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Asymmetric Non-Abelian Orbifolds and Model Building

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Abstract

The rules for the free fermionic string model construction are extended to include general non-abelian orbifold constructions that go beyond the real fermionic approach. This generalization is also applied to the asymmetric orbifold rules recently introduced. These non-abelian orbifold rules are quite easy to use. Examples are given to illustrate their applications.

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I. INTRODUCTION

This paper presents the generalization of the rules for the free fermionic and asymmetric abelian orbifold constructions to include non-abelian orbifolds. The orbifold construction [1], both abelian and non-abelian, is an algebraic approach to string model building. The rules presented here are particularly useful for asymmetric orbifolds [2], where the geometric approach is less powerful.

Consider a typical compactification of an n-dimensional space R^n. An n-dimensional orbifold Ω is constructed by identifying points of R^n under a space group S of rotations and translations. The subgroup of S formed by pure translations is referred to as the lattice Λ, since identification of points of R^n under Λ defines the torus Γ^n. In this case, Ω can be obtained by identifying points of Γ^n under the point group P, which is the discrete subgroup of only rotations in S. This may be summarized by

$$\Omega = R^n/S = \Gamma^n/P \ . \tag{1.1}$$

Of course, the lattice Λ in a consistent string model is Lorentzian. Also, it is understood that appropriate twisted sectors must be included in the orbifold model. More generally, one may construct the same model Ω by starting with another model $\Omega^{'}$ modded (or orbifolded) by an appropriate group G of translations and rotations. In general, there are more than one choice of $(\Omega^{'},\ G)$ that will reproduce the same Ω. For example, we may start with an appropriate torus $\overline{\Gamma}$ and identify its points under a specific group $\overline{P}$ of translations and rotations. Here $\overline{P}$ is a subgroup of the isometry group of the torus $\overline{\Gamma}$.

$$\Omega = \Omega^{'}/G = \overline{\Gamma}/\overline{P} = \cdots \ . \tag{1.2}$$

A priori, the group G may be either abelian or non-abelian. (Here, a typical $\overline{P}$ is non-abelian.) An orbifold that involves a (non-)abelian G is referred to as a (non-)abelian orbifold. An asymmetric orbifold refers to an orbifold when the group actions on the left movers and on the right movers are different. So it may happen that the same model Ω can be reached via an abelian orbifold or via a non-abelian orbifold, either symmetric or asymmetric, all depending on which $\Omega^{'}$ one starts with.

Let us recall the situation with the free fermionic string model construction [3]. A particular free fermionic string model is dictated by the choice of spin structures and their correlations. Since the spin structures are generically left-right asymmetric, these models are asymmetric orbifolds in the orbifold language. However, the present rules for the free fermionic string construction do not allow general types of non-abelian orbifolds. This is unnecessarily restrictive, since in this construction, we must always start from a unique $\Omega^{'}$. For example, in the 4-dimensional heterotic string case, $\Omega^{'}$ is simply the $SO(44)$ tachyonic string model. Although the set of spin structures is obviously commuting, in the real world-sheet fermionic basis, they do yield models that correspond to a special type of non-abelian orbifolds (that involve non-commuting $\mathbf{Z}_2$ twists) in the usual orbifold language. However, the present rules do not allow the construction of other types of non-abelian orbifolds. A simple example will illustrate this point.

Consider the $\mathbf{Z}_2$ orbifold of a single compactified boson, which can have arbitrary radius. However, if we use the present free fermionic string construction, we must start with the boson at radius 1. The rules allow us to construct either

(1) a $\mathbf{Z}_2$ twisted boson at radius 1 by giving the two real world-sheet fermions different spin structures (one periodic and one anti-periodic), or

(2) a boson with radius at any rational value, by giving the single complex fermion appropriate spin structures.

But the present free fermionic string rules do not permit the construction of a twisted boson at a radius away from 1, since the two sets of spin structures mentioned above are not compatible with each other. In the orbifold language, the above construction involves $G = \overline{P}$ which is non-abelian.

Fortunately, the generalization of the rules needed is not difficult to obtain. In this paper, we shall extend the free fermionic construction rules to include orbifold actions of non-abelian groups. Some of the important steps in reaching our rules were already developed in the literature. Besides the original papers by Dixon, Harvey, Vafa and Witten [1], we refer the readers to the work by Ginsparg [4], and that by Li and Lam [5]. We believe the rules presented in this paper are relatively easy to use, especially for intricate models. With these new rules, we expect the free fermionic construction to be closer to the usual orbifold constructions. This should not be surprising, since the equivalence of world-sheet fermions and bosons is well-known.

Recently the free fermionic string construction rules have been generalized to include world-sheet bosons [6]. These relatively simple rules are useful in the construction of asymmetric orbifolds, since they allow us to consider twists other than $\mathbf{Z}_2$ twists. In this construction, the starting Ω' is quite flexible. Generically, it is chosen to be an $N = 4$ supersymmetric Narain model [7]. The generalization of these rules to include non-abelian orbifolds is essentially identical to that for the free fermionic case. Using these rules, it is quite straightforward to construct general non-abelian orbifolds, including models with non-abelian point groups. The readers who are interested only in the bosonic formulation may go directly to section V, then to section III and VI.

We shall start with the free fermionic string construction. In section II, we briefly review its rules. In section III, we give the generalized rules for the construction of consistent non-abelian orbifold models within the framework of the free fermionic string construction. In section IV, we illustrate these rules by giving some explicit examples. We consider examples in both ten and four dimensions. In section V, we review the rules for asymmetric orbifolds. The generalization of these rules to the non-abelian case is identical to that given for the free fermionic case in section III. To illustrate the use of the rules for non-abelian asymmetric orbifold constructions, we give two examples in section VI. The first example involves a non-abelian point group P, namely the permutation group S_3, plus Wilson lines. The second example is the three-family $SO(10)_3$ model recently constructed [8]. Here, we recast that model as a non-abelian orbifold. Appendix A gives a discussion of the $\mathbf{Z}_2$ orbifold of a single compactified boson.

II. PRELIMINARIES

In this section, we briefly review the rules for constructing free fermionic string models as presented in Ref. [9]. To be concrete, let us focus on heterotic strings compactified to four space-time dimensions. In the light cone gauge which we adopt, the world-sheet degrees of freedom consist of two string coordinates and 10 (22) right (left) moving chiral world-sheet complex fermions. A string model is succinctly defined by the choice of spin structures, *i.e.*, a set of basis vectors $\{V_0, V_1, \cdots, V_n\}$. Each V_i is a 32 (rational) component vector with Lorentzian signature $((-1)^{10}, (+1)^{22})$. Let $\alpha_i V_i$ and $\beta_i V_i$ be two linear combinations that specify the boundary conditions of the 32 complex fermions ψ^ℓ, $\ell = 1, 2, \cdots, 32$ on the torus,

$$\begin{aligned} \psi^\ell(\sigma_1 + 2\pi, \sigma_2) &= -e^{-2\pi i \alpha_i V_i^\ell} \psi^\ell(\sigma_1, \sigma_2) \ , \\ \psi^\ell(\sigma_1, \sigma_2 + 2\pi) &= -e^{-2\pi i \beta_i V_i^\ell} \psi^\ell(\sigma_1, \sigma_2) \ . \end{aligned} \tag{2.1}$$

It is convenient to choose $-\frac{1}{2} \leq V_i^\ell < \frac{1}{2}$, and we define $\overline{\alpha V} = \alpha V - \Delta(\alpha)$ where $\Delta^l(\alpha) \in \mathbf{Z}$, so that $-\frac{1}{2} \leq \overline{\alpha V^\ell} < \frac{1}{2}$. Consistency requires the presence of $V_0 = (-\frac{1}{2}(-\frac{1}{2} - \frac{1}{2} - \frac{1}{2})^3 | (-\frac{1}{2})^{22})$ where the first component labels the boundary condition of the superpartner of the two right-moving transverse space coordinates.

We impose three consistency requirements on the one-loop string partition function Z: (1) World-sheet supersymmetry, *i.e.*, the triplet constraint on each vector V_i

$$\begin{aligned} s_i \equiv V_i^1 &= V_i^{2+3n} + V_i^{3+3n} + V_i^{4+3n} \quad (\text{mod } 1) \quad n = 0, 1, 2 \\ &= 0 \text{ or } -\frac{1}{2} \ . \end{aligned} \tag{2.2}$$

This ensures the proper boundary conditions for the world-sheet supercurrent and hence space-time Lorentz invariance in the covariant gauge.
(2) One-loop modular invariance, and
(3) Physically sensible projection, *i.e.*, every space-time degree of freedom contributes to the partition function with the proper weight, $+1$ for a space-time boson (with $\alpha s = 0$ (mod 1)) and -1 for a space-time fermion (with $\alpha s = -1/2$ (mod 1)).

A consistent string model is defined by the set $\{V_0, V_1, \cdots, V_n\}$ and the structure constants $\{k_{ij} : i, j = 0, 1, \cdots, n\}$ which satisfy

$$k_{ij} + k_{ji} = V_i \cdot V_j \quad (\text{mod } 1) \ , \tag{2.3}$$

$$m_j k_{ij} = 0 \quad (\text{mod } 1) \ , \tag{2.4}$$

$$k_{ii} + k_{i0} + s_i - \frac{1}{2} V_i \cdot V_i = 0 \quad (\text{mod } 1) \ , \tag{2.5}$$

where all the dot products are defined with Lorentzian signature, *i.e.*, left movers minus right movers. Here, m_j is the smallest positive integer such that $m_i V_i^\ell \in \mathbf{Z}$ for all ℓ.

The world-sheet fermionic contribution to the one-loop partition function is then given by

$$Z = \frac{1}{\prod_i m_i} \sum_{\alpha,\beta} \mathrm{Tr}\Big((-1)^{\alpha s} \, q^{H^L_{\overline{\alpha V}}} \, \overline{q}^{H^R_{\overline{\alpha V}}} \, e^{-2\pi i \beta_i (V_i \cdot N_{\overline{\alpha V}} - \sum k_{ij} \alpha_j - s_i - V_i \cdot \overline{\alpha V})} \Big) \tag{2.6}$$

where $\alpha s \in \mathbf{Z}$ ($\mathbf{Z} + \frac{1}{2}$) for space-time bosons (fermions). The partition function is divided into different sectors labeled by the vectors $\overline{\alpha V}$, with hamiltonian $H^L_{\overline{\alpha V}}$ ($H^R_{\overline{\alpha V}}$) for the left (right) movers, and a vector of fermion number operators, $N_{\overline{\alpha V}}$, for the 32 complex fermions. A precise definition of $N_{\overline{\alpha V}}$ is given in Ref. [9]. The states that are kept in Z satisfy

$$V_i \cdot N_{\overline{\alpha V}} = \sum_j k_{ij}\alpha_j + s_i - V_i \cdot \overline{\alpha V} \quad (\text{mod } 1) \ . \tag{2.7}$$

To establish a dictionary between the present construction and the orbifold approach, let us rewrite the partition function as

$$Z = \sum_\alpha Tr\Big((-1)^{\alpha s}\ q^{H^L_{\overline{\alpha V}}}\ \overline{q}^{H^R_{\overline{\alpha V}}}\ \prod_{i=0}^{n} P^i_{\alpha V}\Big) \tag{2.8}$$

where the sum is over all the sectors $\overline{\alpha V}$ and

$$P^i_{\alpha V} = \frac{1}{m_i}\sum_{\beta_i=0}^{m_i-1} e^{-2\pi i\beta_i(V_i\cdot N_{\overline{\alpha V}}\ -\sum k_{ij}\alpha_j - s_i - V_i\cdot\overline{\alpha V})} \tag{2.9}$$

are projection operators. The states that survive in Z are invariant under all the projections $P^i_{\alpha V}$. On the other hand, the partition function of an orbifold by the action of a discrete abelian group G is

$$Z = \sum_{g\in G} \mathrm{Tr}\Big((-1)^{F_g}\ q^{H^L_g}\ \overline{q}^{H^R_g} P_g\Big) = \frac{1}{|G|}\sum_{g,h\in G} Z(g,h) \tag{2.10}$$

$$\text{where} \quad P_g = \frac{1}{|G|}\sum_{h\in G} h \ . \tag{2.11}$$

Therefore, for each model constructed from complex world-sheet fermions, we can associate an abelian orbifold with orbifold group $G = \mathbf{Z}_{m_0} \times \mathbf{Z}_{m_1} \times \cdots \times \mathbf{Z}_{m_n}$ (where $\mathbf{Z}_{m_0}$ is always $\mathbf{Z}_2$). To make the correspondence more explicit, we denote $V_g = \overline{\alpha V}$ and $V_h = \overline{\beta V}$. The $\mathbf{0}$ sector ($\overline{\alpha V} = 0$) is the identity sector while all the others are the twisted (or shifted) sectors.

As is clear from the rules, given the orbifold group G, there are distinct choices of the structure constants (or torsions) $\{k_{ij}\}$, corresponding to different choices of projections in the various sectors. If the boundary conditions are always periodic or antiperiodic, the complex fermion may be split into 2 real fermions, which may then have different boundary conditions. When we allow real fermions with different boundary conditions, the set $\{V_i\}$ must satisfy the cubic constraint

$$4\sum_{l:real} V^l_i\ V^l_j\ V^l_k = 0 \quad (\text{mod } 1) \quad \text{for all } i,j,k \ . \tag{2.12}$$

This implies that we can find a complex fermion basis for any three vectors. Typically, there is no complex fermion basis for the complete set $\{V_i\}$. In order to stay within the framework of complex world-sheet fermions, one has to introduce the notion of non-abelian orbifolds. As we shall see, the free fermionic string model construction covers only a special type of non-abelian orbifolds. In the next section, we will further generalize the above formulation to include other types of non-abelian orbifolds.

III. NON-ABELIAN ORBIFOLDS

We are now ready to discuss the general case of non-abelian orbifolds [1,4,5]. Starting with a consistent string model, whose Hilbert space admits a discrete symmetry G, modding out the theory by the action of G generally results in a new string model. As discussed before, the consistency conditions for abelian G can be expressed in terms of $\{k_{ij}, V_i\}$. In this section, we will examine the constraints when G is non-abelian. Our approach follows that of Ref. [5]. The main improvement is the inclusion of a careful treatment of the compactified 6 dimensions here, and the generalization to the bosonic formulation later.

It turns out that the consistency constraints can be inferred from the associated abelian orbifolds $Z(G_a)$ where G_a are abelian subgroups of G. To see this, let us go back to Eq. (2.10). For abelian G, the operator interpretation of Eq. (2.10) is clear. The Hilbert space decomposes into a set of twisted (and/or shifted) sectors labeled by g, and in each sector, there is a projection onto G invariant states. In taking the trace, we can always find a basis that is simultaneously diagonal in H^L_g and H^R_g and h. The projection in the case of non-abelian G requires a more careful treatment. In each twisted sector g, the sum over $h \in G$ includes only elements that commute with g. The group action, however, mixes states in conjugate sectors. This follows because, if

$$g\psi^\ell g^{-1} = -e^{-2\pi i \overline{\alpha V}^l} \psi^l \ , \tag{3.1}$$

then for any $c \in G$,

$$(cgc^{-1})(c\psi^\ell c^{-1})(cgc^{-1})^{-1} = -e^{-2\pi i \overline{\alpha V}^l} (c\psi c^{-1}) \ . \tag{3.2}$$

As a result,

$$Z(g, h) = Z(cgc^{-1}, chc^{-1}) \ . \tag{3.3}$$

The partition function may be expressed in terms of conjugacy classes C_i:

$$\begin{aligned} Z(G) &= \frac{1}{|G|} \sum_{g\in G} \sum_{gh=hg} \mathrm{Tr}\Big((-1)^{F_g}\ q^{H^L_g}\ \overline{q}^{H^R_g}\ h\Big) \\ &= \sum_i \frac{1}{|N_i|} \sum_{h\in N_i} \mathrm{Tr}\Big((-1)^{F_{C_i}}\ q^{H^L_{C_i}}\ \overline{q}^{H^R_{C_i}}\ h\Big) \end{aligned} \tag{3.4}$$

where the group N_i is the stabilizer (or little) group of the conjugacy class C_i and is defined only up to conjugation. In general, the summation in Eq. (3.4) decomposes into distinct modular orbits, *i.e.*, distinct subsets each of which is modular invariant. However, individual modular orbit does not satisfy physically sensible projection. In the full summation in Eq. (3.4), physically sensible projection is explicitly satisfied since in each sector labeled by C_i, the summation over h is a properly normalized projection onto states invariant under the stabilizer group N_i. To see that modular invariance is satisfied, let us consider G_a (the maximal abelian subgroups of G), $G_{ab} = G_a \cap G_b$ and $G_{abc} = G_a \cap G_b \cap G_c$ etc. It is easy to see that $Z(G)$ may be rewritten as a weighted sum of the abelian orbifolds $Z(G_a), Z(G_{ab}), Z(G_{abc}), \cdots$

$$Z(G) = \frac{1}{|G|}\Big\{\sum_a |G_a|\ Z(G_a) - \sum_{a<b} |G_{ab}|\ Z(G_{ab}) + \sum_{a<b<c} |G_{abc}|\ Z(G_{abc}) - \cdots\Big\} \tag{3.5}$$

where $Z(G_a) = \sum_{g,h\in G_a} Z(g, h)$. Since each of the $Z(G_{a\cdots})$ is a modular invariant partition function, modular invariance of $Z(G)$ is automatic. Notice that the coefficients in Eq.(3.5) always add up to 1. Each partition function is properly normalized; this guarantees that the the final partition function $Z(G)$ contains exactly one graviton.

Nevertheless, the above reasoning does not imply that all non-abelian orbifolds obtained this way are consistent. Notice that each abelian orbifold $Z(G_a)$ is specified by $\{k^a_{ij}, V_i^a\}$ (as well as the others $Z(G_{ab})\cdots$). For a given G_a (*i.e.* the set $\{V_a\}$), there are, in general, inequivalent $Z(G_a)$ models depending on the choices of $\{k^a_{ij}\}$. In order to determine Eq. (3.5) precisely, we must find the appropriate $\{k^a_{ij}\}$ for each $Z(G_a)$. It turns out that the non-abelian group properties of G impose stringent constraints on the choices of k^a_{ij}. A consistent non-abelian orbifold model exists only if all the k_{ij}'s satisfy all the constraints. These constraints are divided into four types:

1) Between different abelian orbifolds $Z(G_a)$ and $Z(G_b)$

Consider the g twisted sector in $Z(G_a)$. Suppose there is $c \in G$ such that cgc^{-1} is outside G_a, say cgc^{-1} belongs to G_b. Since g and cgc^{-1} belong to the same conjugacy class, we have from (3.3), $Z_a(g,h) = Z_b(cgc^{-1}, chc^{-1})$, *i.e.*, $Z(G_a)$ and $Z(G_b)$ are isomorphic. With an appropriate choice of the bases ($V_i^a = V_i^b$), we have the constraints relating $Z(G_a)$ and $Z(G_b)$,

$$k^a_{ij} = k^b_{ij} \ . \tag{3.6}$$

We shall impose this constraint in the outset.

2) Within the same abelian orbifold $Z(G_a)$

Next, suppose that the isomorphism is inner, *i.e.*, both $g, cgc^{-1} \in G_a$. Let $g_i \in G_a$ be the generators of G_a, *i.e.*,

$$V_{g_i} = V_i \quad \text{and} \quad V_{cgc^{-1}} = \overline{\lambda_{il} V_l} \ .$$

Let us consider specific terms in $Z(G_a)$. From Eq. (3.3), $Z_a(g,h) = Z_a(cgc^{-1}, chc^{-1})$, where

$$V_g = \overline{\alpha V} \ , \qquad V_h = \overline{\beta V} \ , \qquad V_{cgc^{-1}} = \overline{\alpha\lambda V} \ , \qquad V_{chc^{-1}} = \overline{\beta\lambda V} \ .$$

Since these two sectors are isomorphic, if we confine ourselves to the $N_{\overline{\alpha V}} = N_{\overline{\alpha\lambda V}} = 0$ states in the trace, we have

$$e^{2\pi i\beta_i(\sum k_{ij}\alpha_j + s_i - V_i\cdot\overline{\alpha V})} = e^{2\pi\beta_i\lambda^i_l(\sum k_{ln}\lambda^j_n\alpha_j + s_i - V_l\cdot\overline{\alpha\lambda V})} \ .$$

Choosing $V_g = V_j$, $V_h = V_i$, we obtain the constraint within $\{k_{ij}, V_i\}_a$,

$$k_{ij} = \lambda_{il}k_{ln}\lambda_{nj} - \Delta_i \cdot \lambda_{jn}V_n \pmod 1 \tag{3.7}$$

where $\Delta_i = \lambda_{il}V_l - \overline{\lambda_{il}V_l}$. Notice that $s_i = \lambda_{il}s_l \pmod 1$ since conjugation does not mix space-time bosons and fermions.

3) Between $Z(G_a)$ and $Z(G_{ab})$

Then, we need to find the relations between $\{k_{ij}, V_i\}_a$ and $\{k_{ln}, V_l\}_{ab}$ where G_{ab} is a subgroup of G_a. Clearly, we can write, for each generator $g \in G_{ab}$,

$$V_l^{ab} = \overline{\gamma_{li} V_i^a}$$

and

$$\begin{aligned} k_{ln}^{ab} + k_{nl}^{ab} = V_l^{ab} \cdot V_n^{ab} &= \overline{\gamma_{li} V_i^a} \cdot \overline{\gamma_{nj} V_j^a} \\ &= (\gamma_l V^a - \Delta_l)(\gamma_n V^a - \Delta_n) \\ &= \gamma_l^i \gamma_n^j (k_{ij}^a + k_{ji}^a) - \Delta_l \cdot \gamma_n V^a - \Delta_n \cdot \gamma_l V^a \ . \end{aligned}$$

Consistency with (3.7) gives

$$k_{ln}^{ab} = \gamma_l^i k_{ij}^a \gamma_n^j - \Delta_l \cdot \gamma_n V^a \ . \tag{3.8}$$

4) Between three different abelian orbifolds $Z(G_a)$, $Z(G_b)$ and $Z(G_c)$

Consider an element g that is in G_a, G_b and G_c. Suppose $h_1 \in G_a$, $h_2 \in G_b$. To obtain non-trivial constraints, $h_1 h_2 \notin G_a, G_b$, *i.e.*, $[h_1 h_2, h_1] \neq 0$, $[h_1 h_2, h_2] \neq 0$. Let $h_1 h_2 \in G_c$ and $V_g = \overline{\alpha^a V^a} = \overline{\alpha^b V^b} = \overline{\alpha^c V^c}$. Although the same g twisted sector is in $Z(G_a)$, $Z(G_b)$ and $Z(G_c)$, it is expressed in different bases. Fortunately, the operators that relate these different bases commute with $q^{H_g^L} \overline{q}^{H_g^R}$. If we restrict ourselves to the $N_{\overline{\alpha^a V^a}} = N_{\overline{\alpha^b V^b}} = N_{\overline{\alpha^c V^c}} = 0$ states, the projection operators h_1, h_2 and $h_1 h_2$ are purely phases, and their group relation yields,

$$e^{2\pi i \left[\beta_i^a (V_i^a \cdot \overline{\alpha^a V^a} - \sum k_{ij}^a \alpha_j^b - s_i^a) + \beta_i^b (V_i^b \cdot \overline{\alpha^b V^b} - \sum k_{ij}^b \alpha_j^b - s_i^b)\right]} = e^{2\pi i \beta_i^c (V_i^c \cdot \overline{\alpha^c V^c} - \sum k_{ij}^c \alpha_j - s_i^c)} \ .$$

Since $\beta^a s^a + \beta^b s^b = \beta^c s^c \pmod 1$, we have the relation between $\{k_{ij}, V_i\}_a$, $\{k_{ij}, V_i\}_b$ and $\{k_{ij}, V_i\}_c$,

$$\beta_i^a (V_i^a \cdot \overline{\alpha^a V^a} - k_{ij}^a \alpha_j^a) + \beta_i^b (V_i^b \cdot \overline{\alpha^b V^b} - k_{ij}^b \alpha_j^b) = \beta_i^c (V_i^c \cdot \overline{\alpha^c V^c} - k_{ij}^c \alpha_j) \pmod 1 \tag{3.9}$$

where $V_g = \overline{\alpha^a V^a} = \overline{\alpha^b V^b} = \overline{\alpha^c V^c}$ and $V_{h_1} = \overline{\beta^a V^a}$, $V_{h_2} = \overline{\beta^b V^b}$, $V_{h_1 h_2} = \overline{\beta^c V^c}$. In the examples given in the next section, we shall see that Eq. (3.9) is a stringent constraint for consistent non-abelian orbifolds, especially when the non-abelian orbifold is also asymmetric.

Let us summarize the steps required to construct consistent non-abelian orbifolds:
(1) Rewrite the partition function Z as a sum of abelian orbifold partition functions (Eq. (3.5)).
(2) Check that the free fermionic construction rules are satisfied in each abelian orbifold $\{k_{ij}, V_i\}_A$.
(3) Impose extra constraints (3.6),(3.7),(3.8) and (3.9) between various $\{k_{ij}, V_i\}_A$. With an appropriate choice of the basis vectors, Eq. (3.6) is satisfied automatically.

IV. EXAMPLES

In this section, we demonstrate how to apply our rules by constructing some explicit examples. Before we focus on four-dimensional models, let us begin our discussion with ten-dimensional examples that exhibit some of the basic features of our construction.

A. Ten-dimensional Models

There are only nine known consistent heterotic string models in ten dimensions [10]. Among them, eight have rank 16 gauge group and can be realized in terms of complex world-sheet fermions. The ninth has gauge group a single E_8 realized at level 2, and was first constructed using *real* fermion basis. In the bosonic formulation, this higher level model can be obtained by modding out the $E_8 \times E_8$ model by the combined action of a 2π spatial rotation and a $\mathbf{Z}_2$ permutation of the two E_8's [11]. We now illustrate how the same model can be constructed by a D_4 orbifold using the rules just given. We are using this well-known example to clarify our notation.

The group D_4 is generated by two non-commuting elements r and θ with the defining relations,

$$r^2 = \theta^4 = 1, \qquad r\theta^3 = \theta r \ . \tag{4.1}$$

It has 8 elements and is divided into 5 conjugacy classes: $C_1 = \{1\}$, $C_{\theta^2} = \{\theta^2\}$, $C_r = \{r, r\theta^2\}$, $C_{r\theta} = \{r\theta, r\theta^3\}$, and $C_\theta = \{\theta, \theta^3\}$. The stabilizer groups are given by $N_1 = N_{\theta^2} = D_4$, $N_r = \{1, \theta^2, r, r\theta^2\}$, $N_{r\theta} = \{1, \theta^2, r\theta, r\theta^3\}$, and $N_\theta = \{1, \theta, \theta^2, \theta^3, \theta^4\}$. It is straightforward to express the full partition function as a sum of abelian orbifolds,

$$Z = \frac{1}{2}\ Z_\theta + \frac{1}{2}\ Z_r + \frac{1}{2}\ Z_{r\theta} - \frac{1}{2}\ Z_{\theta^2} \tag{4.2}$$

where Z_θ is a $\mathbf{Z}_4$ ($\cong N_\theta$) orbifold, Z_r and $Z_{r\theta}$ are $\mathbf{Z}_2 \times \mathbf{Z}_2$ ($\cong N_r$ and $N_{r\theta}$ respectively) orbifolds, Z_{θ^2} is a $\mathbf{Z}_2$ ($\cong \{1, \theta^2\}$) orbifold.

We have not specified the representations of the internal fermions under the action of D_4. There are 5 irreducible representations of D_4: 4 one-dimensional representations defined by $\theta = \pm 1$, $r = \pm 1$ and only one two-dimensional representation:

$$\theta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \qquad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

where we have chosen the basis such that θ is diagonal.

Let us start with the simplest model, *i.e.*, the model generated by a single basis vector $V_0 = ((-\frac{1}{2})^4 \mid (-\frac{1}{2})^{16})$. This model is non-supersymmetric with 32 tachyons and gauge group $SO(32)$. We may define the D_4 action by embedding eight copies of the two-dimensional representation on the left movers while leaving the right movers untouched. In other words, the left movers transform under θ and r as follows:

$$\begin{aligned} \theta: \qquad & \psi^{2\ell-1} \to \ \ i\ \psi^{2\ell-1} \\ & \psi^{2\ell} \to -i\ \psi^{2\ell} \\ r: \qquad & \psi^{2\ell-1} \to \psi^{2\ell} \\ & \psi^{2\ell} \to \psi^{2\ell-1} \end{aligned} \tag{4.4}$$

for $\ell = 1, \cdots, 8$. The generating vectors when expressed in their own diagonal bases are:

$$
\begin{aligned}
V_\theta &= \left(0^4 \mid (\frac{1}{4}\; -\frac{1}{4})^8\right) , \\
V_r &= \left(0^4 \mid (-\frac{1}{2}\; 0)^8\right) , \\
V_{r\theta} &= \left(0^4 \mid (-\frac{1}{2}\; 0)^8\right) , \\
V_{\theta^2} &= \left(0^4 \mid (-\frac{1}{2})^{16}\right) .
\end{aligned}
\tag{4.5}
$$

We now examine the abelian models in some detail. To begin, let us consider the Z_r model generated by the abelian set $\{V_0, V_{\theta^2}, V_r\}$. The matrix of dot products $V_i \cdot V_j$ and the structure constants k_{ij} are given by:

$$
V_i \cdot V_j = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix} , \qquad k_{ij} = \begin{pmatrix} k_{00} & k_{\theta^2 0} & k_{r0} \\ k_{\theta^2 0} & k_{\theta^2 0} & k_{r\theta^2} \\ k_{r0} & k_{r\theta^2} & k_{r0} \end{pmatrix} , \tag{4.6}
$$

where $k_{ij} = 0, \frac{1}{2} \pmod 1$. There are six potentially massless sectors: $\mathbf{0}$, $V_0 + V_{\theta^2}$, V_r, $V_r + V_{\theta^2}$, $V_0 + V_r$ and $V_0 + V_r + V_{\theta^2}$. The spectrum generating formula in the $\mathbf{0}$ sector (with vacuum energy $[-\frac{1}{2}, -1]$) reads,

$$
\begin{aligned}
\frac{1}{2} \sum_{i=1}^{4} N_i + \frac{1}{2} \sum_{i=1}^{16} \tilde{N}_i &= \frac{1}{2} \pmod 1 , \\
\frac{1}{2} \sum_{i=1}^{16} \tilde{N}_i &= 0 \pmod 1 , \\
\frac{1}{2} \sum_{i=1}^{8} \tilde{N}_{2i-1} &= 0 \pmod 1 .
\end{aligned}
\tag{4.7}
$$

The tachyons are projected out and the lowest energy states include the graviton and gauge bosons in the adjoint of $SO(16) \times SO(16)$. On the other hand, the $V_0 + V_{\theta^2}$ sector (with vacuum energy $[0, -1]$) provides the accompanying superpartners if and only if $k_{r0} = k_{r\theta^2}$. This follows from the constraints,

$$
\begin{aligned}
\frac{1}{2} \sum_{i=1}^{4} N_i + \frac{1}{2} \sum_{i=1}^{16} \tilde{N}_i &= k_{00} + k_{\theta^2 0} + \tfrac{1}{2} \pmod 1 , \\
\frac{1}{2} \sum_{i=1}^{16} \tilde{N}_i &= 0 \pmod 1 , \\
\frac{1}{2} \sum_{i=1}^{8} \tilde{N}_{2i-1} &= k_{r0} + k_{r\theta^2} \pmod 1 .
\end{aligned}
\tag{4.8}
$$

The gravitino ($\tilde{N}_i = 0$ for all i) survives the projection if $k_{r0} = k_{r\theta^2}$. In addition, we have space-time fermions in the adjoint representation of $SO(16) \times SO(16)$. If $k_{r0} \neq k_{r\theta^2}$, the gravitino is projected out and the space-time fermions form the $(\mathbf{16}, \mathbf{16})$ representation of $SO(16) \times SO(16)$.

There may be additional gauge bosons coming from the sectors V_r and $V_r + V_{\theta^2}$. In the V_r sector (with vacuum energy $[-\frac{1}{2}, 0]$), the spectrum generating formula gives,

$$
\begin{aligned}
\frac{1}{2}\sum_{i=1}^{4} N_i + \frac{1}{2}\sum_{i=1}^{16} \tilde{N}_i &= k_{r0} + \tfrac{1}{2} \quad (\text{mod } 1) \ , \\
\frac{1}{2}\sum_{i=1}^{16} \tilde{N}_i &= \quad k_{r\theta^2} \qquad (\text{mod } 1) \ , \\
\frac{1}{2}\sum_{i=1}^{8} \tilde{N}_{2i-1} &= \quad k_{r0} \qquad (\text{mod } 1) \ .
\end{aligned} \tag{4.9}
$$

Provided that $k_{r0} = k_{r\theta^2}$, there are extra gauge bosons in the **128** spinor representation of the first $SO(16)$. The $V_r + V_{\theta^2}$ sector differs from the V_r sector by an interchange of left moving NS and R fermions. Therefore, it provides gauge bosons in the spinor representation of the second $SO(16)$. Together with the gauge bosons from the **0** sector, they furnish the adjoint representation of $E_8 \times E_8$.

Finally, the sectors $V_0 + V_r$ and $V_0 + V_r + V_{\theta^2}$ contribute space-time fermions in the $(\mathbf{128}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{128})$ representation of $SO(16) \times SO(16)$ for all choices of k_{ij}.

To summarize, modding out the non-supersymmetric $SO(32)$ model by $\mathbf{Z}_2 \times \mathbf{Z}_2$ may result in two completely different models: a supersymmetric $E_8 \times E_8$ model if $k_{r0} = k_{r\theta^2}$ and a non-supersymmetric $SO(16) \times SO(16)$ model otherwise.

The above analysis also applies to the $Z_{r\theta}$ model as it has the same generating vectors (although in different bases). The Z_{θ^2} model is generated by $\{V_0, V_{\theta^2}\}$ and so there are only two constraints in each sector. The **0** sector provides the graviton and gauge bosons in the adjoint of $SO(32)$ while their superpartners reside in the $V_0 + V_1$ sector.

What remains is Z_θ generated by $\{V_0, V_\theta\}$. The matrix of dot product $V_i \cdot V_j$ and the structure constants k_{ij} are:

$$
V_i \cdot V_j = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad k_{ij} = \begin{pmatrix} k_{00} & k_{\theta 0} \\ k_{\theta 0} & k_{\theta 0} + \frac{1}{2} \end{pmatrix} \ , \tag{4.10}
$$

where $k_{ij} = 0, \frac{1}{2}$ (mod 1). There are 6 potentially massless sectors: **0**, $V_0 + 2V_\theta$, V_θ, $3V_\theta$, $V_0 + V_\theta$ and $V_0 + 3V_\theta$. The **0** sector contributes 256 gauge bosons in the adjoint representation of $U(16)$. The sectors $V_0 + V_\theta$ and $V_0 + 3V_\theta$ together provide 240 space-time fermions. If $k_{\theta 0} = 0$, the $V_0 + 2V_\theta$ sector contributes the superpartners of the **0** sector. In addition, there are extra gauge bosons coming from the sectors V_θ and $3V_\theta$ (both with vacuum energy $[-\frac{1}{2}, -\frac{1}{2}]$) that complete the adjoint representation of $SO(32)$. As a result, we simply recover the Z_{θ^2} model. However, if $k_{\theta 0} = \frac{1}{2}$, the gravitino is projected out and there are 240 space-time fermions from the $V_0 + 2V_\theta$ sector. The massless states in the sectors V_θ and $3V_\theta$ are projected out but the 2 tachyons survive the projection. The model has gauge group $U(16)$.

What is the resultant D_4 orbifold? *A priori*, there are 6 possibilities corresponding to different choices of k_{ij} in Z_r, $Z_{r\theta}$ and Z_θ (after taking into account the fact that $Z_r \cong Z_{r\theta}$). However, not all of them are consistent. The non-abelian rules further restrict the number of possibilities to three. First consider the constraints between conjugate sectors V_r and $V_{r\theta^2} = \overline{V_r + V_{r\theta^2}}$. From definition, $\lambda_{rr} = \lambda_{r\theta^2} = 1$ and $\Delta_r = V_r + V_{\theta^2} - V_{r\theta^2} = (0^4 \mid (-1)^8 \ 0^8)$ From Eq. (3.7),

$$
\begin{aligned}
k_{rr} &= k_{rr} + k_{r\theta^2} + k_{\theta^2 r} + k_{\theta^2\theta^2} - \Delta_r \cdot (V_r + V_{\theta^2}) \ , \\
k_{rj} &= k_{rj} + k_{\theta^2 j} - \Delta_r \cdot V_j \ , \\
k_{jr} &= k_{jr} + k_{j\theta^2} \ ,
\end{aligned} \tag{4.11}
$$

where $j = 0, \theta^2$. It follows that the above equations give rise to a single constraint,

$$k_{\theta^2 0} = 0 \ . \tag{4.12}$$

The other conjugate sectors are $\{V_{r\theta}, V_{r\theta^3}\}$ and $\{V_\theta, V_{\theta^3}\}$. The former gives the same constraint as above. In the latter case, notice that $V_{\theta^3} = \overline{3V_\theta}$. Therefore, $\lambda_{\theta\theta} = 3$ and $\Delta_\theta = (0^4 \mid 1^8(-1)^8)$. The constraints become,

$$\begin{aligned} k_{\theta\theta} &= 9k_{\theta\theta} - \Delta_\theta \cdot 3V_\theta = 9k_{\theta\theta} \ , \\ k_{\theta 0} &= 3k_{\theta 0} - \Delta_\theta \cdot V_0 = 3k_{\theta 0} \ , \\ k_{0\theta} &= 3k_{0\theta} \ , \end{aligned} \tag{4.13}$$

which are automatically satisfied.

We now turn to the constraints between three different abelian orbifolds. Let us take $V_{h_1} = V_r$, $V_{h_2} = V_\theta$ and $V_{h_1 h_2} = V_{r\theta}$. The common sector V_g can be either V_0 or V_{θ^2}. From Eq. (3.9),

$$\begin{aligned} V_r \cdot V_0 - k_{r0} + V_\theta \cdot V_0 - k_{\theta 0} &= V_{r\theta} \cdot V_0 - k_{r\theta,0} \ , \\ V_r \cdot V_{\theta^2} - k_{r\theta^2} + V_\theta \cdot \overline{2V_\theta} - 2k_{\theta\theta} &= V_{r\theta} \cdot V_{\theta^2} - k_{r\theta,\theta^2} \ , \end{aligned} \tag{4.14}$$

which give

$$\begin{aligned} k_{r0} + k_{\theta 0} &= k_{r\theta,0} \ , \\ k_{r\theta^2} &= k_{r\theta,\theta^2} \ . \end{aligned} \tag{4.15}$$

As a consequence, the parameters k_{ij} in the abelian models are not independently chosen. If $k_{\theta 0} = 0$ (*i.e.* Z_θ is the $SO(32)$ model), Eq. (4.15) implies that Z_r and $Z_{r\theta}$ both have gauge group $E_8 \times E_8$ or $SO(16) \times SO(16)$. It is obvious from Eq. (4.2) that the resultant D_4 orbifold is either the $E_8 \times E_8$ or the $SO(16) \times SO(16)$ model. On the other hand, if $k_{\theta 0} = \frac{1}{2}$ (*i.e.* Z_θ is the $U(16)$ model), Z_r and $Z_{r\theta}$ have gauge group $E_8 \times E_8$ and $SO(16) \times SO(16)$ respectively. From Eq. (4.2),

$$Z = \frac{1}{2} \, Z_{E_8 \times E_8} + \frac{1}{2} \, Z_{SO(16) \times SO(16)} + \frac{1}{2} \, Z_{U(16)} - \frac{1}{2} \, Z_{SO(32)} \ . \tag{4.16}$$

By simply counting the number of gauge bosons (248), space-time fermions (496) and tachyon (1) in the partition function Z, one can easily identify the resultant D_4 orbifold as the only rank 8 model in ten dimensions with gauge group E_8 realized at level 2. This can be confirmed by a careful construction of the physical states.

We have seen how the rules provide non-trivial constraints between the abelian orbifolds. There are cases in which the constraints cannot be satisfied and as a result, no consistent model can be obtained. Let us take the quaternion group Q as an example [5]. It is generated by two order 4 elements p and q ($p \neq q$) with the defining relations,

$$p^4 = q^4 = 1, \qquad p^2 = q^2, \qquad qp = p^3 q \ . \tag{4.17}$$

There are 5 conjugacy classes: $C_1 = \{1\}$, $C_{p^2} = \{p^2\}$, $C_p = \{p, p^3\}$, $C_q = \{q, q^3\}$ and $C_{pq} = \{pq, (pq)^3\}$. The stabilizer groups are $N_1 = Q$, $N_{p^2} = Q$, $N_p = \{1, p, p^2, p^3\}$, $N_q = \{1, q, q^2, q^3\}$ and $N_{pq} = \{1, pq, (pq)^2, (pq)^3\}$. Therefore, the full partition function is

$$Z = \frac{1}{2}\, Z_p + \frac{1}{2}\, Z_q + \frac{1}{2}\, Z_{pq} - \frac{1}{2}\, Z_{p^2}\ . \tag{4.18}$$

The one-dimensional representations are the same as that of D_4 while the two-dimensional representation is

$$p = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\ , \qquad q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{4.19}$$

where we have chosen the basis such that p is diagonal. Both q and pq take the same diagonal form as that of p in their diagonal bases. Therefore, Z_p, Z_q and Z_{pq} are isomorphic and in what follows, we will only consider Z_p.

Again, we start with the model generated by V_0. We may embed four copies of the two-dimensional representation on the first eight left movers, leaving all the other internal fermions untouched. The generating vector V_p is thus

$$V_p = \left(0^4 \mid (\frac{1}{4}\ -\frac{1}{4})^4\ 0^8\right)\ . \tag{4.20}$$

The matrix of dot products $V_i \cdot V_j$ and the structure constants k_{ij} of Z_p are given by

$$V_i \cdot V_j = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\ , \qquad k_{ij} = \begin{pmatrix} k_{00} & k_{p0} \\ k_{p0} & k_{p0} + \frac{1}{4} \end{pmatrix}\ , \tag{4.21}$$

where $k_{00}, k_{p0} = 0, \frac{1}{2} \pmod 1$. The group multiplication (Eq. (3.9)) is not satisfied. To see this, take $g = p^2$, $h_1 = p$, $h_2 = q$ and $h_1 h_2 = pq$. On one hand,

$$V_p \cdot 2V_p - 2k_{pp} + V_q \cdot 2V_q - 2k_{qq} = 0\ , \tag{4.22}$$

but on the other hand,

$$V_{pq} \cdot 2V_{pq} - 2k_{pq,pq} = \frac{1}{2}\ . \tag{4.23}$$

However, it is straightforward to see that if we embed eight copies of the two-dimensional representation on the left movers, *i.e.*,

$$V_p = \left(0^4 \mid (\frac{1}{4}\ -\frac{1}{4})^8\right)\ , \tag{4.24}$$

there are two consistent models via this Q orbifold: the supersymmetric $SO(32)$ model and the $U(16)$ model.

In a similar fashion, one can show that the constraint Eq. (3.7) is not satisfied when the orbifold group is the extra-special p-group (for any odd $p \geq 5$) [5,12].

B. Four-dimensional Models

We have taken our exercise on ten-dimensional examples far enough, let us proceed to the more interesting four-dimensional models. A new ingredient in four dimensions is the world-sheet supersymmetry which is encoded in the triplet constraint (Eq. (2.2)).

Let us start with a free fermionic string model defined by the set of basis vectors:

$$
\begin{aligned}
V_0 &= \Big(-\frac{1}{2}\ (-\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2})^3 \mid (-\frac{1}{2})^{22}\Big) \ , \\
V_1 &= \Big(-\frac{1}{2}\ (-\frac{1}{2}\ 0\ 0)^3 \mid 0^{22}\Big) \ , \\
V_2 &= \Big(-\frac{1}{2}\ (-\frac{1}{2}\ 0\ 0)(0\ -\frac{1}{2}\ 0)^2 \mid (-\frac{1}{2})^{10}\ 0^{10}\ (-\frac{1}{2})^2\Big) \ , \\
V_3 &= \Big(0^7\ (0\ -\frac{1}{2}\ -\frac{1}{2}) \mid 0^{10}(-\frac{1}{2})^2\ 0^5\ (-\frac{1}{2})^4\ 0\Big) \ .
\end{aligned}
\tag{4.25}
$$

The matrix $V_i \cdot V_j$ and the structure constants k_{ij} are given by,

$$
V_i \cdot V_j = \begin{pmatrix} 3 & -1 & 2 & 1 \\ -1 & -1 & -\frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \ , \qquad k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & k_{21}+\frac{1}{2} & k_{31} \\ 0 & k_{21} & \frac{1}{2} & k_{32} \\ 0 & k_{31} & k_{32} & \frac{1}{2} \end{pmatrix} \ . \tag{4.26}
$$

In general, the particle content depends on the choices of k_{ij}. In this model, however, $\{k_{0j} : j = 0, \cdots, 3\}$ do not affect the massless spectrum and we may simply set them to zero. V_1 introduces gravitinos and hence space-time supersymmetry. V_2 serves the dual purpose of cutting the supersymmetry from $N = 4$ to $N = 2$, and the gauge group from $SO(44)$ to $[SO(20)]^2 \times SO(4)$. V_3 breaks the gauge group further to $SO(20) \times SO(4) \times SO(10) \times SO(6) \times [U(1)]^2$. If $k_{31} = \frac{1}{2}$, V_3 removes all the gravitinos , otherwise $k_{31} = 0$ and the model remains $N = 2$ supersymmetric.

Taking the N=2 supersymmetric model (*i.e.*, choosing $k_{31} = 0$) as our starting point, we can reduce the supersymmetry to $N = 1$ as well as obtain a higher level gauge group via non-abelian orbifold. Again, take D_4 as an example. We define the D_4 action by choosing the following generating vectors,

$$
\begin{aligned}
V_\theta &= \Big(-\frac{1}{2}\ (0\ -\frac{1}{2}\ 0)(0\ -\frac{1}{2}\ 0)(-\frac{1}{2}\ 0\ 0) \mid (\frac{1}{4}\ -\frac{1}{4})^8(-\frac{1}{2})^2\ 0^2(-\frac{1}{2})^2\Big) \ , \\
V_r &= \Big(-\frac{1}{2}\ (0\ -\frac{1}{2}\ 0)(-\frac{1}{2}\ 0\ 0)(0\ 0\ -\frac{1}{2}) \mid (-\frac{1}{2}\ 0)^8\ 0^6\Big) \ , \\
V_{r\theta} &= \Big(\ 0^4\ (-\frac{1}{2}\ -\frac{1}{2}\ 0)(-\frac{1}{2}\ 0\ -\frac{1}{2}) \mid (-\frac{1}{2}\ 0)^8(-\frac{1}{2})^2\ 0^2(-\frac{1}{2})^2\Big) \ , \\
V_{\theta^2} &= \Big(\ 0^{10} \mid (-\frac{1}{2})^{16}\ 0^6\Big) \ .
\end{aligned}
\tag{4.27}
$$

In other words, we assign two-dimensional representations only on the first 16 left movers and one-dimensional representations on the others.

Before we work out the complete spectrum, let us gain some insights from the individual abelian models. V_{θ^2} does not break supersymmetry and so Z_{θ^2} remains $N = 2$. On the other hand, both V_r and V_θ break the supersymmetry to $N = 1$. Using simple counting argument, physically sensible projection for the gravitinos can only be satisfied if $Z_{r\theta}$ has $N = 0$ or 2. As we shall see, $V_{r\theta}$ either removes all the gravitinos or leaves them untouched, depending on whether $k_{r\theta 1} = k_{21}$.

Chiral fermions can appear only in $N = 1$ models. *A priori*, both Z_θ and Z_r may contain chiral fermions. However, in order for Z_θ to be part of the D_4 orbifold, Z_θ is necessarily non-chiral. To see this, let us assume that Z_θ is chiral. A particle from the θ sector together with an antiparticle from the θ^3 sector form a single chiral multiplet. On the other hand, θ and θ^3 belong to the same conjugacy class and hence half of the states from these two sectors are projected out. Physically sensible projection is not satisfied because the chiral multiplet cannot be divided into half.

Let us examine the abelian models in more detail. First consider Z_{θ^2} generated by the vectors $\{V_0, \cdots, V_3, V_{\theta^2}\}$. The matrix of dot products $V_i \cdot V_j$ and the structure constants k_{ij} are given by:

$$V_i \cdot V_j = \begin{pmatrix} \cdot & & & & 4 \\ & \cdot & & & 0 \\ & & \cdot & & \frac{5}{2} \\ & & & \cdot & \frac{1}{2} \\ 4 & 0 & \frac{5}{2} & \frac{1}{2} & 4 \end{pmatrix} , \qquad k_{ij} = \begin{pmatrix} \cdot & & & & k_{\theta^2 0} \\ & \cdot & & & k_{\theta^2 1} \\ & & \cdot & & k_{\theta^2 2} + \frac{1}{2} \\ & & & \cdot & k_{\theta^2 3} + \frac{1}{2} \\ k_{\theta^2 0} & k_{\theta^2 1} & k_{\theta^2 2} & k_{\theta^2 3} & k_{\theta^2 0} \end{pmatrix} . \tag{4.28}$$

The gauge bosons only come from the $\mathbf{0}$ sector. With the addition of V_{θ^2}, the gauge group is broken to $SO(20) \times SO(4) \times SO(8) \times U(1) \times SO(6) \times [U(1)]^2$. The 2 gravitinos in the V_1 sector are not projected out and the model remains $N = 2$ supersymmetric.

Next, Z_r is generated by $\{V_0, \cdots, V_3, V_{\theta^2}, V_r\}$ and therefore

$$V_i \cdot V_j = \begin{pmatrix} \cdot & & & & & 1 \\ & \cdot & & & & -\frac{1}{2} \\ & & \cdot & & & 1 \\ & & & \cdot & & 0 \\ & & & & \cdot & 2 \\ 1 & -\frac{1}{2} & 1 & 0 & 2 & 1 \end{pmatrix} , \qquad k_{ij} = \begin{pmatrix} \cdot & & & & & k_{r0} \\ & \cdot & & & & k_{r1+\frac{1}{2}} \\ & & \cdot & & & k_{r2} \\ & & & \cdot & & k_{r3} \\ & & & & \cdot & k_{r\theta^2} \\ k_{r0} & k_{r1} & k_{r2} & k_{r3} & k_{r\theta^2} & k_{r0} \end{pmatrix} . \tag{4.29}$$

The gauge group is broken further by V_r to $[SO(10)]^2 \times [U(1)]^2 \times [SO(4)]^4 \times U(1) \times SO(6) \times [U(1)]^2$. The supersymmetry is reduced to $N = 1$ due to the extra constraint:

$$V_r \cdot N_{V_1} = \frac{1}{2}\,(N_1 + N_5) = k_{r1} + \frac{1}{2} - V_r \cdot V_1 = k_{r1} \ . \tag{4.30}$$

The V_r sector has vacuum energy $[0, 0]$. The spectrum generating formula reads,

$$\begin{aligned} \frac{1}{2}\,N_1 + \frac{1}{2}\,(\tilde{N}_1 + \tilde{N}_3 + \cdots + \tilde{N}_9) = &\quad k_{2r} - \tfrac{1}{2} && \pmod 1 , \\ \frac{1}{2}\,(\tilde{N}_1 + \tilde{N}_3 + \cdots \tilde{N}_{15}) = &\quad k_{\theta^2 r} && \pmod 1 , \\ \frac{1}{2}\,(N_1 + N_5) = &\quad k_{1r} && \pmod 1 , \\ \frac{1}{2}\,(N_{10} + \tilde{N}_{11}) = &\quad k_{3r} && \pmod 1 , \\ \frac{1}{2}\,(N_1 + N_3 + N_5 + N_{10}) = &\ k_{\theta^2 r} + k_{0r} - \tfrac{1}{2} && \pmod 1 . \end{aligned} \tag{4.31}$$

It contributes chiral fermions in the representation $(\mathbf{16}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{16}, \mathbf{1}, \overline{\mathbf{2}}, \mathbf{1})$ or $(\overline{\mathbf{16}}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\overline{\mathbf{16}}, \mathbf{1}, \overline{\mathbf{2}}, \mathbf{1})$ of $[SO(10)]^2 \times [SO(4)]^2$ (the first two $SO(4)$) depending on the space-time helicity. The $V_r + V_{\theta^2}$ sector differs from the V_r sector by an interchange of the first 16 left moving NS and R fermions and so contributes chiral fermions in the representation $(\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{16},\mathbf{1},\overline{\mathbf{2}})$ or $(\mathbf{1},\overline{\mathbf{16}},\mathbf{1},\mathbf{2})+(\mathbf{1},\overline{\mathbf{16}},\mathbf{1},\overline{\mathbf{2}})$. Altogether, there are eight families of chiral fermions in the $\mathbf{16}$ (or $\overline{\mathbf{16}}$) of $SO(10)$.

The analysis of $Z_{r\theta}$ is similar and we shall be brief. With the generating vectors $\{V_0,\cdots,V_3,V_{\theta^2},V_{r\theta}\}$,

$$V_i\cdot V_j=\begin{pmatrix} \cdot &&&&& 2\\ & \cdot &&&& -\frac{1}{2}\\ && \cdot &&& \frac{3}{2}\\ &&& \cdot && \frac{1}{2}\\ &&&& \cdot & 2\\ 2 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 2 & 2 \end{pmatrix},\qquad k_{ij}=\begin{pmatrix} \cdot &&&&& k_{r\theta,0}\\ & \cdot &&&& k_{r\theta,1}+\frac{1}{2}\\ && \cdot &&& k_{r\theta,2}+\frac{1}{2}\\ &&& \cdot && k_{r\theta,3}+\frac{1}{2}\\ &&&& \cdot & k_{r\theta,\theta^2}\\ k_{r\theta,0} & k_{r\theta,1} & k_{r\theta,2} & k_{r\theta,3} & k_{r\theta,\theta^2} & k_{r\theta,0} \end{pmatrix}. \tag{4.32}$$

Again, gauge bosons only come from the $\mathbf{0}$ sector and with the addition of $V_{r\theta}$, the gauge group is broken to $[SO(10)]^2\times[U(1)]^2\times[SO(4)]^2\times[U(1)]^2\times SO(4)\times[U(1)]^2$. The 2 gravitinos in the V_1 sector are either all projected out (if $k_{r\theta,1}\neq k_{21}$) or remain untouched (if $k_{r\theta,1}=k_{21}$).

We finally come to Z_θ which is generated by $\{V_0,\cdots,V_3,V_\theta\}$. Therefore,

$$V_i\cdot V_j=\begin{pmatrix} \cdot &&&& 0\\ & \cdot &&& -\frac{1}{2}\\ && \cdot && 0\\ &&& \cdot & \frac{1}{2}\\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix},\qquad k_{ij}=\begin{pmatrix} \cdot &&&& k_{\theta0}\\ & \cdot &&& k_{\theta1}+\frac{1}{2}\\ && \cdot && k_{\theta2}\\ &&& \cdot & k_{\theta3}+\frac{1}{2}\\ k_{\theta0} & k_{\theta1} & k_{\theta2} & k_{\theta3} & k_{\theta0} \end{pmatrix}. \tag{4.33}$$

The gauge group is broken to $U(10)\times U(2)\times U(4)\times[U(1)]^2\times SO(4)\times[U(1)]^2$. In contrast to V_r, V_θ breaks the supersymmetry to $N=1$ but does not introduce chiral fermions.

What can we say about the resultant D_4 orbifold simply by counting? If we choose $k_{r\theta,1}=k_{21}$, there are 2 gravitinos in $Z_{r\theta}$ and the resultant model has $N=1$ supersymmetry. In addition, there are 68 gauge bosons and four families of chiral fermions in the $\mathbf{16}$ of $SO(10)$.

A thorough examination of the spectrum (such as identification of the gauge group) requires a more careful treatment. There are 27 potentially massless sectors, 13 of which are projected out by the physically sensible projection. The model is supersymmetric and so it suffices to list only half of the remaining sectors (the others can be obtained by adding V_1). The untwisted sectors include: $\mathbf{0}$, $V_2+V_{\theta^2}$ and $V_2+V_3+V_{\theta^2}$ while the twisted sectors include: V_r, $V_r+V_{\theta^2}$, V_θ and $3V_\theta$.

The 68 gauge bosons all come from the $\mathbf{0}$ sector. In the basis such that θ is diagonal, the spectrum generating formula gives,

$$\begin{aligned} \frac{1}{2}\left(N_1+\cdots+N_{10}\right)+\frac{1}{2}\left(\tilde{N}_1+\cdots+\tilde{N}_{22}\right)&=\frac{1}{2}\quad(\text{mod }1)\ ,\\ \frac{1}{2}\left(N_1+N_2+N_5+N_8\right)&=\frac{1}{2}\quad(\text{mod }1)\ , \end{aligned}$$

$$\frac{1}{2}\,(N_1+N_2+N_6+N_9)+\frac{1}{2}\,(\tilde{N}_1+\cdots+\tilde{N}_{10}+\tilde{N}_{21}+\tilde{N}_{22})=\frac{1}{2}\pmod 1\,, \tag{4.34}$$
$$\frac{1}{2}\,(N_9+N_{10})+\frac{1}{2}\,(\tilde{N}_{11}+\tilde{N}_{12}+\tilde{N}_{18}+\cdots+\tilde{N}_{21})=0\pmod 1\,,$$
$$\frac{1}{2}\,(N_1+N_3+N_6+N_8)+\frac{1}{4}\,(\tilde{N}_1-\tilde{N}_2+\cdots+\tilde{N}_{15}-\tilde{N}_{16})$$
$$+\frac{1}{2}\,(\tilde{N}_{17}+\tilde{N}_{18}+\tilde{N}_{21}+\tilde{N}_{22})=\frac{1}{2}\pmod 1\,.$$

The states that survive the projection are invariant under θ. As mentioned before, the gauge bosons form the adjoint representation of $U(10)\times U(2)\times U(4)\times[U(1)]^2\times SO(4)\times U(1)^2$. Take the $U(10)$ bosons for example. They are obtained by exciting $\tilde{N}_i=(\delta_{ij}-\delta_{ik})$ for $j,k=1,3,\cdots,9$ or $2,4,\cdots,10$ or $\tilde{N}_i=\pm(\delta_{ij}+\delta_{ik})$ for $j=1,3,\cdots,9$ and $k=2,4,\cdots,10$.

These states, however, are not invariant under the full non-abelian group D_4. Under the action of r, $\tilde{N}_{2i-1}$ and $\tilde{N}_{2i}$ for $i=1,\cdots 8$ are interchanged. Nevertheless, we can obtain group invariant states by forming linear combinations of θ invariant states. In other words, let $|\theta\rangle$ be an θ invariant state, then the combination $|\theta\rangle+r\,|\theta\rangle$ is invariant under D_4. It follows that the gauge bosons invariant under D_4 form the adjoint representation of $SO(10)_2\times U(1)\times SO(4)_2\times[U(1)]^2\times SO(4)\times[U(1)]^2$. The rank of the gauge group is reduced to 14. It can easily be verified that the $SO(10)$ and the $SO(4)$ are realized at level 2 by examining the OPE of their Kac-Moody currents. We may regard the $SO(10)_2$ as the observable sector and the rest as the hidden sector. In what follows, we will only consider states that are not singlets under $SO(10)$.

Besides gauge bosons, the **0** sector also contributes scalars. For example take $N_i=\pm\delta_{i2}$ and $\tilde{N}_i$ the same as that of the gauge bosons, we have two families of scalars in the **45** of $SO(10)$. In addition, there are scalars in the $(\mathbf{10},\mathbf{4})$ of $SO(10)\times SO(4)$.

The analysis of the twisted sectors is similar except that group invariance requires mixing of states from conjugate sectors. To illustrate this, consider a physical state in the V_r sector. It is invariant under r and θ^2 and we denote it as $|r,\theta^2\rangle$. The group invariant combination is thus,

$$\left|r,\theta^2\right\rangle+\theta\left|r,\theta^2\right\rangle\ . \tag{4.35}$$

Here $\theta\,|r,\theta^2\rangle$ is in the conjugate sector $V_r+V_{\theta^2}$ because $\theta r\theta^3=r\theta^2$. As a result, the conjugate sectors together contribute four families of chiral fermions in the **16** of $SO(10)$.

The other sectors only contribute $SO(10)$ singlets. To summarize, the resultant model has observable gauge group $SO(10)$ realized at level 2, $N=1$ supersymmetry, four chiral families of $SO(10)$ and massless scalars in the adjoint of $SO(10)$.

So far we have limited ourselves to only one-dimensional representations on the right movers. With the $[SU(2)]^6$ form of the supercurrent,

$$T_F=i\psi^u\partial X_u+i\sum_{\ell=1}^{6}\psi_1^\ell\psi_2^\ell\psi_3^\ell\ , \tag{4.36}$$

a general automorphism that respects world-sheet supersymmetry is a product of an inner automorphism for the individual $SU(2)$ algebras and an outer automorphism that permutes the different $SU(2)$. This allows a truly non-abelian action on the right movers since in general, two such automorphisms do not commute.

Consider two non-commuting automorphisms θ and r of $[SU(2)]^2$ defined by

$$\begin{array}{ll} \theta : & \psi_j^1 \to \psi_j^2 \\ & \psi_j^2 \to \psi_j^1 \sigma_j \\ r : & \psi_j^1 \to \psi_j^2 \\ & \psi_j^2 \to \psi_j^1 \end{array} \tag{4.37}$$

where $\sigma_1 = \sigma_2 = -1$, $\sigma_3 = 1$. It is obvious that $\theta^4 = r^2 = 1$ and $r\theta^3 = \theta r$. Therefore, the group generated by θ and r is D_4.

The eigenstates and the corresponding eigenvalues of θ are:

$$\begin{array}{lll} \chi_r^1 = & \psi_3^1 + \psi_3^2 \ , & \theta = 1 \ ; \\ \chi_r^2 = & -\psi_3^1 + \psi_3^2 \ , & \theta = -1 \ ; \\ \chi^3 = & i\psi_1^1 - \psi_1^2 \ , & \theta = -i \ ; \\ \chi^4 = & -i\psi_2^1 - \psi_2^2 \ , & \theta = i \ ; \end{array} \tag{4.38}$$

where χ with the subscript r are real fermions. Similarly, for r:

$$\begin{array}{lll} \chi_r^1 = & \psi_3^1 + \psi_3^2 \ , & r = 1 \ ; \\ \chi_r^2 = & -\psi_3^1 + \psi_3^2 \ , & r = -1 \ ; \\ \chi^3 = & \psi_1^1 + \psi_1^2 \ , & r = 1 \ ; \\ \chi^4 = & -\psi_2^1 + \psi_2^2 \ , & r = -1 \ . \end{array} \tag{4.39}$$

To illustrate how to construct non-abelian orbifolds with non-trivial $SU(2)$ outer automorphisms, let us consider a symmetric D_4 orbifold of the model generated by $\{V_0, V_1\}$. The D_4 action is specified by the generating vectors:

$$\begin{aligned} V_\theta &= \Big(-\frac{1}{2}\ \{0_r, (-\frac{1}{2})_r, -\frac{1}{4}, \frac{1}{4}\}^2\ (0 - \frac{1}{2}\ 0) \mid -\frac{1}{2}\ \{0_r, (-\frac{1}{2})_r, -\frac{1}{4}, \frac{1}{4}\}^2\ (0 - \frac{1}{2}\ 0)\ 0^{12}\Big) \ , \\ V_r &= \Big(-\frac{1}{2}\ \{0_r, (-\frac{1}{2})_r, 0, -\frac{1}{2}\}^2\ (0 - \frac{1}{2}\ 0) \mid -\frac{1}{2}\ \{0_r, (-\frac{1}{2})_r, 0, -\frac{1}{2}\}^2\ (0 - \frac{1}{2}\ 0)\ 0^{12}\Big) \ . \end{aligned} \tag{4.40}$$

In other words, two copies of $[SU(2)]^2$ transform under θ and r defined above. The remaining fermions are simply one-dimensional representations of D_4.

In order that the abelian models are well defined, the original set of generating vectors must commute with the group action. It is easy to see that V_0 and V_1 commute with both V_θ and V_r. The non-abelian rules impose constraints on the structure constants:

$$\begin{aligned} k_{\theta^2 j} &= 0 \ , \\ k_{rj} + k_{\theta j} &= k_{r\theta, j} \ , \\ k_{r\theta^2} &= k_{r\theta,\theta^2} \ , \end{aligned} \tag{4.41}$$

where $j = 0, 1$. It follows immediately from $k_{\theta^2 1} = 0$ that Z_{θ^2} remains $N = 4$ supersymmetric. The gauge group is broken to $SO(8) \times SO(36)$. Both Z_r and Z_θ have $N = 2$ supersymmetry with gauge groups $SO(4) \times SO(4) \times SO(6) \times SO(30)$ and $SO(4) \times SO(4) \times SO(36)$ respectively. $Z_{r\theta}$ has gauge group $U(4) \times SO(6) \times SO(30)$. Depending on $k_{r\theta,1}$, the $N = 4$

supersymmetry in $Z_{r\theta}$ is unbroken ($k_{r\theta,1} = 0$) or completely broken ($k_{r\theta,1} = \frac{1}{2}$). As usual, rank reduction and higher level gauge group come with the non-abelian orbifold. The resultant D_4 orbifold has rank 20 gauge group $SO(4)_2 \times SO(6) \times SO(30)$ and $N = 2$ (or 0) supersymmetry.

Similar analysis can be carried out for non-abelian groups generated by other automorphisms of $[SU(2)]^6$. We may also start with a model generated by more basis vectors, provided that they commute with the automorphisms. A complete classification of the $[SU(2)]^6$ automorphisms as well as the sets of mutually commuting basis vectors have been studied in Ref. [13].

V. BOSONIC FORMULATION

The above rules can be easily generalized to the general orbifold case. This generalization is easiest if we start with the recent rules for asymmetric orbifold [6], since they are brought to a form very similar to that for the free fermionic string construction. Because the generalization in this formulation is completely analogous to that for the free fermionic construction, we shall be brief.

Let us first review the abelian orbifold case. Again, let us consider heterotic strings compactified to four space-time dimensions. In the light-cone gauge, which we adopt, we have the following world-sheet degrees of freedom: One complex boson ϕ^0 (corresponding to two transverse space-time coordinates); three right-moving complex bosons ϕ^ℓ_R, $\ell = 1,2,3$ (corresponding to six internal coordinates); four right-moving complex fermions ψ^r, $r = 0,1,2,3$ (ψ^0 is the world-sheet superpartner of the right-moving component of ϕ^0, whereas ψ^ℓ are the world-sheet superpartners of ϕ^ℓ_R, $\ell = 1,2,3$); eleven left-moving complex bosons ϕ^ℓ_L, $\ell = 4,5,...,14$ (corresponding to twenty-two internal coordinates). Before orbifolding, the corresponding string model has $N = 4$ space-time supersymmetry and the internal momenta span an even self-dual Lorentzian lattice $\Gamma^{6,22}$.

It is convenient to organize the string states into sectors labeled by the monodromies of the string degrees of freedom. These monodromies can be combined into a single vector. For each complex boson we will specify either T^ℓ_i (twist), or U^ℓ_i (shift). To keep track of whether a given entry corresponds to a twist or a shift, it is convenient to introduce *auxiliary* vectors

$$W_i = (0(0\ W^1_i)(0\ W^2_i)(0\ W^2_i)||W^4_i \cdots W^{14}_i)\ . \tag{5.1}$$

The double vertical line separates the right- and left-movers. The entries W^ℓ_i are defined as follows: $W^\ell_i = \frac{1}{2}$ if $T^\ell_i \neq 0$; $W^\ell_i = 0$, otherwise. For example,

$$V_i = (V^0_i(V^1_i\ T^1_i)(V^2_i\ T^2_i)(V^3_i\ T^3_i)||U^4_i \cdots U^{13}_i\ T^{14}_i)\ , \tag{5.2}$$

$$W_i = (0(0\ \frac{1}{2})^3||0^{10}\ \frac{1}{2})\ , \tag{5.3}$$

where T^1_i, T^2_i, T^3_i and T^{14}_i correspond to twists, U^4_i,...,U^{13}_i correspond to shifts, and V^r_i, $r = 0,1,2,3$, specify the fermionic spin structures. Without loss of generality we can restrict the values of V^r_i and T^ℓ_i as follows: $-\frac{1}{2} \leq V^r_i < \frac{1}{2}$; $0 \leq T^\ell_i < 1$ (A complex boson (fermion) with boundary condition T^ℓ_i (V^r_i) $= 0$ or $\frac{1}{2}$ can be split into two real bosons (fermions)).

The shifts U_i^ℓ can be combined into a real $(6,22)$ dimensional Lorentzian vector $\vec{U}_i$ defined up to the identification $\vec{U}_i \sim \vec{U}_i + \vec{P}$, where $\vec{P}$ is an arbitrary vector of $\Gamma^{6,22}$.

The notation we have introduced proves convenient in describing the sectors of a given string model based on the orbifold group G. For G to be a finite abelian group, the element $g(V_i)$ must have a finite order $m_i \in \mathbf{N}$, *i.e.* $g^{m_i}(V_i) = 1$. This implies that V_i^r and T_i^ℓ must be rational numbers, and the shift vector $\vec{U}_i$ must be a rational multiple of a vector in $\Gamma^{6,22}$; that is, $m_i V_i^r, m_i T_i^\ell \in \mathbf{Z}$, and $m_i \vec{U}_i \in \Gamma^{6,22}$. To describe all the elements of the group G, it is convenient to introduce the set of generating vectors $\{V_i\}$ such that $\overline{\alpha V} = \mathbf{0}$ if and only if $\alpha_i \equiv 0$. Here $\mathbf{0}$ is the null vector: $\mathbf{0} = (0(0\ 0)^3||0^{11})$. Also, $\alpha V \equiv \sum_i \alpha_i V_i$ (the summation is defined as $(V_i + V_j)^\ell = V_i^\ell + V_j^\ell$), α_i being integers that take values from 0 to $m_i - 1$. The overbar notation is defined as follows: $\overline{\alpha V} \equiv \alpha V - \Delta(\alpha)$, and the components of $\overline{\alpha V}$ satisfy $-\frac{1}{2} \leq \overline{\alpha V}^r < \frac{1}{2}$, $0 \leq \overline{\alpha T}^\ell < 1$; here $\Delta^r(\alpha), \Delta^\ell(\alpha) \in \mathbf{Z}$. So the elements of the group G are in one-to-one correspondence with the vectors $\overline{\alpha V}$ and will be denoted by $g(\overline{\alpha V})$. It is precisely the abelian nature of G that allows this correspondence (by simply taking all possible linear combinations of the generating vectors V_i). So the sectors of the model are labeled by the vectors $\overline{\alpha V}$. To maintain world-sheet supersymmetry, each vector must satisfy the following *supercurrent* constraint

$$V_i^\ell + T_i^\ell = V_i^0 \equiv s_i \ (\mathrm{mod}\ 1)\ , \quad \ell = 1,2,3\ . \tag{5.4}$$

Here s_i is the monodromy of the supercurrent $\overline{S}(\overline{z}e^{-2\pi i}) = \exp(2\pi i s_i)\overline{S}(\overline{z})$, which must satisfy $s_i \in \frac{1}{2}\mathbf{Z}$. Then the sectors with $\overline{\alpha V}^0 = 0$ give rise to space-time bosons, while the sectors with $\overline{\alpha V}^0 = -\frac{1}{2}$ give rise to space-time fermions. This is the bosonic equivalence of the triplet constraint (2.2).

The rules for model building is quite similar to that for the free fermionic string construction. To obtain relatively simple rules for building asymmetric orbifold models, we will confine our attention to the orbifolds with twists whose orders are co-prime numbers. (The order of a twist generated by a vector $\overline{\alpha V}$ is defined as the smallest positive integer $t(\overline{\alpha V})$, such that $\forall \ell\ t(\overline{\alpha V})\overline{\alpha T}^\ell \in \mathbf{Z}$; note that $t(\overline{\alpha V})$ is a divisor of $m(\overline{\alpha V})$). In this case, a given model can be viewed as being generated by a single twist V^* of order $t^* = \prod_i t_i$, such that $\overline{\alpha V} = \overline{\alpha^* V^*}$, where $\alpha^*/t^* = \sum_i \alpha_i/t_i$ (mod 1), and α^* takes values $0, 1, ..., t^* - 1$. We may work in the V^* basis or the $\{V_i\}$ basis.

In a given sector $\overline{\alpha V}$, the right- and left-moving Hamiltonians are given by the corresponding sums of the Hamiltonians for individual string degrees of freedom. The Hilbert space in the $\overline{\alpha V}$ sector is given by the momentum states $|\vec{P}_{\overline{\alpha V}} + \alpha\vec{U}, \mathbf{n}\rangle$, and also the states obtained from these states by acting with the fermion and boson creation operators (oscillator excitations). $\mathbf{n}$ is a collective notation for $n_i = 0, 1, ..., t_i - 1$. In the untwisted sectors, that is, sectors $\overline{\alpha V}$ with $t(\overline{\alpha V}) = 1$, we have $\vec{P}_{\overline{\alpha V}} \in \Gamma^{6,22}$. In the twisted sectors $\overline{\alpha V}$ with $t(\overline{\alpha V}) \neq 1$, we have $\vec{P}_{\overline{\alpha V}} \in \tilde{I}(\overline{\alpha V})$, where $\tilde{I}(\overline{\alpha V})$ is the lattice dual to the lattice $I(\overline{\alpha V})$, which in turn is the sublattice of $\Gamma^{6,22}$ invariant under the action of the twist part of the group element $g(\overline{\alpha V})$. The ground states $|\vec{0}, \mathbf{n}\rangle$ in the $\overline{\alpha V}$ sector appear with certain degeneracies $\xi(\overline{\alpha V}, \mathbf{n})$, which are non-negative integers. In the untwisted sectors $\xi(\overline{\alpha V}, \mathbf{n}) = 1$ if all $n_i = 0$, and zero, otherwise. In the twisted sectors, the situation is more involved. Here, the states with quantum numbers $\mathbf{n}$ appear with the multiplicity $\xi(\overline{\alpha V}, \mathbf{n}))$, with fixed point phase $f_i(\overline{\alpha V}, \mathbf{n})$ given by

$$f_i(\overline{\alpha V}, \mathbf{n}) = n_i/t_i \ , \tag{5.5}$$

The requirements of modular invariance and physically sensible projection unambiguously fix the degeneracies $\xi(\overline{\alpha V}, \mathbf{n})$. Now, the sum of $\xi(\overline{\alpha V}, \mathbf{n})$ over all $\mathbf{n}$ is the number of fixed points $\xi(\overline{\alpha V})$ in the $\overline{\alpha V}$ sector [2]:

$$\sum_{\mathbf{n}} \xi(\overline{\alpha V}, \mathbf{n}) = \xi(\overline{\alpha V}) = (M(\overline{\alpha V}))^{-1/2} \prod_{\ell} [2\sin(\pi \overline{\alpha T}^{\ell})] \ , \tag{5.6}$$

where the product over ℓ does not include the terms with $\overline{\alpha T}^{\ell} = 0$, and $M(\overline{\alpha V})$ is the determinant of the metric of $I(\overline{\alpha V})$. We also have the following constraints,

$$\xi(\overline{\alpha V}) \geq \sum_{\mathbf{n}} \xi(\overline{\alpha V}, \mathbf{n}) \exp(-2\pi i \beta f(\overline{\alpha V}, \mathbf{n})) \tag{5.7}$$

where the sum must be a non-zero integer. These constraints essentially fix all the $\xi(\overline{\alpha V}, \mathbf{n})$.

Since the momentum in the twisted sectors belongs to a shifted $\tilde{I}(\overline{\alpha V})$ lattice, the level matching condition requires that this lattice must have a prime $N(\overline{\alpha V})$, where $N(\overline{\alpha V})$ is the smallest positive integer such that for all vectors $\vec{P} \in \tilde{I}(\overline{\alpha V})$, $N(\overline{\alpha V})\vec{P}^2 \in 2\mathbf{Z}$; moreover, for the corresponding characters to have the correct modular transformation properties (*i.e.*, so that they are permuted under both S- and T-transformations), either $N(\overline{\alpha V}) = 1$ (in which case $I(\overline{\alpha V})$ is an even self-dual lattice), or $N(\overline{\alpha V}) = t(\overline{\alpha V})$ (in which case $I(\overline{\alpha V})$ is even but not self-dual).

Imposing modular invariance and physical sensible projections, the following constraints on a consistent model emerge:

$$\begin{aligned}
k_{ij} + k_{ji} &= V_i \cdot V_j \quad (\mathrm{mod}\ 1)\ , \quad i \neq j\ , \\
k_{ii} + k_{i0} + s_i - \frac{1}{2} V_i \cdot V_i &= \quad 0 \quad\ \ (\mathrm{mod}\ 1)\ , \\
(k_{ij} - W_i \cdot V_j) m_j &= \quad 0 \quad\ \ (\mathrm{mod}\ 1)\ , \\
m_i V_i \cdot (W(\overline{\alpha V}) - \sum_j \alpha_j W_j) &= \quad 0 \quad\ \ (\mathrm{mod}\ 1)\ .
\end{aligned} \tag{5.8}$$

Here $V_i \cdot V_j$ is generalized to

$$V_i \cdot V_j = \vec{U}_i \cdot \vec{U}_j - \sum_r V_i^r V_j^r + \sum_{\ell:\ right} T_i^{\ell} T_j^{\ell} - \sum_{\ell:\ left} T_i^{\ell} T_j^{\ell}\ . \tag{5.9}$$

The spectrum generating formula reads:

$$V_i \cdot \mathcal{N}_{\overline{\alpha V}} + f_i(\overline{\alpha V}, \mathbf{n}) + \frac{\alpha^* t^*}{2t_i} \vec{Q}^2(\vec{P}_{\overline{\alpha V}}) = \sum_j k_{ij}\alpha_j + s_i - V_i \cdot (\overline{\alpha V} - W(\overline{\alpha V})) \quad (\mathrm{mod}\ 1)\ . \tag{5.10}$$

Here, $\vec{Q}(\vec{P}_{\overline{\alpha V}}) \in \tilde{I}(\overline{\alpha V})$ is an arbitrary vector such that $\vec{P}_{\overline{\alpha V}} - \alpha^* \vec{Q}(\vec{P}_{\overline{\alpha V}}) \in I(\overline{\alpha V})$. For example, for $\alpha^* = 1$, we can choose $\vec{P}_{\overline{\alpha V}} = \vec{Q}(\vec{P}_{\overline{\alpha V}})$.

$$V_i \cdot \mathcal{N}_{\overline{\alpha V}} \equiv \vec{U}_i \cdot \vec{P}_{\overline{\alpha V}} - \sum_r V_i^r N_{\overline{\alpha V}}^r + \sum_{\ell:\ right} T_i^{\ell} J_{\overline{\alpha V}}^{\ell} - \sum_{\ell:\ left} T_i^{\ell} J_{\overline{\alpha V}}^{\ell}\ . \tag{5.11}$$

where $N^r_{\overline{\alpha V}}$ are the fermion number operators, and $J^\ell_{\overline{\alpha V}}$ the boson number operators. Both have integer eigenvalues.

The states that satisfy the spectrum generating formula include both on- and off-shell states. The on-shell states must satisfy the additional constraint that the left- and right-moving energies are equal. In the $\overline{\alpha V}$ sector they are given by:

$$
\begin{aligned}
E^L_{\overline{\alpha V}} = & -1 + \sum_{\ell:\ left} \{\frac{1}{2}\overline{\alpha T}^\ell(1-\overline{\alpha T}^\ell) + \sum_{q=1}^{\infty}[(q+\overline{\alpha T}^\ell - 1)n^\ell_q + (q-\overline{\alpha T}^\ell)\overline{n}^\ell_q]\} \\
& + \sum_{q=1}^{\infty} q(n^0_q + \overline{n}^0_q) + \frac{1}{2}(\vec{P}^L_{\overline{\alpha V}} + \alpha\vec{U}^L)^2 \ , \qquad (5.12) \\
E^R_{\overline{\alpha V}} = & -\frac{1}{2} + \sum_{\ell:\ right} \{\frac{1}{2}\overline{\alpha T}^\ell(1-\overline{\alpha T}^\ell) + \sum_{q=1}^{\infty}[(q+\overline{\alpha T}^\ell - 1)m^\ell_q + (q-\overline{\alpha T}^\ell)\overline{m}^\ell_q]\} \\
& + \sum_{q=1}^{\infty} q(m^0_q + \overline{m}^0_q) + \frac{1}{2}(\vec{P}^R_{\overline{\alpha V}} + \alpha\vec{U}^R)^2 \\
& + \sum_r \{\frac{1}{2}(\overline{\alpha V}^r)^2 + \sum_{q=1}^{\infty}[(q+\overline{\alpha V}^r - \frac{1}{2})k^r_q + (q-\overline{\alpha V}^r - \frac{1}{2})\overline{k}^r_q]\} \ . \qquad (5.13)
\end{aligned}
$$

Here n^ℓ_q and $\overline{n}^\ell_q$ are occupation numbers for the left-moving bosons ϕ^ℓ_L, whereas m^ℓ_q and $\overline{m}^\ell_q$ are those for the right-moving bosons ϕ^ℓ_R. These take non-negative integer values. k^r_q and $\overline{k}^r_q$ are the occupation numbers for the right-moving fermions, and they take only two values: 0 and 1. The occupation numbers are directly related to the boson and fermion number operators. For example, $N^r_{\overline{\alpha V}} = \sum_{q=1}^{\infty}(k^r_q - \overline{k}^r_q)$.

To obtain non-abelian orbifold models, with the non-abelian group, we consider the abelian orbifolds, whose abelian groups are subgroups of the non-abelian group. We follow the same approach given in section III to obtain the consistency constraints on the sets of $\{k_{ij}, V_i\}$. Since the derivation and the rules are identical, we shall not repeat it here.

VI. EXAMPLES

The simpliest example is the $\mathbf{Z}_2$ orbifold of a single boson. This well known example can also be recast in this formalism and is given in Appendix A. Here we give two orbifold models. The first model has the permutation group S_3 as the non-abelian point group. The second one is simply the 3-family $SO(10)_3$ grand unification model given in Ref. [8], recast in the non-abelian orbifold language. Its isometry group is $T \times \mathbf{Z}_2$, where T is the tetrahedral group.

(1) The S_3 Model

Let us start with the following 4-dimensional Narain model, with $\Gamma^{6,22} = \Gamma^{6,6} \otimes \Gamma^{16}$, where Γ^{16} is the Spin(32)/$\mathbf{Z}_2$ lattice, and $\Gamma^{6,6} = (\Gamma^{1,1} \otimes \Gamma^{2,2})^2$ where both $\Gamma^{1,1}$ and $\Gamma^{2,2}$ are specific even self-dual lattices: $\Gamma^{1,1} = \{(p_R||p_L) \mid p_R = \frac{1}{\sqrt{2}}(m-n),\ p_L = \frac{1}{\sqrt{2}}(m+n)\ ;\ m,n \in \mathbf{Z}\}$, *i.e.*, an $SU(2)$ lattice and $\Gamma^{2,2} = \{(\vec{p}_R||\vec{p_L}) \mid \vec{p}_{R,L} = \frac{1}{2R}\tilde{e}^i m_i \mp Re_i n^i\}$ where e_i and $\tilde{e}^i$ are the $SU(3)$ simple roots and their duals and R is the compactification radius. This $N=4$ model (called $N4$) has $SU(2) \times SU(2) \times U(1)^2 \times U(1)^2 \times SO(32)$ gauge group.

Let us first consider the $\mathbf{Z}_3$ orbifold:

$$
\begin{aligned}
V_0 &= (-\frac{1}{2}(-\frac{1}{2}\ 0)_r^2\ (-\frac{1}{2}\ 0)^2||0_r^2\ 0^2|0_r^{16})\ , \\
V_1 &= (0(0\ 0)_r^2\ (-\frac{1}{3}\frac{1}{3})(\frac{1}{3}\ \frac{2}{3})||(-\frac{\sqrt{2}}{3})_r(\frac{\sqrt{2}}{3})_r\ \frac{1}{3}\ \frac{2}{3}|(\frac{1}{3})_r^8\ (\frac{2}{3})_r^2\ 0_r^6)\ , \\
W_1 &= (0(0\ 0)_r^2\ (0\ \frac{1}{2})^2||0_r^2\ (\frac{1}{2})^2|0_r^{16})\ ,
\end{aligned}
\tag{6.1}
$$

where the subscript r labels real fermions and/or bosons. Since this is a symmetric $\mathbf{Z}_3$ twist, $V_1 \cdot W_1 = 0$,

$$
V_i \cdot V_j = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\ , \qquad k_{ij} = 0\ . \tag{6.2}
$$

Using the spectrum generating formula (5.10), it is straightforward to work out the massless spectrum of this model. This orbifold yields a $N = 2$ model (called $S3$) which has the gauge symmetry $U(1) \times U(1) \times U(10) \times SO(12)$. Here, the $U(1)$ factors come from $\Gamma^{1,1}$s.

Next we consider a $\mathbf{Z}_2$ model. Let us consider the two bosons ϕ_1, ϕ_2 in $\Gamma^{2,2}$. Under this $\mathbf{Z}_2$ twist,

$$
\begin{aligned}
&\phi_2 \to -\phi_2 \\
&\phi_1 \to \phi_1 + \phi_2
\end{aligned}
\tag{6.3}
$$

i.e., it is a reflection that leaves $2\phi_1 + \phi_2$ invariant (Note that since this reflection acts on both left- and right-moving momenta, it can be viewed as a rotation in a four dimensional *Euclidean* lattice spanned by the vectors $(p_R||p_L)$. In the example below, we could alternatively combine two reflections acting on the left-moving momenta into a $\mathbf{Z}_2$ rotation). In fact, ϕ_2 and $\frac{1}{\sqrt{5}}\ (2\phi_1 + \phi_2)$ form the diagonal basis of this $\mathbf{Z}_2$ twist. Using this basis for both $\Gamma^{2,2}$, the $\mathbf{Z}_2$ orbifold is given by

$$
\begin{aligned}
V_0^{'} &= (-\frac{1}{2}(-\frac{1}{2}\ 0)_r^6||0_r^6|0_r^{16})\ , \\
V_1^{'} &= (-\frac{1}{2}(-\frac{1}{2}\ \frac{1}{2})_r^2\ [(-\frac{1}{2}\ \frac{1}{2})_r(00)_r]^2||(\frac{1}{2})_r\ (\frac{1}{2}\ 0)_r^2|(\frac{1}{2})_r^{12}\ 0_r^4)\ , \\
W_1^{'} &= (0\ (0\ \frac{1}{2})_r^2\ [(0\ \frac{1}{2})_r\ (0\ 0)_r]^2||(\frac{1}{2})^2\ (\frac{1}{2}\ 0)_r^2|(\frac{1}{2})_r^{12}\ 0_r^4)\ .
\end{aligned}
\tag{6.4}
$$

Here, $V_1^{'} \cdot W_1^{'} = \frac{3}{2}$,

$$
V_i^{'} \cdot V_j^{'} = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\ , \qquad k_{ij}^{'} = \begin{pmatrix} 0 & k_{10}^{'} + \frac{1}{2} \\ k_{10}^{'} & k_{10}^{'} \end{pmatrix}\ . \tag{6.5}
$$

Here the two $\Gamma^{1,1}$ as well as Γ^{16} are twisted. This $\mathbf{Z}_2$ orbifold yields a $N = 2$ model (called $S2$) with gauge symmetry $U(1)^4 \times SO(12) \times SO(20)$.

A comment on our choice of $\Gamma^{2,2}$ is necessary here. The invariant sublattice I of $\Gamma^{2,2}$ under the $\mathbf{Z}_3$ twist consists of the origin $(\vec{0}\ ||\ \vec{0})$ only. On the other hand, the invariant sublattice $I^{'}$ of $\Gamma^{2,2}$ under the $\mathbf{Z}_2$ twist is non-trivial:

$$I^{'} = \{\frac{m}{2R}\ (\tilde{e}^1\ ||\ \tilde{e}^1) + 3Rn\ (-\tilde{e}^1\ ||\ \tilde{e}^1)\ ;\ m,n \in \mathbf{Z}\} \tag{6.6}$$

where $\tilde{e}^1$ is the dual of e_1. Here $I^{'}$ is even but not self-dual. It is easy to find its dual $\tilde{I}^{'}$,

$$\tilde{I}^{'} = \{\frac{m}{4R}\ (\tilde{e}^1\ ||\ \tilde{e}^1) + \frac{3}{2}Rn\ (-\tilde{e}^1\ ||\ \tilde{e}^1)\ ;\ m,n \in \mathbf{Z}\}\ . \tag{6.7}$$

Recall $N_{I^{'}}$, where $N_{I^{'}}\vec{P}^2 \in 2\mathbf{Z}$ for $\vec{P} \in \tilde{I}^{'}$. In this case, $N_{I^{'}} = t_1^{'} = 2$, which is consistent. If we turn on appropriate antisymmetric background field and choose $R = 2$ to get an enhanced $SU(3)$ gauge symmetry, we will find that the corresponding $N_{I^{'}} = 4$, which is unacceptable according to our rules. In fact, we must have zero antisymmetric background field. As a consequence, we have only $U(1)^2$ gauge symmetry from each $\Gamma^{2,2}$.

Notice that the $\mathbf{Z}_3$ twist and the $\mathbf{Z}_2$ twist on $\Gamma^{2,2}$ do not commute; in fact, they generate the permutation group S_3 as the point group. So the final orbifold involves a symmetric S_3 and an asymmetric $\mathbf{Z}_2$ twists. Since the $\mathbf{Z}_3$ shifts and the $\mathbf{Z}_2$ twist on part of Γ^{16} do not commute, they generate the isometry group D_3, which is the same as S_3. The group S_3 has 2 generators θ and r satisfying $\theta^3 = r^2 = 1$ and $r\theta^2 = \theta r$. It is divided into 3 conjugacy classes: $C_1 = \{1\}$, $C_\theta = \{\theta, \theta^2\}$ and $C_r = \{r, r\theta, r\theta^2\}$. The stabilizer groups are given by $N_1 = S_3$, $N_\theta = \{1, \theta, \theta^2\}$ and $N_r = \{1, r\}$. It is straightforward to check that the non-abelian constraints given in Section III are easily satisfied. The resulting orbifold has the partition function

$$Z = \frac{1}{2}\ Z_{S3} + Z_{S2} - \frac{1}{2}\ Z_{N4}\ . \tag{6.8}$$

It is not difficult to work out the final massless spectrum of this $N = 1$ supersymmetric model. The gauge symmetry, coming only from Γ^{16}, is $U(1) \times SO(10)_2 \times SO(10)_1$, where the subscripts indicate the levels of the current algebra $SO(10)$.

(2) The 3-family $SO(10)_3$ model [8]

We start with a $N = 4$ supersymmetric Narain model, which was referred to as $N0$, with the lattice $\Gamma^{6,22} = \Gamma^{2,2} \otimes \Gamma^{4,4} \otimes \Gamma^{16}$. Here $\Gamma^{2,2} = \{(p_R||p_L)\}$ is an even self-dual Lorentzian lattice with $p_R, p_L \in \tilde{\Gamma}^2$ ($SU(3)$ weight lattice), $p_L - p_R \in \Gamma^2$ ($SU(3)$ root lattice). Similarly, $\Gamma^{4,4} = \{(P_R||P_L)\}$ is an even self-dual Lorentzian lattice with $P_R, P_L \in \tilde{\Gamma}^4$ ($SO(8)$ weight lattice), $P_L - P_R \in \Gamma^4$ ($SO(8)$ root lattice). Γ^{16} is the Spin(32)/$\mathbf{Z}_2$ lattice. This model has $SU(3) \times SO(8) \times SO(32)$ gauge group.

Next, consider the model generated by the following vectors:

$$\begin{aligned}
V_0 &= (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^3|0_r^{16})\ ,\\
V_1 &= (0(0\ \frac{1}{2}e_1)(0\ a_1)(0\ b_1)||0^3|(\frac{1}{2})_r^5\ 0_r^5\ 0_r^5\ (-\frac{1}{2})_r)\ ,\\
V_2 &= (0(0\ \frac{1}{2}e_2)(0\ a_2)(0\ b_2)||0^3|0_r^5\ (\frac{1}{2})_r^5\ 0_r^5\ (-\frac{1}{2})_r)\ .
\end{aligned} \tag{6.9}$$

Here e_1 and e_2 are the simple roots of $SU(3)$ ($e_1 \cdot e_1 = e_2 \cdot e_2 = -2e_1 \cdot e_2 = 2$), whereas the four-dimensional real vectors $\mathbf{s} = (a_1, b_1)$ and $\mathbf{c} = (a_2, b_2)$ are respectively the spinor and conjugate weight vectors of $SO(8)$. The generating vectors V_1 and V_2 are order two shifts ($m_1 = m_2 = 2$). They play the role of Wilson lines. The auxiliary vectors $W(\overline{\alpha V})$ are therefore null as there is no twisting of the lattice.

The matrix of the dot products $V_i \cdot V_j$ reads

$$V_i \cdot V_j = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{6.10}$$

$$k_{ij} = 0 \ . \tag{6.11}$$

The resulting model, which was referred to as $N1$, is a Narain model with $N = 4$ space-time supersymmetry and the gauge group $SU(3) \times SO(8) \times SO(10)^3 \times SO(2)$. The details of this construction can be found in Ref. [6,8].

Now, we consider the asymmetric orbifold model generated by the following vectors:

$$\begin{aligned} V_0 &= (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^3|0^5\ 0_r^5\ 0_r) \ , \\ V_1^{'} &= (0(-\frac{1}{3}\ \frac{1}{3})^3||0\ (\frac{1}{3})^2|(\frac{1}{3})^5\ 0_r^5\ (\frac{2}{3})_r) \ , \\ W_1^{'} &= (0(0\ \frac{1}{2})^3||0\ (\frac{1}{2})^2|(\frac{1}{2})^5\ 0_r^5\ 0_r) \ . \end{aligned} \tag{6.12}$$

The matrix of the dot products $V_i^{'} \cdot V_j^{'}$ reads

$$V_i^{'} \cdot V_j^{'} = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} \end{pmatrix} \ . \tag{6.13}$$

The structure constants k_{ij} are then given by

$$k_{ij} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \ . \tag{6.14}$$

Suppose we start from the $N1$ model, with the vectors (6.12) acting on it. This is the abelian orbifold construction. Here the right- and left-moving bosons corresponding to the $SU(3) \times SO(8)$ subgroup are complexified, whereas the single boson corresponding to the $SO(2)$ subgroup is real, so are the five bosons $\varphi^I \equiv \frac{1}{\sqrt{3}}(\phi_1^I + \phi_2^I + \phi_3^I)$; the other ten real bosons are complexified via linear combinations $\Phi^I \equiv \frac{1}{\sqrt{3}}(\phi_1^I + \omega\phi_2^I + \omega^2\phi_3^I)$ and $(\Phi^I)^\dagger \equiv \frac{1}{\sqrt{3}}(\phi_1^I + \omega^2\phi_2^I + \omega\phi_3^I)$, where $\omega = \exp(2\pi i/3)$ (The real bosons ϕ_p^I, $I = 1, ..., 5$, correspond to the $p^{\rm th}$ $SO(10)$ subgroup, $p = 1, 2, 3$). Thus, the orbifold group element described by the V_1 vector permutes the real bosons corresponding to the three $SO(10)$s, which in turn is equivalent to modding out by their outer automorphism. The resulting model (called $A1$) is simply the $SO(10)$ grand unification, with gauge symmetry $SU(3)_1 \times SU(3)_3 \times SO(10)_3 \times U(1)$ and 9 chiral $SO(10)$ families. The details can be found in Ref. [6,8].

Notice that the world-sheet basis that is diagonal under the action of either of the Wilson lines (6.9) is different from the basis that is diagonal under the $\mathbf{Z}_3$ twist. This implies that the $\mathbf{Z}_3$ twist action does not commute with the Wilson line action. So if we want to start from the original $N0$ model to reach the $A1$ model, we must perform a non-abelian orbifold, *i.e.*, an isometry $\overline{P}$ composed of the Wilson lines (6.9) and the $\mathbf{Z}_3$ twist plus shifts (6.12).

Since the Wilson lines correspond to a $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ (shift) action, we see that $\overline{P} = T$, the tetrahedral group.

Let the resulting model coming from the above $\mathbf{Z}_3$ orbifold (*i.e.*, V_0, V_1' of Eq(6.12)) on the original $N0$ model be referred to as the $B1$ model. Again, it is easy to work out the massless spectrum of the $B1$ model. It has $N=1$ supersymmetry and gauge symmetry $SU(3)_1 \times SU(3)_3 \times SU(11)_1 \times SO(10)_1 \times U(1)$. Besides the usual gravity and gauge supermultiplets, this $B1$ model also has, in its massless part of the spectrum, three copies of $(\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1})(0)_L$, $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{45})(0)$, $(\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{16})(-1)_L$, $(\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{10})(+2)_L$, and $(\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})(-4)_L$. We see that all the massless states are singlets of $SU(3)_1 \times SU(11)_1$. Note that the $SU(3)_3$ chiral anomaly of the chiral field $(\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1})(0)_L$ is cancelled by that of the other fields, as a $\mathbf{10}_L$ of $SU(3)$ has 27 times the anomaly contribution of a $\mathbf{3}_L$. The model is also $U(1)$ anomaly-free due to the underlying E_6 structure of the $SO(10)\times U(1)$ matter fields as can be seen from the branching $\mathbf{27} = \mathbf{16}(-1) + \mathbf{10}(+2) + \mathbf{1}(-4)$ under $E_6 \supset SO(10)\times U(1)$.

Now the $A1$ model may be reached as the non-abelian orbifold of the $N0$ model, whose partition function is a linear combination of those of the $N0$, the $N1$ and the $B1$ models. It is easy to see that the non-abelian rules given in Section III are satisfied. The tetrahedral group T has 12 elements generated by r and θ satisfying $r^2 = \theta^3 = (r\theta)^3 = 1$. There are 4 conjugacy classes: $C_1 = \{1\}$, $C_r = \{r, r', rr'\}$, $C_\theta = \{\theta, r\theta, r'\theta, rr'\theta\}$ and $C_{\theta^2} = \{\theta^2, (r\theta)^2, (r'\theta)^2, (rr'\theta)^2\}$ where $r' = \theta r \theta^{-1}$. The stabilizer groups are given by $N_1 = T$, $N_r = \{1, r, r'\}$ and $N_\theta = \{1, \theta, \theta^2\}$. So the partition function of the $A1$ model is given by

$$Z_{A1} = Z_{B1} + \frac{1}{3}\, Z_{N1} - \frac{1}{3}\, Z_{N0} \ . \tag{6.15}$$

As mentioned above, the $A1$ model has gauge symmetry $SU(3)_1 \times SU(3)_3 \times SO(10)_3 \times U(1)$. Besides the usual gravity and gauge supermultiplets, the massless spectrum of the resulting $A1$ model has three copies of $(\mathbf{1},\mathbf{10},\mathbf{1})(0)_L$, $(\mathbf{1},\mathbf{1},\mathbf{45})(0)$, $(\mathbf{1},\overline{\mathbf{3}},\mathbf{16})(-1)_L$, $(\mathbf{1},\overline{\mathbf{3}},\mathbf{10})(+2)_L$, and $(\mathbf{1},\overline{\mathbf{3}},\mathbf{1})(-4)_L$. Comparing the $A1$ and the $B1$ models, it seems that the Wilson lines (6.9) removes the $SU(11)$ gauge bosons in the $B1$ model while raising the $SO(10)$ current algebra level from 1 to 3. This happens because the $\mathbf{Z}_3$ twist on the $SO(32)$ in the $N0$ model is an inner-automorphism. So this $\mathbf{Z}_3$ twist is equivalent to some $\mathbf{Z}_3$ shifts, which cannot change the current algebra level or cut the rank of the gauge group. On the other hand, the same $\mathbf{Z}_3$ twist on the $SO(10)^3$ in the $N1$ model is an outer-automorphism, which converts $SO(10)^3$ to $SO(10)_3$.

In the above example, one may prefer the two-step abelian orbifold approach over the non-abelian orbifold approach, since starting from the $N1$ model is almost as easy as starting from the $N0$ model. However, in the case where the point group is non-abelian, such as the S_3 model discussed earlier, we do not see an obvious alternative.

To reach the final 3-family $SO(10)$ model, one must reduce the number of chiral families from 9 to 3 by further orbifolding the $A1$ model by a $\mathbf{Z}_2$ twist. This $\mathbf{Z}_2$ orbifold is generated by the following vectors:

$$\begin{aligned}
V_0 &= (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^3|0^5\ 0_r^5\ 0_r) \ , \\
V_2' &= (0(0\ 0)(-\frac{1}{2}\ \frac{1}{2})^2||(\frac{1}{2}e_1)(\frac{1}{2})^2|0^5\ 0_r^5\ 0_r) \ , \\
W_2' &= (0(0\ 0)(0\ \frac{1}{2})^2||0(\frac{1}{2})^2|0^5\ 0_r^5\ 0_r) \ .
\end{aligned} \tag{6.16}$$

The final model is the 3-family $SO(10)$ grand unified model presented in Ref. [8], with gauge symmetry $SU(2)_1 \times SU(2)_3 \times SO(10)_3 \times U(1)^3$. Now, this $\mathbf{Z}_2$ twist commutes with the $\mathbf{Z}_3$ twist (6.12) and the Wilson lines (6.9). So, starting from the $N0$ model, the orbifold group is $T \times \mathbf{Z}_2$.

There are alternative routes to the same final 3-family $SO(10)$ model. We may start with the $\mathbf{Z}_2$ orbifold (6.16) of the $N0$ model, which we refer to as the $N2$ model. If we now orbifold this $N2$ model by the tetrahedral group T, the partition function of the final model is given by

$$Z_F = Z_{A3} + \frac{1}{3}\, Z_{N3} - \frac{1}{3}\, Z_{N2} \tag{6.17}$$

where $N3$ is the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold (*i.e.*, $\{V_0, V_1, V_2\}$) of the $N2$ model, and $A3$ is the $\mathbf{Z}_3$ orbifold (6.12) of the $N2$ model. The final model is again the same three-family $SO(10)$ grand unified model.

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APPENDIX A: THE $\mathbf{Z}_2$ ORBIFOLD OF A SINGLE BOSON

The $\mathbf{Z}_2$ orbifold of a single boson compactified at arbitrary radius is well understood. In this case, the orbifold involves a non-abelian isometry group. It is therefore a good example to illustrate how the rules for constructing free fermionic string models and general abelian orbifolds are generalized to include the construction of non-abelian orbifolds. In particular, it will help clarify our notation. This example also illustrates the following point: how the same model Ω can be reached via an abelian or a non-abelian orbifold depending on the model Ω' that we start with.

In the bosonic formulation, one can start with a torus Γ of any desired radius and identify points on Γ under an abelian point group $P = \mathbf{Z}_2$. We can also start with a torus $\overline{\Gamma}$ of fixed radius 1 and obtain the same model by the combined action of (1) changing the radius, and (2) identifying points on the new torus Γ under $P = \mathbf{Z}_2$. These two operations do not commute and generate the non-abelian isometry group $\overline{P}$.

To be explicit, let us recall that $\overline{P}$ is a subgroup of the space group S of rotations and translations. An element of $\overline{P}$ therefore takes $X \to \theta X + v$ and is denoted by (θ, v). The multiplication law for elements of $\overline{P}$ is given by $(\theta, v)(\omega, u) = (\theta\omega, v + \theta u)$.

As an example, let us construct the $\mathbf{Z}_2$ orbifold at radius $\frac{1}{n}$ (for simplicity, we take n to be odd). Starting from a torus of radius 1, we can change the radius to $\frac{1}{n}$ by modding out the lattice by a $\mathbf{Z}_n$ shift generated by $p = (1, \frac{1}{n})$. On the other hand, modding out the torus by a twist $q = (-1, 0)$ results in a $\mathbf{Z}_2$ orbifold. It is easy to verify that $p^n = q^2 = 1$ and $pq = qp^{n-1}$. Therefore, the group generated by p and q is D_n which is divided into $\frac{1}{2}(n+3)$ conjugacy classes: $C_{p^m} = \{p^m, p^{-m}\}$ for $m = 0, \cdots, \frac{1}{2}\,(n-1)$, and $C_q = \{q, pq, \cdots p^{n-1}q\}$.

The stabilizer groups are $N_{p^m} = \{1, p, \cdots, p^{n-1}\}$ and $N_{p^m q} = \{1, p^m q\}$ for $m = 0, \cdots, n-1$. Therefore, the full partition function can be expressed as a sum of the three abelian orbifolds,

$$\begin{aligned} Z &= \frac{1}{2}\, Z_p + \frac{1}{n} \sum_{m=0}^{n-1} Z_q^m - \frac{1}{2}\, Z_1 \\ &= \frac{1}{2}\, Z_p + Z_q - \frac{1}{2}\, Z_1 \end{aligned} \tag{A1}$$

where Z_p is the partition function of a single boson at radius $\frac{1}{n}$, each Z_q^m is equal to Z_q, a $\mathbf{Z}_2$ orbifold partition function at radius 1, and Z_1 is the original partition function of a single boson at radius 1, *i.e.*, that of the starting model Ω'.

We now turn to the free fermionic construction. Consistency requires the presence of V_0. In the case of one complex world-sheet fermion, we always start from the model Ω', which is equivalent to a single boson at radius 1. Its partition function Z_1 is generated by a single vector

$$V_0 = (-\frac{1}{2}\,|-\frac{1}{2})\ . \tag{A2}$$

It is easy to change the radius to $\frac{1}{n}$ by introducing an additional basis vector V_p. The model Z_p generated by

$$\begin{aligned} V_0 &= (-\frac{1}{2}\,|-\frac{1}{2})\ , \\ V_p &= (\frac{1}{n}\,|\,\frac{1}{n})\ , \end{aligned} \tag{A3}$$

corresponds to a single boson at radius $\frac{1}{n}$. We can also construct a $\mathbf{Z}_2$ orbifold by assigning different boundary conditions to the two real world-sheet fermions. The model Z_q generated by

$$\begin{aligned} V_0 &= ((-\frac{1}{2})^2\,|\,(-\frac{1}{2})^2)_r\ , \\ V_q &= (-\frac{1}{2}\ 0\,|-\frac{1}{2}\ 0)_r\ , \end{aligned} \tag{A4}$$

is a $\mathbf{Z}_2$ orbifold at radius 1. However, the vectors V_p and V_q are not compatible because V_p must be in the complex fermion basis while V_q is in the real fermion basis. So the rules given in Ref. [3,9] cannot give the $\mathbf{Z}_2$ orbifold at radius $\frac{1}{n}$. To construct this model, we must use the non-abelian orbifold rules given in Section III. From Eq. (3.6), we see that $k_{ij}^{q,m}$ are the same for all m. Eq. (3.7) implies that $k_{01}^p = k_{10}^p = k_{11}^p = 0$ which are automatically satisfied. There is no new constraint from Eq. (3.8) and Eq. (3.9). Therefore, V_p and V_q define a consistent model, and the resulting Z is given by Eq. (A1).

By now, it is almost trivial to write this in the bosonic formulation. Let us start with the single boson at radius 1 (*i.e.* model Z_1). In the real boson notation, the Z_p model is given by the following set of vectors acting on the Z_1 model,

$$\begin{aligned} V_0 &= (0||0)_r\ , \\ V_p &= (\frac{1}{n}||\frac{1}{n})_r\ , \\ W_p &= (0||0)_r\ , \end{aligned} \tag{A5}$$

and the Z_q model is given the following set of vectors acting on the Z_1 model,

$$\begin{aligned} V_0 &= (0||0)_r \ , \\ V_q &= (\frac{1}{2}||\frac{1}{2})_r \ , \\ W_q &= (\frac{1}{2}||\frac{1}{2})_r \ . \end{aligned} \tag{A6}$$

Since this is a symmetric orbifold, the abelian orbifold rules are trivial to apply, and we may choose $k_{ij} = 0$. Here, the shift V_p and the twist V_q are the two generators of D_n, the isometry group $\overline{P}$. There is no simultaneously diagonalizable basis for V_p and V_q because the two operations do not commute. In the same fashion, we may apply our non-abelian orbifold rules in Section III to deduce the consistency constraints on this D_n orbifold. From Eq. (3.6), we again infer that $k_{ij}^{q;m}$ are the same for all m. Eq. (3.7) implies that $k_{01}^p = k_{10}^p = k_{11}^p = 0$ which are also automatically satisfied. There is no new constraint from Eq. (3.8) and Eq. (3.9). Therefore, V_p and V_q define a consistent model provided that we interpret the $\mathbf{Z}_n$ shift (V_p) and the $\mathbf{Z}_2$ twist (V_q) as generators of the non-abelian group D_n. The resulting Z is again given by Eq. (A1).

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