{\sinh(3\big(\theta-\alpha)\big)}\;\log\Big(\tanh\big(3(\theta-\alpha)/4\big)+e^{-\epsilon_1(\theta)}\Big)=\\ =\pi(1+4N)\ , \tag{4.39}$$

where N is some integer and $⨍_{-\infty}^{\infty}$ denotes the principal value of the singular integral. We have used (4.37) and explicit form of the kernel $\Phi(\theta)$ (4.22) to obtain (4.39). We studied the equations (4.37) and (4.39) numerically. Our analysis suggests that these equations admit real solution only if N in (4.39) is zero, in which case

$$\alpha+\log\kappa=0.49577315\ldots\ . \tag{4.40}$$

The corresponding function

$$Y_1(\theta)=\sigma_0(\theta-\alpha)\,e^{\epsilon_1(\theta)} \tag{4.41}$$

is plotted in Fig.1. In this case the asymptotic value of $Y_1(\theta)$ at $\theta\to-\infty$ is $2\cos(2\pi p_1)=(1-\sqrt{5})/2=-0.61803398\ldots$. Few first coefficients of the expansion (2.26a) calculated from this numerical solution together with their exact values (when available) are presented in Table 3.

Exact values		Numerical values
$G_0^{vac}(\frac{3}{10})=$	$-0.6180339887\ldots$	$-0.618033989\ldots$
$G_1^{vac}(\frac{3}{10})=$	0	$3.6\ldots\,10^{-5}$
$G_2^{vac}(\frac{3}{10})=$	$2.94659\ldots10^{2}$	$2.9467\ldots\,10^{2}$
$G_3^{vac}(\frac{3}{10})=$	$5.80714\ldots\,10^{3}$	$5.80705\ldots\,10^{3}$
$G_4^{vac}(\frac{3}{10})=$	$6.2826\ldots\,10^{4}$	$6.281\ldots\,10^{4}$
$G_5^{vac}(\frac{3}{10})=$	$4.729\ldots\,10^{5}$	$4.75\ldots\,10^{5}$
$G_6^{vac}(\frac{3}{10})=$	$2.72\ldots\,10^{6}$	$2.6\ldots\,10^{6}$
$G_7^{vac}(\frac{3}{10})=$	(unknown)	$1.7\ldots\,10^{7}$

Table 3. The exact eigenvalues of a few first NIM for the $\Delta=0$ vacuum state in the Lee-Yang CFT (as given by (A.2), (A.7), (A.8), (A.10) with $a=2/5$ and $p=3/10$) and numerical values of the same NIM obtained with a polynomial fit of $Y_1(\theta)$ determined from the numerical solution of the integral equation (4.37). The quantity $G_0^{vac}(p)$ denotes the constant term in power series expansion (2.26a)

To obtain the asymptotic expansion for $\log Y_1(\theta)$ similar to (4.32) one could try again to substitute the expansion (4.31) into (4.37). However in this case the log factor in the integrand in (4.37) does not decay sufficiently fast as $\theta' \to \infty$ and so the integrals appearing after this substitution typically diverge. To handle this problem it is convenient to rewrite the equation (4.37) as

$$\begin{aligned} \log |\, Y_1(\theta)\,| = \kappa\ e^{\theta} - \frac{1}{2}\ \log\big(S(\theta-\alpha+i\pi/3)\, S(\theta-\alpha+2i\pi/3)\big) + \\ \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta-\theta')\ \log\big(\,|\,1+Y_1^{-1}(\theta')\,|\,\big)\ . \end{aligned} \tag{4.42}$$

Now using (4.31) we obtain the asymptotic series expansion

$$\log Y_1(\theta) \simeq \kappa\ e^{\theta} - \frac{2}{\pi}\ \sum_{n=1}^{\infty} \sin\frac{2\pi(n+1)}{3}\ \tilde{\chi}_n\ e^{(1-2n)\theta}\ , \tag{4.43}$$

where

$$\tilde{\chi}_n = \frac{2\pi}{2n-1}\ \sin\frac{2\pi(n+1)}{3}\ e^{\alpha(2n-1)} + \int_{-\infty}^{\infty} d\theta\ e^{(2n-1)\theta} \log\big(\,|\,1+Y_1^{-1}(\theta)\,|\,\big)\ . \tag{4.44}$$

Exact values	Numerical values
$I_1(0) = \frac{11}{60} \quad = \quad 1.833333\ldots 10^{-1}$	$1.83332\ldots 10^{-1}$
$I_5(0) = \frac{341}{756000} \quad = \quad 4.51058\ldots 10^{-4}$	$4.5106\ldots 10^{-4}$
$I_7(0) = -\frac{451}{8100000} \quad = -5.567901\ldots 10^{-5}$	$-5.56789\ldots 10^{-5}$
$I_{11}(0) = \frac{3373117}{464373000000} \quad = \quad 7.26380\ldots 10^{-6}$	$7.2631\ldots 10^{-6}$
$I_{13}(0) = -\frac{825517}{174960000000} \quad = -4.7183\ldots 10^{-6}$	$-4.717\ldots 10^{-6}$
$I_{17}(0) = \text{(unknown)}$	$4.67\ldots 10^{-6}$

Table 4. The exact vacuum eigenvalues of a few first (non-vanishing) LIM given by (B.1)-(B.8) with $c = -22/5$ and $\Delta = 0$ and numerical values for the same LIM obtained from (4.46) with the numerical solution of the integral equation (4.37).

Like in the previous example, for $n = 1$ the integral in (4.44) can be calculated analytically using appropriate modification of the "dilogarithm trick" (see Appendix C),

$$\tilde{\chi}_1 = -\frac{11\,\pi^2}{15}\ \kappa^{-1}\ . \tag{4.45}$$

The vacuum eigenvalues $I_{2n-1}^{vac}(0)$ of the local IM are expressed through $\tilde{\chi}_n$ as

$$I_{2n-1}^{vac}(0) = \frac{2}{\pi C_n} \sin \frac{2\pi(n+1)}{3} \, \tilde{\chi}_n \ , \tag{4.46}$$

where C_n are given by (4.35), so that (4.45) agrees exactly with $I_1^{vac}(0) = 11/60$. The values of few further local IM obtained from the above numerical solution are compared with the exact values in Table 4.

5. Excited States in Scaling Lee-Yang Model

In this section we apply the approach outlined in Sect.4 to the finite size energy spectrum of the SLYM (1.3). The QFT (1.3) is particular case of (3.4) and so it possesses infinitely many local IM (3.8). In fact, due to the strong degeneracy of the representations entering (4.12) in this case some of these IM vanish. Only the IM $\mathbb{I}_{2n-1}$ with $2n-1 \neq 0 \,(\mathrm{mod}\, 3)$ are nonzero. As is mentioned in the Introduction the finite-size ground state energy $E_0(R)$ of this model was studied by TBA in [4]. In this section we show that the approach based on the operators (3.22) and the fusion relations (3.34) makes it possible to study the excited state energies in the finite size system as well.

As was explained above, the perturbed theory inherits the space of states from the CFT $\mathcal{M}_{2/5}$, that is it has the form

$$\mathcal{H}_{SLYM} = \mathcal{H}_{p_0} \oplus \mathcal{H}_{p_1} \ , \tag{5.1}$$

where $p_0 = 1/10$ and $p_1 = 3/10$. Consider the operator $\mathbb{T}(\mu|\lambda) \equiv \mathbb{T}_{1/2}(\mu|\lambda)$, as defined in (3.22). This operator invariantly act in the space (5.1) and when specialized to this space it satisfies the truncated fusion relation

$$\mathbb{T}(\mu|q^{\frac{1}{2}}\lambda)\, \mathbb{T}(\mu|q^{-\frac{1}{2}}\lambda) = \mathbb{I} + \mathbb{T}(\mu|\lambda) \ , \tag{5.2}$$

and therefore the eigenvalues $T(\mu|\lambda)$ of $\mathbb{T}(\mu|\lambda)$ in the space (5.1) satisfy exactly the same functional equation

$$T(\mu|q^{\frac{1}{2}}\lambda)\, T(\mu|q^{-\frac{1}{2}}\lambda) = 1 + T(\mu|\lambda) \tag{5.3}$$

as the eigenvalues of the operator $\mathbf{T}$ in the space (4.12). However, in the perturbed case ($\mu \neq 0$) we are interested in the solutions to (5.3) which have significantly different analytic properties as the functions of λ^2. As was explained in Sect.3, the eigenvalues $T(\mu|\lambda)$ have essential singularities both at $\lambda^2 = \infty$ and $\lambda^2 = 0$, and they exhibit the asymptotic behavior dictated by (3.28) and (3.30) near these singularities.

Like in the Sect.4 it is convenient here to introduce the rapidity

$$\theta = \frac{5}{6} \log\left(\lambda^2/\mu\right) . \tag{5.4}$$

Note that this definition differs from (4.15) by a constant shift $\log(\mu^{\frac{5}{6}} R/2\pi)$. Also, we will use the dimensionless variable

$$r = m\, R \equiv \left(\kappa\, \mu^{\frac{5}{6}}/\pi\right) R \tag{5.5}$$

instead of R. Here m is the mass related to the coupling parameter $\hat{\mu}$ (1.3) as in (1.5). The eigenvalues $T(\mu|\lambda)$ can be considered as the functions of θ and r. Like in (4.15) we denote

$$Y(r|\theta) = T(\mu|\lambda)\,; \qquad r = \kappa\, \mu^{\frac{5}{6}}\, R/\pi\,, \quad e^{\theta} = \lambda^{\frac{5}{3}}\, \mu^{-\frac{5}{6}} . \tag{5.6}$$

In this notations the functional equation (5.3) takes the form (4.14), i.e.

$$Y(r|\theta + i\pi/3)\, Y(r|\theta - i\pi/3) = 1 + Y(r|\theta) . \tag{5.7}$$

We are interested now in the solutions which are analytic everywhere in the complex θ-plane, satisfy the periodicity condition (4.17) and have the asymptotic

$$\log Y(r|\theta) \sim r\ e^{|\theta|}/2\ , \quad \theta \to \pm\infty . \tag{5.8}$$

Like in the conformal case discussed in Sect.3 the solution to (5.7) with the asymptotic conditions (5.8) is completely determined by the pattern of zeroes of $Y(r|\theta)$ in the strip $-\pi/3 < \Im m\, \theta < \pi/3$. The same transformations as in Sect.3 lead to the integral equation

$$\begin{aligned}\epsilon(r|\theta) =& r\cosh\theta + \sum_{b=1}^{M} \log\sigma_1\big(\theta - \beta_b, \gamma_b\big) + \\ & \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta - \theta') \log\ \Big(\prod_{a=1}^{N} \sigma_0(\theta' - \alpha_a) + e^{-\epsilon(r|\theta')}\Big)\ ,\end{aligned} \tag{5.9}$$

where

$$\epsilon(r|\theta) = \log Y(r|\theta) - \sum_{a=1}^{n} \log\sigma_0(\theta - \alpha_a) \tag{5.10}$$

and we have assumed that there are $N+2M$ zeroes in the physical strip, N real zeroes $\alpha_a(r)$ and M complex-conjugated pairs $\beta_b(r) \pm i\,\gamma_b(r)$; here we have shown explicitly that the positions of these zeroes depend on the parameter r [7]. And again, like in Sect.4 these positions are determined by the equations

$$Y(r|\alpha_a + i\pi/3) = Y(r|\alpha_a - i\pi/3) = -1, \quad a = 1, ..., N\ ; \tag{5.11}$$

$$Y(r|\beta_b + i\gamma_b - i\pi/3) = Y(r|\beta_b - i\gamma_b + i\pi/3) = -1, \quad b = 1, ..., M\ . \tag{5.12}$$

Before studying particular examples of the eigenvalues $Y(r|\theta)$ let us show how the mass spectrum and factorizable S-matrix of this QFT can be recovered from these equations in the limit $r \to \infty$. Simple estimate shows that for $r >> 1$ the first term in the r.h.s. of Eq.(5.9) dominates everywhere in the strip $\Im m\, \theta \in (-\pi/3+\varepsilon\,,\, \pi/3-\varepsilon)$ except in the small disks of the size ε around the zeroes α_a and $\beta_b \pm i\,\gamma_b$, where ε is some number exponentially small in r. In this domain the first term in the r.h.s of (5.7) can be neglected and hence the function $Y(r|\theta)$ can be written as

$$\begin{aligned} &Y(r|\theta) \simeq \prod_a S(\theta - \beta_a - i\pi/2)\exp(r\cosh\theta)\ , \\ &\qquad r >> 1; \qquad |\theta - \beta_a \pm i\pi/6| > \varepsilon; \qquad \Im m\, \theta \in (-\pi/3+\varepsilon, \pi/3-\varepsilon)\ , \end{aligned} \tag{5.13}$$

where $S(\theta)$ is the S-matrix defined by (1.4) and β_a are some real numbers. In writing (5.13) we have taken into account the asymptotic conditions (5.8) and the fact that $Y(r|\theta)$ must be real at real θ [8]. The form (5.13) shows in particular that for r sufficiently large there are no real zeroes α_a in (5.9) while the complex zeroes have the form

$$\beta_b(r) \pm i\left(\pi/6 - \nu_b(r)\right)\ , \tag{5.14}$$

where $\beta_a(r)$ are real and $\nu_a(r)$ are exponentially small in r. The equation (5.12) then takes the form

$$\exp(i\,r\,\sinh\beta_b)\ \prod_{b\neq a} S\big(\beta_b - \beta_a\big) = 1, \quad b = 1, 2, ..., M\ , \tag{5.15}$$

---

[7] We will see later that when r change some complex-conjugate zeroes can collide and then become a pair of the real zeroes, or vice versa, so that in different domains of r the numbers N and M may vary, while the total number of zeroes, of course, remains the same.

[8] The factors of the form $S(\theta + i\pi/6)\, S(\theta + 5i\pi/6)$ also satisfy (5.7) with the first term in the r.h.s. in this equation omitted, and they agree with the real analyticity condition. More detailed analysis shows however that these factors would be incompatible with (5.11), (5.12).

where we have taken into account that $S(0) = -1$. This equations shows that for $r >> 1$ the eigenstate associated with the eigenvalue (5.13) is interpreted as M-particle state, the parameters $\beta_b = \beta_b(r)$ being the rapidities of the particles, so that the equations (5.15) have the meaning of the Bethe Ansatz equations for M scalar particles with the S-matrix (1.4) [4] . We should stress here that in the field theory the Bethe Ansatz equations of the form (5.15) make sense only in the limit $r \to \infty$. For finite r these equations acquire corrections due to the vacuum polarization.

Here we consider in details only two simplest eigenvalues of the operator $\mathbb{T}$ of the Lee-Yang model.

I. The simplest eigenvalue is of course the ground state eigenvalue, i.e. the eigenvalue associated with the lowest eigenstate of the Hamiltonian (3.10) in the space (5.1). We denote here this state as $\mid \Psi_0\rangle$ and use the notation $Y_0(r|\theta)$ for the corresponding eigenvalue of $\mathbb{T}$. For the ground state eigenvalue $Y_0(r|\theta)$ there are no zeroes in the strip $\Im m\ \theta \in (-\pi/3, \pi/3)$ at all and the equation (5.9) takes the form

$$\epsilon_0(r|\theta) = r\cosh\theta + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta-\theta')\ \log(1+e^{-\epsilon_0(r|\theta')})\ , \tag{5.16}$$

where $\epsilon_0(r|\theta) = \log Y_0(r|\theta)$. This is exactly the TBA equation for the Lee-Yang model obtained in [4]; here we have arrived at (5.16) by entirely different route, completely bypassing any reference to the on-shell solution of the model. The equation (5.16) was studied in very details in [4] and we can add very little to that analysis. Let us just briefly summarize the results of [4]. For large r the first term in the r.h.s. of (5.16) dominates and so $\epsilon_0(r|\theta) \sim r\cosh\theta;\quad r >> 1$. In the opposite limit $r \to 0$ the function $Y_0(r|\theta)$ becomes nearly constant equal to $\frac{\sqrt{5}+1}{2}$ in the wide range of θ, $\log r < \theta < -\log r$, while for $\theta \gtrsim -\log r$ this function coincides with the "kink solution" $Y_0\big(\theta+\log\big(\frac{r}{2\kappa}\big)\big)$ considered in Sect.4; similarly, for $\theta \lesssim \log r,\ \ Y_0(r|\theta) \to Y_0\big(-\theta-\log\big(\frac{r}{2\kappa}\big)\big)$. In view of (3.26) and (3.27) this is just the manifestation of the fact that for $r \to 0$ the ground state $\mid \Psi_0\rangle$ approaches the conformal vacuum state $\mid \Delta(p_0)\rangle$ (3.13) in (5.1). According to (3.31) the ground state eigenvalues of the local IM (3.8)

$$\begin{aligned} \mathbb{I}_{2n-1} \mid \Psi_0\rangle &= I^{(0)}_{2n-1}(R)\ \mid \Psi_0\rangle \\ \bar{\mathbb{I}}_{2n-1} \mid \Psi_0\rangle &= \bar{I}^{(0)}_{2n-1}(R)\ \mid \Psi_0\rangle \end{aligned} \tag{5.17}$$

can be obtained from $\epsilon_0(r|\theta)$ as the coefficients of the asymptotic expansion

$$\begin{aligned}\epsilon_0(r|\theta) &\simeq r\, e^{\theta}/2 - \sum_{n=1}^{\infty} C_n\ \mu^{5(1-2n)/6}\ e^{(1-2n)\theta}\ I_{2n-1}^{(0)}(R)\,, \quad \theta\to+\infty\ ,\\ \epsilon_0(r|\theta) &\simeq r\, e^{-\theta}/2 - \sum_{n=1}^{\infty} C_n\ \mu^{5(1-2n)/6}\ e^{(2n-1)\theta}\ \bar{I}_{2n-1}^{(0)}(R)\,, \quad \theta\to-\infty\ .\end{aligned} \tag{5.18}$$

It follows from (5.16) that

$$\begin{aligned}I_1^{(0)}(R) &= -\frac{\sqrt{3}}{24}\, m^2 R - \frac{m}{4\pi}\int_{-\infty}^{\infty} d\theta\ e^{\theta}\log\big(1+e^{-\epsilon_0(r|\theta)}\big)\ ,\\ I_{2n-1}^{(0)}(R) &= \frac{2}{\pi C_n}\ \Big(\frac{\pi m}{\kappa}\Big)^{2n-1}\ \sin\frac{2\pi(n+1)}{3}\ \int_{-\infty}^{\infty} d\theta\ e^{(2n-1)\theta}\ \log(1+e^{-\epsilon_0(r|\theta)})\ ,\end{aligned} \tag{5.19}$$

where $n=2,3,\ldots$. The similar relations hold for the "right" local IM $\bar{I}_{2n-1}^{(0)}(R)$. Then, for the ground-state energy

$$\mathbb{H}\mid\Psi_0\rangle = E_0(R)\mid\Psi_0\rangle\ ,$$

we have

$$E_0(R) \equiv I_1^{(0)}(R) + \bar{I}_1^{(0)}(R) = -\frac{\sqrt{3}}{12}\, m^2 R - \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta\ \cosh\theta\ \log\big(1+e^{-\epsilon_0(r|\theta)}\big)\ . \tag{5.20}$$

Obviously it is the first term in (5.20) that dominates at large R and therefore it gives the bulk vacuum energy of SLYM while the second term represents the finite-size corrections. Finally, the leading $R\to 0$ asymptotic

$$E_0(R) \sim -\frac{\pi}{15\, R} \tag{5.21}$$

can be deduced from (5.20) in full agreement with expected form $E_0(R) = -(\pi/6R)(c - 24\Delta(p_0))$.

II. Let us consider now the first excited state of the SLYM. We will denote this state and associated eigenvalue of $\mathbb{T}$ as $\mid\Psi_1\rangle$ and $Y_1(r|\theta)$, respectively. At $R\to\infty$ this state is interpreted as one-particle state with the particle momentum (and rapidity) equal zero. Therefore, according to our discussion above, for large r the eigenvalue $Y_1(r|\theta)$ has two zeroes in the physical strip located at

$$\theta = \pm\, i\,\gamma(r) = \pm\, i\,\big(\pi/6 - \nu(r)\big)\ , \tag{5.22}$$

where $\nu(r) \to 0$ as $r \to \infty$ with exponential accuracy. Correspondingly, in this case the integral equation (5.9) for the function $\epsilon_1(r|\theta) = \log Y_1(r|\theta)$ takes the form

$$\epsilon_1(r|\theta) = r\cosh\theta + \log\sigma_1\big(\theta, \frac{\pi}{6} - \nu\big) + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta - \theta')\ \log(1 + e^{-\epsilon_1(r|\theta')})\ , \qquad (5.23)$$

where $\sigma_1(\theta, \eta)$ is given in (4.24) and the function $\nu(r)$ which describes the deviation of zeroes from their limiting positions at $\pm i\pi/6$ is determined from the equation (5.12)

$$e^{\epsilon_1(r|\,\pm i(\pi/6+\nu))} = -1\ . \qquad (5.24)$$

In the limit $r >> 1$ the term with the integral in (5.23) is exponentially small and with this accuracy one has

$$Y_1(r|\theta) = \frac{\cosh\theta - \cos(\frac{\pi}{6} - \nu)}{\cosh\theta + \cos(\frac{\pi}{6} - \nu)}\ \frac{\cosh\theta - \sin\nu}{\cosh\theta + \sin\nu}\ e^{r\cosh\theta}\ \big(1 + O(e^{-r})\big)\ , \qquad (5.25)$$

where $\nu = \nu(r)$ is derived from (5.24)

$$\nu(r) = \sqrt{3}\ e^{-\frac{\sqrt{3}}{2}r} + O\big(re^{-\sqrt{3}r}\big)\ . \qquad (5.26)$$

For $r \sim 1$ the equations (5.23), (5.24) can be solved numerically. The resulting function $\gamma(r)$ is given in Table 5 for $2.6 < r < 8.0$ and the typical form of the function $Y_1(r|\theta)$ in this range of r is shown in Fig.2.

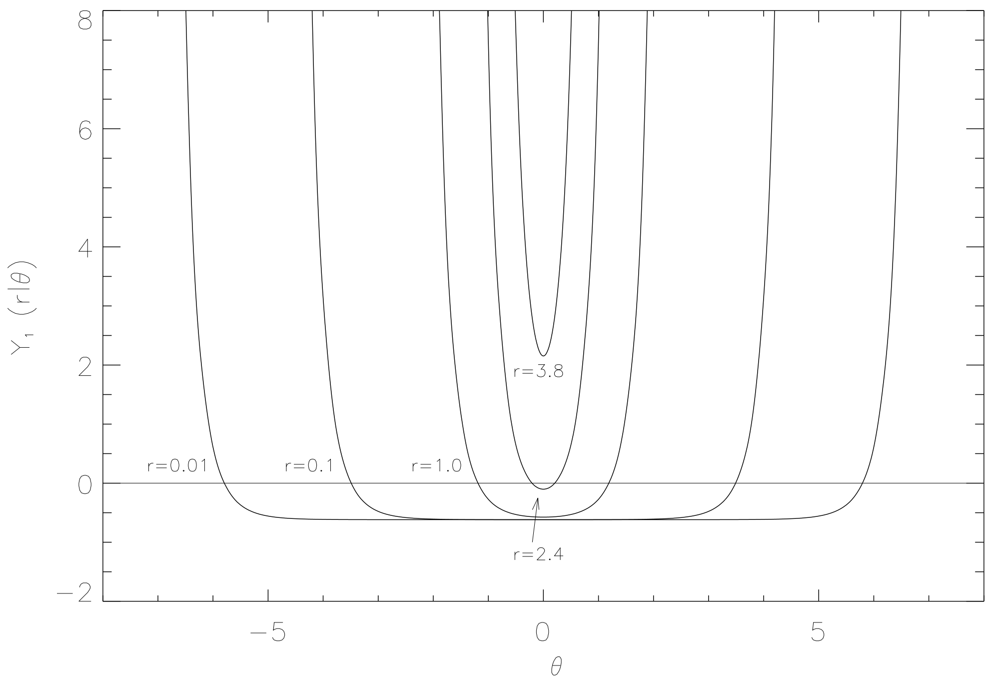

Fig.2. The function $Y_1(r|\theta)$ determined from the numerical solution of the eqations (5.23), (5.24) (for $r > r_0$) and (5.30), (5.31) (for $r < r_0$).

r	$\gamma(r)$	$\Delta E(r)/m$	(TCSA)
2.600000	0.133055	0.238050	0.23805
2.800000	0.256131	0.197403	0.19741
3.000000	0.322401	0.164321	0.16432
3.200000	0.367132	0.137217	0.13722
3.400000	0.399638	0.114891	0.11489
3.600000	0.424188	0.096412	0.096411
3.800000	0.443176	0.081057	0.081055
4.000000	0.458106	0.068256	0.068251
4.200000	0.469989	0.057552	0.057552
4.400000	0.479533	0.048579	0.048583
4.600000	0.487254	0.041042	0.041040
4.800000	0.493538	0.034700	0.034699
5.000000	0.498676	0.029356	0.029352
5.200000	0.502895	0.024846	0.024847
5.400000	0.506370	0.021037	0.021034
5.600000	0.509241	0.017816	0.017812
5.800000	0.511619	0.015092	0.015091
6.000000	0.513592	0.012785	0.012782
6.200000	0.515233	0.010832	0.010837
6.400000	0.516598	0.009178	0.009177
6.600000	0.517737	0.007775	0.0077767
6.800000	0.518688	0.006587	0.0065876
7.000000	0.519482	0.005581	0.0055816
7.200000	0.520147	0.004727	0.0047264
7.400000	0.520703	0.004003	0.0040045
7.600000	0.521169	0.003389	0.0033970
7.800000	0.521559	0.002870	0.0028753
8.000000	0.521886	0.002429	0.0024364

Table 5. The data obtained from the numerical solution of (5.23) and (5.24). The third column contains the excess energy $\Delta E_1(r)$ defined by (5.36) (given in units of the mass m) while the forth column contains the TCSA data for the same quantity computed in [13].

As r decreases the zeroes (5.22) get closer together and at

$$r = r_0 \simeq 2.53576\ldots \qquad (5.27)$$

they collide on the real θ-axis. For $r < r_0$ the function $Y_1(r|\theta)$ has two real zeroes

$$\theta = \pm\, \alpha(r)\ , \tag{5.28}$$

and one can represent it in the form:

$$Y_1(r|\theta) = \sigma_0\big(\theta - \alpha(r)\big)\, \sigma_0\big(\theta + \alpha(r)\big)\ e^{\epsilon_1'(r|\theta)}\ , \qquad r < r_0\ , \tag{5.29}$$

where the function $\sigma_0(\theta)$ is given by (4.24). Now we should solve the equation

$$\epsilon_1'(r|\theta) = r\cosh(\theta) + \int_{-\infty}^{\infty} d\theta'\ \Phi(\theta-\theta')\log\big(\sigma_0(\theta'-\alpha(r))\,\sigma_0(\theta'+\alpha(r)) + e^{-\epsilon_1'(r|\theta')}\big)\ , \tag{5.30}$$

together with

$$i\tanh\big(\frac{3\alpha(r)}{2} + i\frac{\pi}{4}\big)\, e^{\epsilon_1'(\alpha(r)+i\pi/3)} = -i\tanh\big(\frac{3\alpha(r)}{2} - i\frac{\pi}{4}\big)\, e^{\epsilon_1'(\alpha(r)-i\pi/3)} = -1\ , \tag{5.31}$$

instead of (5.23) and (5.24).

r	$\alpha(r)$	$\Delta E(r)/m$	(TCSA)	(CPT)
$1.0\,10^{-4}$	10.39926	23037.48		23037.348
$1.0\,10^{-3}$	8.096681	2302.837		2302.834
$1.0\,10^{-2}$	5.794092	229.3856		229.3849
0.100	3.49150	22.0527		22.0527
0.500	1.88203	3.67974		3.67977
1.000	1.18800	1.44738		1.44736
2.000	0.464431	0.430871	0.43088	0.430197
2.400	0.206153	0.288398	0.28839	0.286762
2.500	0.103278	0.261863	$*$0.26189	0.259921
2.510	0.0875353	0.259338	$*$0.25939	0.257393
2.520	0.0684264	0.256826	$*$0.25691	0.254891
2.525	0.0565558	0.255570	$*$0.25569	0.253650
2.530	0.0414288	0.254311	$*$0.25447	0.252416
2.535	0.0150627	0.253036	$*$0.25325	0.251188

Table 6. The data obtained from the numerical solution of (5.30), (5.31). The third column contains the excess energy $\Delta E_1(r)$ defined by (5.36) (given in units of the mass m) while the forth column contains avaliable TCSA data for the same quantity computed in [13](the numbers marked with an asterisk obtained by the qubic spline interpolation of the resutls of [13]). The fifth column contains CPT results for the same quantity obtained from (5.37).

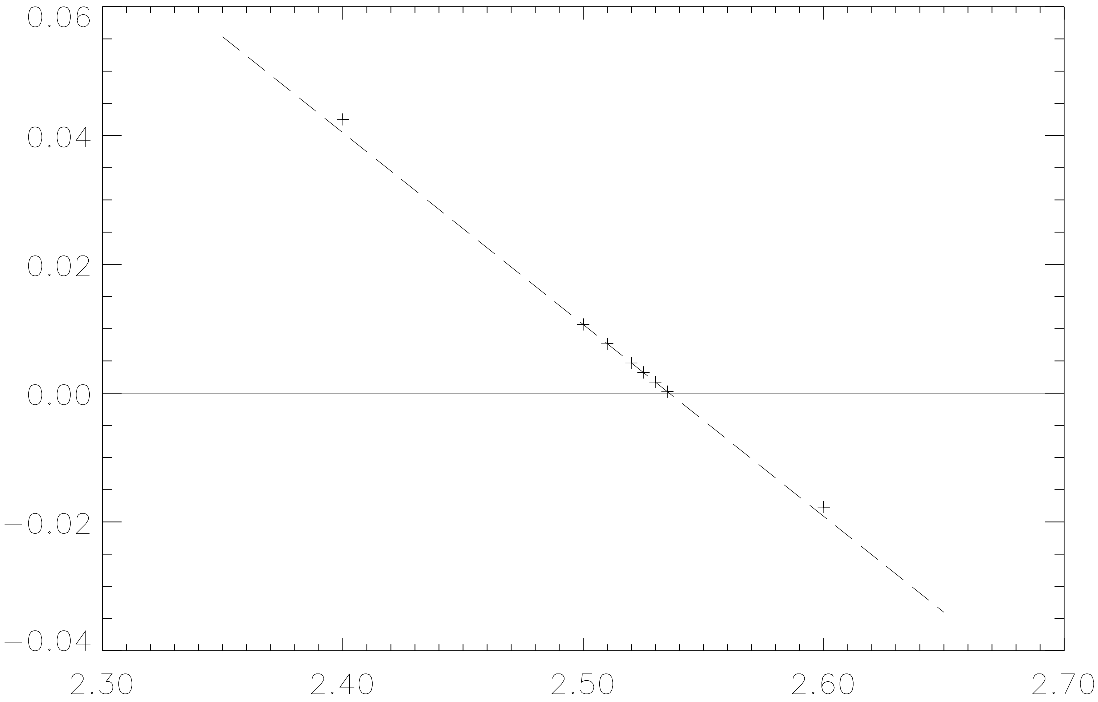

Fig.3. The values of $\alpha^2(r)$ and $-\gamma^2(r)$ for $r \simeq r_0$ and the linear fit (5.32) (dashed line).

The function $\alpha(r)$ obtained by numerical integration of these equations is given in Table 6 for $10^{-4} \leq r < r_0$. Note, that the location of zeroes of $Y_1(r|\theta)$ for $r \simeq r_0$ both for $r > r_0$ and $r < r_0$ is described by the same law (see Fig.3)

$$\left.\begin{array}{c} \alpha^2(r) \\ -\gamma^2(r) \end{array}\right\} \sim A\,(r_0 - r), \qquad r \simeq r_0, \tag{5.32}$$

with the numerical value of constant $A = 0.297893\ldots$.

For $r \to 0$ the zeroes (5.28) depart to $\pm\infty$ as

$$\alpha(r) \sim -\log\big(\frac{r}{2\kappa}\big) + \alpha\,, \quad r \to 0\;, \tag{5.33}$$

where the constant α is the same as in (4.40). In this limit the function $Y_1(r|\theta)$ approaches the constant $2\cos(2\pi p_1) = (1-\sqrt{5})/2$ in the wide range of θ, $\log r < \theta < -\log r$, while $Y_1(r|\theta) \sim Y_1\big(\theta + \log\big(\frac{r}{2\kappa}\big)\big)$ for $\theta \gtrsim -\log r$ and $Y_1(r|\theta) \sim Y_1\big(-\theta - \log\big(\frac{r}{2\kappa}\big)\big)$ for $\theta \lesssim \log r$. Here $Y_1(\theta)$ is the "kink solution" discussed in Sect.4. A typical form of the function $Y_1(r|\theta)$ for $r < r_0$ is shown in Fig.2.

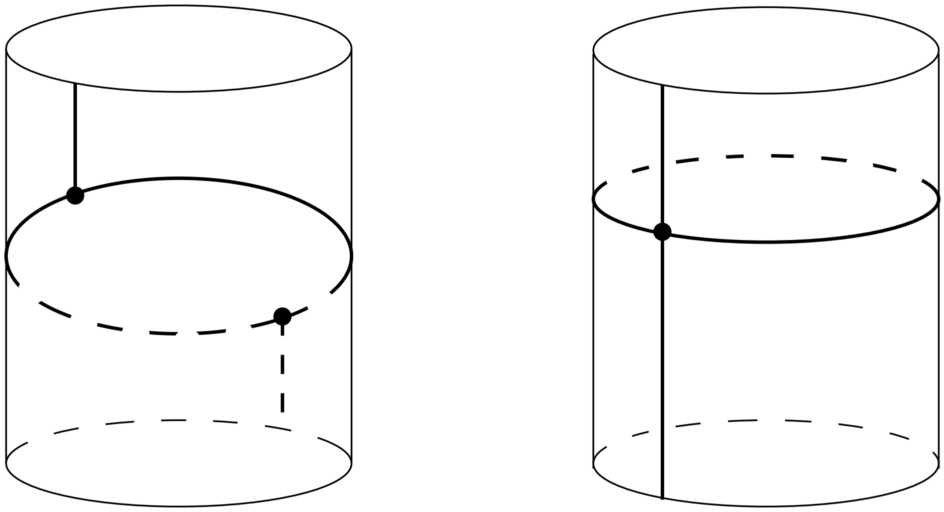

Fig.4. Diagramms representing two leading contributions to the large r asymptotics of $\Delta E_1(r)$. Diagramms (a) and (b) coreresponds to the third and fourth terms in (5.35), respectively.

The eigenvalues of the local IM $\mathbb{I}_{2n-1}$ associated with the excited state $\mid \Psi_1\rangle$ can be calculated from the asymptotic expansion of $\epsilon_1(r|\theta)$ for $r > r_0$ and $\epsilon_1'(r|\theta)$ for $r < r_0$ similar to (5.18). In particular, for the energy $E_1(R)$,

$$\mathbb{H} \mid \Psi_1\rangle = E_1(R) \mid \Psi_1\rangle \ ,$$

this gives

$$E_1(R) = -\frac{\sqrt{3}}{12}\ mr + 2\, m\ \sin\big(\frac{\pi}{6} + \nu(r)\big) - \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta\ \cosh\theta\ \log(1 + Y_1^{-1}(r|\theta)),\ r > r_0\ ,$$
$$E_1(R) = -\frac{\sqrt{3}}{12}\ mr + \sqrt{3}\, m\ \cosh\alpha(r) - \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta\ \cosh\theta\ \log|1 + Y_1^{-1}(r|\theta)|,\ r < r_0\ , \tag{5.34}$$

Substituting (5.25) and (5.26) into the first of the above formulae one obtains a few first terms of the large $r = mR$ expansion,

$$E_1(R) = -\frac{\sqrt{3}}{12}\ mr + m + 3\, m\, e^{-\frac{\sqrt{3}}{2}\, r} - \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta\ \cosh\theta\ S(\theta + i\pi/2)\ e^{-r\cosh\theta} + O\big(re^{-\sqrt{3}\, r}\big)\ , \tag{5.35}$$

where $S(\theta)$ is given by (1.4). Here the first term is the bulk vacuum energy, the second term makes it manifest that for $R \to \infty$ the state $\mid \Psi_1\rangle$ describes one particle with zero

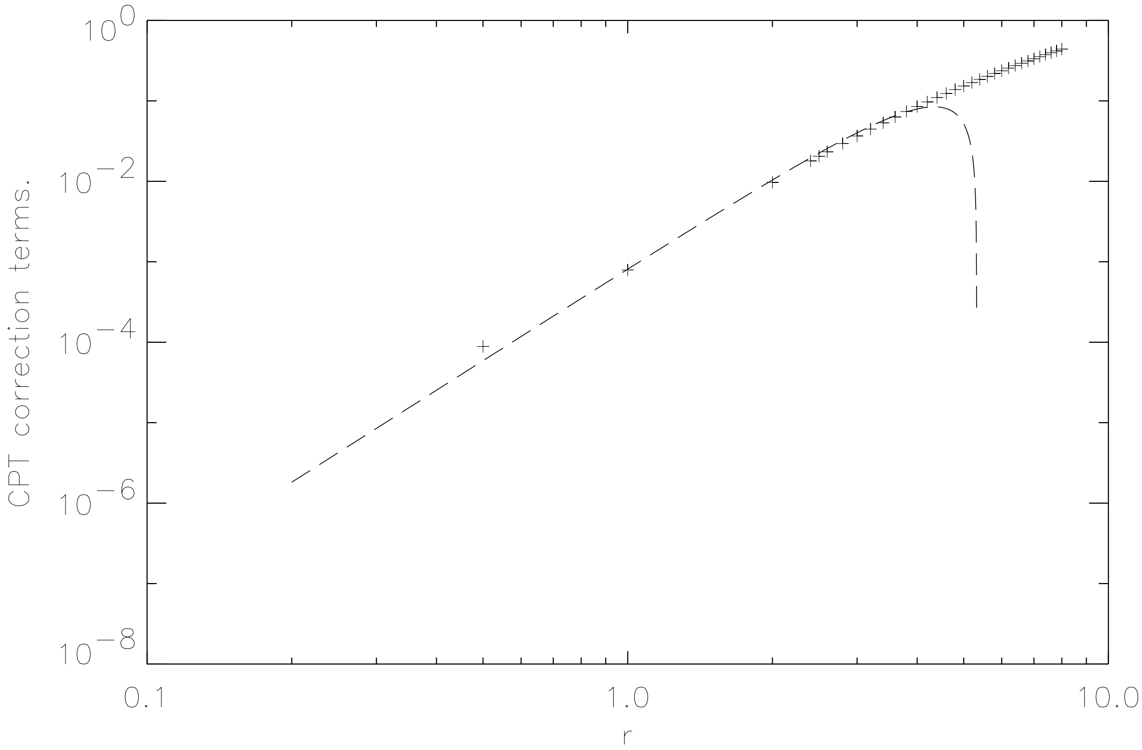

Fig.5. The dashed line represents the CPT correction terms (the second and third terms in (5.37) with negated signs given in units of the mass m) and the points represent the values for $11\pi/(15r) - E_1(R)/m$ obtained from numerical integration in (5.34).

momentum, while the third and the fourth terms are the corrections coming from the diagrams in Fig.4. These terms in (5.35) are in full agreement with the results of [4], [13] and [27].

In the opposite limit $R \to 0$ the state $\mid \Psi_1\rangle$ approaches the CFT highest weight state $\mid \Delta(p_1)\rangle$ (3.13) in (5.1). The later has the dimensions $(\Delta, \bar{\Delta}) = (0,0)$ and corresponds to the identity operator in this CFT. Indeed the calculation in Appendix C shows that $E_1(R) \sim 11\pi/(15\,R)$, as $R \to 0$. In Tables 5 and 6 the function $E_1(R)$ obtained by numerical integration in (5.34) is compared with the first excited state energy obtained in [3] by Truncated Conformal Space Method. To facilitate this comparison we present the data for an "excess energy"

$$\Delta E_1(r) = E_1(R) + \frac{\sqrt{3}}{12} mr - m, \tag{5.36}$$

where the bulk and mass terms are subtracted. Also, in Table 6 we compare our numerical results with the first terms of the short-distance expansion

$$E_1(R) = m\,\Big(\,\frac{11\pi}{15\,r} + e_2\,r^{\frac{19}{5}} + e_3\,r^{\frac{31}{5}} + \cdots\,\Big)\ ; \tag{5.37}$$

$$e_2 = -(1+\sqrt{5})^{\frac{3}{2}}\,\frac{5^{\frac{17}{4}}}{2^{14}}\,\frac{2^{\frac{1}{10}}}{\pi^{\frac{36}{5}}}\,\frac{\Gamma^4(2/5)}{\Gamma^2(1/5)}\,\big(\Gamma(2/3)\,\Gamma(5/6)\big)^{\frac{24}{5}};$$

$$e_3 = 3\,(\sqrt{5}-1)^{\frac{1}{2}}\,\frac{5^{\frac{29}{4}}}{2^{25}}\,\frac{2^{\frac{9}{10}}}{\pi^{\frac{74}{5}}}\,\Gamma^4(2/5)\,\Gamma^6(1/5)\,\frac{\Gamma^3(9/10)\Gamma(3/10)}{\Gamma^3(1/10)\Gamma(7/10)}\,\big(\Gamma(2/3)\Gamma(5/6)\big)^{\frac{36}{5}};$$

obtained by Conformal Perturbation Theory from (1.3) (see Appendix D). In Fig.5 we plot numerical results for $E_1(R)/m - 11\pi/(15r)$ against the correction terms in (5.37)[9].

6. Discussion

In Sect.3 we have constructed the commuting family of $\mathbb{T}$-operators for perturbed CFT (3.4). As is well known the IQFT can be obtained by so called "quantum group restriction" of the sine-Gordon model [28] [29] . The sine-Gordon model is described by the action

$$\mathcal{A}_{SG} = \int \Big[\frac{1}{16\pi}\,(\partial_a\varphi)^2 - 2\widetilde{\mu}\,\cos(\beta\varphi)\Big]\,d^2x\ , \tag{6.1}$$

where $\varphi(z,\bar{z})$ is a scalar field and β and $\widetilde{\mu}$ are parameters, the latter one carrying the dimension [10]

$$\widetilde{\mu} \sim \big[\,length\,\big]^{2\beta^2-2}. \tag{6.2}$$

This QFT is integrable, in particular it possesses infinite set of local IM of the form (3.8) where now $T_{2k+2},\ \Theta_{2k}$ and $\bar{T}_{2k+2},\ \bar{\Theta}_{2k}$ are certain local fields of the sine-Gordon model (and $\hat{\mu}$ is replaced by $\widetilde{\mu}$). The on-shell solution of (6.1) contains two "topologically charged" particles — soliton and antisoliton — and a number of the neutral particles with the masses

$$m_j = 2\,M\,\sin(\pi j\xi),\quad j = \frac{1}{2}, 1, \ldots < \frac{1}{2\xi}\ , \tag{6.3}$$

---

[9] Note, that for $r \leq 0.1$ the correction terms in (5.37) are quite small (less than 10^{-8} of the magnitude of the leading $1/r$ term) and are beyond the precision of our numerical calculation.

[10] In fact the value and even the dimension of the parameter $\tilde{\mu}$ depends on the precise way one defines the composite field $\cos(\beta\varphi)$ in (6.1). Here we assume that this field is canonocally normalized with respect to its short-distance asymptotic, i.e. $\cos(\beta\varphi)(z,\bar{z})\,\cos(\beta\varphi)(z',\bar{z}') \sim \frac{1}{2}\,|z-z'|^{-4\beta^2}\ ,\ (z,\bar{z}) \to (z',\bar{z}')$. In this normalization the field $\cos(\beta\varphi)$ carries the dimension $[\,length\,]^{-2\beta^2}$ and hence the $\tilde{\mu}$ in (6.1) has the dimension (6.2).

where ξ is defined in (2.24) and M is the soliton mass which in turn is related to the parameter $\widetilde{\mu}$ in (6.1)as [14]

$$M = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{\xi}{2})}{\Gamma(\frac{1+\xi}{2})} \left[\pi \tilde{\mu} \frac{\Gamma(\frac{1}{1+\xi})}{\Gamma(\frac{\xi}{1+\xi})} \right]^{\frac{1+\xi}{2}} . \tag{6.4}$$

The factorizable S-matrix of the sine-Gordon model can be found in [30] .

In the case of infinite space the model (6.1) exhibits the symmetry with respect to the quantum group $U_{\tilde{q}}(SL(2))$ [28] [29] where

$$\tilde{q} = e^{\frac{i\pi}{\beta^2}} . \tag{6.5}$$

Namely, the soliton and the antisoliton transform as the two-dimensional representation of this quantum group and the local IM (3.8) and the S-matrix commute with the generators $\tilde{E}, \tilde{F}$ and $\tilde{H}$ of the associated quantum algebra $U_{\tilde{q}}(sl(2))$. Naturally, the Hilbert space $\mathcal{H}_\infty$ of the infinite-space sine-Gordon model containes the subspace $\mathcal{H}_\infty^{singlet}$ of the states anihilated by the above generators. The remarkable fact discovered in [28] , [29] is that this subspace $\mathcal{H}_\infty^{singlet}$ can be interpreted as complete space of states of certain local QFT. It turnes out that this "restricted sine-Gordon model" coinsides with the perturbed CFT (3.4), the parameter $\hat{\mu}$ in (3.4) being related to the constant $\tilde{\mu}$ in (6.1) as [14]

$$\widetilde{\mu}^2 = \frac{(1-2\beta^2)(3\beta^2-1)}{\pi} \left[\frac{\Gamma^3(\beta^2)\Gamma(1-3\beta^2)}{\Gamma^3(1-\beta^2)\Gamma(3\beta^2)} \right]^{\frac{1}{2}} \hat{\mu}^2 \tag{6.6}$$

Although the action of the generators $\tilde{E}, \tilde{F}, \tilde{H}$ can be defined also in appropriate sectors of the finite-size sine-Gordon model with the spatial coordinate compactified on a circle, the quantum group symmetry does not survive this compactification[11]. Nontheless the notion of singlet states still exists, and the subspace $\mathcal{H}_R^{singlet} \in \mathcal{H}_R$ still admits an interpretation as the space of states of the perturbed CFT (3.4) in finite size geometry

$$\mathcal{H}_R^{singlet} \simeq \mathcal{H}_{PCFT} . \tag{6.7}$$

[11] The generators $\tilde{E}, \tilde{F}, \tilde{H}$ can be defined only in particular "good" sectors of the space of states $\mathcal{H}_R$ of the finite-size system, with appropriately chosen "twists" (i.e. the eigenvalues of the operators $\int_0^R \partial_x \varphi$ and $\int_0^R \partial_y \varphi$. This is similar to the supersymmetric QFT where in the finite-size case the SUSY generators exist only in the Ramond sector. However unlike the SUSY theories in general case of $\tilde{q}$ the generators of $U_{\tilde{q}}(sl(2))$ change the "twists" and do not invariantly act in the "good" sectors.

It is clear from the structure of the operators $\mathbb{T}_j(\mu|\lambda)$ defined in (2.13), (3.18) and (3.22) that their action extends to the full sine-Gordon space $\mathcal{H}_R$ where they also satisfy the commutativity conditions (3.25) and all further relations in Sect.3 including (3.31) and (3.34) hold in the full space. In other words the operators $\mathbb{T}_j(\mu|\lambda)$ can be considered as the QFT "transfer-matrices" of the full unrestricted sine-Gordon theory. From this point of view the fields $\phi(z)$ and $\bar{\phi}(\bar{z})$ admit very simple interpretation in terms of the sine-Gordon field, namely ϕ and $\bar{\phi}$ are just the values of the field $\varphi(z, \bar{z})$ on the "left" and "right" components of the light cone, respectively, i.e.

$$\phi(z) = \frac{\beta}{2}\varphi(z, 0), \qquad \bar{\phi}(\bar{z}) = -\frac{\beta}{2}\varphi(R, R - \bar{z}); \tag{6.8}$$

here we find it more convenient to think in terms of the Minkowski space-time, where

$$z = x + t, \qquad \bar{z} = x - t, \tag{6.9}$$

are the real light-cone coordinates. Then according to (3.22) the operator $\mathbb{T}_j(\mu|\lambda)$ is obtained by integrating the Lax flat connection associated with sine-Gordon model (see e.g. [31]) around the space-time cilinder, the integration being done in two stages — first one integrates along the "left" light cone from $z = 0$ to $z = R$ with $\bar{z}$ kept equal to 0, and then the integration along the "right" light cone from $\bar{z} = 0$ to $\bar{z} = R$ is performed,[12] as is shown in Fig.6.

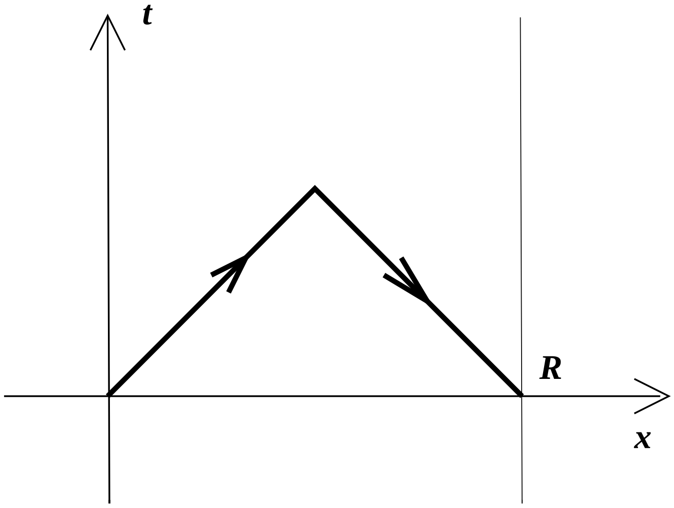

Fig.6. The integration path corresponding to the form (3.22) of the sine-Gordon "transfer matrix".

[12] This choice of the integration path makes calculations simpler. As the connection is flat one could have chosen any other closed contour which wraps around the cilinder.

It is possible to check that if β^2 takes special values such that (4.1) is satisfied the truncation relation (4.2) holds in the full sine-Gordon space of states (the relation (4.7) is specific to the restricted theory) and so an appropriate version of the method outlined in the Sectins 4 and 5 above can be applied to the sine-Gordon model at these values of β^2. However for generic values of β^2 the approach based on the QFT version of the Baxter's Q-operator is expected to be more powerfull. We have shown in [20] how to construct the $\mathbf{Q}$ operator in CFT, as the trace similar to (2.18) but this time taken not over a finite-dimensional reppresentation of $U_q(sl(2))$ but over certain infinite-dimensional representation of so-called q-oscillator algebra. Clearly, the operator $\mathbb{Q}(\mu|\lambda)$ for the sine-Gordon model can also be defined by the formula similar to (3.22). As we observed in [20] the asymptotic expansion of $\log \mathbf{Q}(\lambda)$ in CFT contains the contributions of the "dual nonlocal IM" in addition to the local IM. It is easy to see that the "dual nonlocal IM" are closely related to the nonlocal IM discovered in sine-Gordon theory in [29] . Therefore the operators $\mathbb{Q}(\mu|\lambda)$ and associated Baxter's equations are expected to be a powerfull tools in calculationg the spectra of both local and nonlocal IM in sine-Gordon theory [13]. Our progress in this direction is under way [32].

Our construction for the $\mathbb{T}$-operators can be extended to the cases when higher rank quantum algebras replace $U_q(sl(2))$. In the case of CFT the operators $\mathbf{T}$ introduced in [1] was recently generalized to the case of q-deformed twisted Kac-Moody algebra $A_2^{(2)}$ in [33]. The "massive" versions of these operators similar to (3.22) would apply to another class of IQFT — the CFT perturbed with the operator $\Phi_{1,2}$ — which includes in particular the $T = T_c$ Ising QFT with nonzero magnetic field [11] .

[13] In fact, the relation (3.32) in our Conjecture 2 in Sect.3 is obtained by combining (2.23), (6.4) and (6.6).

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Appendix A.

In this Appendix we use the notation

$$a = \beta^2 \ . \tag{A.1}$$

While the simplest of the integrals (2.27) $G_1^{vac}(p)$ is easily evaluated,

$$\begin{aligned} G_1^{vac}(p) &= \int_0^{2\pi} dx \int_0^x dy \left(2 \sin\big(\frac{x-y}{2}\big)\right)^{-2a} 2 \cos\big(2p\,(x-y-\pi)\big) \\ &= 4\pi \sin\big(\pi(a-2p)\big) \int_0^1 (1-t)^{-2a}\, t^{a-2p-1}\, dt \ = \ \frac{4\pi^2\,\Gamma(1-2a)}{\Gamma(1-a-2p)\,\Gamma(1-a+2p)} \ , \end{aligned} \tag{A.2}$$

no explicit expression for these integrals with $k > 1$ is known for generic p and a. The integral $G_2^{vac}(p)$ can be simplified as

$$\begin{aligned} G_2^{vac}(p) &= (2\pi)^2 \cos(2\pi p)\, \frac{\Gamma^2(1-2a)\,\Gamma(a-2p)\,\Gamma(a+2p)}{\Gamma(1-a-2p)\,\Gamma(1-a+2p)} \\ &- 2\pi\,\Gamma^2(1-a) \left(\frac{\sin\pi(a-2p)\ \Gamma(2p)}{(a-2p)\ \Gamma(1-2a+2p)}\ F(a,p) + \frac{\sin\pi(a+2p)\ \Gamma(-2p)}{(a+2p)\ \Gamma(1-2a-2p)}\ F(a,-p) \right) , \end{aligned} \tag{A.3}$$

where the function $F(a,p)$ is given by the series

$$F(a,p) = \Gamma(1+2a) \sum_{n=0}^{\infty} \frac{\Gamma(a-2p+n)}{n!\,\Gamma(3a-2p+n+1)}\ f_n(a,p) \ , \tag{A.4}$$

with

$$f_0(a,p) = 1 \ , \qquad f_1(a,p) = 8a\ \frac{a-p}{1+a-2p} \ , \tag{A.5}$$

and the rest of the coefficients $f_n(a,p)$ with $n > 1$ determined recurrently through the relation

$$A_n(a,p)\ f_{n+1}(a,p) - B_n(a,p)\ f_n(a,p) + C_n(a,p)\ f_{n-1}(a,p) = 0\ , \tag{A.6}$$

where

$$A_n(a,p) = (1+n+a-2p)\,(1+n-4p);$$

$$B_n(a,p) = 2n^3 + 12n^2\,(a-p) + 2n\,(8p^2+12a^2-4ap-7a) + 8a\,(4p^2+a-4ap-p);$$

$$C_n(a,p) = n\,(n+4a-1)\,(n+3a-2p-1)\,(n+4a-4p-1)\,.$$

This expression is very efficient for numerical evaluation of $G_2^{vac}(p)$ with high precision. Another convenient expression for this integral which also generalizes to higher $G_k^{vac}(p)$ is given in [34].

In particular case $2p = 1-a$ the integrals $G_2^{vac}(p)$ and $G_3^{vac}(p)$ can be evaluated analytically

$$G_2^{vac}(\frac{1-a}{2}) = \frac{2\pi^2}{1-2a}\ \frac{\Gamma^3(1-a)\,\Gamma(3-4a)}{\Gamma^2(2-2a)\,\Gamma(2-3a)}\ , \tag{A.7}$$

$$G_3^{vac}(\frac{1-a}{2}) = \frac{4\pi^3}{1-2a}\ \frac{\Gamma^3(1-a)\,\Gamma^2(1-2a)\,\Gamma(\frac{5}{2}-3a)}{\Gamma(2-4a)\,\Gamma(2-3a)\,\Gamma^3(\frac{3}{2}-a)}\ . \tag{A.8}$$

Finally, note that when q is a root of unity the functional equations (4.5) or (4.10) impose certain algebraic relations between the nonlocal IM, which allow one to express some higher nonlocal IM through the lower ones. In particular, using the functional (4.15) one can show that

$$\begin{aligned} G_2^{vac}(\tfrac{1}{10}) &= \frac{5-\sqrt{5}}{10}\Big(G_1^{vac}(\tfrac{1}{10})\Big)^2 \\ G_3^{vac}(\tfrac{1}{10}) &= \frac{\sqrt{5}-2}{5}\Big(G_1^{vac}(\tfrac{1}{10})\Big)^3 \end{aligned} \tag{A.9}$$

for the $p = 1/10$, $\Delta = -1/5$ vaccuum state in the Lee-Yang CFT and

$$\begin{aligned} G_4^{vac}(\tfrac{3}{10}) &= \frac{5+\sqrt{5}}{10}\Big(G_2^{vac}(\tfrac{3}{10})\Big)^2 \\ G_5^{vac}(\tfrac{3}{10}) &= \frac{5-\sqrt{5}}{10}\,G_2^{vac}(\tfrac{3}{10})\,G_3^{vac}(\tfrac{3}{10}) \\ G_6^{vac}(\tfrac{3}{10}) &= \frac{5+\sqrt{5}}{10}\Big(G_3^{vac}(\tfrac{3}{10})\Big)^2 - \frac{2+\sqrt{5}}{5}\Big(G_2^{vac}(\tfrac{3}{10})\Big)^3 \end{aligned} \tag{A.10}$$

for the $p = 3/10$, $\Delta = 0$ state.

Appendix B.

Here we present eigenvalues of the local IM $\mathbf{I}_{2n-1}$ $(n = 1, 2, ..., 8)$ with $R = 2\pi$ on the highest Virasoro vector with the central charge c and conformal dimension Δ

$$I_1^{vac}(\Delta) = \Delta - \frac{c}{24} \,, \tag{B.1}$$

$$I_3^{vac}(\Delta) = \Delta^2 - \frac{(c+2)}{12}\,\Delta + \frac{c\,(5c+22)}{2880} \,, \tag{B.2}$$

$$I_5^{vac}(\Delta) = \Delta^3 - \frac{(c+4)}{8}\,\Delta^2 + \frac{(c+2)\,(3c+20)}{576}\,\Delta - \frac{c\,(3c+14)\,(7c+68)}{290304} \,, \tag{B.3}$$

$$\begin{aligned} I_7^{vac}(\Delta) = \Delta^4 &- \frac{(c+6)}{6}\,\Delta^3 + \frac{(15c^2+194c+568)}{1440}\,\Delta^2 - \frac{(c+2)\,(c+10)\,(3c+28)}{10368}\,\Delta \\ &+ \frac{c\,(3c+46)\,(25c^2+426c+1400)}{24883200} \,, \end{aligned} \tag{B.4}$$

$$\begin{aligned} I_9^{vac}(\Delta) = \Delta^5 &- \frac{5\,(c+8)}{24}\,\Delta^4 + \frac{(c+8)\,(5c+46)}{288}\,\Delta^3 \\ &- \frac{(35c^3+990c^2+9048c+23488)}{48384}\,\Delta^2 \\ &+ \frac{(c+2)\,(175c^3+7134c^2+96168c+392000)}{11612160}\,\Delta \\ &- \frac{c\,(5c+22)\,(11c+232)\,(7c^2+274c+1960)}{3065610240} \,, \end{aligned} \tag{B.5}$$

$$\begin{aligned} I_{11}^{vac}(\Delta) = \Delta^6 &- \frac{c+10}{4}\,\Delta^5 + \frac{15c^2+322c+1808}{576}\,\Delta^4 \\ &- \frac{105c^3+3700c^2+44612c+165984}{72576}\,\Delta^3 \\ &+ \frac{525c^4+27908c^3+548508c^2+4248784c+10147200}{11612160}\,\Delta^2 \\ &- \frac{(c+2)\,(315c^4+24604c^3+676548c^2+7298480c+25872000)}{418037760}\,\Delta \\ &+ \frac{c\,(13c+350)\,(11025c^4+1160780c^3+25741404c^2+198779728c+470870400)}{27389834035200} \,, \end{aligned} \tag{B.6}$$

$$
\begin{aligned}
I^{vac}_{13}(\Delta) = {}& \Delta^7 - \frac{7\,(c+12)}{24}\ \Delta^6 + \frac{7\,(15c^2+386c+2680)}{2880}\ \Delta^5 \\
& - \frac{(105c^3+4430c^2+66264c+319552)}{41472}\ \Delta^4 \\
& + \frac{(525c^4+33364c^3+818172c^2+8304848c+27924096)}{4976640}\ \Delta^3 \\
& - \frac{(3465c^5+331192c^4+12012668c^3+190959296c^2+1291772608c+2856307200)}{1313832960}\ \Delta^2 \\
& + (c+2)\,\big(121275c^5+18838640c^4+922934036c^3+18803208352c^2+166162628800c \\
& \qquad\qquad + 517957440000\big)\ \Delta/3310859059200 \\
& - c\,(5c+164)\,\big(3465c^5+1026934c^4+39009476c^3+568047656c^2+3512182240c \\
& \qquad\qquad\quad + 7399392000\big)/79460617420800\ ,
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
I^{vac}_{15}(\Delta) = {}& \Delta^8 - \frac{(c+14)}{3}\ \Delta^7 + \frac{7\,(c^2+30c+248)}{144}\ \Delta^6 \\
& - \frac{(7c^3+344c^2+6140c+36304)}{1728}\ \Delta^5 \\
& + \frac{(35c^4+2588c^3+76020c^2+952528c+4102528)}{165888}\ \Delta^4 \\
& - \frac{(231c^5+25654c^4+1122476c^3+22259304c^2+195500000c+597940480)}{32845824}\ \Delta^3 \\
& + \big(21021c^6+3835642c^5+236079916c^4+6438951928c^3+84201249920c^2 \\
& \qquad + 500631533696c+1026155648000\big)\ \Delta^2/143470559232 \\
& - (c+2)\big(3003c^6+1137814c^5+89028148c^4+2852262856c^3+44100366464c^2 \\
& \qquad\qquad + 324626771840c+897792896000\big)\ \Delta/1721646710784 \\
& + \frac{c\,(5c+22)\,(7c+68)\,(17c+658)}{28097274319994880}\times \\
& \quad (429c^4+495724c^3+20021388c^2+285527760c+1457456000)\ .
\end{aligned}
\tag{B.8}
$$

For $c=-2$ the eigenvalues of the all local IM is known [20]:

$$
I^{vac}_{2n-1}(\Delta) = 2^{-n}\ B_{2n}(2p+1/2)\ , \quad \text{with}\ \ \Delta = 2p^2 - 1/8\ , \tag{B.9}
$$

where $B_k(x)$ is the Bernoulli polynomials [35].

One can also find the eigenvalues $I^{vac}_{2n-1}(\Delta)$ for $c = 1/2$ and $\Delta = 0, 1/2, 1/16$:

$$
\begin{aligned}
I^{vac}_{2n-1}(0)\big|_{c=\frac{1}{2}} &= -c_n \, \left(1 - 2^{1-2n}\right) \, \frac{B_{2n}}{4\,n} \,, \\
I^{vac}_{2n-1}(1/2)\big|_{c=\frac{1}{2}} &= c_n \, \left(2^{1-2n} - \left(1 - 2^{1-2n}\right) \, \frac{B_{2n}}{4\,n}\right) \,, \\
I^{vac}_{2n-1}(1/16)\big|_{c=\frac{1}{2}} &= c_n \, \frac{B_{2n}}{4\,n} \,,
\end{aligned}
\tag{B.10}
$$

Here B_k is the Bernoulli numbers [35] and

$$
c_n = \frac{(6n-3)!! \; n! \; 2^{n-1}}{(2n-1)\,(4n-3)! \; 3^n} \,.
$$

Appendix C.

In this Appendix we present the calculation of first integrals in (4.33) and (4.44) (corresponding to the ground and the first excited state in the Lee-Yang CFT) using the so-called "dilogarithm trick". Although for the ground state case such calculations are well known in the literature on TBA (see e.g. [36]) for the most transparent presentation) we include them here for a completeness.

Introducing the notation

$$
L_0(\theta) = \log\left(1 + e^{-\epsilon_0(\theta)}\right) \,, \tag{C.1}
$$

we can write the first integral in (4.33) as

$$
\chi_1 = \int_{-\infty}^{\infty} d\theta \; e^{\theta} L_0(\theta) \,. \tag{C.2}
$$

Differentiating the integral equation (4.30) with respect to θ, expressing therefrom the e^{θ} term and substituting the result into (C.2) one obtains

$$
\chi_1\,\kappa = \int_{-\infty}^{\infty} d\theta \; \partial_\theta \epsilon_0(\theta)\, L_0(\theta) - \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\theta' \; \Phi(\theta - \theta') \; L_0(\theta)\, \partial_{\theta'} L_0(\theta') \,. \tag{C.3}
$$

Using now the integral equation (4.30) to simplify the second term in the last formula one can rewrite it as

$$
\chi_1\,\kappa = \int_{-\infty}^{\infty} d\theta \; \big(\partial_\theta \epsilon_0(\theta)\, L_0(\theta) - \epsilon_0(\theta)\, \partial_\theta L_0(\theta) \big) - \kappa \int_{-\infty}^{\infty} d\theta \; e^{\theta} L_0(\theta) \,. \tag{C.4}
$$

Thus,

$$\chi_1\,\kappa = \frac{1}{2}\int_{-\infty}^{\infty} d\theta\, \left(\,\partial_\theta \epsilon_0(\theta)\, L_0(\theta) - \epsilon_0(\theta)\, \partial_\theta L_0(\theta)\,\right)\,. \tag{C.5}$$

A change to a new integration variable

$$x(\theta) = \left(1 + e^{\epsilon_0(\theta)}\right)^{-1} \tag{C.6}$$

brings this this integral to the form

$$\chi_1\,\kappa = -\mathrm{Li}\left(x(\infty)\right) + \mathrm{Li}\left(x(-\infty)\right)\,, \tag{C.7}$$

where $\mathrm{Li}\,(x)$ is the Rogers dilogarithm function

$$\mathrm{Li}\,(x) = -\frac{1}{2}\int_0^x dy\,\left(\frac{\log(1-y)}{y} + \frac{\log(y)}{1-y}\right) \tag{C.8}$$

and

$$x(-\infty) = \left(\frac{\sqrt{5}-1}{2}\right)^2, \qquad x(\infty) = 0\,,$$

as it follows from (4.18) and (4.19). Here and below we will need the following special values[14] of $\mathrm{Li}\,(x)$

$$\mathrm{Li}\,(1) = \frac{\pi^2}{6}, \qquad \mathrm{Li}\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10}, \qquad \mathrm{Li}\left(\left(\frac{\sqrt{5}-1}{2}\right)^2\right) = \frac{\pi^2}{15}\,. \tag{C.9}$$

Using this in (C.7) one arrives to the result (4.36) given in the main text.

Let us now turn to the calculation of the first integral in (4.44). It can be conveniently rewritten as

$$\tilde{\chi}_1 = \int_{-\infty}^{\infty} d\theta\; e^{\theta}\, L_1(\theta)\,, \tag{C.10}$$

where

$$L_1(\theta) = \log\left(\sigma_0(\theta-\alpha) + e^{-\epsilon_1(\theta)}\right)\,. \tag{C.11}$$

with $\sigma_0(\theta)$ defined in (4.24). Proceeding as above we can bring (C.10) to the form similar to (C.5)

$$\tilde{\chi}_1\,\kappa = \frac{1}{2}\int_{-\infty}^{\infty} d\theta\, \left(\partial_\theta \epsilon_1(\theta)\, L_1(\theta) - \epsilon_1(\theta)\, \partial_\theta L_1(\theta)\right)\,. \tag{C.12}$$

[14] See, e.g. [37] for a comprehensive review on the dilogarithm function.

Defining new functions

$$\begin{aligned}\tilde{\epsilon}(\theta) &= \epsilon_1(\theta) + \log|\sigma_0(\theta-\alpha)| = \log|Y_1(\theta)|\,,\\ \tilde{L}(\theta) &= L_1(\theta) - \log|\sigma_0(\theta-\alpha)| = \log|1+Y_1^{-1}(\theta)|\,,\end{aligned} \tag{C.13}$$

one can further rewrite (C.12) as

$$\tilde{\chi}_1\,\kappa = I + \frac{1}{2}\int_{-\infty}^{\infty} d\theta\,\Big(\partial_\theta\tilde{\epsilon}(\theta)\,\tilde{L}(\theta) - \tilde{\epsilon}(\theta)\,\partial_\theta\tilde{L}(\theta)\Big)\,, \tag{C.14}$$

where I is determined as the principal value of the singular integral

$$I = -\fint_{-\infty}^{\infty} d\theta\,\big(L_1(\theta)+\epsilon_1(\theta)\big)\,\partial_\theta\log|\sigma_0(\theta-\alpha)|\,. \tag{C.15}$$

The integral in (C.14) is calculated similarly to that in (C.5). Splitting the interval of integration into two parts, $(-\infty,\alpha)$ and (α,∞) and changing to a new integration variable

$$\tilde{x}(\theta) = \begin{cases} e^{\tilde{\epsilon}(\theta)}, & \text{for } \theta<\alpha\\ \big(1+e^{\tilde{\epsilon}(\theta)}\big)^{-1}, & \text{for } \theta>\alpha\,,\end{cases} \tag{C.16}$$

one can bring (C.14) to the form

$$\tilde{\chi}_1\,\kappa = I + \mathrm{Li}\,(1) + \mathrm{Li}\,\big(\tilde{x}(-\infty)\big)\,, \tag{C.17}$$

where $\mathrm{Li}\,(x)$ is the dilogarithm function (C.8) and

$$\tilde{x}(-\infty) = \frac{\sqrt{5}-1}{2} \tag{C.18}$$

as it follows from (4.19). The remaining integral (C.15) is calculated by using the integral equation (4.37) and the condition (4.39) determining the root α. In fact, using (4.37) first write (C.15) in the form

$$I = -\pi\sqrt{3}\,\kappa\,e^{\alpha} - \fint_{-\infty}^{\infty}\fint_{-\infty}^{\infty} d\theta d\theta'\,L_1(\theta)\,\big(\Phi(\theta-\theta')+\delta(\theta-\theta')\big)\,\partial_{\theta'}\log|\sigma_0(\theta'-\alpha)|\,, \tag{C.19}$$

where we have used the elementary integral

$$\fint_{-\infty}^{\infty} d\theta\,e^{\theta}\partial_\theta\log|\sigma_0(\theta-\alpha)| = \pi\sqrt{3}\,e^{\alpha}\,. \tag{C.20}$$

The integral over θ' in (C.19) is also elementary and can be easily calculated. After that the r.h.s. of (C.19) becomes (up to a simple factor) identical to the l.h.s. of (4.39)

$$I = -\pi\sqrt{3}\,\kappa\,e^{\alpha} - 3\fint_{-\infty}^{\infty} d\theta\,\frac{\cosh 2(\theta-\alpha)}{\sinh 3(\theta-\alpha)}\,L_1(\theta) = -\pi^2\,(1+4N)\,. \tag{C.21}$$

Collecting together (C.17), (C.21), (C.9) one obtains

$$\tilde{\chi}_1\,\kappa = -\Big(\frac{22}{5}+24\,N\Big)\,\frac{\pi^2}{6}\,, \tag{C.22}$$

which for $N=0$ precisely gives (4.45). The above calculations can easily be generalized for the solutions of (4.28) and (4.29) with arbitrary number of (real and complex) zeroes. Similar calculations for in lattice models can be found in [9].

Appendix D.

One obtains Conformal Perturbation Theory (CPT) by expanding in the interaction parameter $\hat{\mu}^2$ in (3.4). The CPT for the ground-state energy $E_0(R)$ of (1.3) was discussed in [4]. It can be easily modified for the excited-state energies. The excited -state energy $E_1(R)$ in Sect.5 expands as

$$\begin{aligned} E_1(R) = -\frac{\pi c}{6R} - R\sum_{k=2}^{\infty}\frac{\big(-\hat{\mu}^2\big)^k}{k!}\,\Big(\frac{2\pi}{R}\Big)^{2k\,(\Delta-1)+2}\times \\ \int\prod_{i=2}^{k}\big((z_i\bar{z}_i)^{\Delta-1}\,d^2z_i\big)\ \langle\Phi(1,1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\ldots\Phi(z_k,\bar{z}_k)\rangle_{CFT}^{conn}\ , \end{aligned} \tag{D.1}$$

where $d^2z\equiv dx\,dy$ for $z=x+iy$, $c=-22/5$ is the central charge of CFT $\mathcal{M}_{2/5}$, $\Delta=-1/5$ is the conformal dimension of the field Φ in (1.3) and $\langle\ldots\rangle_{CFT}^{conn}$ are the (connected) correlation functions of the this CFT [15]. The first terms in (D.1) contain

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\rangle_{CFT}^{conn} = |z_1-z_2|^{-4\Delta}\ , \tag{D.2}$$

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\rangle_{CFT}^{conn} = \mathbb{C}_{\Phi\Phi\Phi}\,|z_1-z_2|^{-2\Delta}\,|z_2-z_3|^{-2\Delta}\,|z_3-z_1|^{-2\Delta}\ . \tag{D.3}$$

Here

$$\mathbb{C}_{\Phi\Phi\Phi} = i\,\frac{5^{\frac{1}{4}}}{10\pi}\ \frac{\Gamma^2(1/5)\Gamma(2/5)}{\Gamma(4/5)}$$

is the triple-Φ OPE structure constant [19]. The most important difference in (D.1) as compared with the similar formula for $E_0(R)$ in [4] is that the integrals in (D.1) diverge as $(z_i,\bar{z}_i)\to 0,\infty$ and therefore the expression (D.1) can not be taken literally. This divergence is easily traced down to the fact that $|\,\Psi_1\rangle$ is not a ground state. As is well known in ordinary quantum mechanical perturbation theory, in doing calculations for the excited states in each order one has to take care of the orthogonality between the state under consideration and all the lower-energy states. This condition brings into (D.1) additional terms which exactly provide the subtractions of the above divergences. So in (D.1) appropriate subtractions of these divergences are implied. Equivalent way to

[15] For generic excited-state energy $E_n(R)$ the expansion similar to (D.1) would contain the insertion $V_n(0,0)V_n(\infty,\infty)$ in the correlation functions, where $V_n(z,\bar{z})$ is the CFT fields associated with the CFT limit of the excited state $|\,\Psi_n\rangle$ under consideration. For the state $|\,\Psi_1\rangle_{R=0}$ the associated field is the identity operator and therefore this insertion is ignored in (D.1).

define the integrals in (D.1) (which is similar to "dimensional regularization" of Feynman diagrams) is to evaluate the integrals in (D.1) for $\Delta < 1/2$ where the integrals converge and then analytically continue to $\Delta = -1/5$, see e.g. [38]. With this prescription the integrals in the first two terms of the series in (D.1) can be evaluated explicitly [19],

$$\int d^2 z \; |z|^{2\Delta-2} \, |1-z|^{-4\Delta} = \pi \, \frac{\Gamma^2(\Delta)\, \Gamma(1-2\Delta)}{\Gamma^2(1-\Delta)\, \Gamma(2\Delta)} \; , \tag{D.4}$$

$$\begin{gathered}\int d^2 z_1 \, d^2 z_2 \; |z_1|^{2\Delta-2} \, |z_2|^{2\Delta-2} \, |1-z_1|^{-2\Delta} \, |1-z_2|^{-2\Delta} \, |z_1-z_2|^{-2\Delta} = \\ \frac{\pi^2}{4} \, \frac{\Gamma^3(\frac{\Delta}{2})\, \Gamma(1-\frac{3\Delta}{2})}{\Gamma^3(1-\frac{\Delta}{2})\, \Gamma(\frac{3\Delta}{2})}\end{gathered} \tag{D.5}$$

and one arrives at (5.37).

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