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Exact Friedel oscillations in the $g = \frac{1}{2}$ Luttinger liquid.

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Abstract: A single impurity in the 1D Luttinger model creates a local modification of the charge density analogous to the Friedel oscillations. In this paper, we present an exact solution of the case $g = \frac{1}{2}$ (the equivalent of the Toulouse point) at any temperature T and impurity coupling, expressing the charge density in terms of a hypergeometric function. We find in particular that at $T = 0$, the oscillatory part of the density goes as $\ln x$ at small distance and $x^{-1/2}$ at large distance.

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1. Introduction

The Luttinger model, describing the low-energy excitations of an interacting one-dimensional fermion gas, is one of the simplest non-Fermi-liquid metals. Experimental observations of this non-Fermi state in 1D quantum wires are difficult, since disorder tends to localize the excitations. The model has also been proposed to describe the edge states in fractional quantum Hall devices [1]. Tunneling through a point contact is then a practically ideal situation for comparing theory with experiments [2], [3], [4].

While most of the attention has focussed on transport properties so far, static properties are also of interest. In particular, the effect of the impurity on the charge density, the equivalent of Friedel oscillations [5], has recently been considered [6]. A very similar problem, in antiferromagnetic spin chains, was also discussed in [7]. See also [8] for other related work.

The 1D Luttinger model with an impurity is integrable, and ultimately, using advanced techniques of quantum field theory, [9], these Friedel oscillations should be exactly computable for all couplings, but there are important technical difficulties. The particular point $g = 1/2$, the equivalent of the "Toulouse point" in the anisotropic Kondo problem, is equivalent to free fermions [10]. It should thus be amenable by more elementary techniques. However, even in that case, exact expressions have not yet been obtained [6] because the density operator is not local in the Fermion basis. We show in this paper how to circumvent this difficulty, largely based on a work of Chatterjee and Zamolodchikov [11].

2. Generalities

Let us start from a description of the quantities involved in the Luttinger liquid, following closely [6]. Using standard bosonization formula [12], one can write the electron creation operator in the spinless case - to which we restrict here - as a combination of two bosonic fields, $\phi(x), \theta(x)$. Decomposing into left and right moving parts, one has :

$$\psi(x)^\dagger (\text{ resp. } \bar{\psi}(x)^\dagger) \propto \exp\left[\pm i k_f x + i\sqrt{\pi}(\theta(x) \pm \phi(x))\right]. \tag{2.1}$$

The canonical momentum for ϕ is $\partial_x \theta$ and they obey the commutation relations :

$$[\theta(x), \phi(x')] = -\frac{i}{2} sgn(x - x'). \tag{2.2}$$

The Friedel oscillations is describing the charge or density oscillations of the electrons in the presence of a barrier. The density operator [13] :

$$\rho(x) = \rho_0 + \frac{1}{\sqrt{\pi}}\partial_x\phi + \frac{k_f}{\pi}\cos[2k_f x + 2\sqrt{\pi}\phi(x)] \tag{2.3}$$

with $\rho_0 = \frac{k_f}{\pi}$ the background charge. The hamiltonian for these fields is :

$$H = \frac{v_f}{2}\int_{-\infty}^{\infty} dx\ [\Pi^2 + (\partial_x\phi)^2] + H_{int}. \tag{2.4}$$

The term H_{int} describes a screened Coulomb interaction. The impurity will be coupled to the fields at one point $x = 0$ by the term :

$$H_{imp} = \frac{\sqrt{\pi}\lambda}{k_F}\partial_x\phi(0) + \lambda\cos[2\sqrt{\pi}\phi(0)]. \tag{2.5}$$

We will restrict to the case where H_{int} is short range, leading to a Luttinger liquid. In that case, the effect of the interaction is to renormalize the fields. The hamiltonian can then be brought into the usual form (setting $v_F = g$) :

$$H = \int_{-\infty}^{\infty} dx\ \left[8\pi g\Pi^2 + \frac{1}{8\pi g}(\partial_x\phi)^2\right] + \lambda\cos[\phi(0)], \tag{2.6}$$

while the Friedel oscillations, subtracting the background charge density, are encoded into :

$$\frac{< \rho(x) - \rho_0 >}{\rho_0} = \cos\left(2k_F x + \eta_F\right) < \cos\phi(x) > . \tag{2.7}$$

Here, the additional phase shift $\eta_F = -\frac{g\pi\lambda}{k_F}$ in the cosine term arises from the unitary transformation that cancels the $\partial_x\phi(0)$ term in (2.5) to get (2.6).

To proceed, we perform some manipulations. Decompose the field ϕ into its left and right components, $\phi = \phi_L + \phi_R$. Introduce then the left movers :

$$\phi^e(x+t) = \frac{1}{\sqrt{2}}\left[\phi_L(x,t) + \phi_R(-x,t)\right]$$
$$\phi^o(x+t) = \frac{1}{\sqrt{2}}\left[\phi_L(x,t) - \phi_R(-x,t)\right]$$

Now the impurity interaction reads $H_{imp} = \lambda\cos[\sqrt{2}\phi^e(x=0)]$, while the observable we are studying is proportional to :

$$\cos\phi(x,t) = \cos\left\{\frac{1}{\sqrt{2}}\left[\phi^e + \phi^o\right](x+t) + \frac{1}{\sqrt{2}}\left[\phi^e - \phi^o\right](-x+t)\right\}$$

Set now :

$$\Phi^e_L = \sqrt{2}\phi^e(x+t),\ x<0 \quad \Phi^e_R = \sqrt{2}\phi^e(-x+t),\ x<0$$
$$\Phi^o_L = \sqrt{2}\phi^o(x+t),\ x<0 \quad \Phi^o_R = -\sqrt{2}\phi^o(-x+t),\ x<0$$

We then fold the system and recombine these left and right components into a single field to get the hamiltonian :

$$H = H^o + H^e, \tag{2.8}$$

with :

$$H^o = \frac{1}{2}\int_{-\infty}^{0} dx\ \left[8\pi g(\Pi^o)^2 + \frac{1}{8\pi g}(\partial_x\Phi^o)^2\right], \quad \Phi^o(0) = 0, \tag{2.9}$$

and :

$$H^e = \frac{1}{2}\int_{-\infty}^{0} dx\ \left[8\pi g(\Pi^e)^2 + \frac{1}{8\pi g}(\partial_x\Phi^e)^2\right] + \lambda\cos\frac{1}{2}\Phi^e(0). \tag{2.10}$$

We will often refer to (2.10) as the boundary sine-Gordon hamiltonian. Using parity together with the decoupling of the odd and even fields we have :

$$\frac{<\rho(x)-\rho_0>}{\rho_0} = \cos(2k_F x+\eta_F) < \cos\frac{1}{2}\Phi^o(x) >< \cos\frac{1}{2}\Phi^e(x) > . \tag{2.11}$$

We shall obtain results in the continuum limit. Then, since there is no boundary coupling for the odd field, one has [14] :

$$< \cos\frac{1}{2}\Phi^o(x) > \propto \left(\frac{a}{x}\right)^{g/2}, \tag{2.12}$$

where a is a lattice coupling , $x >> a$ (a is defined by the propagator $< \Phi^o(x)\Phi^o(0) >= -4g\ln(x/a)$ for the bulk theory. The exact proportionality factor in (2.12) will be worked out later). Regarding the even field, things are more complicated. On general grounds, one expects :

$$< \cos\frac{1}{2}\Phi^e(x) > \propto \left(\frac{a}{x}\right)^{g/2} F\left[\lambda a\left(\frac{x}{a}\right)^{1-g}\right], \tag{2.13}$$

where the function F goes to one for large values of the argument, and vanishes linearly at small values of the argument (this latter property follows from the perturbative analysis in the regime $g > 1/2$). Continuum limit results will hold for $x >> a$ and thus $\lambda << 1/a$, while the product $\lambda a\left(\frac{x}{a}\right)^{1-g}$ is kept constant. For simplicity we set $a = 1$ in what follows.

3. Exact Friedel oscillations at $g = 1/2$ and $T = 0$.

The point $g = 1/2$ corresponds to free fermions in the bulk, the analog of the Toulouse limit in the Kondo problem. To compute the universal function F exactly in that case, we will proceed in two steps: (i) We will show that $\cos\frac{\Phi^e}{2}$ can be expressed as the product of spin operators in two decoupled massless Ising models, one of them having a boundary magnetic field, the other having fixed boundary conditions. (ii) These spin correlators in the boundary Ising models will be computed based on a method due to Chatterjee and Zamolodchikov.

In the following we do computations for the even field (ie the field having boundary interaction); results for the odd field follow simply by taking the limit of Dirichlet boundary conditions in the former solution.

A natural way to relate the boundary sine-Gordon model to two copies of the Ising model is by using bosonization. At the free fermion point this was considered in [15], where it was shown that the boundary interaction $\lambda\cos\frac{1}{2}\Phi^e(0)$ becomes linear in the complex Dirac fermion fields, $\psi_\pm, \overline{\psi}_\pm$. (In the conformal limit, $\psi_\pm$ and $\overline{\psi}_\pm$ are the left and right components and $\pm$ is the $U(1)$ charge.) The boundary equations of motion at $x = 0$ take the form [15]:

$$\begin{aligned}
\psi_+ + \psi_- - \overline{\psi}_- - \overline{\psi}_+ &= 0 \\
i\partial_t(\psi_- - \overline{\psi}_+) - \tilde{\lambda}^2(\overline{\psi}_- - \psi_-) &= 0 \\
i\partial_t(\psi_+ - \overline{\psi}_-) + \tilde{\lambda}^2(\psi_+ - \overline{\psi}_+) &= 0,
\end{aligned} \tag{3.1}$$

where $\tilde{\lambda} \propto \lambda$ and λ is defined in (2.10). Note that these equations are valid regardless of whether there is a bulk mass term or not.

Defining real components of the fermi fields as $\psi_\pm = \psi_1 \pm i\psi_2$, $\overline{\psi}_\pm = -(\overline{\psi}_1 \pm i\overline{\psi}_2)$, one finds that (3.1) is equivalent to

$$\begin{aligned}
\psi_1 &= -\overline{\psi}_1 \\
i\partial_t(\psi_2 - \overline{\psi}_2) + \tilde{\lambda}^2(\psi_2 + \overline{\psi}_2) &= 0.
\end{aligned} \tag{3.2}$$

One can now compare these equations with the boundary Ising equations of motion [11], one sees that at the free fermion point, the boundary sine-Gordon theory is equivalent to two boundary Ising models, one with fixed boundary conditions (equivalently a magnetic field $h = \infty$), the other with a varying magnetic field $h \propto \lambda$. The precise relation between h and λ is however more delicate to obtain, see below.

A less natural, but more precise way to relate boundary sine-Gordon model to two copies of the Ising model is to use a scattering description, based on the boundary S-matrix and form factors. To clarify some of the following discussion, let us for a while add to H^e a term proportional to $\int_{-\infty}^{0} \cos\Phi^e(x)$, generalizing it into a massive boundary sine-Gordon model, which is well known to be integrable [16]. At $g = 1/2$, to which we restrict, the spectra of the theory is composed of solitons and anti-solitons (there is no breathers) and the boundary reflexion matrix is described by the two terms, P and Q given by [16] :

$$R^{\pm}_{\pm} = P, \quad R^{\mp}_{\pm} = Q, \tag{3.3}$$

where the labels $\pm$ refer to the soliton and antisoliton of the sine-Gordon spectrum. For our purposes it is much more convenient to take the following combinations :

$$\begin{aligned} P+Q &= i\tanh\left(i\frac{\pi}{4} - \frac{\theta}{2}\right), \\ P-Q &= i\tanh\left(i\frac{\pi}{4} - \frac{\theta}{2}\right)\frac{\kappa - i\sinh\theta}{\kappa + i\sinh\theta}. \end{aligned} \tag{3.4}$$

The key remark here is that the combination given are just reflection matrices for the fermion of the Ising model in a low temperature phase in a boundary magnetic field h, with $\kappa = 1 - \frac{4\pi h^2}{m}$. ($\kappa = \frac{1}{k}$ in notations of [16]). One combination describes fixed Ising spin boundary conditions (or infinite boundary magnetic field), and the other is at finite Ising boundary magnetic field. This suggests that in the basis of symmetric and antisymmetric combinations of solitons and anti-solitons, the theory decouples to two Ising models with boundary magnetic fields.

This can be confirmed for instance by considering form factors, ie matrix elements of the operators in the quasiparticle basis [17]. The form factors for the operators $e^{\pm i\Phi/2}$ in the sine-Gordon model at $g = 1/2$ are given by :

$$f^{\pm}(\theta_1, ...\theta_{2n})_{\epsilon_1,...,\epsilon_{2n}} = (\pm 2i)^n \sqrt{u_1...u_{2n}} \prod_{i=1}^{2n} u_i^{\mp\epsilon_i/2} \prod_{i<j} (\epsilon_i u_i - \epsilon_j u_j)^{\epsilon_i\epsilon_j} \tag{3.5}$$

with the notation $u_i = e^{\theta_i}$, θ the usual quasi particle rapidity [17]. In order to show the relation to the Ising model, we change the basis from $|A>$, $|S>$ to $\frac{1}{\sqrt{2}}(|S> + |A>)$ and $\frac{1}{\sqrt{2}}(|S> - |A>)$. The explicit calculation is done in the appendix, in the new basis the form factors are :

$$f^{\pm}_{\epsilon'_1,...,\epsilon'_{2n}}(\theta_1, ..., \theta_{2n}) = i^n (-1)^{n-/2} \prod_{i<j} \tanh\left(\frac{\theta_i - \theta_j}{2}\right)^{|\epsilon'_i + \epsilon'_j|/2}, \tag{3.6}$$

where the isotopic indices refer now to the sign of the combination between solitons and anti-solitons and n_- denotes the number of ϵ_i' indices having value $-$. This formula holds only for n_- even, which will be the case for the correlator of interest. Recall now the expression of form factors for spin operator σ in the Ising model [18] :

$$f^\sigma(\theta_1,\ldots,\theta_{2n}) = i^n \prod_{i<j} \tanh\frac{\theta_i-\theta_j}{2}, \tag{3.7}$$

the form factors of μ on an even number of particles vanishing.

In the presence of the boundary, the one point function of of interest is obtained as $< B|\cos\frac{\Phi^e(x)}{2}|0>$, where $|B>$ is the boundary state, $|0>$ the ground state. The boundary state can be expressed in terms of multiparticle states following [16]. In the new basis mixing solitons and anti-solitons, this boundary state actually factorizes :

$$\begin{aligned} |B> &= |B>_+ \otimes |B>_-, \\ |B>_\pm &= \exp\left[\int \frac{d\beta}{2\pi} A^*_\pm(-\beta)A^*_\pm(\beta)K_\pm(\beta)\right] \end{aligned} \tag{3.8}$$

where $A^*_{\epsilon'}$ denote the creation operators in the new basis, and $K_\pm(\theta) \equiv \pm(P\pm Q)(\frac{i\pi}{2}-\theta)$. Here, the additional $\pm$ sign occurs because the K matrix is obtained through the general formula [16] $K^{ab}(\theta) = R^b_{\bar{a}}\left(\frac{i\pi}{2}-\theta\right)$; soliton and antisoliton are conjugate in the sine-Gordon model, while the Ising fermion is self conjugate. Expanding these boundary states, using the form factors (3.6) and (3.7), we see that the correlation function becomes indeed a product of two spin operators (in the low temperature phase of the Ising models, where the one point function of the disorder operator vanishes) :

$$< \cos\frac{\Phi^e}{2} > = < \sigma >_h < \sigma >_\infty . \tag{3.9}$$

Having shown (3.9), we may now let the mass of the sine-Gordon model (the amplitude of the bulk $\cos\Phi^e$ term) go to zero to recover the original problem. The correspondence between the variable λ in the boundary sine-Gordon, and the field h in the boundary Ising model, can be found in that case using results of [19]. Using the conventions of [11] for the boundary Ising model one finds $T_B = 8\pi h^2 = 4\pi\lambda^2$ so $h = \frac{\lambda}{\sqrt{2}}$. This completes our derivation of the relation between the Friedel oscillations and the Ising model. We can now concentrate on finding the correlation of the spin operator in the Ising model with a boundary magnetic field.

At $T = 0$, the one point function of the spin with a boundary has been evaluated in the very interesting paper [11]. There it is shown that (we trade the variable λ of the original action for h), introducing the variable $X = 4\pi h^2 x$, and setting $\bar{\sigma}(X) = \langle\sigma(x)\rangle_h$, the following holds :

$$\left[4\frac{d^2}{dX^2} + \left(\frac{1}{X} - 8\right)\frac{d}{dX} + \left(-\frac{1}{X} + \frac{9}{16}\frac{1}{X^2}\right)\right]\bar{\sigma}(X) = 0. \tag{3.10}$$

The solution of this equation that describes the appropriate physics is :

$$\langle\sigma(x)\rangle_h = 2^{13/8}\sqrt{\pi} h x^{3/8}\Psi(1/2, 1; 8\pi h^2 x), \tag{3.11}$$

where σ is normalized as usual in the bulk. Here, Ψ is a degenerate hypergeometric function. It is simply expressed in terms of Bessel functions as $\Psi(1/2, 1; 2x) = \frac{e^x}{\sqrt{\pi}} K_0(x)$. At short distance one finds that $\sigma \approx -2^{13/8} h\, x^{3/8} \ln x$ and at large distances $\sigma \approx 2^{1/8} x^{-1/8}$. Taking into account the fact that the physical observable in the case of Friedel oscillations is (2.11) one finds :

$$\frac{< \rho(x) - \rho_0 >}{\rho_0} = 4\sqrt{\pi} h\ \cos(2k_F x + \eta_F)\Psi(1/2, 1; 8\pi h^2 x), \quad h = \frac{\lambda}{\sqrt{2}}. \tag{3.12}$$

The asymptotic behaviour is $\frac{<\rho(x)-\rho_0>}{\rho_0} \propto -\ln x\ \cos(2k_F x + \eta_F)$ as $x \to 0$ and $\frac{<\rho(x)-\rho_0>}{\rho_0} \propto x^{-1/2}\ \cos(2k_F x + \eta_F)$ as $x \to \infty$.

Our result agrees with the numerical simulations in [6], except at very small x, where it was found that $\frac{<\rho(x)-\rho_0>}{\rho_0}$ behaves as a power law, with a small negative exponent. This discrepancy is very likely due to fact that, for stability reasons, the true UV region is difficult to access numerically - indeed, at intermediate values of x, our result does behave like a power law [20].

4. Exact Friedel oscillations at $g = 1/2$ and $T \neq 0$.

We can also extend the computation of [11] to finite temperature. The main idea is still that the boundary magnetic field does not destroy the free field structure of the Ising model [21], leading to a determination of the correlator by elementary considerations [11].

For convenience, we first rotate the geometry so now the boundary lies along the x axis. As mentioned earlier (3.1), the equations of motion for the Majorana fermion of the Ising model with boundary magnetic field are (in imaginary time) :

$$[(\partial_z + 4i\pi h^2)\psi(z) - (\partial_{\bar{z}} - 4i\pi h^2)\bar{\psi}(\bar{z})]_{z=\bar{z}} = 0, \tag{4.1}$$

where $z = x + iy$. Having this condition, the idea is to introduce a fermion field $\chi(z) = (\partial_z + 4i\pi h^2)\psi(z)$ and its conjugate $\bar{\chi}(\bar{z})$, and realize that the previous boundary condition states that $\bar{\chi}(\bar{z})$ is the analytic continuation of $\chi(z)$ to the lower half plane. Hence the correlator $< \chi(z)\mu(w,\bar{w}) >$ is analytic in the **full** z plane with two square root branch points at $z = w$ and $z = \bar{w}$.

With a finite temperature, the argument is the same apart from the periodicity in the imaginary time direction, which after having rotated the system is the x direction, is $z \to z + \frac{1}{\beta}$. The boundary condition, a local statement, remains the same.

The next step is to write a global form for the correlator $< \chi(z)\mu(w,\bar{w}) >$, and this is where the effect of temperature will be seen. On the cylinder, one requires the right hand side to have square root branch points at $w + nT, \bar{w} + mT$, n, m integers, and to be periodic. One can therefore write :

$$
\begin{aligned}
< \chi(z)\mu(w,\bar{w}) > \left(\frac{\sin \pi T(z-w)}{\pi T}\right)^{1/2} \left(\frac{\sin \pi T(z-\bar{w})}{\pi T}\right)^{1/2} \\
= \pi T \ \cot[\pi T(z-w)] \ A(w,\bar{w},T) + \pi T \ \cot[\pi T(z-\bar{w})] \ \bar{A}(w,\bar{w},T) + B(w,\bar{w},T).
\end{aligned}
\tag{4.2}
$$

Observe how the right hand side is periodic in $z \to z + T$. In this form, the coefficients, $A, \bar{A}, B$ are unknown that need to be fixed. We can now use the operator product expansions, for example in $(z-w)$ $(\omega = e^{i\pi/4})$:

$$
\begin{aligned}
\chi(z)\mu(w,\bar{w}) = \frac{\bar{\omega}}{\sqrt{2}}(z-w)^{-3/2}\{-\frac{1}{2}\sigma(w,\bar{w}) + (z-w)(2\partial_w + 4i\pi h^2)\sigma(w,\bar{w}) \\
+(z-w)^2(4\partial_w^2 + 16i\pi h^2\partial_w)\sigma(w,\bar{w}) + \ldots\}.
\end{aligned}
\tag{4.3}
$$

and a similar relation for $\bar{\chi}$. The reader might fear that (4.3) which holds in the plane might be changed by β dependent terms when the geometry is compactified. Actually, it is well known [22] that short distance expansions are invariant in conformal mappings. This constraint for instance determines the coefficients $4, 8/3$ in the operator product [11] :

$$
\psi(z)\mu(w,\bar{w}) = \frac{\bar{\omega}}{\sqrt{2}}(z-w)^{-1/2}\left[\sigma(w,\bar{w}) + 4(z-w)\partial_w\sigma(w,\bar{w}) + \frac{8}{3}(z-w)^2\partial_w^2\sigma(w,\bar{w}) + \ldots\right].
\tag{4.4}
$$

Now taking the global form (4.2) and expanding in $(z-w)$ we get relations between the coefficients $A, \bar{A}, B$ and the spin function and its derivatives (similarly expanding in $(\bar{z}-\bar{w})$ and comparing with the OPE of $\bar{\chi}\mu$ we get more relations). Then, by matching (4.2) and

(4.3) and eliminating the unknown coefficients one finds that the one point function of the spin satisfies the equation :

$$\begin{aligned} &\left\{ \frac{d^2}{(dy)^2} + \left(\frac{\pi T}{2} \coth 2\pi T y - 8\pi h^2 \right) \frac{d}{dy} \right. \\ &\left. + \left[\frac{9}{16} (\pi T)^2 (\coth 2\pi T y)^2 - \frac{4\pi^2 h^2 T}{2} \coth 2\pi T y - \frac{\pi^2 T^2}{2} \right] \right\} < \sigma(y) >= 0. \end{aligned} \tag{4.5}$$

In particular, this equation reproduces eq. (30) of [11] in the limit $T \to 0$. To solve this equation, let us set $< \sigma(y) >= [\sinh(2\pi T y)]^{-1/8} f(y)$. Introduce then the new variables $\Lambda = \frac{2h^2}{T}$ and $Y = (1 - \coth 2\pi T y)/2$, and set $f(y) = \bar{f}(Y)$. We then find an hypergeometric equation :

$$\left\{ (Y - Y^2) \frac{d^2}{(dY)^2} + \left(1 + \frac{\Lambda}{2} - 2Y \right) \frac{d}{dY} - \frac{1}{4} \right\} \bar{f}(Y) = 0. \tag{4.6}$$

The physical solution of the problem , going back to the original x variable, is then :

$$< \sigma(x) >_h = \left(\frac{4\pi T}{\sinh 2\pi T x} \right)^{1/8} F\left(\frac{1}{2}, \frac{1}{2}; 1 + 2\frac{h^2}{T}, \frac{1 - \coth 2\pi T x}{2} \right). \tag{4.7}$$

To select this solution, we observe that the one point function at large x can should expand, by conformal invariance, as a sum of exponential terms $\exp(-4\pi T x \Delta)$, with Δ conformal weights of the (central charge) $c = \frac{1}{2}$ conformal field theory. This excludes the other hypergeometric function solution of (4.6). The normalisations are fixed by looking at the case $h \to \infty$. If $h \to \infty$, one has fixed boundary conditions for the spin. The one point function can then be obtained by conformal mapping from the half plane where it is known to be [23] $< \sigma(x) >_{hp} = 2^{1/8} x^{-1/8}$ (with $< \sigma(x)\sigma(0) >= x^{-1/4}$ in the full plane). Setting $z' = \frac{1}{2\pi T} \ln \frac{z-1}{z+1}$ maps the half plane on the semi-infinite cylinder, the line boundary $Rez = 0$ becoming the circle boundary $Rez' = 0, Imz' \in [-1/2T, 1/2T]$. In this transformation one finds for the semi-infinite cylinder $< \sigma(x) >_{sic} = \left(\frac{4\pi T}{\sinh 2\pi T x} \right)^{1/8}$, in agreement with (4.7). Also, as $x \to \infty$, we expect to recover the same result since at large distance the spin sees fixed boundary conditions. This fixes the normalization constant in (4.7).

Finally, we can study the limit $T \to 0$. By using the standard transformation formulas $z \to 1/z$ for the argument of the hypergeometric functions, together with the definition of Ψ in terms of the basic degenerate hypergeometric function Φ (all notations are those of [24]), one finds, as $\gamma, z \to \infty$, γ/z finite

$$F(\alpha, \alpha; \gamma, z) \approx \left(\frac{-\gamma}{z} \right)^{\alpha} \Psi(1/2, 1; -\gamma/z),$$

where we used

$$F(\alpha,\alpha+1-\gamma;1,1/z)\approx\Phi(\alpha,1;-\gamma/z),$$

which can be proved using the series representation of the involved functions. Therefore, as $T\to 0$, one finds $<\sigma(x)>_h\approx 2^{1/8}x^{-1/8}(8\pi h^2x)^{1/2}\Psi(1/2,1;8\pi h^2x)$, in agreement with the foregoing results [11].

The qualitative effect of the temperature can be seen on the small and large x limits of the one point function. One has

$$\begin{aligned}
&<\sigma(x)>_h\approx -2^{9/8}T^{1/2}\frac{\Gamma\left(1+\frac{2h^2}{T}\right)}{\Gamma\left(\frac{1}{2}+\frac{2h^2}{T}\right)},\quad x<<1/T\\
&<\sigma(x)>_h\approx(8\pi)^{1/8}e^{-\pi Tx/4},\quad x>>1/T.
\end{aligned}\tag{4.8}$$

We can now come back to our original problem, the Friedel oscillations. Using (2.11) with the previous solution we find the final expression :

$$\frac{<\rho(x)-\rho_0>}{\rho_0}=\cos(2k_Fx+\eta_F)\left(\frac{4\pi T}{\sinh 2\pi Tx}\right)^{1/2}F\left(\frac{1}{2},\frac{1}{2};1+2\frac{h^2}{T},\frac{1-\coth 2\pi Tx}{2}\right),\tag{4.9}$$

where as before $h=\frac{\lambda}{\sqrt{2}}$. The periodicity $x\to x-i/T$ of the final solution might appear a bit surprising since it is the imaginary time y which is compactified on a circle of radius $1/T$, not x. It can be understood from the fact that one is dealing with a *massless* theory in the bulk. In fact, this periodicity can also be seen by taking the massless limit of the expressions for finite temperature correlators obtained in [25] by using the form factors.

We finally observe that the same technique could be applied to study the screening cloud at the Toulouse point of the anisotropic Kondo problem [26]. This will be described elsewhere.

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Appendix A. Change of basis for the sine-Gordon form factors.

In this appendix we want to show how the one particle form factor for the operator $e^{\pm i\phi/2}$ in the sine-Gordon model at the free fermion point is related to spin form-factor in the Ising model. The form factor is a tensor function in the space of isotopic indices and in the new basis described by $\frac{1}{\sqrt{2}}(|S> + |A>)$ and $\frac{1}{\sqrt{2}}(|S> - |A>)$ (which we will denote by $+$ and $-$ in the following), the form factor is simply given by :

$$f_{\epsilon'_1,\dots,\epsilon'_{2n}}(\theta_1,\dots,\theta_{2n}) = A^{\epsilon_1}_{\epsilon'_1}\cdots A^{\epsilon_{2n}}_{\epsilon'_{2n}} f_{\epsilon_1,\dots,\epsilon_{2n}}(\theta_1,\dots,\theta_{2n}), \tag{A.1}$$

where the primed indices denote the new basis and with :

$$A^{\epsilon}_{\epsilon'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{A.2}$$

describing the change of basis. The forms factor in the $|S>, |A>$ basis was given before :

$$f^{\pm}(\theta_1,\dots\theta_{2n})_{\epsilon_1,\dots,\epsilon_{2n}} = (\pm 2i)^n \sqrt{u_1\dots u_{2n}} \prod_{i=1}^{2n} u_i^{\mp\epsilon_i/2} \prod_{i<j}(\epsilon_i u_i - \epsilon_j u_j)^{\epsilon_i\epsilon_j} \tag{A.3}$$

where the non-zero form factors are those preserving charge and we used the variables $u_i = e^{\theta_i}$.

Let us first look at the form factor with all primed indices chosen to be $+$, it is given by :

$$\begin{aligned} f^{\pm}_{+\dots+}(\theta_1,\dots,\theta_{2n}) =& (\pm i)^n \\ & \sum_{\sum\epsilon_i=0} \sqrt{u_1\dots u_{2n}} \prod_{i=1}^{2n} u_i^{\mp\epsilon_i/2} \prod_{i<j}(\epsilon_i u_i - \epsilon_j u_j)^{\epsilon_i\epsilon_j}. \end{aligned} \tag{A.4}$$

By making a dilatation one sees hat this is an homogeneous function of degree zero. This function has poles for all values of $u_i = -u_j$ and zeroes for $u_i = u_j$. Only the sign has to be worked out, the final form factor is :

$$f^{\pm}_{+\dots+}(\theta_1,\dots,\theta_{2n}) = i^n \prod_{i<j} \tanh\left(\frac{\theta_i - \theta_j}{2}\right). \tag{A.5}$$

Imagine now that some of the primed indices are $-$, then the function is still homegeneous of degree zero but if let say ϕ_i are the rapidities of the particles having indices $-$, it is not difficult to show that there are now no zeroes and poles between terms involving θ_i and ϕ_j. There are still zeroes and poles between each type of particle and we find that the form factor is given by :

$$f^{\pm}_{\epsilon'_1,\dots,\epsilon'_{2n}}(\theta_1,\dots,\theta_{2n}) = i^n(-1)^{n_-/2} \prod_{i<j} \tanh\left(\frac{\theta_i - \theta_j}{2}\right)^{|\epsilon'_i+\epsilon'_j|/2}. \tag{A.6}$$

In this expresion, n_- is the number of indices ϵ'_i of type $-$ ((A.6) holds only when n_- is even, which is the case for our computation).

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