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Finite temperature correlations in the one-dimensional quantum Ising model.

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Abstract: We extend the form-factors approach to the quantum Ising model at finite temperature. The two point function of the energy is obtained in closed form, while the two point function of the spin is written as a Fredholm determinant. Using the approach of [1], we obtain, starting directly from the continuum formulation, a set of six differential equations satisfied by this two point function. Four of these equations involve only spacetime derivatives, of which three are equivalent to the equations obtained earlier in [2],[3]. In addition, we obtain two new equations involving a temperature derivative. Some of these results are generalized to the Ising model on the half line with a magnetic field at the origin.

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1. Introduction

The form-factors approach has been a surprisingly efficient tool for computing correlation functions in $1+1$ quantum integrable theories at zero temperature (T) (see eg. [4],[5],[6],[7],[8]). Recall that a quantum integrable theory in $1+1$ is conveniently described in terms of particles (massive or massless), with a factorized scattering encoded in the S matrix (solution of the Yang-Baxter equation). Multiparticle states provide a natural basis for the space of states of the theory [9],[6]. The form-factors are the matrix elements of the physical operators in this basis. In most theories, they can be obtained by solving a set of axioms generalizing Watson's equation [4]. Correlation functions are then expressed as infinite series of products of these matrix elements. While exact, re-summed expressions are hard to obtain, the approach is usually very satisfactory from a practical point of view: the series converge rather quickly [6],[7], allowing in particular potential comparison with numerical simulations.

So far, only integrable quantum theories at $T=0$ have been considered (equivalently, statistical mechanics models on the plane). For the purpose of comparison with experiments however, it is crucial to obtain results at finite temperature (which corresponds in statistical mechanics to models on a cylinder). This seems to be a rather challenging subject. In [10], a related problem has been considered, that of $T=0$ but a non-vanishing chemical potential. The main difficulty encountered there was that the non vanishing potential changes the nature of the ground state, and to use the form-factor approach one has to compute matrix elements of operators between states with a finite number of particles per unit length. This is rather delicate; in fact, simply computing the scalar product of such states has been a major achievement in the study of integrable systems [11], [1].

In the present paper, we present a first step towards finite temperature results by treating in detail the quantum Ising model in one dimension ($1d$). Many of the difficulties expected in general integrable systems, disappear in that case due to the simplicity of the S matrix, $S=-1$. Some of them remain however, especially for the spin operator which is non local in terms of fermions.

The $1d$ quantum Ising model also presents physical interest on its own. It presents a quantum-critical point between an ordered ground state with a broken symmetry and a quantum paramagnetic phase, and is the simplest system for which finite temperature crossovers near quantum critical points can be exactly computed. The crossovers are non-trivial and most of their qualitative features do generalize to higher dimensions. These exact results are therefore an important testing ground for our current understanding of these crossovers [12] [13] [14]. Indeed, as was recently noted [15], there are a number of striking similarities between the crossovers of the $d = 1$ quantum Ising model and those of the $d = 2$ $O(3)$ quantum rotor model (the latter is a useful model of anti-ferromagnetic spin fluctuations in the cuprate superconductors [13],[14].) Finally, it is not unreasonable to hope that detailed low temperature studies of experimental realizations of the $d = 1$ Ising model will emerge in the future.

The paper is organized as follows. In sections 2 and 3 we develop a method to handle correlators at finite temperature. This method focuses onto excitations over the "thermal ground state" (rather than the usual vacuum of the theory), and seems the most suitable to generalizations to truly interacting theories [10]. The method is explained in more detail in Section 2, where we derive form-factors expansions for the $d = 1$ quantum Ising correlators at finite temperature. The two point function of the energy is explicitly computed, and a Fredholm determinant expression is obtained for the two point function of the spin. In Section 3, the analysis is extended to the model on the half line with a magnetic field at the boundary. Again, the one point function of the energy is computed, and a Fredholm determinant expression for the one point function of the spin is obtained. In section 4, we return to the bulk problem, and obtain differential equations satisfied by the two-point spin correlators. Earlier work [2] had obtained differential equations for the $d = 2$ classical Ising model in the infinite plane (equivalent to the $d = 1$ quantum Ising model at $T = 0$), and precisely the same equations were later shown in [3] to also apply to the $d = 1$ quantum Ising model at non-zero T. Among our results is a set of similar equations which are consistent with, but slightly stronger than, those of [3], and obtained by a very different method. In addition, we will obtain entirely new equations which have the novel feature of involving derivatives with respect to temperature, and which are therefore special to the non-zero T case. Differential equations for correlators with temperature derivatives were obtained earlier in [1] for the non-relativistic dilute Bose gas, and the methodology of Section 4 is similar. In the conclusion, we comment on the differential equations that should be satisfied by the one point function of the spin with a boundary. We also put our results for the two-point spin correlator in the bulk in a more physical perspective.

2. The form-factor approach to correlators of the Ising model at finite T.

2.1. Generalities

We describe the quantum Ising model in 1d in terms of massive fermions. The associated creation and annihilation operators have the algebra:

$$\begin{aligned} A(\beta_1)A(\beta_2) &= -A(\beta_2)A(\beta_1) \\ A^\dagger(\beta_1)A^\dagger(\beta_2) &= -A^\dagger(\beta_2)A^\dagger(\beta_1) \\ A(\beta_1)A^\dagger(\beta_2) &= -A^\dagger(\beta_2)A(\beta_1) + 2\pi\delta(\beta_1-\beta_2). \end{aligned} \tag{2.1}$$

In these expressions, the β_i's are the usual rapidity variables parameterizing momentum and energy as $p(\beta) = m\sinh\beta$, $e(\beta) = m\cosh\beta$.

Let us now consider the question of evaluating the two point function of some operator $\mathcal{O}$ in the Ising model at finite temperature T. The space coordinate is x, $-\infty < x < \infty$, and we call t the time coordinate. Formally one has:

$$< \mathcal{O}(x,t)\mathcal{O}(0,0) >= \frac{1}{Z}\sum_{\psi,\psi'} e^{-E_\psi/T}\, \frac{< \psi|\mathcal{O}(x,t)|\psi' >< \psi'|\mathcal{O}(0,0)|\psi >}{< \psi|\psi >< \psi'|\psi' >}, \tag{2.2}$$

where the states $|\psi >$ are multiparticle states, the matrix elements of $\mathcal{O}$ are the so-called form-factors, and Z is the partition function which formally corresponds to $Z = \sum_\psi e^{-E_\psi/T}$.

There are divergences in the expression (2.2) that are a result of working with multiparticle states that are defined in infinite volume (the x-direction). These infinite volume divergences manifest themselves in the norms of the states, e.g. $< \beta|\beta >= 2\pi\delta(0)$. These can be regulated by letting the cylinder have a finite length L and letting $L \to \infty$. The $\delta(0)$ are then regularized to $\frac{mL}{2\pi}\cosh\beta$. With this explicit regularization, it is natural to consider a thermodynamic approach to the expression (2.2), which we describe in the remainder of this section. In this free field situation, it is possible to deal with the expression (2.2) directly, and this provides a check on the thermodynamic approach; this is presented in Appendix A.

2.2. Thermodynamic approach

If we regularize the norms as explained above, the sum over intermediate states is a discrete sum over states with allowed momenta. On general grounds, we expect the partition function to go like $\exp(Lf)$ where f is the dimensionless free energy per unit length.

That Z behaves exponentially means that the sum is dominated by terms where the energy goes like L, the size of the system. In other words, the sum is dominated by multiparticle states that have a number of particles proportional to the size of the system L. These states we call **macroscopic**, and for our purpose they will be completely characterized by non-vanishing densities of particles (per unit length) as $L \to \infty$. This situation is to be contrasted with computations at $T = 0$, where one considers only microscopic states, that have a finite number of particles as $L \to \infty$.

Since microscopic states give a negligible contribution to (2.2) for $T \neq 0$, we can replace $\sum_\psi$ by a functional integral :

$$\sum_\psi \to \int [d\rho], \tag{2.3}$$

where ρ denotes the density of particles per unit length (the exact measure to put in (2.3) deserves some discussion, but it will not matter in what follows). The denominator and numerator become respectively

$$\begin{aligned} Z &= \int [d\rho] e^{-\beta\{E([\rho])+S([\rho])\}} \\ \text{Num} &= \int [d\rho] e^{-\beta\{E([\rho])+S([\rho])\}} \sum_{\psi'} \frac{< [\rho]|\mathcal{O}(x,t)|\psi' >< \psi'|\mathcal{O}(0,0)|[\rho] >}{< \psi'|\psi' >}. \end{aligned} \tag{2.4}$$

Here $E[\rho] = mL \int d\theta \rho(\theta) \cosh\theta$, the entropy follows from Stirling's formula $S = L\int d\theta[(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h \ln\rho^h]$, where ρ^h is the density of holes, ρ the density of particles, with $\rho(\beta) + \rho^h(\beta) = \frac{1}{2\pi} m \cosh\beta$. The point now is that the argument of the numerator and the denominator (2.4) differ only by the correlator, which is of order one. In the limit $L \to \infty$, both are dominated by the same saddle point, characterized by $\rho/\rho^h = \exp(-m\cosh\beta/T)$. From now on we reserve the notation ρ for the saddle point density. Then the exponential contributions of the numerator and denominator cancel out, and we get the simple result (a similar result appears in a different context in [1]) :

$$< \mathcal{O}(x,t)\mathcal{O}(0,0 >= \frac{1}{< 0_T|0_T >} \sum_{\psi'} \frac{< 0_T|\mathcal{O}(x,t)|\psi' >< \psi'|\mathcal{O}(0,0)|0_T >}{< \psi'|\psi' >}, \tag{2.5}$$

where we denote by $|0_T >$ **any** multiparticle state that is characterized by the macroscopic density $[\rho]$ (which one does not matter). We will refer to this state, a bit incorrectly, as the "thermal ground state".

Now, the remaining task is to evaluate the matrix elements. There are several ways to proceed, - here, we present what seems the most physical approach, and probably the most suitable to generalizations.

2.3. Excitations over the thermal ground state

The idea is not to use the usual form-factors, which correspond to excitations above the vacuum $|0>$, but rather to introduce new form-factors appropriate for the excitations above $|0_T>$. Consider therefore a multiparticle state that realizes the density ρ. The excitations above this state are of two kinds: one can add a particle, or one can create a hole. We will take into account these two possibilities by introducing two types of "thermal" particles, characterized by a label $\epsilon = \pm 1$. Observe that the sets of rapidities where a thermal particle with $\epsilon = 1$ and $\epsilon = -1$ can be created are separate. The creation/annihilation operators of thermal particles with the same ϵ satisfy (2.1) while operators with different ϵ anticommute. The excitations therefore still have factorized scattering with $S = -1$. We now define the form-factors of an operator $\mathcal{O}$ as

$$F(\beta_1, \beta_2, ..., \beta_n)_{\epsilon_1,...,\epsilon_n} \equiv \frac{1}{\sqrt{< 0_T|0_T >}} {}^{\epsilon_n,...,\epsilon_1} < \beta_n \ldots \beta_1|\mathcal{O}|0_T >, \tag{2.6}$$

where the subscript T is implicit for multiparticle states. These form-factors have to satisfy relations that are very similar to the ones for excitations over the ground state. One has

$$F(\beta_1, \beta_2, ..., \beta_n)_{\epsilon_1,...,\epsilon_n} = -F(\beta_2, \beta_1, ..., \beta_n)_{\epsilon_2,\epsilon_1,...,\epsilon_n}, \tag{2.7}$$

from the fact that $S = -1$. CPT invariance gives :

$$F(\beta_1, \beta_2, ..., \beta_n + 2\pi i)_{\epsilon_1,...,\epsilon_n} = F(\beta_n, \beta_1, ..., \beta_{n-1})_{\epsilon_n,\epsilon_1,...,\epsilon_{n-1}}. \tag{2.8}$$

The third axiom is slightly more complicated. First, one expects as usual an annihilation pole at $\beta_i = \beta_j - i\pi$ when $\epsilon_i = \epsilon_j$. In addition, remembering the origin of $|0_T>$, and the fact that a hole and a particle can annihilate each other, there should also be a pole at $\beta_i = \beta_j$ when $\epsilon_i = -\epsilon_j$. This leads to

$$\begin{aligned} \mathrm{Res}_{\beta_1=\beta_2-i\pi} F^{(n)}(\beta_1, \beta_2, ..., \beta_n)_{\epsilon_1,\epsilon_1,...\epsilon_n} &= F^{(n-2)}(\beta_3, ..., \beta_n)_{\epsilon_3,...,\epsilon_n} \\ \mathrm{Res}_{\beta_1=\beta_2} F^{(n)}(\beta_1, \beta_2, ..., \beta_n)_{\epsilon_1,-\epsilon_1,...\epsilon_n} &= F^{(n-2)}(\beta_3, ..., \beta_n)_{\epsilon_3,...,\epsilon_n} \end{aligned} \tag{2.9}$$

There are two minimal solutions to these equations. The one corresponding to the energy operator is

$$\begin{aligned} F(\beta_1, \beta_2)_{++} &= m \sinh(\frac{\beta_1 - \beta_2}{2}) \\ F(\beta_1, \beta_2)_{+-} &= im \cosh(\frac{\beta_1 - \beta_2}{2}) \\ F(\beta_1, \beta_2)_{-+} &= -im \cosh(\frac{\beta_1 - \beta_2}{2}) \\ F(\beta_1, \beta_2)_{--} &= m \sinh(\frac{\beta_1 - \beta_2}{2}) \end{aligned} \tag{2.10}$$

The solution corresponding to the spin σ and disorder operators μ is

$$F(\beta_1, ..., \beta_n)_{\epsilon_1,...,\epsilon_n} = i^{n/2} \prod_{i<j} \tanh\left(\frac{\beta_i - \beta_j}{2}\right)^{\epsilon_i \epsilon_j}, \tag{2.11}$$

where n is even for σ, with $< 0_T|\sigma|0_T > / < 0_T|0_T >= 1$, and n is odd for μ. Note that we have chosen to work in the ordered phase here, and that the form-factors approach requires us to consider the subtracted energy operator (of vanishing one-point function), while the spin is not subtracted. In (2.10) as well as (2.11), the overall normalization is largely arbitrary.

In this free field situation, the particle and hole form factors are deduced from the usual ones for particles only by shifting the hole rapidities by $i\pi$. This hints at a more direct way of obtaining expressions for the correlators, which is explained in appendix A.

Knowing the form-factors, the two point function of the operator $\mathcal{O}$ can finally be computed. A little subtlety appears in the sum over intermediate states. While multi-thermal particle states are normalized as usual, their allowed rapidities are distributed according to the filling fractions

$$f_\epsilon(\beta) = \frac{1}{1 + e^{-\epsilon m \cosh \beta / T}}. \tag{2.12}$$

One finds therefore

$$\begin{aligned} < \mathcal{O}(x,t)\mathcal{O}(0,0) >= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_i} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left\{ \frac{d\beta_i}{2\pi} f_{\epsilon_i}(\beta_i) e^{-i\epsilon_i [mt \cosh \beta_i - mx \sinh \beta_i]} \right\} \\ \times |F(\beta_1, \ldots, \beta_n)_{\epsilon_1,\ldots,\epsilon_n}|^2. \end{aligned} \tag{2.13}$$

It is not difficult to verify that the above expression satisfies the Martin-Schwinger-Kubo relation:

$$< \mathcal{O}(x, t - i/T)\mathcal{O}(0) >=< \mathcal{O}(0)\mathcal{O}(x,t) > . \tag{2.14}$$

2.4. The energy correlations

As a check, it is useful to consider the conformal limit of the energy two-point function, which can be compared with standard results. We obtain the conformal correlator by the limit

$$\begin{aligned} \beta = \pm\beta_0 \pm \theta, \quad \beta_0 \to \infty \\ m \to 0, \frac{m}{2} e^{\beta_0} \to \mu. \end{aligned} \tag{2.15}$$

Here μ is a parameter adjusting the normalization. The spectrum separates into left and right moving particles with $e = \mu e^{\theta} = \pm p$. In this limit, only the left-right and right-left form factors will be non zero. They are simply :

$$
\begin{aligned}
F(\theta_1, \theta_2)^{RL}_{++} &= \mu e^{(\theta_1+\theta_2)/2} \\
F(\theta_1, \theta_2)^{RL}_{+-} &= i\mu e^{(\theta_1+\theta_2)/2} \\
F(\theta_1, \theta_2)^{RL}_{-+} &= -i\mu e^{(\theta_1+\theta_2)/2} \\
F(\theta_1, \theta_2)^{RL}_{--} &= \mu e^{(\theta_1+\theta_2)/2}.
\end{aligned}
\tag{2.16}
$$

The two point function of the energy factorizes as the product of a right and a left moving part. For the right moving part we find :

$$
\begin{aligned}
< e(x,t)e(0,0) >_R &= \mu^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} [f_+(\theta) e^{-2z_R \mu e^{\theta}} + f_-(\theta) e^{2z_R \mu e^{\theta}}] e^{\theta} \\
&= \mu \int_0^{\infty} \frac{du}{2\pi} \left[\frac{e^{-2uz_R}}{1+e^{-u/T}} + \frac{e^{2uz_R}}{1+e^{u/T}} \right] \\
&= \frac{\mu T}{2\sin(2\pi T z_R)},
\end{aligned}
\tag{2.17}
$$

where we have set $z_R = (t - ix)/2$ (and similarly $z_L = (t + ix)/2$). This is the well known conformal result for the fermions correlator, and multiplying by its conjugate gives of course the correct energy correlator.

In the massive case, the holomorphic and anti-holomorphic parts will not be decoupled and the solution will be more involved. Equation (2.13) gives, restricting to $x = 0$ for simplicity :

$$
\begin{aligned}
&\frac{(2\pi)^2}{m^2} < e(0,t)e(0,0) >= \\
&\int_0^{\infty} du_1 du_2 \frac{e^{imt(u_1+1/u_1)/2}}{1+e^{m(u_1+1/u_1)/2T}} \frac{e^{imt(u_2+1/u_2)/2}}{1+e^{m(u_2+1/u_2)/2T}} \left(\frac{1}{u_1} - \frac{1}{u_2}\right)^2 \\
&+ \left(\frac{1}{u_1} - \frac{1}{u_2}\right)^2 + \frac{e^{imt(u_1+1/u_1)/2}}{1+e^{m(u_1+1/u_1)/2T}} \frac{e^{-imt(u_2+1/u_2)/2}}{1+e^{-m(u_2+1/u_2)/2T}} \left(\frac{1}{u_1} + \frac{1}{u_2}\right)^2 \\
&+ \frac{e^{-imt(u_1+1/u_1)/2}}{1+e^{-m(u_1+1/u_1)/2T}} \frac{e^{imt(u_2+1/u_2)/2}}{1+e^{m(u_2+1/u_2)/2T}} \left(\frac{1}{u_1} + \frac{1}{u_2}\right)^2 \\
&+ \frac{e^{-imt(u_1+1/u_1)/2}}{1+e^{-m(u_1+1/u_1)/2T}} \frac{e^{-imt(u_1+1/u_2)/2}}{1+e^{-m(u_1+1/u_1)/2T}} \left(\frac{1}{u_1} - \frac{1}{u_2}\right)^2,
\end{aligned}
\tag{2.18}
$$

where we have introduced the variable $u = e^{\beta}$. The integrated result is given by :

$$
\begin{aligned}
&\frac{(2\pi)^2}{m^2} < e(0,t)e(0,0) >= \\
&\left\{\sum_{n=0}^{\infty}(-1)^n\left[K_1(imt+nm/T)+K_1(nm/T+m/T-imt)\right]\right\}^2 \\
-&\left\{\sum_{n=0}^{\infty}(-1)^n\left[K_0(imt+nm/T)-K_0(nm/T+m/T-imt)\right]\right\}^2 .
\end{aligned}
\tag{2.19}
$$

Here K are Bessel functions of the second kind. It reproduces the conformal result in the limit $m \to 0$. In the limit $T \to 0$ it also reproduces the result of [16].

2.5. The spin-disorder correlations

The conformal spin correlator cannot be naively recovered by the limit (2.15), the integrals being infrared divergent. In fact, the understanding of the spin correlator is considerably more involved, and will be the subject of most of this paper.

Let us continue to Euclidean space space $t \to -it$, and define $z = (t - ix)/2$, $\overline{z} = (t + ix)/2$. Define

$$
\tau_{\pm} = \langle \sigma(z,\overline{z})\sigma(0)\rangle \pm \langle \mu(z,\overline{z})\mu(0)\rangle. \tag{2.20}
$$

It is convenient to express everything in terms of the variable u:

$$
u = e^{\theta}. \tag{2.21}
$$

Then, the formula (2.13) leads to

$$
\begin{aligned}
\tau_{\pm} = \sum_{N=0}^{\infty}\frac{(\pm 1)^N}{N!}\sum_{\epsilon_1,\epsilon_2,..=\pm 1}\int_0^{\infty}\frac{du_1}{2\pi u_1}\cdots\frac{du_N}{2\pi u_N}&\left[\prod_{i=1}^{N} f_{\epsilon_i}(u_i)e^{-\epsilon_i(mzu_i+m\overline{z}/u_i)}\right] \\
\times &\left[\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^{2\epsilon_i\epsilon_j}\right],
\end{aligned}
\tag{2.22}
$$

where it should be clear from the context that the thermal filling factors have changed their definition from (2.12) to one appropriate for an energy expressed in terms of the u variable

$$
f_{\epsilon}(u) = \frac{1}{1+e^{-\epsilon m(u+1/u)/2T}}. \tag{2.23}
$$

At $T = 0$ (2.22) reduces to well known expressions (see *e.g.* [17], [18]).

The poles in (2.22), which are due to particle hole annihilation, are now clearly seen as an artifact of the continuum limit. Consider the case where L is finite and the particles have quantized momenta. Then, a particle and a hole cannot have the same rapidity: a state is either filled or empty. When replacing discrete sums by integrals, this exclusion disappeared, and has to be reinstated by hand. The prescription will be to move slightly one of the rapidities off the real axis. This is what will be understood in the following.

The functions $\tau_\pm$ can be expressed as a Fredholm determinant of a 2×2 matrix of integral operators

$$\tau_\pm = \mathrm{Det}(1 \pm \mathbf{W}), \tag{2.24}$$

where

$$\mathbf{W}(u,v) = \begin{pmatrix} W_{++}(u,v) & W_{+-}(u,v) \\ W_{-+}(u,v) & W_{--}(u,v) \end{pmatrix}. \tag{2.25}$$

The two dimensional vector space structure arises from the ϵ indices distinguishing particles and holes. Extending the Fredholm theory to this case, one has by definition:

$$\mathrm{Det}(1 \pm \mathbf{W}) = \sum_{N=0}^{\infty} \frac{(\pm 1)^N}{N!} \sum_{\epsilon_1 \cdots \epsilon_N = \pm} \int_0^\infty du_1 \cdots du_N \; \det\{W_{\epsilon_i,\epsilon_j}(u_i,u_j)\}, \tag{2.26}$$

where $\det\{W_{\epsilon_i,\epsilon_j}(u_i,u_j)\}$ is an ordinary determinant of the finite $N \times N$ matrix with $W_{\epsilon_i,\epsilon_j}(u_i,u_j)$ as entries:

$$\det\left\{W_{\epsilon_i,\epsilon_j}(u_i,u_j)\right\} = \begin{vmatrix} W_{\epsilon_1\epsilon_1}(u_1,u_1) & W_{\epsilon_1\epsilon_2}(u_1,u_2) & \cdots & W_{\epsilon_1,\epsilon_N}(u_1,u_N) \\ W_{\epsilon_2\epsilon_1}(u_2,u_1) & \cdot & \cdots & W_{\epsilon_2\epsilon_N}(u_2,u_N) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ W_{\epsilon_N\epsilon_1}(u_N,u_1) & \cdot & \cdots & W_{\epsilon_N\epsilon_N}(u_N,u_N) \end{vmatrix}. \tag{2.27}$$

Let $\{\frac{1}{u_i+v_j}\}$ for $1 \le i,j \le n$ denote the $n \times n$ matrix with $1/(u_i+v_j)$ as entries. Then using the identity

$$\det\{\frac{1}{u_i+v_j}\} = \frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(u_i+v_j)}, \tag{2.28}$$

one finds that the sum (2.22) can be reproduced with the following choice of kernel:

$$\mathbf{W}(u,v) = \begin{pmatrix} \frac{e_+(u)e_+(v)}{u+v} & \frac{e_+(u)e_-(v)}{u-v} \\ \frac{e_-(u)e_+(v)}{u-v} & \frac{e_-(u)e_-(v)}{u+v} \end{pmatrix}, \tag{2.29}$$

where

$$\begin{aligned} e_+(u) &= \sqrt{\frac{f_+(u)}{\pi}} \exp\left(-\frac{m}{2}(zu+\overline{z}/u)\right) \\ e_-(u) &= \sqrt{\frac{f_-(u)}{\pi}} \exp\left(+\frac{m}{2}(zu+\overline{z}/u)\right). \end{aligned} \tag{2.30}$$

In principle, all of our subsequent results for $\tau_\pm$ can be derived from the above formulation in terms of Fredholm determinants. However, the 2×2 matrix structure makes the manipulations quite cumbersome, and the following trick for dispensing with the 2×2 structure is quite useful. Notice that in (2.22) the double sum over particles and holes can be traded for an integral over u running over the whole real axis

$$\begin{aligned}\tau_\pm = \sum_{N=0}^{\infty} \frac{(\pm 1)^N}{N!} \int_{-\infty}^{\infty} \frac{du_1}{2\pi|u_1|} \cdots \frac{du_N}{2\pi|u_N|} \left[\prod_{i=1}^{N} f(u_i) e^{-(mzu_i + m\bar{z}/u_i)}\right] \\ \times \left[\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2\right].\end{aligned} \tag{2.31}$$

Here, we introduced

$$f(u) = \frac{1}{1+e^{-m(u+1/u)/2T}}. \tag{2.32}$$

Now, one has

$$\tau_\pm = \mathrm{Det}(1 \pm \mathbf{K}), \tag{2.33}$$

where $\mathbf{K}$ is a scalar kernel

$$\mathbf{K}(u,v) = \frac{e(u)e(v)}{u+v}, \tag{2.34}$$

with

$$e(u) = \sqrt{\frac{f(u)\mathrm{sgn}(u)}{\pi}} \exp\left(-\frac{m}{2}(zu+\bar{z}/u)\right), \tag{2.35}$$

and it is implied that the Fredholm determinant in (2.33) is taken over functions in the domain $-\infty < u < \infty$. There is a non-analyticity in the kernel at $u=0$, and any subtleties associated with this have to be resolved by regarding the above formalism simply as a convenient shorthand for the better-defined 2×2 matrix integral operators in (2.24). In practice, we will find that one can simply ignore the singular behavior at $u=0$: particles (or holes) with small u, have a very large energy ($\sim 1/u$) and their contribution is exponentially suppressed.

In Section 4 we will derive some differential equations that are obeyed by (2.31).

3. Ising model with a boundary and a temperature.

It is also interesting to include a finite temperature in the quantum Ising model on the half line $x \in [-\infty, 0]$ with a boundary magnetic field. This provides a good example of the convenience of the thermal approach.

It will be useful to think of this problem from an Euclidean point of view, where the theory at finite temperature is interpreted as a theory on a cylinder of radius $R = 1/T$. One can then consider both possible directions as imaginary time direction; we will refer to the picture where the hamiltonian evolves in the x direction along the length of the cylinder as the L channel, and the crossed channel where the evolution is along the circumference of the cylinder as the R-channel.

At $T = 0$, the boundary problem is very conveniently addressed in the L-channel. In this picture the boundary interactions are contained in a boundary state $|B>$ at x=0. Introduce the quantity [19]

$$\widehat{R}(\beta) = i\tanh(\beta/2)\frac{\kappa+\cosh\beta}{\kappa-\cosh\beta}, \tag{3.1}$$

which is related to the reflection matrix by $\widehat{R}(\beta) = R(\frac{i\pi}{2}-\beta)$. The constant κ is related to the boundary magnetic field h by $\kappa = 1 - h^2/2m$. One has then [19]

$$|B>= \exp\left[\int_{-\infty}^{\infty}\frac{d\beta}{4\pi}\widehat{R}(\beta)A^\dagger(-\beta)A^\dagger(\beta)\right]|0>. \tag{3.2}$$

Suppose now that we include a temperature. As we have seen in section 2, in the crossed R-channel, the effect of this temperature can be absorbed by defining a dressed theory, with two sorts of particles, new form-factors, together with an integration metric f_ϵ. If one formally carries out the same arguments as in [19], one ends up with a new boundary state that is very similar to (3.2), for the dressed theory (that is in particular the theory involving dressed form-factors):

$$|B_T>= \exp\left[\sum_{\epsilon=\pm}\int_{-\infty}^{\infty}\frac{d\beta}{4\pi}\widehat{R}_\epsilon(\beta)f_\epsilon\left(i\frac{\pi}{2}-\beta-i\eta\right)A_\epsilon^\dagger(-\beta)A_\epsilon^\dagger(\beta)\right]|0_T>, \tag{3.3}$$

where

$$\widehat{R}_+(\beta) = \widehat{R}(\beta), \widehat{R}_-(\beta) = \widehat{R}(\beta - i\pi), \tag{3.4}$$

and the $A_\epsilon^\dagger$ operators as the same as in section 2. In (3.3), $\eta > 0$ is a shift in the imaginary rapidity direction necessary to suppress the poles, and ensure the right $T \to 0$ limit.

For completeness, formula (3.3) is proven in appendix B using a direct approach.

As an application, let us look at the energy in the Ising model with boundary and temperature. Previous formula give

$$< 0_T|\epsilon(x)|B_T>= \frac{m}{2}\sum_{\epsilon=\pm}\int_{-\infty}^{\infty}\frac{d\beta}{4\pi}\widehat{R}_\epsilon(\beta)f_\epsilon\left(\frac{i\pi}{2}-\beta-i\eta\right) e^{2mx\epsilon\cosh\beta}(e^\beta - e^{-\beta}). \tag{3.5}$$

Observe that the integrals involving holes with $\epsilon = -1$ in (3.5) converge only when $2|x| < \eta/T$, which implies $T|x| << 1$. The correlators are better defined by changing the contour of integration, which is actually equivalent to a computation in the crossed channel. For brevity, we shall not do so here.

We can similarly consider the spin correlator. One finds

$$
\begin{aligned}
&< 0_T|\sigma| -\beta_1, \beta_1, \cdots, -\beta_n, \beta_n >_{\epsilon_1,\ldots,\epsilon_n} = \\
&\qquad (-i)^n \prod_{i=1}^{n} \tanh\beta_i \prod_{i<j} \tanh\left(\frac{\beta_i - \beta_j}{2}\right)^{2\epsilon_i\epsilon_j} \tanh\left(\frac{\beta_i + \beta_j}{2}\right)^{2\epsilon_i\epsilon_j} .
\end{aligned}
\tag{3.6}
$$

From this and (3.3) it follows that

$$
\begin{aligned}
&< 0_T|\sigma(x)|B_T >= \\
&\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{\epsilon_i} \int_{-\infty}^{\infty} \prod_i \left\{ \frac{d\beta_i}{4\pi} \widehat{R}_{\epsilon_i}(\beta_i) f_{\epsilon_i}\left(\frac{i\pi}{2} - \beta_i - i\eta\right) e^{2mx\epsilon_i \cosh\beta_i} \right\} \\
&\qquad \times \prod_{i=1}^{n} \tanh\beta_i \prod_{i<j} \tanh\left(\frac{\beta_i - \beta_j}{2}\right)^{2\epsilon_i\epsilon_j} \tanh\left(\frac{\beta_i + \beta_j}{2}\right)^{2\epsilon_i\epsilon_j} .
\end{aligned}
\tag{3.7}
$$

The reality of the one point function of σ can be checked on (3.7) by observing that $\widehat{R}^*_\epsilon(\beta) = \widehat{R}_\epsilon(-\beta), R^*_\epsilon(\beta) = R_\epsilon(-\beta)$.

Once again it is convenient to re-express (3.7) into a simpler form by introducing the variable $u = e^\beta$ and trading the sum over ϵ for an extended integral of u over the whole real axis. One finds then

$$
\begin{aligned}
< 0_T|\sigma(x)|B_T > &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \prod_i \left\{ \frac{du_i}{4\pi|u_i|} \widehat{R}(u_i) f(i/u_i) e^{mx(u_i+1/u_i)} \frac{u_i^2 - 1}{u_i^2 + 1} \right\} \\
&\qquad \times \prod_{i<j} \left[\frac{u_i - u_j}{u_i + u_j} \quad \frac{u_i u_j - 1}{u_i u_j + 1} \right]^2 ,
\end{aligned}
\tag{3.8}
$$

where $f(u)$ is as before (2.23).

As was done for zero temperature in [20], one can express (3.8) as a Fredholm determinant. One has the identity,

$$
\begin{aligned}
\prod_{i<j} \left[\frac{u_i - u_j}{u_i + u_j} \quad \frac{u_i u_j - 1}{u_i u_j + 1} \right]^2 &= \prod_{i<j} \left(\frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2 \\
&= \det \left\{ \frac{2\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j} \right\} ,
\end{aligned}
\tag{3.9}
$$

where $\mu \equiv (u+1/u)/2$. Thus,

$$
\begin{aligned}
< 0_T|\sigma(x)|B_T > &= \mathrm{Det}(1+\mathbf{V}) \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left(\prod_i du_i \right) \mathrm{det} V(u_i, u_j),
\end{aligned}
\tag{3.10}
$$

where $\mathbf{V}$ is the scalar kernel:

$$
\begin{aligned}
\mathbf{V}(u,u') &= \frac{e(u)e(u')}{u+1/u+u'+1/u'} \\
e(u) &= \sqrt{\frac{(u-1/u)(-i\widehat{R}(u))f(i/u)}{2\pi|u|}} e^{mx(u+1/u)/2}.
\end{aligned}
\tag{3.11}
$$

Using the techniques of this paper it is also possible to compute correlation functions in boundary field theory where the finite size is in the x-direction, i.e. on a infinitely long, finite width strip. This is described in Appendix C.

4. Differential Equations for Bulk Correlators

In this section we return to a study of the bulk two-point functions of spin and disorder operators at finite temperature. Recall that in Section 2 we had obtained an expression for these as a Fredholm determinant of a scalar kernel $\mathbf{K}$ defined in (2.34). In this section we will use this expression to derive a set of non-linear, integrable, partial differential equations for the two-point functions. The general method we follow is that reviewed in [1] for the case of the non-relativistic Bose gas, although we shall take a somewhat more algebraic point of view. The computations are somewhat involved, and readers not interested in the methodology can skip ahead to Section 4.5, where we give a self-contained presentation of the main results.

4.1. The Resolvent

The first step in deriving differential equations for (2.34) is to find the "resolvent" for the kernel $\mathbf{K}$. We will need the resolvents $\mathbf{R}_\pm$ defined by

$$
(1-\mathbf{R}_\pm)(1\pm\mathbf{K}) = 1. \tag{4.1}
$$

The resolvents $\mathbf{R}_\pm$ are also integral operators, e.g.

$$
(\mathbf{R}\mathbf{K})(u,v) = \int_{\infty}^{\infty} dw \; \mathbf{R}(u,w)\mathbf{K}(w,v). \tag{4.2}
$$

The resolvent for a kernel of the type (2.34) was described in [21]. Let $[V_{\pm}]$ denote the class of scalar integral operators of the type $e(u)e'(v)/(u \pm v)$. As explained in[21], these operators form a group with a Z_2 graded multiplication law:

$$[V_{(-)}] \times [V_{(-)}] = [V_{(-)}], \qquad [V_{(+)}] \times [V_{(-)}] = [V_{(+)}], \qquad [V_{(+)}] \times [V_{(+)}] = [V_{(-)}]. \tag{4.3}$$

It turns out that $\mathbf{R}_{\pm}$ has both $[V_+], [V_-]$ pieces, so let us write

$$\mathbf{R}_{\pm} = \mathbf{H} \pm \mathbf{F}, \qquad \mathbf{H} \in [V_-], \ \mathbf{F} \in [V_+], \tag{4.4}$$

which turns out to be consistent with the Z_2 gradings. Define functions $g_{\pm}(u)$ as follows:

$$(1 - \mathbf{R}_{\pm})e = g_{\pm}. \tag{4.5}$$

Above, $(\mathbf{R}e)(u)$ is shorthand for $\int dv \mathbf{R}(u,v)e(v)$. Then the definition (4.1) and the fact that $\mathbf{K}^T = \mathbf{K}$, $\mathbf{R}^T = \mathbf{R}$ also give us the relations

$$\begin{aligned} &(1 \pm \mathbf{K})g_{\pm} = e \\ &e(1 - \mathbf{R}_{\pm}) = g_{\pm} \\ &g_{\pm}(1 \pm \mathbf{K}) = e \end{aligned} \tag{4.6}$$

.

Then the resolvent can be expressed in terms of the g functions:

$$\begin{aligned} \mathbf{H}(u,v) &= \frac{1}{2(u-v)} \left(g_-(u)g_+(v) - g_+(u)g_-(v) \right) \\ \mathbf{F}(u,v) &= \frac{1}{2(u+v)} \left(g_-(u)g_+(v) + g_+(u)g_-(v) \right). \end{aligned} \tag{4.7}$$

4.2. Intertwining Relations

A central property of the integral operators $\mathbf{K}$ and $\mathbf{R}_{\pm}$ is a set of intertwining relations between them and a family of diagonal operators. Essentially all of the differential equations and constraints that we shall derive in subsequent sections are a simple consequence of these intertwining relations. These relations constitute a succinct algebraic encapsulation of the integrability of the $\mathbf{K}$ kernel.

The simplest of the intertwining relations are with the diagonal operators $\mathbf{d}_n(u,v)$ (n integer) which multiply the operand by u^n when $u = v$; explicitly the $\mathbf{d}_n$ are defined by

$$\begin{aligned} (\mathbf{O}_1 \mathbf{d}_n \mathbf{O}_2)(u,v) &= \int_{-\infty}^{\infty} dw_1 dw_2 \mathbf{O}_1(u,w_1) \mathbf{d}_n(w_1,w_2) \mathbf{O}_2(w_2,v) \\ &= \int_{-\infty}^{\infty} dw \mathbf{O}_1(u,w) w^n \mathbf{O}_2(w,v) \end{aligned} \tag{4.8}$$

where $\mathbf{O}_1$ and $\mathbf{O}_2$ are arbitrary operators. Now, using the definition (2.29) , we can easily obtain our first set of intertwining relations:

$$
\begin{aligned}
(\mathbf{d}_1\mathbf{K}+\mathbf{K}\mathbf{d}_1)(u,v) &= e(u)e(v)\\
(\mathbf{d}_{-1}\mathbf{K}+\mathbf{K}\mathbf{d}_{-1})(u,v) &= \frac{e(u)}{u}\frac{e(v)}{v}
\end{aligned}
\tag{4.9}
$$

Notice that the right-hand-sides are all projection operators *i.e.* they are all products of functions of u and v. The intertwining of $\mathbf{d}_n$ for general n with the $\mathbf{K}_\pm$ now follows by repeated application of the above and the identity $\mathbf{d}_{n+m}=\mathbf{d}_n\mathbf{d}_m$. One can also check that the operator $\mathbf{d}_1\mathbf{d}_{-1}=1$ commutes with the $\mathbf{K}_\pm$.

In a similar manner we can obtain the intertwining of $\mathbf{R}_\pm$ with the $\mathbf{d}_n$ by using (4.4) and (4.7):

$$
\begin{aligned}
(\mathbf{d}_1\mathbf{R}_\pm-\mathbf{R}_\mp\mathbf{d}_1)(u,v) &= \pm g_\mp(u)g_\pm(v)\\
(\mathbf{d}_{-1}\mathbf{R}_\pm-\mathbf{R}_\mp\mathbf{d}_{-1})(u,v) &= \pm\frac{g_\pm(u)}{u}\frac{g_\mp(v)}{v}
\end{aligned}
\tag{4.10}
$$

Again, the right-hand-sides are projectors.

We now introduce our second set of diagonal operators: $\mathbf{d}_z$ and $\mathbf{d}_{\overline{z}}$, which take derivatives with respect to z and $\overline{z}$ respectively. They are defined by

$$
\begin{aligned}
(\mathbf{O}_1\mathbf{d}_z\mathbf{O}_2)(u,v) &= \int_{-\infty}^{\infty} dw\,\mathbf{O}_1(u,w)\partial_z\mathbf{O}_2(w,v)\\
(\mathbf{O}_1\mathbf{d}_{\overline{z}}\mathbf{O}_2)(u,v) &= \int_{-\infty}^{\infty} dw\,\mathbf{O}_1(u,w)\partial_{\overline{z}}\mathbf{O}_2(w,v)
\end{aligned}
\tag{4.11}
$$

Notice that the derivatives act on everything to their right *i.e.* on the $\mathbf{O}_2$ and on all operands the left-hand-side may act on. Again, using the defining relations (2.29), (2.35), we can easily obtain the intertwining of the $\mathbf{d}_z$, $\mathbf{d}_{\overline{z}}$ with $\mathbf{K}$:

$$
\begin{aligned}
(\mathbf{d}_z\mathbf{K}-\mathbf{K}\mathbf{d}_z)(u,v) &= -\frac{m}{2}e(u)e(v)\\
(\mathbf{d}_{\overline{z}}\mathbf{K}-\mathbf{K}\mathbf{d}_{\overline{z}})(u,v) &= -\frac{m}{2}\frac{e(u)}{u}\frac{e(v)}{v}
\end{aligned}
\tag{4.12}
$$

As usual, the right-hand-sides are projectors. The intertwining of the $\mathbf{R}_\pm$ with the $\mathbf{d}_z$, $\mathbf{d}_{\overline{z}}$ can be deduced from (4.12) and (4.1), but as we shall not need it in this paper, we will refrain from displaying it.

Finally, we will need the intertwining of $\mathbf{K}_\pm$ with an operator involving temperature derivatives. We shall follow the idea of [1], and choose an operator which commutes with the thermal factor (2.23). The appropriate operator, $\mathbf{d}_T$ is defined by

$$
(\mathbf{O}_1\mathbf{d}_T\mathbf{O}_2)(u,v) = \int_{-\infty}^{\infty} dw\,\mathbf{O}_1(u,w)\left(\left(w-\frac{1}{w}\right)T\frac{\partial}{\partial T}+\left(w+\frac{1}{w}\right)w\frac{\partial}{\partial w}\right)\mathbf{O}_2(w,v).
\tag{4.13}
$$

Again all derivatives act on all operators to their right. This particular combination of derivatives was chosen because

$$\left(\left(u-\frac{1}{u}\right)T\frac{\partial}{\partial T}+\left(u+\frac{1}{u}\right)u\frac{\partial}{\partial u}\right)\Omega\left((u+1/u)/T\right)=0 \tag{4.14}$$

where $\Omega(y)$ is an arbitrary function of the combination $y=(u+1/u)/T$ alone. This is precisely the combination in which T occurs in $\mathbf{K}_\pm$, and (4.14) raises the possibility that the intertwining of $\mathbf{d}_T$ with $\mathbf{K}_\pm$ may be simple. It turns out that the result is somewhat more complicated than those obtained earlier, but is nevertheless a sum of a small number of projection operators. A lengthy, but ultimately straightforward, computation gave us

$$\begin{aligned}(\mathbf{d}_T\mathbf{K}+\mathbf{K}\mathbf{d}_T)(u,v) = e(u)T\frac{\partial}{\partial T}e(v) &- \frac{e(u)}{u}T\frac{\partial}{\partial T}\frac{e(v)}{v} - e(u)e(v)\\ &- \frac{m}{2}(u-v)\left(z+\frac{\overline{z}}{u^2v^2}\right)e(u)e(v)\end{aligned} \tag{4.15}$$

All derivatives, including the ones appearing on the right-hand-sides, act on all operators to their right. This equality holds in the sense that if both sides are multiplied by an arbitrary function of $v, T, z, \overline{z}$, and v is then integrated over, the results are equal up to a surface term which is presumed to vanish. Notice especially the location of the T derivatives on the right-hand-side – this leads to the most compact form of the above equation, and is also the most convenient in subsequent manipulations; when the above equation acts on the g function, this form immediately gives T derivatives of the "potentials" which we will introduce in the following subsection.

The identities (4.9), (4.10), (4.12), and (4.15) are the main results of this section. Our results in the remainder of the paper for the relations obeyed by $\tau_\pm$ (except for those in Appendix D) follow by simple, repeated applications of these intertwining relations.

4.3. Definition of the potentials

We need to introduce some additional formalism and notation before turning to the study of the differential equations.

We consider here the so-called "potentials", which are the set of possible scalar overlaps among the e and g functions introduced in Section 4.1. These occur, for instance, when we consider derivatives of the $\tau_\pm$ correlators. Thus from (2.33)

$$\begin{aligned}\partial_z \ln\tau_+ &= \partial_z \ln \mathrm{Det}(1+\mathbf{K})\\ &= \mathrm{Tr}\left((1+\mathbf{K})^{-1}\partial_z\mathbf{K}\right)\\ &= -\frac{m}{2}\mathrm{Tr}\left((e(1-\mathbf{R}_+)e\right)\\ &= -\frac{m}{2}\mathrm{Tr}\left(eg_+\right).\end{aligned} \tag{4.16}$$

In the last step we used the defining relation (4.5). The expression $\mathrm{Tr}\,(eg_+)$ is one of the potentials; notice that the variable u has been integrated over, and the potential is an implicit function of z, $\overline{z}$, and T alone. We will also need potentials defined by inserting various powers of the variable u in the scalar product:

$$\mathrm{Tr}\,(e\mathbf{d}_n g_+) = \int_{-\infty}^{\infty} du u^n e(u) g_+(u). \tag{4.17}$$

To make things compact, we introduce some notation here for the potentials that turn up in our analysis. Our main results will be expressed in terms of the following potentials

$$\begin{aligned} b_\pm &= \mathrm{Tr}\,(eg_\pm) \\ c_\pm &= \mathrm{Tr}\,(e\mathbf{d}_{-1}g_\pm) \\ d_\pm &= (1 \pm c_\mp)\mathrm{Tr}\,(e\mathbf{d}_{-2}g_\pm) \end{aligned} \tag{4.18}$$

The reason for the additional factors in the definitions of the $d_\pm$ potentials will become clear shortly. Not all of these potentials are independent; the intertwining relations (4.9) and (4.10) leads to a constraint between them:

$$\begin{aligned} c_+ = \mathrm{Tr}\,(e\mathbf{d}_{-1}g_+) &= \mathrm{Tr}\,(e\mathbf{d}_{-1}(1-\mathbf{R}_+)e) \qquad \text{from}(4.5) \\ &= \mathrm{Tr}\,(e(1-\mathbf{R}_-)\mathbf{d}_{-1}e) - c_+c_- \quad \text{from}(4.10) \\ &= \mathrm{Tr}\,(g_-\mathbf{d}_{-1}e) - c_+c_- \quad \text{from}(4.6)\ , \end{aligned} \tag{4.19}$$

which using (4.17) becomes finally

$$c_- - c_+ = c_+c_-. \tag{4.20}$$

We solve this constraint by henceforth parameterizing

$$c_+ = 1 - e^{\phi} \quad , \quad c_- = e^{-\phi} - 1. \tag{4.21}$$

Let us also generalize (4.16), and tabulate all the derivatives of the $\tau_\pm$:

$$\begin{aligned} \partial_z \ln \tau_\pm &= \mp \frac{m}{2} b_\pm \\ \partial_{\overline{z}} \ln \tau_\pm &= \mp \frac{m}{2} d_\mp \end{aligned} \tag{4.22}$$

Our choices in the definitions of the $d_\pm$ earlier were made so that the above equations came out in this symmetrical form.

In our subsequent analysis, some additional potentials also arise at intermediate stages. To handle these, it is convenient to introduce here some additional notation

$$\begin{aligned} a_1 &= \mathrm{Tr}\,(e\mathbf{d}_1 g_+) + \mathrm{Tr}\,(e\mathbf{d}_1 g_-) \\ a_2 &= e^{-\phi}\mathrm{Tr}\,(e\mathbf{d}_{-3}g_+) + e^{\phi}\mathrm{Tr}\,(e\mathbf{d}_{-3}g_-)\,. \end{aligned} \tag{4.23}$$

Both a_1, a_2 will drop out of our final results.

4.4. Differential Equations

We will first consider the differential equations which involve only derivatives with respect to z and $\overline{z}$. These will turn out to be closely related to those derived earlier by Perk *et. al.* by a very different computation on a lattice model. Then we will turn to a new set of equations involving temperature derivatives.

4.4.1 Spacetime derivatives

We begin by turning the intertwining relations (4.12) into differential equations for the f functions. Begin with the equation $(1+\mathbf{K})g_+ = e$ ((4.6)) and act on both sides with the operator $(1-\mathbf{R}_+)\mathbf{d}_z$. Simplifying this and related equations by repeated use of all the intertwining relations gives us

$$\begin{aligned} \mathbf{d}_z \begin{pmatrix} g_+ \\ g_- \end{pmatrix}(u) &= -\frac{m}{2}\begin{pmatrix} -b_+ - b_- & u \\ u & b_+ + b_- \end{pmatrix}\begin{pmatrix} g_+(u) \\ g_-(u) \end{pmatrix} \\ \mathbf{d}_{\overline{z}} \begin{pmatrix} g_+ \\ g_- \end{pmatrix}(u) &= -\frac{m}{2}\begin{pmatrix} 0 & e^{2\phi}/u \\ e^{-2\phi}/u & 0 \end{pmatrix}\begin{pmatrix} g_+(u) \\ g_-(u) \end{pmatrix}. \end{aligned} \tag{4.24}$$

These are a set of equations in the Lax form, and integrability condition $\mathbf{d}_z\mathbf{d}_{\overline{z}} = \mathbf{d}_{\overline{z}}\mathbf{d}_z$ leads to differential equations among the potentials alone. However, a richer set of equations is obtained by acting on the definitions of the potentials (4.18) by $\mathbf{d}_z$ and $\mathbf{d}_{\overline{z}}$, using (4.24), and the derivatives of the e functions which follow from their definition in (2.35). This gives us

$$\begin{aligned} \partial_z\phi &= \frac{m}{2}(b_+ + b_-) \\ \partial_{\overline{z}}\phi &= \frac{m}{2}(d_+ + d_-) \\ \partial_{\overline{z}} b_\pm &= me^{\pm\phi}\sinh\phi \\ \partial_z d_\pm &= me^{\mp\phi}\sinh\phi. \end{aligned} \tag{4.25}$$

With these equations in hand, we can now express (4.24) in a more symmetrical form. First define the functions $h_\pm(u)$ by

$$h_\pm(u) = e^{\mp\phi/2} g_\pm(u). \tag{4.26}$$

Note that $e(u)$, $g_\pm(u)$, and $h_\pm(u)$ are the only functions which depend upon the spectral parameter u; all other functions are independent of u. Now it is easy to show from (4.24) and (4.25) that

$$\begin{aligned} \mathbf{d}_z \begin{pmatrix} h_+ \\ h_- \end{pmatrix}(u) &= -\frac{1}{2}\begin{pmatrix} -\partial_z\phi & mue^{-\phi} \\ ue^{\phi} & \partial_z\phi \end{pmatrix}\begin{pmatrix} h_+(u) \\ h_-(u) \end{pmatrix} \\ \mathbf{d}_{\overline{z}} \begin{pmatrix} h_+ \\ h_- \end{pmatrix}(u) &= -\frac{1}{2}\begin{pmatrix} \partial_{\overline{z}}\phi & me^{\phi}/u \\ me^{-\phi}/u & -\partial_{\overline{z}}\phi \end{pmatrix}\begin{pmatrix} h_+(u) \\ h_-(u) \end{pmatrix}. \end{aligned} \tag{4.27}$$

From (4.22) and (4.25) , it can be shown that the second derivatives of the τ functions are characterized by the compact equations:

$$\begin{aligned} \partial_z \partial_{\overline{z}} \ln \tau_\pm &= \mp \frac{m^2}{2} e^{\pm\phi} \sinh\phi \\ \partial_z \partial_{\overline{z}} \phi &= \frac{m^2}{2} \sinh 2\phi. \end{aligned} \tag{4.28}$$

The second of these equations is the compatibility condition $\mathbf{d}_z \mathbf{d}_{\overline{z}} = \mathbf{d}_{\overline{z}} \mathbf{d}_z$ of (4.27).

Another simple consequence of (4.25) follows from combining it with the definitions (4.22); one notices easily that

$$\partial_z \left(\ln(\tau_-/\tau_+) - \phi\right) = 0 \qquad \partial_{\overline{z}} \left(\ln(\tau_-/\tau_+) - \phi\right) = 0, \tag{4.29}$$

which implies that $\phi = \ln(\tau_-/\tau_+) +$ a dimensionless constant dependent only on the ratio T/m. In Appendix D we show that the integration constant is in fact also independent of T/m, and then show by an explicit computation at $T = 0$ that its value is zero. Therefore we have

$$e^\phi = \frac{\tau_-}{\tau_+}. \tag{4.30}$$

It is also useful to consider differential equations satisfied by the $a_{1,2}$ potentials defined in (4.23); as we will now show, these equations lead to additional differential equations involving only the $b_\pm$ and $d_\pm$. Using the same methods as those used for (4.25) we can obtain

$$\begin{aligned} \partial_z b_\pm &= -\frac{m}{2} \left(a_1 \mp b_\pm (b_+ + b_-)\right) \\ \partial_{\overline{z}} d_\pm &= -\frac{m}{2} \left(a_2 \pm d_\pm (d_+ + d_-)\right) \\ \partial_{\overline{z}} a_1 &= -m (b_+ e^{-\phi} + b_- e^{\phi}) \cosh\phi \\ \partial_z a_2 &= -m (d_+ e^{\phi} + d_- e^{-\phi}) \cosh\phi \end{aligned} \tag{4.31}$$

The last two equations above are in fact not new; we can solve the first two for $a_{1,2}$ and then they follow from applications of (4.25). We can also eliminate $a_{1,2}$ between the first two equations of (4.31) and obtain

$$\begin{aligned} \partial_z b_+ - \partial_z b_- &= \frac{m}{2} \left(b_+ + b_-\right)^2 \\ \partial_{\overline{z}} d_+ - \partial_{\overline{z}} d_- &= -\frac{m}{2} \left(d_+ + d_-\right)^2 \end{aligned} \tag{4.32}$$

We will not use the $a_{1,2}$ any more in this paper, although we will implicitly use the first two equations in (4.31) in the derivation of some equations in the next section.

4.4.2 Temperature derivatives

We will now follow the same procedure as in 4.4.1, but with the $\mathbf{d}_T$ operator replacing the $\mathbf{d}_z$ and $\mathbf{d}_{\overline{z}}$ operators.

First, it is useful to notice that

$$(\mathbf{d}_T e)(u) = -\frac{m}{2}\left(u+\frac{1}{u}\right)\left(zu-\frac{\overline{z}}{u}\right)e(u). \tag{4.33}$$

We can now get the $\mathbf{d}_T$ derivatives of the f functions by acting on the $(1\pm\mathbf{K})g_\pm = e$ equations by $(1-\mathbf{R}_\mp)\mathbf{d}_T$ on the left; this gives

$$\mathbf{d}_T\begin{pmatrix} g_+ \\ g_- \end{pmatrix}(u) = \begin{pmatrix} A & B_+ \\ B_- & -A \end{pmatrix}\begin{pmatrix} g_+(u) \\ g_-(u) \end{pmatrix} \tag{4.34}$$

where

$$\begin{aligned} A &= -\frac{T}{u}\partial_T\phi + \frac{muz}{2}(b_+ + b_-) + \frac{m\overline{z}}{2u}(d_- + d_+) \\ B_+ &= (1 + z\partial_z - T\partial_T)\, b_+ - \frac{mz}{2}\left(u^2+1\right) + \frac{m\overline{z}}{2u^2}\left(u^2+e^{2\phi}\right) \\ B_- &= -(1 + z\partial_z - T\partial_T)\, b_- - \frac{mz}{2}\left(u^2+1\right) + \frac{m\overline{z}}{2u^2}\left(u^2+e^{-2\phi}\right); \end{aligned} \tag{4.35}$$

we have used (4.31) to eliminate the a_1 potential which arises in the above derivation. We will write this equation in a symmetrical form momentarily.

First, we derive equations relating the T derivatives of the potentials; we will do this by evaluating

$$\mathrm{Tr}\left(\mathbf{d}_{-1}\mathbf{d}_T(eg_\pm)\right) \tag{4.36}$$

in two different ways. For the first we use $\mathbf{d}_T(eg_\pm) = g_\pm\mathbf{d}_T e + e\mathbf{d}_T g_\pm$ and evaluate the right hand side using (4.33) and (4.34); alternatively, we can write, using the definition (4.13),

$$\mathrm{Tr}\left(\mathbf{d}_{-1}\mathbf{d}_T(eg_\pm)\right) = (T\partial_T - 1)\,\mathrm{Tr}\left(e(1-\mathbf{d}_{-2})g_\pm\right), \tag{4.37}$$

where we have integrated by parts over the u integral. Comparing the results of these two evaluations, we obtain

$$e^{\mp\phi}\left(1 + z\partial_z - T\partial_T\right) b_\pm - e^{\pm\phi}\left(1 + \overline{z}\partial_{\overline{z}} - T\partial_T\right) d_\pm = m(\overline{z}-z)\sinh\phi \tag{4.38}$$

Again, we have used (4.31) to eliminate the $a_{1,2}$ potentials at intermediate stages.

Now we can combine (4.26)and (4.38) to put (4.34) into a more symmetrical form:

$$\mathbf{d}_T\begin{pmatrix} h_+ \\ h_- \end{pmatrix}(u) = \begin{pmatrix} \widetilde{A} & \widetilde{B}_+ \\ \widetilde{B}_- & -\widetilde{A} \end{pmatrix}\begin{pmatrix} h_+(u) \\ h_-(u) \end{pmatrix} \tag{4.39}$$

where

$$\widetilde{A} = -\frac{1}{2}\left(u+\frac{1}{u}\right)T\partial_T\phi + \frac{muz}{2}(b_+ + b_-) + \frac{m\overline{z}}{2u}(d_- + d_+)$$

$$\begin{aligned}\widetilde{B}_\pm &= \pm\frac{e^{\mp\phi}}{2}\left(1 + z\partial_z - T\partial_T\right)b_\pm \pm \frac{e^{\pm\phi}}{2}\left(1+\overline{z}\partial_{\overline{z}} - T\partial_T\right)d_\pm \\ &\quad - \frac{m}{2}\left(zu^2e^{\mp\phi} - \frac{\overline{z}e^{\pm\phi}}{u^2}\right) - \frac{m}{2}(z-\overline{z})\cosh\phi.\end{aligned} \tag{4.40}$$

This is another set of equations in the Lax form, and like (4.27) they behave simply under the $z \leftrightarrow \overline{z}$, $u \leftrightarrow 1/u$, $b_\pm \leftrightarrow d_\pm$ transformations. We can examine the consequences of the compatibility conditions $\mathbf{d}_T\mathbf{d}_z = \mathbf{d}_z\mathbf{d}_T$ and $\mathbf{d}_T\mathbf{d}_{\overline{z}} = \mathbf{d}_{\overline{z}}\mathbf{d}_T$ on (4.40) and (4.27); this generates equations for the potentials which turn to be already implied by the equations in Section 4.4.1 and (4.38).

4.5. Recapitulation

Our derivation of the differential equations has been rather circuitous, so it seems useful here to present a self-contained review of the final results. The discussion here is also designed to be accessible to readers who have skipped Sections 4.1-4.4.

The main quantities of interest are the correlators $\tau_\pm$ defined in (2.20). We now introduce the variables $b_\pm$, $d_\pm$ and ϕ by

$$\begin{aligned} b_\pm &= \mp\frac{2}{m}\partial_z \ln\tau_\pm \\ d_\pm &= \pm\frac{2}{m}\partial_{\overline{z}}\ln\tau_\mp \\ \phi &= \ln(\tau_-/\tau_+). \end{aligned} \tag{4.41}$$

These variables were introduced earlier as "potentials", but here we shall simply consider (4.41) as their defining relations.

A complete, mutually-independent, set of non-linear partial differential equations obeyed by these quantities is

$$\partial_z\partial_{\overline{z}}\ln\tau_\pm = \mp\frac{m^2}{2}e^{\pm\phi}\sinh\phi \tag{4.42a}$$

$$\partial_z b_+ - \partial_z b_- = \frac{m}{2}\left(b_+ + b_-\right)^2 \tag{4.42b}$$

$$\partial_{\overline{z}} d_+ - \partial_{\overline{z}} d_- = -\frac{m}{2}\left(d_+ + d_-\right)^2 \tag{4.42c}$$

$$e^{\mp\phi}\left(1+z\partial_z - T\partial_T\right)b_\pm - e^{\pm\phi}\left(1+\overline{z}\partial_{\overline{z}} - T\partial_T\right)d_\pm = m(\overline{z}-z)\sinh\phi. \tag{4.42d}$$

The equations (4.42b), (4.42c), and the difference of the two equations in (4.42a) precisely exhaust the set of equations obtained earlier in [2],[3]. Note that these equations do not involve T derivatives, and we have one more equation of this type (one of the two in (4.42a)) than those obtained earlier. The equations in (4.42d) are new.

The equations in (4.42a) and (4.42d) can also be written in the Lax form[22] *i.e.* as the integrability conditions of linear equations obeyed by auxilliary functions dependent upon an additional spectral parameter. We need 2 auxilliary functions $h_\pm(u)$, where u is the spectral parameter. The linear equations satisfied by $h_\pm(u)$ are (4.27) and (4.39), where we note that on the left-hand-sides of these equations one can replace $\mathbf{d}_z \to \partial_z$, $\mathbf{d}_{\overline{z}} \to \partial_{\overline{z}}$ and $\mathbf{d}_T \to (u - 1/u)T\partial_T + (u + 1/u)u\partial_u$.

4.6. Massless Limit

In this section we will examine the behavior of the non-linear partial differential equations in the limit of small m/T. We recall that all correlators are smooth functions of m at $m = 0$ for finite T [23],[15], so the $\tau_\pm$ obey the following series expansions

$$\begin{aligned} \left(\frac{m}{T}\right)^{1/4} \tau_+ &= \sum_{n=0}^{\infty} \left(\frac{m}{T}\right)^{2n} \tau_{2n}(zT, \overline{z}T) \\ \left(\frac{m}{T}\right)^{1/4} \tau_- &= \sum_{n=0}^{\infty} \left(\frac{m}{T}\right)^{2n+1} \tau_{2n+1}(zT, \overline{z}T). \end{aligned} \tag{4.43}$$

The prefactor of $(m/T)^{1/4}$ follows from the overall normalization condition for $\tau_\pm$ implicit in (2.22), and the behavior of the correlators at $m = 0$. The coefficients τ_n in (4.43) are dimensionless functions only of zT and $\overline{z}T$. Indeed, the τ_n are clearly spacetime integrals over correlators in the massless conformal field theory of two σ or μ operators and n thermal (energy) operators. These correlators possess a holomorphic/anti-holomorphic factorization property, but it is not expected that the result of a spacetime integral over them in a cylinder geometry will be in a factorized form. Let us stress here that conformal perturbation theory expansions such as (4.43) as possible only because $T \neq 0$, so there are no infrared divergences [24].

An interesting question is the convergence of expansions such as (4.43). Of course, the absence of any phase transitions at finite temperature in one-dimensional physical systems with short-range interactions implies that there can be no singularity at any finite real value of m/T. The only singularities on the real axis consist of powers of $\exp(-m/T)$ as $m/T \to \infty$ (this is clear from the results of [15]). However, phase transitions are possible in one-dimensional systems with unphysical complex couplings. In the present case, the largest eigenvalue in the spin even sector crosses the largest eigenvalue in the spin odd sector when $m = i\pi T$, and presumably this determines the radius of convergence of (4.43).

We can now insert (4.43) into the definitions (4.41) and the differential equations (4.42.) All the equations organize themselves neatly in series of integer powers of m, and demanding that they are obeyed at each order leads to a hierarchy of differential equations obeyed by the τ_n. The zeroth order equations in this hierarchy involve only τ_0 and they are

$$\begin{aligned} \partial_z \partial_{\overline{z}} \ln \tau_0 &= 0 \\ (1 + z\partial_z - T\partial_T)\partial_z \ln \tau_0 &= 0 \\ (1 + \overline{z}\partial_{\overline{z}} - T\partial_T)\partial_{\overline{z}} \ln \tau_0 &= 0. \end{aligned} \tag{4.44}$$

As τ_0 is proportional to the spin correlators of the massless, conformal, theory, it is expected to factorize into holomorphic/anti-holomorphic parts:

$$\tau_0(w, \overline{w}) = \Psi_0(w)\Psi_0(\overline{w}), \tag{4.45}$$

where we have introduced $w \equiv zT$, $\overline{w} = \overline{z}T$. Substituting (4.45) for τ_0, we find that (4.44) are obeyed for *any* function $\Psi_0(w)$. The form of Ψ_0 follows from a, now standard, argument based on conformal invariance [25], and its overall scale was determined in [15]:

$$\Psi_0(w) = \frac{2^{5/12}\pi^{1/8}e^{1/8}A_G^{-3/2}}{(\sin(2\pi w))^{1/8}}, \tag{4.46}$$

where A_G is Glaisher's constant ($\ln A_G = 1/12 - \zeta'(-1)$). Notice that, apart from the scale, Ψ_0 is determined by the requirement that its only singularities be $\Psi_0(w \to 0) \sim w^{-1/8}$ and its periodic images under $w \to w + 1/2$, and that it have no zeros.

5. Conclusion

In conclusion, we would like to stress that the differential equations involving only spatial coordinates (the equations (4.42a–c)) would hold for any filling function f in (2.22). This is because these equations follow only from the monodromy relations (4.9),(4.12), that are insensitive to f. The last equation in (4.42)would hold for any filling function f that depends only on $(u + 1/u)/T$ and satisfies $f(\infty) = 1, f(-\infty) = 0$. This is because the differential equation follows from (4.15) that in turns follows from (4.14), together with (4.30) that depends on the $T = 0$ behavior of f. This does not mean that the differential equations are too general to be useful: they can be quite constraining after imposing some limiting behavior of the correlators (like (4.46)) determined by other means (similar features are known for the Bose gas [1]).

While we have not derived the differential equations for the boundary problem, we can straightforwardly obtain some of them. Indeed, the kernel (3.11) differs from the zero temperature one in [20] by the filling function f only. Thus, as for the bulk two point function, the differential equations derived in [26][20] involving derivatives with respect to κ and x still apply for the non-zero temperature correlator.

The presence of a non-zero temperature also introduces a number of new physical phenomena, not found at $T = 0$, which are of considerable theoretical interest. Foremost among these is the appearance of irreversible, thermal relaxational behavior in the long-time spin correlations (for field-theorists, it is perhaps useful to note here that we are referring to correlations in *real* time; this relaxational behavior has no simple characterization in Euclidean time where correlations are periodic with period $1/T$). While this relaxational behavior is expected to occur at all non-zero T, there are some important distinctions in its physical nature between high T ($T \gg m$) and low T ($T \ll m$), where m is mass gap above the ground state of the continuum Ising model.
(*i*) $T \gg m$: The presence of thermal relaxational behavior has been established in this limit, and a simple argument based on conformal invariance also allowed exact computation of a relaxation rate constant [27]. In this high T regime, the fluctuations are dominated by states with energy $\hbar\omega \sim k_B T$, and the spin dynamics are therefore not amenable to a description by some effective classical model of dissipative dynamics [14].
(*ii*) $T \ll m$: The physical situation is quite different at low T just above the ordered state (we will not comment here on the behavior above the quantum paramagnetic state). Now the most important excitations have an energy $\hbar\omega \ll k_B T$ [15], and their de-Broglie wavelength is therefore much smaller than their mean spacing. This reasoning led to the conjecture [15] that the long-time relaxational spin dynamics is described by a classical Fokker-Planck-like equation. Symmetry and detailed-balance considerations completely specify the form of the Fokker-Planck equation [28], and its exact solution leads to a prediction for long-time, large-distance two spin correlation function in terms of a single unknown damping constant [15].

The study in this paper is a step towards understanding the finite temperature dynamics of the continuum quantum Ising model, as it crosses over from the high T limit (where the dynamic correlations are known in closed form) to its conjectured classical behavior at low T. In all our computations, time evolution is specified by the usual unitary Heisenberg operator $e^{iHt/\hbar}$, where H is the Hamiltonian on the quantum Ising model, and there is no coupling to a heat bath. The temperature appears solely in specifying an initial density matrix $e^{-H/T}$, which is also stationary under the time evolution. Contrast this with the dynamics of the effective low T classical spin model, which is driven entirely by an ever-present, random fluctuating force from a heat bath. The conjecture of [15] asserts that these two dynamics become equivalent for $T \ll m$. While we have not determined the validity of this conjecture in this paper, we have taken several significant steps towards obtaining dynamic correlation functions under the unitary Heisenberg time evolution, and the final objective now appears within reach.

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Appendix A. Direct Approach

In this appendix, we would like to give a different derivation of the result (2.13).

To start, we consider the one point function of some operator O:

$$\langle \mathcal{O} \rangle_T = \frac{1}{Z} \sum_{\psi} e^{-E_\psi/T} \frac{\langle \psi | \mathcal{O} | \psi \rangle}{\langle \psi | \psi \rangle}. \tag{A.1}$$

In a multiparticle basis for the states $|\psi >$, one has the following resolution of the identity when $L = \infty$:

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d\beta_1 \cdots d\beta_n}{(2\pi)^n} \; |\beta_1, \ldots, \beta_n\rangle \langle \beta_n, \ldots, \beta_1|. \tag{A.2}$$

Thus,

$$\langle \mathcal{O} \rangle_T = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta_1 \cdots d\beta_n \; \left(\prod_{i=1}^{n} e^{-e(\beta_i)/T} \right) \langle \beta_n, \ldots, \beta_1 | \mathcal{O} | \beta_1, \ldots, \beta_n \rangle. \tag{A.3}$$

The matrix elements of $\mathcal{O}$ are the so called form-factors. As we will explain below, one finds that the sum (A.3) has two kinds of singularities that must be dealt with before one can obtain a meaningful expression. The first kind of singularity is a result of working with multiparticle states that are defined in infinite volume, i.e. for $L = \infty$. These infinite volume singularities manifest themselves as $\delta(0)$, i.e. $\delta(\beta - \beta')$ with $\beta = \beta'$. As expected on general grounds, these infinite volume singularities are removed upon dividing by Z. The second kind of singularity arises from poles in the form factors. These 'pole singularities' can be removed by a renormalization of the one-point functions.

One first needs some general properties of the form factors. We specialize to a theory with S-matrix equal to -1. In terms of the operators defined in (2.1), the multiparticle states have the following representation:

$$\begin{aligned} |\beta_1, \ldots, \beta_n\rangle &= A^\dagger(\beta_1) \cdots A^\dagger(\beta_n) |0\rangle \\ \langle \beta_n, \ldots, \beta_1| &= \langle 0 | A(\beta_n) \cdots A^\dagger(\beta_1). \end{aligned} \tag{A.4}$$

Using the algebra of the operators $A, A^\dagger$, one can evaluate the inner products $\langle \beta_n, ... | \beta_1 .... \rangle$.

Let $\mathcal{A}, \mathcal{B}$ denote some ordered sets of rapidities, $\mathcal{A} = \{\beta_n, ..., \beta_1\}$, $\mathcal{B} = \{\beta'_1, ..., \beta'_m\}$, and consider the form factor $\langle \mathcal{A} | \mathcal{O}(x) | \mathcal{B} \rangle$. If $\mathcal{A}$ and $\mathcal{B}$ have no overlap, i.e. $\langle \mathcal{A} | \mathcal{B} \rangle = 0$, then the form factor obeys the crossing relation (see also next appendix):

$$\langle \mathcal{A} | \mathcal{O}(x) | \mathcal{B} \rangle = \langle 0 | \mathcal{O}(x) | \mathcal{B}, \mathcal{A} - i\pi \rangle, \tag{A.5}$$

where $\mathcal{A} - i\pi$ denotes all rapidities shifted by $-i\pi$. If $\langle\mathcal{A}|\mathcal{B}\rangle \neq 0$, the form factor is expressed as a sum over all ways of breaking up $\mathcal{A}, \mathcal{B}$ into two sets [6]:

$$\langle\mathcal{A}|\mathcal{O}(x)|\mathcal{B}\rangle = \sum_{\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2;\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2} \langle\mathcal{A}_1^+|\mathcal{O}(x)|\mathcal{B}_1\rangle\langle\mathcal{A}_2|\mathcal{B}_2\rangle\ S_{\mathcal{A},\mathcal{A}_1}S_{\mathcal{B},\mathcal{B}_1}. \tag{A.6}$$

In this formula, $\mathcal{A}^+$ denotes rapidities shifted by a infinitesimally small imaginary part $i\eta$, so that the crossing relation is valid:

$$\langle\mathcal{A}^+|\mathcal{O}|\mathcal{B}\rangle = \langle\mathcal{A}+i\eta|\mathcal{O}|\mathcal{B}\rangle = \langle 0|\mathcal{O}|\mathcal{B}, \mathcal{A} - i\pi + i\eta\rangle. \tag{A.7}$$

The S-matrix factors are defined as the product of S-matrix elements required to bring $|\mathcal{A}\rangle$ into the order $|\mathcal{A}_2, \mathcal{A}_1\rangle$, and similarly for $\mathcal{B}$. Namely,

$$\langle\mathcal{A}| = S_{\mathcal{A},\mathcal{A}_1}\langle\mathcal{A}_2, \mathcal{A}_1|, \qquad |\mathcal{B}\rangle = S_{\mathcal{B},\mathcal{B}_1}|\mathcal{B}_1, \mathcal{B}_2\rangle. \tag{A.8}$$

We will refer to the single term in (A.6) with $\mathcal{A}_1 = \mathcal{A}, \mathcal{B}_1 = \mathcal{B}$ as the connected piece and the rest as the disconnected pieces of the form factor.

We will also need the partition function Z. In infinite volume, Z is singular, and it is well known how to regulate it on a cylinder of finite length. It suffices for our purposes however to work with an unregulated expression for Z, since, as we will see, dividing by Z simply cancels similar infinite volume divergences in the sum (A.3). Therefore, we take

$$Z = \sum_{n=0}^{\infty}\frac{1}{n!}\int\frac{d\beta_1\cdots d\beta_n}{(2\pi)^n}\left(\prod_{i=1}^{n}e^{-e(\beta_i)/T}\right)\langle\beta_n,\ldots,\beta_1|\beta_1,\ldots,\beta_n\rangle. \tag{A.9}$$

Up to two particles, using the Faddeev-Zamolodchikov algebra one finds:

$$Z = 1 + \int d\beta e^{-e(\beta)/T} < \beta|\beta > + \frac{1}{2}\int d\beta_1 d\beta_2 e^{-(e(\beta_1)+e(\beta_2))/T} < \beta_2, \beta_1|\beta_1, \beta_2 > + \ldots.. \tag{A.10}$$

One can explicitly evaluate the lowest order terms in the sum (A.3), using (A.6) and (A.10). One finds that the effect of the disconnected pieces of the form factors is two-fold: some of the disconnected terms lead to $\delta(0)$ singularities as in $Z(T)$, whereas others lead to a modification of the integration measure $\int d\beta$. Doing the explicit computation up to 3 particles, one finds,

$$\begin{aligned}\sum_{\psi} e^{-E_\psi/T}\frac{\langle\psi|\mathcal{O}|\psi\rangle}{\langle\psi|\psi\rangle} &= Z(\langle 0|\mathcal{O}|0\rangle \\ &+ \int d\beta e^{-e(\beta)/T}\left(1 - e^{-e(\beta)/T} + e^{-2e(\beta)/T}\right)\langle\beta^+|\mathcal{O}|\beta\rangle \\ &+ \frac{1}{2}\int d1d2\ e^{-(e_1+e_2)/T}\left(1 - e^{-e_1/T} - e^{-e_2/T}\right)\langle 2^+, 1^+|\mathcal{O}|1, 2\rangle \\ &+ \frac{1}{3!}\int d1d2d3\ e^{-(e_1+e_2+e_3)/T}\langle 3^+, 2^+, 1^+|\mathcal{O}|1, 2, 3\rangle + \ldots)\end{aligned} \tag{A.11}$$

($d1 = d\beta_1, e_1 = e(\beta_1)$, etc.) In the above formula, since we did the computation only up to 3 particles, we only verified explicitly the appearance of Z up to the appropriate order, depending on which term Z multiplies.

The above computation, along with some combinatoric checks at higher order, are sufficient to understand that to all orders one will find:

$$\langle \mathcal{O} \rangle_T = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\beta_1 \cdots d\beta_n}{(2\pi)^n} \prod_{i=1}^{n} f(\beta_i) e^{-e(\beta_i)/T} \langle \beta_n^+, \ldots, \beta_1^+ | \mathcal{O} | \beta_1, \ldots, \beta_n \rangle, \qquad \text{(A.12)}$$

where

$$f(\beta) = \frac{1}{1 + \exp(-e(\beta)/T)}. \qquad \text{(A.13)}$$

In summary, the non-trivial effect of the disconnected pieces of the form factors is to modify the $\int d\beta$ integrations by the factor $f(\beta)$. In the thermodynamic approach, $f = \rho^h/(\rho + \rho^h)$.

The expression (A.12) has another kind of singularity arising from poles in the connected piece of the form factor. In fact, one of the axioms of the form factor bootstrap expresses the residue of the pole in $\langle 0|\mathcal{O}|...\beta, \beta' - i\pi, ...\rangle$ as $\beta \to \beta'$ in terms of form factors with lower numbers of particles. We will refer to these as pole singularities. In the Ising model with $S = -1$, for the spin/disorder fields the residue axiom reads

$$\langle 0|\mathcal{O}| \ldots \beta, \beta - i\pi + i\eta \ldots \rangle = \frac{1}{\eta} \langle 0|\mathcal{O}| \ldots, \ldots \rangle. \qquad \text{(A.14)}$$

Thus,

$$\langle \beta_n^+, \ldots, \beta_1^+ | \mathcal{O} | \beta_1, \ldots, \beta_n \rangle = \left(\frac{1}{\eta} \right)^n \langle 0|\mathcal{O}|0 \rangle. \qquad \text{(A.15)}$$

This leads to

$$\langle \mathcal{O} \rangle_T = \lim_{\eta \to 0} \exp \left(\int \frac{d\beta}{2\pi\eta} f(\beta) e^{-e(\beta)/T} \right) \langle \mathcal{O} \rangle_{T=0}. \qquad \text{(A.16)}$$

Note that here we left η inside the integration sign, as it may well have to depend on β when one tries to give a more precise meaning to (A.16)(see the next appendix). Here, our attitude will be to ignore this (infinite) multiplicative renormalization, our goal being to obtain expressions for the two point functions up to an overall normalization, which we set somewhat arbitrarily.

The above procedure leads to meaningful expressions for the two-point correlation functions. One begins with

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle_T = \frac{1}{Z} \sum_{\psi, \psi'} e^{-E_\psi/T} \frac{\langle \psi | \mathcal{O}(x) | \psi' \rangle \langle \psi' | \mathcal{O}(0) | \psi \rangle}{\langle \psi | \psi \rangle \langle \psi' | \psi' \rangle}. \qquad \text{(A.17)}$$

By explicitly evaluating the above sum for low numbers of particles, one reaches the following conclusions. Again, the disconnected terms are of two types, one diverging in infinite volume but being cancelled by $1/Z$, the other leading to modifications of the integration measures as for the 1-point function. We find that the $\int d\beta$ integrations for the sum over states ψ and ψ' are *both* modified. To illustrate this, consider for instance the term:

$$\frac{1}{2}\int d\beta d\beta'_1 d\beta'_2 e^{-e(\beta)/T}\langle\beta|\mathcal{O}(x)|\beta'_1,\beta'_2\rangle\langle\beta'_2,\beta'_1|\mathcal{O}(0)|\beta\rangle. \tag{A.18}$$

One of the disconnected pieces is

$$\begin{aligned}&-\frac{1}{2}\int d\beta d\beta'_1 d\beta'_2 e^{-e(\beta)/T}\delta(\beta-\beta'_1)\delta(\beta-\beta'_2)\langle 0|\mathcal{O}(x)|\beta'_2\rangle\langle\beta'_1|\mathcal{O}(0)|0\rangle\\ &\quad=-\frac{1}{2}\int d\beta e^{-E(\beta)/T}\langle 0|\mathcal{O}(x)|\beta\rangle\langle\beta|\mathcal{O}(0)|0\rangle.\end{aligned} \tag{A.19}$$

This is interpreted as arising from the second term in the expansion of an $f(\beta)$ measure factor in the sum over states ψ'. In addition, the pole singularities can be removed by multiplicative renormalization of the fields as described above. Based on this, one finds the following expression for the 2-point functions (again, up to an overall normalization):

$$\begin{aligned}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_T=&\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\int\frac{d\beta_1...d\beta_n}{(2\pi)^n}\frac{d\beta'_1....d\beta'_m}{(2\pi)^m}\left(\prod_{i=1}^{n}f(\beta_i)e^{-e(\beta_i)/T}\right)\\ &\times\left(\prod_{r=1}^{m}f(\beta'_r)\right)\langle 0|\mathcal{O}(x)|\beta'_1,...,\beta'_m,\beta_n-i\pi,...,\beta_1-i\pi\rangle\\ &\times\langle 0|\mathcal{O}(0)|\beta_1,....,\beta_n,\beta'_m-i\pi,....,\beta'_1-i\pi\rangle.\end{aligned} \tag{A.20}$$

Let us refer to the β' states in (A.20) as particles and the β-states as holes. Introducing an index $\epsilon=1$ for particles and $\epsilon=-1$ for holes, define the particle-hole states as follows:

$$\begin{aligned}|\beta_1,\ldots,\beta_n\rangle_{\epsilon_1\cdots\epsilon_n}&=|\beta_1-\widetilde{\epsilon}_1 i\pi,\ldots,\beta_n-\widetilde{\epsilon}_n i\pi\rangle\\ {}_{\epsilon_n\cdots\epsilon_1}\langle\beta_n,\ldots,\beta_1|&=\langle\beta_n+\widetilde{\epsilon}_n i\pi,\ldots,\beta_1+\widetilde{\epsilon}_1 i\pi|,\end{aligned} \tag{A.21}$$

where $\widetilde{\epsilon}=(\epsilon-1)/2$. The form-factors of the particle-hole states are the usual ones but with the appropriate shifts by $i\pi$, e.g.

$$\langle 0|\mathcal{O}(0)|\beta_1,\ldots,\beta_n\rangle_{\epsilon_1\cdots\epsilon_n}=\langle 0|\mathcal{O}(0)|\beta_1-\widetilde{\epsilon}_1 i\pi,\ldots,\beta_n-\widetilde{\epsilon}_n i\pi\rangle. \tag{A.22}$$

The expansion (A.20) can the be written as a sum over particles and holes:

$$\begin{aligned}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_T=&\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\epsilon=\pm1}\int\frac{d\beta_1...d\beta_N}{(2\pi)^N}\left[\prod_{i=1}^{N}f_{\epsilon_i}(\beta_i)\exp\left(-\epsilon_i(mze^{\beta_i}+m\overline{z}e^{-\beta_i})\right)\right]\\ &\times\left|\langle 0|\mathcal{O}(0)|\beta_1,...,\beta_N\rangle_{\epsilon_1\cdots\epsilon_N}\right|^2,\end{aligned} \tag{A.23}$$

where $f_\pm$ are defined in (2.12). This expression coincides with (2.13).

Appendix B. Derivation of $|B_T>$

We work first at $T=0$. By considering $<0|\mathcal{O}|B>$, i.e. in the L channel, and using the boundary state of [19], we find the expression

$$<0|\mathcal{O}(x)|B>=\sum_n\int_{-\infty}^{\infty}\frac{d\beta_1\dots d\beta_n}{(4\pi)^n n!}\widehat{R}(\beta_1)\dots\widehat{R}(\beta_n)<0|\mathcal{O}|-\beta_1,\beta_1,\dots,-\beta_n,\beta_n> \\ \exp\left[2mx(\cosh\beta_1+\dots+\cosh\beta_n)\right]. \tag{B.1}$$

Now, suppose we want to compute this one point function in the crossed, R channel. It then reads simply $<0_B|\mathcal{O}|0_B>$, where $|0_B>$ is the ground state of the Ising model on the half line with a magnetic field at $x=0$.

On the other hand, let us perform crossing on (B.1). For this, let us first set $\beta\equiv i\frac{\pi}{2}-\beta'$. Using the general expression

$$<\beta_{n+1}\dots\beta_{n+m}|\mathcal{O}|\beta_1\dots\beta_n>=<0|\mathcal{O}|\beta_1,\dots,\beta_n,\beta_{n+m}-i\pi,\dots,\beta_{n+1}-i\pi>, \tag{B.2}$$

one finds

$$\begin{aligned}<0|\mathcal{O}|-\beta_1,\beta_1,\dots,-\beta_n,\beta_n>&=(-1)^n<-\beta'_n,\dots,-\beta'_1|\mathcal{O}|\beta'_1,\dots,\beta'_n>\\ &=<-\beta'_n+i\pi,\dots,-\beta'_1+i\pi|\mathcal{O}|\beta'_1-i\pi,\dots,\beta'_n-i\pi>.\end{aligned} \tag{B.3}$$

The β' integration runs now from $\frac{i\pi}{2}-\infty$ to $\frac{i\pi}{2}+\infty$. Let us move this contour back to the real axis - to do so, we request that $\widehat{R}$ has no pole in the physical strip, which will be the case for appropriate magnetic field [19]. Using that $\widehat{R}(\beta)=R\left(\frac{i\pi}{2}-\beta\right)$, together with $R(\beta)=-[R(\beta-i\pi)]^*$ the one point function (B.1) can be then rewritten as

$$\sum_n\int_{-\infty}^{\infty}\frac{d\beta_1\dots d\beta_n}{(4\pi)^n n!}(-1)^n\left[R(\beta_1-i\pi)\dots R(\beta_n-i\pi)\right]^* \\ <-\beta_n+i\pi,\dots,-\beta_1+i\pi|\mathcal{O}|\beta_1-i\pi,\dots,\beta_n-i\pi> \\ \exp\left\{-2mix[\sinh\beta_1+\dots+\sinh\beta_n]\right\}. \tag{B.4}$$

Expression (B.4) can be explained by realizing that the boundary changes the nature of the Fermi sea of the quantum theory. From (B.4), we see that we can represent the new ground state $|0_B>$ by writing formally as $|0_B>=\otimes_{\beta>0}\left\{|\beta-i\pi>-R(\beta-i\pi)|-\beta-i\pi>\right\}$. Indeed, evaluate now the one point function of $\mathcal{O}$ on the half line , $<0_B|\mathcal{O}|0_B>$. We first have two terms where no R matrix is involved, and which reads naively, using crossing,

$$\left(\otimes_{\beta>0}<\beta|\right)\mathcal{O}(x)\left(\otimes_{\beta>0}|\beta>\right)=\prod\frac{1}{\eta(\beta)}, \tag{B.5}$$

together with a similar expression with $\beta \to -\beta$. Here $\eta(\beta)$ is a cut-off necessary to avoid the divergences due to particle-antiparticle annihilation. The general form of η is easy to obtain: at finite L, require that a particle and an antiparticle cannot occupy the same state: this leads to a a cut-off in rapidity space such that $\delta p(\beta) = m \cosh\beta\delta\beta = \frac{\pi}{L}$, or $\delta(\beta) = \frac{\pi}{mL\cosh\beta}$ [1]. We use this cut-off as the value of $\eta(\beta)$ in what follows.

A renormalization of the operator $\mathcal{O}$ of course suppresses the term on the right hand side of (B.5). Let us set $(\otimes_{\beta>0} < \beta|)\, \mathcal{O}(x)\, (\otimes_{\beta>0}|\beta >) + (\beta \to -\beta) = 1$ (much the same renormalization would occur in the bulk). Note that there is no x dependence so far.

Then, there are also crossed terms in $< 0_B|\mathcal{O}|0_B >$, which involve the matrix R. These are x dependent and cannot be absorbed in a renormalization. Indeed, for a given rapidity β in the decomposition of the ket $|0_B >$, in the corresponding bra we can pick either β or $-\beta$. In that case, compared with (B.5) and its $\beta \to -\beta$ equivalent we get a relative factor

$$\frac{< -\beta|\mathcal{O}|\beta >}{< \beta^+|\mathcal{O}|\beta > + < -\beta^+|\mathcal{O}| - \beta >} = \frac{< -\beta|\mathcal{O}|\beta >}{2\frac{mL\cosh\beta}{\pi}}. \tag{B.6}$$

On the other hand, this particular rapidity where a pairing with a reflected state takes place can be chosen with some multiplicity: there are $\frac{L}{\pi}\cosh\beta d\beta$ states with which one can do so in $[\beta, \beta + d\beta]$ Clearly, the two L-dependent contributions cancel out, leaving an overall factor of 1/2. Generalizing to arbitrary number of pairings and using the fact that $R^*(\beta) = R(-\beta)$, one reproduces the expression (B.4), where the integrals with $\beta > 0$ corresponds to taking the reflected state in the bra, the integral with $\beta < 0$ to taking the reflected state in the ket.

The foregoing form of $|0_B >$ is easily checked (eg by evaluating matrix elements between $|0_B >$ and any other state) to satisfy the relations [19] $A^\dagger(\beta)|0_B >= R(\beta)A^\dagger(-\beta)|0_B >$, using that $-[R(\beta - i\pi)] = \frac{1}{R(\beta)}$.

To extend the result at finite T, we require similarly for an excited state $|\beta_B >$ to satisfy $A(\beta)|\beta_B >= R^*(\beta)A(-\beta)|\beta_B >$. Using $R^*(\beta) = \frac{1}{R(\beta)}$, it follows that an excited state is obtained by replacing some of the $|\beta - i\pi > -R(\beta - i\pi)| - \beta + i\pi >$ by $|\beta > +R(\beta)| - \beta >$. This leads to two types of states in the boundary heat bath $|0_{B,T} >$ characterized by a label $\epsilon = \pm 1$, with relative weights $f_\pm(\beta)$. As a result, the one point function at finite temperature reads then

[1] Here we used the fact that with a boundary, while only positive rapidities are allowed, the density is twice as big

$$< 0_{B,T}|\mathcal{O}(x)|0_{B,T} >= \sum_n \int_{-\infty}^{\infty} \frac{d\beta_1 \dots d\beta_n}{(4\pi)^n n!} \sum_{\epsilon_i} (-1)^{\overline{\epsilon}_i} \left[R(\beta_1 - \overline{\epsilon}_1 i\pi) \dots R(\beta_n - \overline{\epsilon}_n i\pi)\right]^*$$
$$< -\beta_n + \overline{\epsilon}_n i\pi, \dots, -\beta_1 + \overline{\epsilon}_1 i\pi|\mathcal{O}|\beta_1 - \widetilde{\epsilon}_1 i\pi, \dots, \beta_n - \widetilde{\epsilon}_n i\pi >$$
$$\prod_i \exp[2mix\epsilon_i \sinh\beta_i] f_{\epsilon_i}(\beta_i), \tag{B.7}$$

where we set $\overline{\epsilon} \equiv \frac{1+\epsilon}{2}$.

One can then get back to the original channel by following the previous transformations in the opposite way to find (3.3),(3.7).

Appendix C. Boundary Field Theory on a Finite Strip

The techniques of this paper allows us to consider a different geometry, that of an infinitely long strip of finite width R. In this geometry one can consider a quantum field theory with non-trivial boundary interactions at $x = 0, R$, where the strip is defined by $0 \leq x \leq R$. Here, R does not have the interpretation as an inverse temperature. Let us quantize the theory in a way that views x as the imaginary time. In this quantization the Hamiltonian H is the usual bulk hamiltonian with no boundary terms, and the spectrum and form factors are the same as for a bulk theory with hamiltonian H. In this picture, the boundary interactions are contained in a boundary state as in (3.2).

Consider the one-point function of a field $\mathcal{O}(x)$. For x a semi-infinite line with $R = \infty$, this correlator was characterized with form factors in [20]. Due to the translation invariance in the t-direction, this correlator depends only on x. For the finite width strip, one is allowed to put different boundary conditions at each end of the strip. Let the boundary condition at $x = 0$ correspond to the boundary state $|B_b\rangle$, and at $x = R$ the boundary state $|B_a\rangle$. The partition function in this picture is then

$$Z_{ab}(R) = \langle B_a|e^{-RH}|B_b\rangle. \tag{C.1}$$

The one-point function is defined by

$$\langle B_a|\mathcal{O}(x)|B_b\rangle_R = \frac{1}{Z_{ab}(R)} \langle B_a|e^{-H(R-x)}\mathcal{O}(0)e^{-Hx}|B_b\rangle. \tag{C.2}$$

Since the boundary state involves pairs of particles with opposite rapidity, the states which contribute as intermediate states are

$$|2n\rangle = |-\beta_1, \beta_1, \ldots, -\beta_n, \beta_n\rangle. \tag{C.3}$$

Inserting two complete sets of states in (C.2), one on each side of the field $\mathcal{O}$, one obtains

$$\begin{aligned}\langle B_a|\mathcal{O}(x)|B_b\rangle_R = &\sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} \int_{-\infty}^{\infty} \frac{d\beta_1 \cdots d\beta_n}{(4\pi)^n} \int_{-\infty}^{\infty} \frac{d\beta'_1 \cdots d\beta'_m}{(4\pi)^m} \\ &\times \left(\prod_{i=1}^{n} e^{-2(R-x)e(\beta_i)} \widehat{R}_a^*(\beta_i)\right) \left(\prod_{i=1}^{m} e^{-2xe(\beta'_i)} \widehat{R}_b(\beta'_i)\right) \\ &\times \langle \beta_n, -\beta_n, \ldots, \beta_1, -\beta_1|\ \mathcal{O}(0)\ |-\beta'_1, \beta'_1, \ldots, -\beta'_m, \beta'_m\rangle,\end{aligned} \tag{C.4}$$

where as usual $e(\beta) = m\cosh\beta$.

As in appendix A, one finds that the disconnected pieces of the form factors can be incorporated by simply modifying the measures for the integrals over β. Also as before the residue singularities lead to a multiplicative renormalization of the fields, which we again ignore. It is not hard to see that one obtains

$$\begin{aligned}\langle B_a|\mathcal{O}(x)|B_b\rangle_R = &\sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} \int_{-\infty}^{\infty} \frac{d\beta_1 \cdots d\beta_n}{(4\pi)^n} \int_{-\infty}^{\infty} \frac{d\beta'_1 \cdots d\beta'_m}{(4\pi)^m} \\ &\times \left(\prod_{i=1}^{n} e^{-2(R-x)e(\beta_i)} \widehat{R}_a^*(\beta_i) f(\beta_i)\right) \left(\prod_{i=1}^{m} e^{-2xe(\beta'_i)} \widehat{R}_b(\beta'_i) f(\beta'_i)\right) \\ &\times \langle \beta_n^+, -\beta_n^+, \ldots, \beta_1^+, -\beta_1^+|\ \mathcal{O}(0)\ |-\beta'_1, \beta'_1, \ldots, -\beta'_m, \beta'_m\rangle,\end{aligned} \tag{C.5}$$

where now

$$f(\beta) = \frac{1}{1 + e^{-2Re(\beta)} \widehat{R}_b(\beta) \widehat{R}_a^*(\beta)}. \tag{C.6}$$

Result (C.5) can be put in a thermodynamic form by defining $f_+(\beta) = f(\beta)$ as in (C.6), and $f_-(\beta) = 1 - f(\beta)$. Then, using that $\widehat{R}(\beta - i\pi) = -\frac{1}{\widehat{R}(\beta)}$, one finds

$$\begin{aligned}< B_a|\mathcal{O}(x)|B_b >_R = &\sum_{N=0}^{\infty} \sum_{\epsilon=\pm} \frac{1}{N!} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left\{ \frac{d\beta_i}{4\pi} e^{-2x\epsilon_i e(\beta_i)} [\widehat{R}_b]_\epsilon(\beta_i) f_\epsilon(\beta_i \right\} \\ &< 0|\mathcal{O}| -\beta_1, \beta_1, \ldots, -\beta_N, \beta_N >_{\epsilon_1, \ldots, \epsilon_N} .\end{aligned} \tag{C.7}$$

This can easily be recovered in a thermodynamic approach. The TBA was developed for the partition functions Z_{ab} in [29], and one sees that f given in (C.6) corresponds precisely to the appropriate ratio of densities as determined there.

Appendix D. Computations for bulk correlators

In this appendix we shall provide the missing steps required to establish (4.30).

First, we show that the analog of (4.29) with T derivatives also holds. As in (4.16) we have

$$T\partial_T \ln \tau_\pm = \pm \mathrm{Tr}\left((1-\mathbf{R}_\pm)T\partial_T \mathbf{K}\right). \tag{D.1}$$

From the defining relations (2.34) and (2.35) we have

$$T\partial_T \mathbf{K}(u,v) = \frac{T\partial_T f(u)}{2f(u)}\mathbf{K}(u,v) + \mathbf{K}(u,v)\frac{T\partial_T f(v)}{2f(v)}. \tag{D.2}$$

Inserting this into (4.16), and using (4.1) we get

$$T\partial_T \ln \tau_\pm = \mathrm{Tr}\left(\mathbf{R}_\pm(u,u)\frac{T\partial_T f(u)}{f(u)}\right) \tag{D.3}$$

where the symbol Tr implies an integral from $-\infty$ to ∞ over variables $u, v, \ldots$ that appear in its argument. Now using the explicit expression for $\mathbf{R}$ in (4.4) and (4.7) we get

$$T\partial_T \ln \tau_\pm = \frac{1}{2}\mathrm{Tr}\left(\left(\partial_u g_-(u) g_+(u) - g_-(u)\partial_u g_+(u) \pm \frac{g_-(u)g_+(u)}{u}\right)\frac{T\partial_T f(u)}{f(u)}\right). \tag{D.4}$$

In particular, we have

$$T\partial_T \ln(\tau_-/\tau_+) = -\mathrm{Tr}\left(\frac{g_-(u)}{u}\frac{T\partial_T f(u)}{f(u)}g_+(u)\right). \tag{D.5}$$

In the following, we will show that the right hand side of (D.5) is simply related to the potentials which were introduced earlier. We have also examined $T\partial_T \ln(\tau_-\tau_+)$, but its value does not appear to be expressible solely in terms of the potentials of Section 4.3.

Apply the operator $(1-\mathbf{R}_\pm)T\partial_T$ to both sides of the first equation in (4.6). Using (D.2) and simplifying we get

$$T\partial_T g_\pm(u) = \frac{T\partial_T f(u)}{2f(u)}g_\pm(u) - \int_{-\infty}^{\infty} dv \mathbf{R}_\pm(u,v)\frac{T\partial_T f(v)}{f(v)}g_\pm(v). \tag{D.6}$$

Also, from (2.34) we have

$$T\partial_T e(u) = \frac{T\partial_T f(u)}{2f(u)}e(u). \tag{D.7}$$

We are now ready to take T derivatives of the potentials. We have from the definition (4.18) and (D.6), (D.7)

$$\begin{aligned} T\partial_T c_\pm &= T\partial_T \mathrm{Tr}(e\mathbf{d}_{-1}g_\pm) \\ &= \mathrm{Tr}\left(e\mathbf{d}_{-1}(1-\mathbf{R}_\pm)\frac{T\partial_T f}{f}g_\pm\right) \\ &= (1\mp c_\pm)\mathrm{Tr}\left(g_\mp\mathbf{d}_{-1}\frac{T\partial_T f}{f}g_\pm\right), \end{aligned} \tag{D.8}$$

where in the last step we have used the intertwining relations (4.10). Finally, using the parametrization (4.21), and comparing with (D.5), we get

$$T\partial_T\left(\ln(\tau_-/\tau_+)-\phi\right)=0. \tag{D.9}$$

The combination of (4.29) and (D.8) implies that $\tau_-/\tau_+ = Ce^\phi$ with C a pure number independent of z, $\overline{z}$, and T. We now determine C by computing the large z, $\overline{z}$ behavior of $\tau_\pm$ and ϕ at $T=0$ directly from the form factor expansion. Indeed, simply by evaluating the first two terms in the summation in (2.22) we have

$$\tau_\pm = 1\pm\frac{e^{-mr}}{\sqrt{2\pi mr}}+\ldots \qquad T=0,\ r\to\infty \tag{D.10}$$

where $r^2=4z\overline{z}=x^2+t^2$. Inserting (D.10) into the first differential equation in (4.28) we get

$$\phi=-\frac{2e^{-mr}}{\sqrt{2\pi mr}}+\ldots \qquad T=0,\ r\to\infty \tag{D.11}$$

Finally, inserting (D.10), (D.11) into $\tau_-/\tau_+=Ce^\phi$ and matching the large r behavior, we get $C=1$.

References

[1] The review V. E. Korepin, N.M Bogoliubov, A.G. Izergin, "Quantum Inverse Scattering Method and Correlation Functions", Cambridge University Press (1993) and references therein.

[2] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch, Phys. Rev. B **13**, 316 (1976); B.M. McCoy and T.T. Wu, Phys. Rev. Lett. **45**, 675 (1980); J.H.H. Perk, Phys. Lett. **79A**, 1, 3 (1980); B.M. McCoy, J.H.H. Perk, and T.T. Wu, Phys. Rev. Lett. **46**, 757 (1981).

[3] J.H.H. Perk, H.W. Capel, G.R.W. Quispel, and F.W. Nijhoff, Physica **123A**, 1 (1984).

[4] K. M. Watson, Phys. Rev. B**95** (1954) 228.

[5] M. Karowski, P. Weisz, Nucl. Phys. B**139** (1978) 455.

[6] F. Smirnov, "Form factors in completely integrable models of quantum field theory", World Scientific and references therein.

[7] J. Cardy, G. Mussardo, Nucl. Phys. B**410** (1993), 451.

[8] G. Delfino, G. Mussardo, P. Simonetti, Phys. Rev. D**51** (1995), 6620.

[9] A, Zamolodchikov, Al. Zamolodchikov, Ann. Phys. **120** (1979) 253.

[10] F. Lesage, H. Saleur, "Exact results for correlators in quantum impurity problems with an external field", preprint USC-96-09, to appear.

[11] M. Gaudin, "La fonction d'onde de Bethe", Masson (1983)

[12] J.A. Hertz, Phys. Rev. B **14**, 525 (1976).

[13] S. Chakravarty, B.I. Halperin, and D.R. Nelson, Phys. Rev. B **39**, 2344 (1989).

[14] S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992); A.V. Chubukov and S. Sachdev, Phys. Rev. Lett.**71**, 169 (1993); A.V. Chubukov, S. Sachdev and J. Ye, Phys. Rev. B **49**, 11919 (1994).

[15] S. Sachdev, Nucl. Phys. B **464**, 576 (1996); K. Damle and S. Sachdev, Phys. Rev. Lett. **76**, 4412 (1996).

[16] R. Hecht, Phys. Rev. B **138** (1967) 557.

[17] V. P. Yurov, Al. B. Zamolodchikov, Int. J. Mod. Phys. A **6** (1991) 33419.

[18] O. Babelon and D. Bernard, Physics Letters. B **288**, 113 (1992).

[19] S. Ghoshal, A. Zamolodchikov, Int. J. Phys. A **9**, 3841 (1994).

[20] R. Konik, A. LeClair, and G. Mussardo, hep-th/9508099, to appear in Int. J. Mod. Phys.

[21] D. Bernard and A. Leclair, Nucl. Phys. B **426** (1994) 534.

[22] The equations in the Lax form actually contain slightly less information than that in (4.42a) and (4.42d). Let us write $L_\pm = \partial_z \partial_{\overline{z}} \ln \tau_\pm \pm (m^2/2) e^{\pm\phi} \sinh\phi$. Then the Lax form actually only implies $L_+ - L_- = 0$, $(1 + z\partial_z - T\partial_T)L_+ = 0$, and $(1 +$

$\overline{z}\partial_{\overline{z}} - T\partial_T)L_- = 0$, which is slightly weaker than $(4.42a)$ which implies $L_\pm = 0$. The equations $(4.42d)$ however emerge completely from the Lax form.

[23] A. Zamolodchikov, Nucl. Phys. B **348** (1991) 619

[24] C. Itzykson, H. Saleur, J. Stat. Phys. **48** (1987) 449.

[25] J.L. Cardy, J. Phys. A **17**, L385 (1984)

[26] R. Z. Bariev, Teor. Mat. Fiz. **40** (1979), 40; Teor. Mat. Fiz. **42** (1980), 262; Teor. Mat. Fiz. **77** (1988), 1090.

[27] S. Sachdev in *Proceedings of the 19th IUPAP International Conference on Statistical Physics*, Xiamen, China, July 31 - August 4 1995, ed. B.-L. Hao, (World Scientific, Singapore 1996)

[28] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).

[29] A. Leclair, G. Mussardo, H. Saleur, and S. Skorik, Nucl. Phys. B **453** (1995) 581.