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QED for a Fibrillar Medium of Two-Level Atoms

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We consider a fibrillar medium with a continuous distribution of two-level atoms coupled to quantized electromagnetic fields. Perturbation theory is developed based on the current algebra satisfied by the atomic operators. The one-loop corrections to the dispersion relation for the polaritons and the dielectric constant are computed. Renormalization group equations are derived which demonstrate a screening of the two-level splitting at higher energies. Our results are compared with known results in the slowly varying envelope and rotating wave approximations. We also discuss the quantum sine-Gordon theory as an approximate theory.

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1. Introduction

The interaction of radiation with a medium of atoms is an important problem with many applications, in particular to the propagation of optical pulses and to lasers. When studying resonance phenomenon it is natural to approximate the atoms as two-level systems, resulting in optical Bloch equations. For a single atom in an external classical electric field, this problem was solved by Rabi[1]. A collection of atoms coupled to radiation is described by the so-called Maxwell-Bloch equations. Propagation effects in the latter model, were studied by McCall and Hahn[2]. There, a semi-classical approximation was made wherein the electromagnetic field was purely classical. With some additional approximations (see below) it was shown that the equations of motion reduce to the well-known sine-Gordon equation, and the optical soliton solutions were observed experimentally also in [2]. The problem of many-atom spontaneous emission was studied in a simplified model by Dicke[3], which does involve quantized electric fields. We refer the reader to [4] for an excellent account of optical resonance phenomena.

In this paper we study the fully quantum system of radiation in interaction with a continuous distribution of atoms in a fiber geometry. Two different models are studied, one following from taking the interaction hamiltonian to be $-\vec{d} \cdot \vec{E}$, which we refer to as the 'current-model', the other following from the minimal coupling prescription $\vec{p} \to \vec{p} - e\vec{A}$ and referred to as the 'charge-model'. We develop perturbation theory using the current algebra satisfied by the atomic operators. This allows us to easily determine the dependence of various physical quantities on the number of atoms N. In particular by computing the photon self-energy we determine the first quantum corrections to the polariton dispersion relation and dielectric constant. As we will show, the current verses charge models have different different behavior at one-loop, the current-model displaying more desirable features.

In the current model, we show how the quantum corrections imply a renormalization group equation for the two-level energy splitting, this splitting becoming screened at higher energies.

In the last sections of the paper we compare our results with known results obtained under various approximations, namely the slowly-varying envelope and rotating wave approximations. We also argue that the quantum sine-Gordon theory has some validity as an effective quantum field theory.

The two-dimensional quantum field theories we study are interesting in their own right. The more interesting of the two models we study (the current-model) is defined by the hamiltonian

$$H = H_0^\phi + \int dx \left(\frac{\omega_0}{2} S_3(x) + \frac{\beta}{2} \, \partial_t \phi(x) \left(S^+(x) + S^-(x) \right) \right), \tag{1.1}$$

where H_0^ϕ is a free hamiltonian for a scalar field ϕ, $S_3, S^\pm$ satisfy a current algebra (see equation 2.29), ω_0 is the two-level splitting of the atoms, and β is a dimensionless coupling that depends on the strength of the dipole transition. The renormalization group equation for the radiative level shifts is a consequence of the beta-function for ω_0, which we compute to lowest order. To the best of our knowledge, this model is not integrable, though various approximations to it are integrable (see below).

2. Two Models for the Quantum Maxwell-Bloch Theory

There are two related models describing the coupling of two-level atoms to quantized electromagnetic fields. One follows from taking the interaction hamiltonian to be $-\vec{d} \cdot \vec{E}$ where $\vec{d}$ is the dipole moment operator and $\vec{E}$ the electric field. The other follows from the usual minimal coupling prescription, $\vec{p} \to \vec{p} - e\vec{A}$ where $\vec{p}$ is the momentum operator and $\vec{A}$ the vector potential. We will refer to these as the 'current-model' and 'charge-model' respectively. In this section we describe the reduction of both these models to one spacial dimension. We will first consider the case of a single atom, and then extend this to a continuous distribution of atoms. We set $c = 1$ in most places, but keep $\hbar \neq 1$ in some formulas to clarify certain points. All formulas with $c \neq 1$ have both c and $\hbar$ restored.

For both cases, we model a single unperturbed atom as a single electron which has two eigenstates $|2\rangle, |1\rangle$, with energy difference $\hbar\omega_0$. Letting H_0^{atom} denote the unperturbed atomic hamiltonian, one has

$$H_0^{\text{atom}} |1\rangle = -\frac{\hbar\omega_0}{2} |1\rangle, \qquad H_0^{\text{atom}} |2\rangle = +\frac{\hbar\omega_0}{2} |2\rangle. \tag{2.1}$$

In the basis $(|2\rangle, |1\rangle)$,

$$H_0^{\text{atom}} = \frac{\hbar\omega_0}{2} \, \sigma_3, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

2.1 Current-Model

One can couple the 2-level atoms to the electromagnetic field starting from the energy of an electron in an external electric field. Let the single atom be centered at $\vec{x}_0$, and let $\vec{d} = e\,\vec{x}$ denote the electric dipole moment operator. The complete hamiltonian is

$$H = H_0^{\text{field}} + H_0^{\text{atom}} - \vec{d} \cdot \vec{E}(\vec{x}), \tag{2.3}$$

where H_0^{field} is the free Maxwell hamiltonian, and $\vec{x}$ is the position of the electron. Let us further assume that the electric field does not vary significantly over the region where the atomic wavefunction is non-zero, i.e.

$$\langle a|\vec{d} \cdot \vec{E}(\vec{x})|b\rangle \approx \vec{E}(\vec{x}_0)\langle a|\vec{d}|b\rangle, \tag{2.4}$$

for $|a, b\rangle$ unperturbed atomic eigenstates.

The matrix elements of $\vec{d}$ have the following general form

$$\begin{aligned} &\langle 1|\vec{d}|1\rangle = \langle 2|\vec{d}|2\rangle = 0 \\ &\langle 2|\vec{d}|1\rangle = d\,e^{i\alpha}\widehat{n}, \qquad \langle 1|\vec{d}|2\rangle = d\,e^{-i\alpha}\widehat{n}, \end{aligned} \tag{2.5}$$

where d is a real parameter and $\widehat{n}$ is a unit vector that specifies the orientation of the atom in space. $\langle a|\vec{d}|b\rangle = 0$ for $a = b$ since $\vec{d}$ is a vector operator with odd parity and the states $|a\rangle$ are assumed to have definite parity.

Letting $\widehat{E} = \vec{E} \cdot \widehat{n}$, one then has

$$\vec{d} \cdot \vec{E}(\vec{x}) = d\,\widehat{E}(\vec{x}_0)\left(e^{i\alpha}\sigma^+ + e^{-i\alpha}\sigma^-\right), \tag{2.6}$$

where

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We will later need the algebra of the σ-operators:

$$\left[\sigma_3, \sigma^\pm\right] = \pm 2\sigma^\pm, \qquad \left[\sigma^+, \sigma^-\right] = \sigma_3. \tag{2.7}$$

Note that the phase $e^{i\alpha}$ can be removed by letting $\sigma^\pm \to e^{\mp i\alpha}\sigma^\pm$ without affecting the algebra; we henceforth set this phase to one.

Consider a fibrillar geometry, where the atom can be viewed as an impurity in an optical fiber of length L and cross-sectional area $\mathcal{A}$, where $L \gg \sqrt{\mathcal{A}}$. One can perform a reduction to this essentially one-dimensional theory as follows. The free Maxwell action which determines H_0^{field} is

$$S_{\text{Maxwell}} = \frac{1}{4\pi} \int d^3\vec{x}dt \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{2.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Let $\widehat{x}$ denote the direction along the fiber, and $\widehat{y}, \widehat{z}$ the directions transverse to it. One can first consistently set $A_0 = 0$. We also require the energy flux to be along the fiber, so that the Pointing vector is in the $\widehat{x}$ direction. This requires $A_x = 0$ and $\partial_y A_z = \partial_z A_y = 0$. Thus, we only have to deal with the components of $\vec{A}$ transverse to the direction of the fiber. Of these, only $\widehat{A} = \vec{A} \cdot \widehat{n}$ couples to the atom. Assume $\widehat{A}$ is independent of y, z, and let $\int dydz = \mathcal{A}$. One has

$$S_{\text{Maxwell}} = \frac{\mathcal{A}}{4\pi} \int dxdt \frac{1}{2} \left(\partial_t \widehat{A} \partial_t \widehat{A} - \partial_x \widehat{A} \partial_x \widehat{A} \right). \tag{2.9}$$

One can alternatively understand the appearance of the cross-sectional area $\mathcal{A}$ by considering the mode expansion of a free scalar field in finite volume V,

$$\Phi(\vec{x}) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2|\vec{k}|}} \left(a(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \right). \tag{2.10}$$

In the fibrillar geometry, $V = \mathcal{A}L$ and $\sqrt{\mathcal{A}}$ is very small compared to L. As $L \to \infty$, the modes in the $\widehat{x}$-direction are nearly continuous. The transverse modes in the $\widehat{y}, \widehat{z}$ directions have a maximum wavelength on the order of $\sqrt{\mathcal{A}}$ and are thus very energetic in comparison to the low energy modes in the L-direction. Thus, we are assuming these high energy transverse modes are negligibly excited, which is reasonable if $1/\sqrt{\mathcal{A}} \gg \omega_0$. In general $\mathcal{A}$ should be replaced by $\mathcal{A}_{\text{eff}}$, which is the effective cross-sectional area of the fiber as a waveguide.

Rescale $\widehat{A}$:

$$\widehat{A} = \sqrt{\frac{4\pi\hbar}{\mathcal{A}_{\text{eff}}}} \phi. \tag{2.11}$$

Then

$$\frac{S_{\text{Maxwell}}}{\hbar} = \int dxdt \frac{1}{2} \left(\partial_t \phi \partial_t \phi - \partial_x \phi \partial_x \phi \right). \tag{2.12}$$

The field ϕ is a dimensionless scalar field. In the quantum theory it satisfies the commutation relations

$$[\phi(x,t), \partial_t \phi(x',t)] = i\delta(x - x'). \tag{2.13}$$

Using $\widehat{E} = -\partial_t \widehat{A}$, and dividing by $\hbar$ to give the hamiltonian units of 1/time, we obtain the complete hamiltonian of the current-model:

$$H = H_0^{\phi} + H_0^{\rm atom} + H_{\rm int}^{(\rm current)}, \tag{2.14}$$

where $H_0^{\rm atom}$ is defined in (2.2),

$$H_0^{\phi} = \int dx \; \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2, \tag{2.15}$$

$$H_{\rm int}^{(\rm current)} = \frac{\beta}{2} \; \partial_t \phi(x_0) \left(\sigma^+ + \sigma^- \right), \tag{2.16}$$

and

$$\beta = \sqrt{\frac{16\pi}{\hbar c \mathcal{A}_{\rm eff}}} \; d. \tag{2.17}$$

The terminology 'current-model' refers to the fact that the spin operators couple to $\partial_t \phi$ which is the spacial component of the conserved topological current $\epsilon^{\mu\nu}\partial_\nu \phi$, where $\epsilon^{\mu\nu}$ is the anti-symmetric tensor.

The parameter β is the important dimensionless coupling constant of the model. Since $d \sim e R_{\rm atom}$, where $R_{\rm atom}$ is an atomic dimension,

$$\frac{\beta^2}{8\pi} \approx 2 \left(\frac{e^2}{\hbar c} \right) \frac{R_{\rm atom}^2}{\mathcal{A}_{\rm eff}} \approx \frac{2}{137} \frac{R_{\rm atom}^2}{\mathcal{A}_{\rm eff}}. \tag{2.18}$$

Thus, generally, $\beta^2/8\pi$ is very small. An idealized upper limit would correspond to a chain of atoms in a waveguide that is one atom in thickness, so that $R_{\rm atom}^2 \sim \mathcal{A}_{\rm eff}$. This is perhaps nearly realizable with a polymer waveguide. In this situation $\beta^2/4\pi \approx 1/137$. We will refer to this hypothetical limiting case where the quantum effects are strongest as the quantum optical chain.

The parameter $\beta^2/8\pi$ determines the spontaneous decay rate $1/\tau$ of a single excited atom. First order perturbation theory gives

$$|\langle 1, k | 2 \rangle|^2 = \frac{\beta^2 \omega_0}{16\pi} \left(2\pi \delta(\omega_0 - |k|) \right)^2 \tag{2.19}$$

where $|1, k\rangle = |1\rangle_{\rm atom} \otimes |k\rangle$ and $|k\rangle$ is a one-photon state with wave-vector k. This leads to

$$\frac{1}{\tau} = \int dk \, |\langle 1, k | 2 \rangle|^2 \left(2\pi \delta(\omega_0 - |k|) \right)^{-1} = \frac{\beta^2}{8} \omega_0. \tag{2.20}$$

2.2 Charge-Model

For the charge-model we begin with the standard way to couple particles to electromagnetic fields and consider a single electron hamiltonian

$$H = \frac{1}{2m_e}\left(\vec{p} - e\vec{A}(\vec{x})\right)^2 + V(\vec{x}), \tag{2.21}$$

where $\vec{p}$ is the momentum operator and V is the atomic potential. As an approximation we drop the $\vec{A}^2$ term and consider

$$H = H_0^{\rm atom} - \frac{e}{m_e}\vec{p}\cdot\vec{A}, \tag{2.22}$$

where $H_0^{\rm atom} = \vec{p}^2/2m_e + V$. Again, we make a 2-level approximation and consider states $|1,2\rangle$ as in (2.1). As for the current-model, we assume the analog of (2.4). Using

$$\vec{p} = -i\frac{m_e}{\hbar}\left[\vec{x}, H_0^{\rm atom}\right], \tag{2.23}$$

one has

$$-\frac{e}{m_e}\vec{A}(\vec{x}_0)\cdot\langle a|\,\vec{p}\,|b\rangle = \frac{i}{\hbar}(E_b - E_a)\vec{A}(\vec{x}_0)\cdot\langle a|\,\vec{d}\,|b\rangle, \tag{2.24}$$

where E_a is the energy of the state $|a\rangle$, and $\vec{d}$ is again the electric dipole operator $\vec{d} = e\,\vec{x}$.

Using the parameterization (2.5) for the $\vec{d}$ matrix elements, and rescaling $\widehat{A}$ as in (2.11), one obtains a hamiltonian of the form (2.14), the only difference being the interaction term:

$$H_{\rm int}^{\rm (charge)} = -i\frac{\omega_0\beta}{2}\phi(x_0)\left(\sigma^+ - \sigma^-\right). \tag{2.25}$$

We refer to this as the 'charge-model' since $\phi(\infty) - \phi(-\infty)$ is the charge associated to the topological current $\epsilon^{\mu\nu}\partial_\nu\phi$.

The spontaneous decay rate to lowest order for the charge-model is the same as for the current-model (2.20).

2.3 Continuous Distribution of Atoms

Consider now a collection of atoms, with the N atoms positioned at $x = x_i$, $i = 1, .., N$. Also, let $\vec{d}_i = e(\vec{x} - \vec{x}_i)$, $|1,2\rangle_i$ and $\sigma(i)$ denote the dipole moment, 2-level states, and Pauli matrix operators for the i-th atom. Since ${}_i\langle 1|\vec{d}_i|1\rangle_i = {}_i\langle 2|\vec{d}_i|1\rangle_i = 0$, and ${}_i\langle 1|2\rangle_i = 0$, the matrix elements of $\vec{d}_i$ don't depend on $\vec{x}_i$. However, in general the atoms have variable orientations in space and ${}_i\langle 2|\vec{d}_i|1\rangle_i = d\,e^{i\alpha}\widehat{n}_i$, where $\widehat{n}_i$ can vary from atom to atom. To simplify the situation, we assume that all the atoms are somehow aligned, for example by some external electric field or by being embedded in a crystal. Then,

$$
{}_i\langle 2|\vec{d}_i|1\rangle_i = d\, e^{i\alpha}\widehat{n}, \qquad {}_i\langle 1|\vec{d}_i|2\rangle_i = d\, e^{-i\alpha}\widehat{n}. \tag{2.26}
$$

For the current-model, the interaction becomes

$$
H_{\rm int}^{(\rm current)} = \frac{\beta}{2}\sum_{i=1}^{N} \partial_t\phi(x_i)\left(\sigma^+(i)+\sigma^-(i)\right). \tag{2.27}
$$

Introduce space-time dependent spin operators as follows:

$$
S^a(x) = \sum_{i=1}^{N} \sigma^a(i)\delta(x-x_i). \tag{2.28}
$$

These operators satisfy a current algebra

$$
\begin{aligned}
\left[S_3(x,t), S^{\pm}(x',t)\right] &= \pm 2\, S^{\pm}(x,t)\delta(x-x') \\
\left[S^+(x,t), S^-(x',t)\right] &= S_3(x,t)\delta(x-x').
\end{aligned} \tag{2.29}
$$

The hamiltonian for both the current and charge models takes the form (2.14), where now

$$
H_0^{\rm atom} = \frac{\omega_0}{2}\int dx\; S_3(x), \tag{2.30}
$$

and

$$
\begin{aligned}
H_{\rm int}^{(\rm current)} &= \frac{\beta}{2}\int dx\; \partial_t\phi(x,t)\left(S^+(x,t)+S^-(x,t)\right) \\
H_{\rm int}^{(\rm charge)} &= -i\frac{\omega_0\beta}{2}\int dx\; \phi(x,t)\left(S^+(x,t)-S^-(x,t)\right).
\end{aligned} \tag{2.31}
$$

In working with the above formulation, one must impose a further condition that there is a *single* electron bound to each atom. For the one-atom operators, this is manifest in the two-dimensional representation of the σ's, which has the additional relations $(\sigma^{\pm}(i))^2 = 0$, $(\sigma_3(i))^2 = 1$. Note from the definition (2.28) that

$$
\left(S^{\pm}(x,t)\right)^2 \neq 0. \tag{2.32}
$$

These issues are more easily resolved in a fermionic description, which we turn to next.

2.4 Fermionic Description

Consider first the one-atom case. In a second quantized description, one introduces a fermion wavefunction $|\psi\rangle$:

$$|\psi\rangle = b_1|1\rangle + b_2|2\rangle, \qquad \langle\psi| = \langle 1|b_1^\dagger + \langle 2|b_2^\dagger, \tag{2.33}$$

where the b's are fermion operators satisfying

$$\{b_1, b_1^\dagger\} = \{b_2, b_2^\dagger\} = 1. \tag{2.34}$$

The operator $b_1^\dagger$ ($b_2^\dagger$) creates an electron in the lowest (highest) level. The σ operators then have the following well-known representation:

$$\sigma^+ = b_2^\dagger b_1, \qquad \sigma^- = b_1^\dagger b_2, \qquad \sigma_3 = b_2^\dagger b_2 - b_1^\dagger b_1, \tag{2.35}$$

and the algebra (2.7) is a consequence of (2.34). Since each atom has a single electron, one must impose the constraint

$$b_2^\dagger b_2 + b_1^\dagger b_1 = 1. \tag{2.36}$$

This additional algebraic relation leads to the 2-level relations $(\sigma^\pm)^2 = 0, (\sigma_3)^2 = 1$.

In the multi-atom case, one can define

$$\begin{aligned} S^+(x) &= b_2^\dagger(x) b_1(x), \qquad S^-(x) = b_1^\dagger(x) b_2(x) \\ S_3(x) &= b_2^\dagger(x) b_2(x) - b_1^\dagger(x) b_1(x). \end{aligned} \tag{2.37}$$

The algebra (2.29) is then a consequence of the anti-commutation relations:

$$\{b_1(x), b_1^\dagger(x')\} = \{b_2(x), b_2^\dagger(x')\} = \delta(x - x'). \tag{2.38}$$

Define the number operator:

$$\widehat{N} = \int dx \left(b_2^\dagger(x) b_2(x) + b_1^\dagger(x) b_1(x) \right). \tag{2.39}$$

The operator $\widehat{N}$ commutes with the hamiltonian and corresponds to the number of atoms in the sample. We therefore impose the constraint

$$\widehat{N} = N. \tag{2.40}$$

In evaluating quantum transition amplitudes, one doesn't have to take special account of this constraint as long as one deals with initial and final states that satisfy the constraint, since $[\widehat{N}, H] = 0$. Note that

$$\int dx\, S_3(x) = -\widehat{N} + 2\int dx\, b_2^\dagger(x) b_2(x). \tag{2.41}$$

In the sequel, we will work with the perturbative vacuum:

$$|\Omega\rangle = |0\rangle_{\text{photon}} \otimes |\downarrow\rangle_{\text{atom}}, \tag{2.42}$$

where $|\downarrow\rangle$ denotes the atomic state with all N atoms in their lowest energy state, and $|0\rangle$ is the state with no photons. This vacuum state satisfies

$$b_2(x)|\Omega\rangle = S^-(x)|\Omega\rangle = 0, \tag{2.43}$$

and

$$H_0^{\text{atom}}|\Omega\rangle = -\frac{N\omega_0}{2}\,|\Omega\rangle. \tag{2.44}$$

From (2.41), one has

$$\int dx\;\; S_3(x)\;|\Omega\rangle = -N|\Omega\rangle. \tag{2.45}$$

Assuming spacial translation invariance of $|\Omega\rangle$ (we are ignoring boundary effects for L very large), (2.45) implies

$$S_3(x)|\Omega\rangle = -\frac{N}{L}|\Omega\rangle. \tag{2.46}$$

We remark that $|\Omega\rangle$ is not the exact ground state of the theory; see section 5.

3. Perturbation Theory

In this section, we study the perturbative expansion for the general correlation functions of the field ϕ, which are simply related to electric field correlators via (2.11). For convenience of notation, in the charge-model we make the redefinition $S^\pm \to i^{\pm 1} S^\pm$, which does not affect the commutation relations (2.29). The interaction hamiltonian for both models can then be written as

$$H_{\text{int}} = g\int dx\;\mathcal{O}(x,t)\,\big(S^+(x,t) + S^-(x,t)\big) \tag{3.1}$$

where for

$$\begin{aligned}
\text{current}-\text{model}: &\qquad g=\frac{\beta}{2}, \qquad \mathcal{O}=\partial_t\phi \\
\text{charge}-\text{model}: &\qquad g=\frac{\omega_0\beta}{2}, \qquad \mathcal{O}=\phi.
\end{aligned} \tag{3.2}$$

We begin with the partition function, defined as the vacuum to vacuum transition amplitude. In the interaction picture,

$$Z=\langle\Omega|U(\infty,-\infty)|\Omega\rangle, \tag{3.3}$$

where

$$U(\infty,-\infty)=T\exp\left(-ig\int_{-\infty}^{\infty}dxdt\ \mathcal{O}(x,t)(S^{+}+S^{-})\right). \tag{3.4}$$

The correlator $\langle\downarrow|S^{a_1}\cdots S^{a_n}|\downarrow\rangle_{g=0}$ is only non-zero when $\sum_i a_i=0$, thus,

$$\begin{aligned}
Z=\sum_{n=0}^{\infty}(-)^n g^{2n}\int dx_{2n}\cdots dx_1\int_{t_{2n}>\cdots>t_1} dt_{2n}\cdots dt_1\ \langle 0|\mathcal{O}(t_{2n},x_{2n})\cdots\mathcal{O}(x_1,t_1)|0\rangle_{g=0} \\
\times\sum_{a_1,..,a_{2n}=\pm}\langle\downarrow|S^{a_{2n}}(x_{2n},t_{2n})\cdots S^{a_1}(x_1,t_1)|\downarrow\rangle_{g=0}.
\end{aligned} \tag{3.5}$$

The time dependence of the S-correlators is simple, since the hamiltonian at $g=0$ implies

$$S^{\pm}(x,t)=e^{\pm i\omega_0 t}S^{\pm}(x,0). \tag{3.6}$$

Thus,

$$\langle\downarrow|S^{a_{2n}}(x_{2n},t_{2n})\cdots S^{a_1}(x_1,t_1)|\downarrow\rangle_{g=0}=\left[\prod_{i=1}^{2n}e^{ia_i\omega_0 t}\right]\langle\downarrow|S^{a_{2n}}(x_{2n},0)\cdots S^{a_1}(x_1,0)|\downarrow\rangle_{g=0}. \tag{3.7}$$

It will be helpful to pass to momentum space. Our conventions are $\vec{k}=(\omega,k)$, $d^2\vec{k}=d\omega dk$, $\vec{k}^2=\omega^2-k^2$, and $\vec{k}\cdot\vec{x}=\omega t-kx$. Define

$$\langle 0|\mathcal{O}(t_{2n},x_{2n})\cdots\mathcal{O}(x_1,t_1)|0\rangle_{g=0}=\int\frac{d^2\vec{k}_1}{(2\pi)^2}\cdots\frac{d^2\vec{k}_{2n}}{(2\pi)^2}\left(\prod_{i=1}^{2n}e^{-i\vec{k}_i\cdot\vec{x}_i}\right)\ \widetilde{G}_0^{(2n)}(\vec{k}_1,\vec{k}_2,.....,\vec{k}_{2n}). \tag{3.8}$$

Substituting this into (3.5), one can perform the time integrals using

$$\int_{-\infty}^{t}dt' e^{-i\omega t'}=\frac{i}{\omega+i\epsilon}e^{-i\omega t}, \tag{3.9}$$

where here and below ϵ is infinitesimally small and positive. One finds

$$Z = \sum_{n=0}^{\infty} -ig^{2n} \int dx_1 \cdots dx_{2n} \int \frac{d^2\vec{k}_1}{(2\pi)^2} \cdots \frac{d^2\vec{k}_{2n}}{(2\pi)^2} \left(\prod_{i=1}^{2n} e^{ik_i x_i} \right) 2\pi\delta \left(\sum_i \omega_i \right) \widetilde{G}_0^{(2n)}(\vec{k}_1, ..., \vec{k}_{2n})$$
$$\times f_{a_1,\ldots,a_{2n}}(\omega_1, ..., \omega_{2n-1}) \sum_{a_1,..,a_{2n}=\pm} \langle \downarrow | S^{a_{2n}}(x_{2n}, 0) \cdots S^{a_1}(x_1, 0) | \downarrow \rangle_{g=0}, \tag{3.10}$$

with

$$f_{a_1,\ldots,a_{2n}}(\omega_1, ..., \omega_{2n-1}) = \prod_{i=1}^{2n-1} \frac{1}{\sum_{j=1}^{i} (\omega_j - a_j\omega_0) + i\epsilon}. \tag{3.11}$$

The infinite volume singularities in the sum (3.10) must be regulated in order to obtain something meaningful. However, the correlation functions, which are obtained by dividing by Z, are well defined order by order in perturbation theory.

Let $\Phi_1, \Phi_2, ...$ denote some local fields. The same analysis as above leads to

$$\langle \Omega | T \left(\Phi_1(\vec{x}'_1) \cdots \Phi_m(\vec{x}'_m) \right) | \Omega \rangle = \int \frac{d^2\vec{k}'_1}{(2\pi)^2} \cdots \frac{d^2\vec{k}'_m}{(2\pi)^2} \left(\prod_{i=1}^{m} e^{-i\vec{k}'_i \cdot \vec{x}'_i} \right) \widetilde{G}^{(m)}(\vec{k}'_1, \vec{k}'_2, ..., \vec{k}'_m), \tag{3.12}$$

where

$$\widetilde{G}^{(m)}(\vec{k}'_1, ..., \vec{k}'_m) = \frac{1}{Z} \sum_{n=0}^{\infty} -ig^{2n} \int dx_1 \cdots dx_{2n} \int \frac{d^2\vec{k}_1}{(2\pi)^2} \cdots \frac{d^2\vec{k}_{2n}}{(2\pi)^2} \left(\prod_{i=1}^{2n} e^{ik_i x_i} \right) 2\pi\delta \left(\sum_i \omega_i \right)$$
$$\times \widetilde{G}_0^{(2n+m)}(\vec{k}'_1, ..., \vec{k}'_m; \vec{k}_1, ..., \vec{k}_{2n}) f_{a_1,\ldots,a_{2n}}(\omega_1, ..., \omega_{2n-1})$$
$$\times \sum_{a_1,..,a_{2n}=\pm} \langle \downarrow | S^{a_{2n}}(x_{2n}, 0) \cdots S^{a_1}(x_1, 0) | \downarrow \rangle_{g=0}, \tag{3.13}$$

and $\widetilde{G}_0^{(2n+m)}$ is the Fourier transform of

$$\langle 0 | T \left(\Phi_1(\vec{x}'_1) \cdots \Phi_m(\vec{x}'_n) \mathcal{O}(x_1, t_1) \cdots \mathcal{O}(x_{2n}, t_{2n}) \right) | 0 \rangle_{g=0}. \tag{3.14}$$

(For $n = 0$ in (3.13) the $-i$ is omitted.)

The integrands in (3.13) are simple to evaluate. The free Green's functions $\widetilde{G}_0$ are products of free field propagators for the field ϕ. The S-correlation functions can be evaluated using the algebra (2.29) and (2.46). As usual, dividing by Z serves to remove 'vacuum bubbles'. We will illustrate the main features by computing the 2-point function to order β^4 in the next section.

4. Photon Self-Energy and the Dielectric Constant

4.1 Charge-Model Computations

Due to the overall momentum conservation, one can write

$$\langle\Omega|\phi(x,t)\phi(0)|\Omega\rangle = \int \frac{d\omega dk}{4\pi^2}\ \Pi(\omega,k), \tag{4.1}$$

where

$$\Pi(\omega,k) = \int \frac{d^2\vec{k}'}{(2\pi)^2}\ \widetilde{G}^{(2)}(\vec{k},\vec{k}'), \tag{4.2}$$

and $\widetilde{G}^{(2)}$ is defined in (3.13). On general grounds, one expects Π to take the form

$$\begin{aligned}\Pi(\omega,k) &= \frac{i}{\omega^2-k^2-\Sigma(w,k)}\\ &= \frac{i}{\omega^2-k^2}\left(1+\frac{\Sigma}{\omega^2-k^2}+\frac{\Sigma^2}{(\omega^2-k^2)^2}+\dots\right).\end{aligned} \tag{4.3}$$

In quantum electrodynamics, Σ is called the photon self-energy. The dispersion relation is

$$\omega^2-k^2-\Sigma=0. \tag{4.4}$$

In our models, Σ is only a function of ω, so the dielectric constant, defined as $\varepsilon(\omega)=k^2/\omega^2$, is given by

$$\varepsilon(\omega)=1-\frac{\Sigma(\omega)}{\omega^2}. \tag{4.5}$$

The free field ϕ correlators appearing in (3.13) are products of propagators, which follow from the two point function:

$$\langle 0|\phi(x,t)\phi(0)|0\rangle = \int \frac{d^2\vec{k}}{(2\pi)^2}\ \frac{i}{\omega^2-k^2+i\epsilon}\exp(-i\vec{k}\cdot\vec{x}). \tag{4.6}$$

Thus, $\widetilde{G}_0^{(2n+2)}(\vec{k},\vec{k}';\vec{k}_1,...,\vec{k}_{2n})$ is a product of factors $\widetilde{G}_0^{(2)}$ where

$$\widetilde{G}_0^{(2)}(\vec{k}_1,\vec{k}_2)=(2\pi)^2\delta^{(2)}(\vec{k}_1+\vec{k}_2)\frac{i}{\vec{k}_1^2+i\epsilon}. \tag{4.7}$$

For the order β^2 contribution one needs

$$\widetilde{G}_0^{(4)}(\vec{k},\vec{k}';\vec{k}_1,\vec{k}_2)=\widetilde{G}_0^{(2)}(\vec{k},\vec{k}')\widetilde{G}_0^{(2)}(\vec{k}_1,\vec{k}_2)+\widetilde{G}_0^{(2)}(\vec{k},\vec{k}_1)\widetilde{G}_0^{(2)}(\vec{k}',\vec{k}_2)+\widetilde{G}_0^{(2)}(\vec{k},\vec{k}_2)\widetilde{G}_0^{(2)}(\vec{k}',\vec{k}_1). \tag{4.8}$$

In the sum over a_i, only $a_1 = +, a_2 = -$ give a non-zero contribution, and from (2.29), (2.46) one has

$$\langle \downarrow | S^-(x_2, 0) S^+(x_1, 0) | \downarrow \rangle = \frac{N}{L} \delta(x_2 - x_1). \tag{4.9}$$

One sees that the S-correlation function here gives rise to an effective 2-point interaction of the photons, similar to a mass term. The first term in (4.8) gives rise to a vacuum bubble which is subtracted. For the other two terms in (4.8), one finds that all the integrals are saturated with δ-functions, and one simply finds

$$\Pi(\omega, k) = \frac{i}{\omega^2 - k^2 + i\epsilon} + \frac{i\Sigma_2(\omega)}{(\omega^2 - k^2 + i\epsilon)^2}, \tag{4.10}$$

where

$$\Sigma_2(\omega) = \frac{\beta^2}{2} \frac{N}{L} \frac{\omega_0^3}{\omega^2 - \omega_0^2}. \tag{4.11}$$

The order β^4 computation can be carried out explicitly. The two S-operator correlators that contribute to the sum are

$$\langle \downarrow | S^-(x_4, 0) S^+(x_3, 0) S^-(x_2, 0) S^+(x_1, 0) | \downarrow \rangle = \left(\frac{N}{L} \right)^2 \delta(x_{21}) \delta(x_{43}) \tag{4.12}$$

$$\begin{aligned} \langle \downarrow | S^-(x_4, 0) S^-(x_3, 0) S^+(x_2, 0) S^+(x_1, 0) | \downarrow \rangle = &\left(\frac{N}{L} \right)^2 (\delta(x_{23}) \delta(x_{41}) + \delta(x_{13}) \delta(x_{42})) \\ &- 2 \frac{N}{L} \delta(x_{23}) \delta(x_{21}) \delta(x_{41}), \end{aligned} \tag{4.13}$$

where $\delta(x_{12}) = \delta(x_1 - x_2)$, etc. Doing the x integrals one obtains

$$\begin{aligned} \Pi(\omega, k)|_{\beta^4} = &\frac{-i}{Z} \left(\frac{\omega_0 \beta}{2} \right)^4 \int \frac{d^2 \vec{k}'}{(2\pi)^2} \frac{d^2 \vec{k}_1}{(2\pi)^2} \cdots \frac{d^2 \vec{k}_4}{(2\pi)^2} 2\pi \delta(\omega_1 + .. + \omega_4) \widetilde{G}_0^{(6)}(\vec{k}, \vec{k}', \vec{k}_1, ..., \vec{k}_4) \\ &\times \left\{ \left(\frac{N}{L} \right)^2 f_{+-+-}(\omega_1, \omega_2, \omega_3) (2\pi)^2 \delta(k_{12}) \delta(k_{34}) + f_{++--}(\omega_1, \omega_2, \omega_3) \right. \\ &\times \left. \left[\left(\frac{N}{L} \right)^2 (2\pi)^2 \left(\delta(k_{13}) \delta(k_{24}) + \delta(k_{23}) \delta(k_{14}) \right) - 4\pi \frac{N}{L} \delta(k_1 + k_2 + k_3 + k_4) \right] \right\}. \end{aligned} \tag{4.14}$$

($k_{12} = k_1 + k_2$ etc.) The free Green's function one needs is

$$\begin{aligned} \widetilde{G}_0^{(6)}(\vec{k}, \vec{k}'; \vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = &\widetilde{G}_0^{(2)}(\vec{k}, \vec{k}') \left(\widetilde{G}_0^{(2)}(\vec{k}_1, \vec{k}_2) \widetilde{G}_0^{(2)}(\vec{k}_3, \vec{k}_4) + 2 \text{ perm.} \right) \\ &+ \left(\widetilde{G}_0^{(2)}(\vec{k}, \vec{k}_1) \widetilde{G}_0^{(2)}(\vec{k}', \vec{k}_2) \widetilde{G}_0^{(2)}(\vec{k}_3, \vec{k}_4) + 11 \text{ perm.} \right). \end{aligned} \tag{4.15}$$

A diagrammatic technique can be developed for organizing the computation. One first draws a diagram with $n+1$ unconnected lines representing $\widetilde{G}_0^{(2n+2)}$, where each line is a propagator assigned the value $i/(\vec{k}^2+i\epsilon)$, and one repeats this diagram for each possible assignment $(\vec{k},\vec{k}',\vec{k}_1,...,\vec{k}_{2n})$ of the ends of the propagators. One then links them according to the δ-functions in the interactions generated by the S-correlations, and integrates over all remaining momenta including the factors $f(\omega)$. The S-correlation functions are such that they generate new kinds of interactions at each order, so we refrain from outlining a complete set of rules here.

For example, the term proportional to $\delta(k_{12})\delta(k_{34})$ leads to the diagram in figure 1, which equals

$$i\left(\frac{\omega_0\beta}{2}\right)^4\left(\frac{N}{L}\right)^4\frac{1}{(\omega+\omega_0)(\omega^2-k^2)^2}\int\frac{d\omega_2}{2\pi i}\,\frac{1}{(\omega_2-\omega+i\epsilon)(\omega_2-\omega_0+i\epsilon)(\omega_2^2-k^2+i\epsilon)}. \tag{4.16}$$

The labels $1,2..$ in figure 1 refer to $\omega_1,\omega_2...$ and the structure of the diagram implies $\omega_1=-\omega,\omega_2=-\omega_4,\omega_3=\omega$; this leads to the integrand (4.16). The ω_2 integral is easily done. There are poles at $\omega-i\epsilon$, $\omega_0-i\epsilon$, and $\pm(|k|-i\widetilde{\epsilon}^+)$. Closing the contour in the upper half-plane, one only picks up the pole at $\omega_2=-|k|$. The expression (4.16) becomes

$$-i\left(\frac{\omega_0\beta}{2}\right)^4\left(\frac{N}{L}\right)^2\frac{1}{2|k|}\left(\frac{1}{(w^2-k^2)^2(\omega+\omega_0)(\omega+|k|)(\omega_0+|k|)}\right). \tag{4.17}$$

There are a total of 24 terms of this type (after subtracting vacuum bubbles) in (4.14), all proportional to $(\frac{N}{L})^2$.

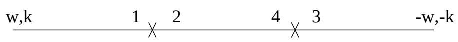

Figure 1. Diagrammatic representation of (4.16).

The term in (4.14) proportional to $\delta(k_1+k_2+k_3+k_4)$ is a new effective 4-point interaction of the photons. In the diagrammatic scheme described above, one contribution for example is shown in figure 2. This has the value

$$\frac{1}{8}\frac{N}{L}(\omega_0\beta)^4\frac{1}{2\omega_0(\omega+\omega_0)(\omega^2-k^2)^2}\int\frac{d^2\vec{k}_3}{(2\pi)^2}\frac{1}{(\vec{k}_3^2+i\epsilon)(\omega_3-\omega_0+i\epsilon)}. \tag{4.18}$$

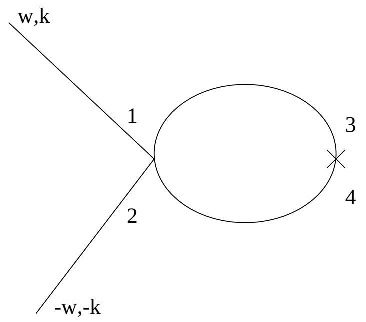

Figure 2. Diagrammatic representation of (4.18).

There are a total of 12 terms of this kind.

Upon summing these contributions, one finds

$$\Pi(\omega,k)|_{\beta^4} = \frac{i}{(\omega^2-k^2)^3}(\Sigma_2(\omega))^2 + \frac{i}{(\omega^2-k^2)^2}\Sigma_4(\omega) \tag{4.19}$$

where

$$\begin{aligned}\Sigma_4(\omega) = -\frac{i}{4}\frac{N}{L}\beta^4\omega_0^5 \int \frac{d\widetilde{\omega}d\widetilde{k}}{(2\pi)^2}\frac{1}{\widetilde{\omega}^2-\widetilde{k}^2+i\epsilon}&\Bigg[\frac{1}{(\omega^2-\omega_0^2)(\widetilde{\omega}-\omega_0+i\epsilon)^2} \\ &+\frac{2\omega_0}{(\omega^2-\omega_0^2)^2(\widetilde{\omega}-\omega_0+i\epsilon)}\Bigg].\end{aligned} \tag{4.20}$$

The first term in (4.19) arises from the sum of all diagrams of the kind shown in figure 1, whereas Σ_4 is the sum of the diagrams of the kind shown in figure 2. Thus we have verified to fourth order the structure anticipated in (4.3), with $\Sigma = \Sigma_2 + \Sigma_4 + \mathcal{O}(\beta^6)$.

In theories with more conventional perturbative expansions, the self-energy is the sum of one-particle irreducible 2-point diagrams. One-particle irreducibility is likely to be more transparent in a perturbative expansion based on the $b_{1,2}$ fermion fields rather than directly on the S operators, but we don't develop this here.

The left-over integrals in Σ_4 identify it as a one-loop contribution. The $\widetilde{k}$ integral can be done using

$$\int dk \frac{1}{\omega^2-k^2+i\epsilon} = -\frac{i\pi}{|\omega|+i\epsilon}. \tag{4.21}$$

The remaining $\widetilde{\omega}$ integral can also be done by safely analytically continuing to the positive imaginary axis avoiding poles. One finds that the $\widetilde{\omega}$ integral is divergent, the divergence arising from the vicinity of the integrand near $|\widetilde{\omega}| = 0$. We regulate this by introducing an infrared cutoff ω_{IR}:

$$\int_0^\infty d\omega \to \int_{\omega_{IR}}^\infty d\omega. \tag{4.22}$$

The natural cutoff is due to the finite size L of the sample, so we take

$$\omega_{IR} = \frac{2\pi}{L}. \tag{4.23}$$

The final result for the self-energy of the charge-model is the following:

$$\Sigma(\omega) = \frac{\beta^2}{2}\frac{N}{L}\frac{\omega_0^3}{\omega^2-\omega_0^2}\left(1+\frac{\beta^2}{8\pi}\left[\frac{3\omega_0^2-\omega^2}{\omega^2-\omega_0^2}\left(\log\left(\frac{\omega_0^2-\omega_{IR}^2}{\omega_{IR}^2}\right)+i\pi\right)+\frac{2\omega_0^2}{\omega_0^2-\omega_{IR}^2}\right]\right). \tag{4.24}$$

The imaginary part arises from the choice of a positive argument of the log, i.e. $\omega_0^2 > \omega_{IR}^2$, and is discussed further below. The size of the one-loop corrections is determined by the dimensionless parameter $\beta^2/8\pi$. The expression (4.24) is the main result of this section.

The self-energy depends on N only through the combination $\beta^2 N\omega_0/L$ with units of mass^2. This has a universal meaning in terms of the volume density η of atoms:

$$m^2 \equiv \frac{\beta^2}{8}\frac{N}{L}\omega_0 = \frac{1}{\tau}\frac{N}{L} = 2\pi\hbar\omega_0\frac{d^2}{c^4}\eta, \tag{4.25}$$

with $\eta = \frac{N}{\mathcal{A}_{\rm eff}L}$. In the limiting case of the quantum optical chain discussed above,

$$m^2 \sim \frac{1}{137}\,2\pi\omega_0\,\rho, \tag{4.26}$$

where ρ is density per unit length of atomic impurities. To obtain an idea of orders of magnitude, for $\hbar\omega_0 = 1$ev, one finds $m = 1$ev for $\rho = 10^5$/cm.

4.2 Current-Model Computations

Consider next the current-model. The photon self-energy should now be defined as follows:

$$\langle\Omega|\partial_t\phi(x,t)\,\partial_t\phi(0)|\Omega\rangle = i\int\frac{d^2\vec{k}}{(2\pi)^2}\;\exp(-i\vec{k}\cdot\vec{x})\frac{\omega^2}{\omega^2-k^2-\Sigma}. \tag{4.27}$$

The general expressions for the charge-model we obtained in this section still apply, except that now $\widetilde{G}_0^{(n)}(\vec{k}_1,...,\vec{k}_n)$ is the Fourier transform of the correlation function $\langle 0|\partial_t\phi(\vec{x}_1)\cdots\partial_t\phi(\vec{x}_n)|0\rangle_{g=0}$. In particular, (4.7) is replaced by

$$\widetilde{G}_0^{(2)}(\vec{k}_1,\vec{k}_2) = (2\pi)^2\delta^{(2)}(\vec{k}_1+\vec{k}_2)\frac{i\omega_1^2}{\vec{k}_1^2+i\epsilon}. \tag{4.28}$$

Repeating the above computations, one finds

$$\Sigma(\omega) = \frac{\beta^2}{2}\frac{N}{L}\frac{\omega_0\omega^2}{\omega^2-\omega_0^2} - \frac{i}{4}\frac{N}{L}\beta^4\omega^2\omega_0 \int \frac{d\widetilde{\omega} d\widetilde{k}}{(2\pi)^2}\frac{\widetilde{\omega}^2}{\widetilde{\omega}^2-\widetilde{k}^2+i\epsilon}\left[\frac{1}{(\omega^2-\omega_0^2)(\widetilde{\omega}-\omega_0+i\epsilon)^2} + \frac{2\omega_0}{(\omega^2-\omega_0^2)^2(\widetilde{\omega}-\omega_0+i\epsilon)}\right]. \tag{4.29}$$

One can perform the integrals as before. An important difference from the charge-model is that here the integrals are ultraviolet rather than infrared divergent. There is a natural u.v. cutoff in the model since it is implicit in (2.4) that the wavelengths of the photons are large compared to the size of the atoms. We introduce a u.v. cutoff μ as follows:

$$\int_0^\infty d\omega \to \int_0^\mu d\omega. \tag{4.30}$$

The final result is

$$\Sigma(\omega) = \frac{\beta^2}{2}\frac{N}{L}\frac{\omega_0\omega^2}{\omega^2-\omega_0^2}\left[1+\frac{\beta^2}{4\pi}-\frac{\beta^2}{8\pi}\left(\frac{\omega^2+\omega_0^2}{\omega^2-\omega_0^2}\right)\left(\log\left(\frac{\mu^2}{\omega_0^2}-1\right)-i\pi\right)\right] \qquad + \mathcal{O}(\beta^6). \tag{4.31}$$

Here, the imaginary part arises from choosing a positive argument of the log, i.e. $\mu^2 > \omega_0^2$, which is physically sensible if resonant photons (with energy ω_0) have a longer wavelength than the size of atoms. Note however that $\mathrm{Im}(\Sigma) = 0$ when $\mu^2 < \omega_0^2$. As we show below, the occurrence of this imaginary part is related to the spontaneous decay lifetime of the atoms.

5. Polaritons and Spontaneous Decay

One sees from the self-energy that the spectrum consists of two 'polariton' branches. To lowest order in β^2, the dispersion relation for the charge-model reads

$$\omega^2 - k^2 = 4m^2\frac{\omega_0^2}{\omega^2-\omega_0^2}, \tag{5.1}$$

which implies

$$\omega^2 = \omega_\pm^2 = \frac{1}{2}\left((\omega_0^2+k^2) \pm \sqrt{(\omega_0^2-k^2)^2+16m^2\omega_0^2}\right). \tag{5.2}$$

As $m \to 0$, the two branches become $\omega^2 = \omega_0^2$ and $\omega^2 = k^2$, i.e. optical phonon-like and photon-like respectively. For finite m, the ω_+ branch is phonon-like for small $|k|$ but photon-like for large $|k|$, and visa versa for the ω_- branch. In general one has quasi-particles with both atomic and photon degrees of freedom. The analagous formula for the current-model is

$$\omega_\pm^2 = \frac{1}{2}\left((\omega_0^2 + k^2 + 4m^2) \pm \sqrt{(\omega_0^2 - k^2 - 4m^2)^2 + 16m^2\omega_0^2} \right). \tag{5.3}$$

In figures 3 and 4 are plots of $\omega(k)$ for both the current and charge models. For both these models one has $\omega_+(k=0) \approx \omega_0 + 2m^2/\omega_0$, and $\omega_-(k=\infty) \approx \omega_0$. Thus there is a gap between the two branches with

$$E_{\rm gap} = \frac{2m^2}{\omega_0} = \frac{\beta^2}{4}\frac{N}{L} = 4\pi d^2 \eta. \tag{5.4}$$

For the idealized quantum optical chain with $\beta^2/4\pi \approx 1/137$, if $N/L = 10^5/\mathrm{cm}$, then $E_{\rm gap} = 2\mathrm{ev}$. Note also that there are no photon-like excitations near $k=0$ for the charge-model, i.e. for $\omega_- > 0$ one needs $|k| > 2m$. This undesirable feature is possibly related to the infrared divergence in the charge-model at one loop, and is perhaps an artifact of our having dropped the A^2 term in (2.21), which spoils the gauge invariance. The current-model on the other hand has a perfectly well-behaved photon branch near $k=0$.

Due to the imaginary part of Σ, and consequently the imaginary part of the dielectric constant, there is attenuation of plane waves in the medium. The origin of this imaginary part is the finite lifetime under spontaneous decay of single excited atomic states. To see this, note that for a small shift of ω_0

$$\frac{\omega_0 + \delta\omega_0}{\omega^2 - (\omega_0 + \delta\omega_0)^2} \approx \frac{\omega_0}{\omega^2 - \omega_0^2}\left(1 + \frac{\delta\omega_0}{\omega_0}\ \frac{\omega^2 + \omega_0^2}{\omega^2 - \omega_0^2}\right). \tag{5.5}$$

Comparing this with (4.31), one sees that the imaginary part of Σ for the current-model can be interpreted as a small imaginary shift to ω_0:

$$\frac{\delta\omega_0}{\omega_0} = i\frac{\beta^2}{8}. \tag{5.6}$$

This leads to the decay of single atoms $e^{i\omega_0 t} \to e^{i\omega_0 t} e^{-t/\tau}$, with τ given in (2.20).

The attenuation constant α is defined such that the intensity of radiation decays as $e^{-\alpha x}$. In terms of the dielectric constant,

$$\alpha = \left| \omega \frac{\mathrm{Im}\varepsilon}{\sqrt{\mathrm{Re}\varepsilon}} \right|. \tag{5.7}$$

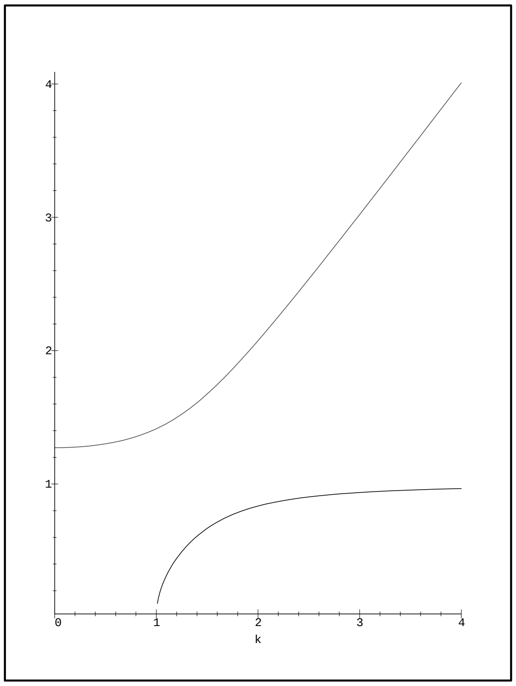

Figure 3. The dispersion relation ω verses k for the charge-model. (For $\omega_0 = 1$, $m = .5$.)

For the charge-model, one finds,

$$\alpha \approx 4\pi m^2 \left(\frac{\beta^2}{8\pi}\right) \frac{\omega_0^2}{(\omega^2 - \omega_0^2)^2} \left|\frac{2\omega_0^2 - \omega^2}{\omega}\right|. \tag{5.8}$$

whereas for the current-model,

$$\alpha \approx 4\pi m^2 \left(\frac{\beta^2}{8\pi}\right) \frac{\omega^2 + \omega_0^2}{(\omega^2 - \omega_0^2)^2} |\omega|. \tag{5.9}$$

6. Renormalization Group for the 2-Level Splitting

We have seen that in the current-model the photon self-energy depends on an ultraviolet cutoff μ. A basic idea of the renormalization group is that couplings also depend on

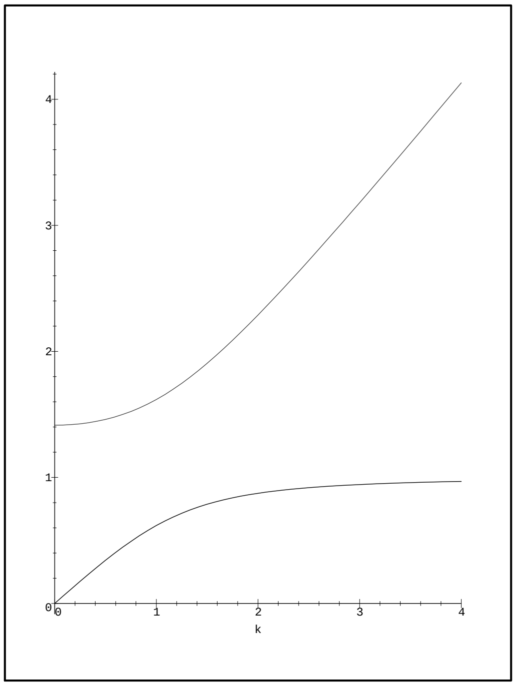

Figure 3. The dispersion relation ω verses k for the current-model. (For $\omega_0 = 1$, $m = .5$.)

the scale μ in such a way that physical quantities are independent of μ. The result (4.31) has precisely a form that allows us to consider ω_0 as a function of μ, since

$$\partial_{\omega_0}\left(\frac{\omega_0}{\omega^2-\omega_0^2}\right) = \frac{\omega^2+\omega_0^2}{(\omega^2-\omega_0^2)^2}. \tag{6.1}$$

The μ-independence of Σ amounts to the renormalization group equation

$$\mu\frac{d}{d\mu}\Sigma = \left(\mu\frac{\partial}{\partial\mu} + \left(\mu\frac{\partial}{\partial\mu}\omega_0\right)\frac{\partial}{\partial\omega_0}\right)\Sigma = 0, \tag{6.2}$$

or equivalently

$$\Sigma(\omega,\omega_0(\mu_2),\mu_2) = \Sigma(\omega,\omega_0(\mu_1),\mu_1). \tag{6.3}$$

To order β^2, (6.2) leads to the beta-function:

$$\mu\frac{\partial}{\partial\mu}\omega_0 = -\frac{\beta^2}{4\pi}\omega_0. \tag{6.4}$$

Since (6.2) could be satisfied assuming no dependence of β on μ we conclude that up to order β^4, β is unrenormalized.

The beta-function (6.4) means that the parameter ω_0, which has engineering dimension 1/time, has an additional anomalous dimension of $-\beta^2/4\pi$. Integrating (6.4),

$$\frac{\omega_0(\mu_2)}{\omega_0(\mu_1)} = \left(\frac{\mu_1}{\mu_2}\right)^{\beta^2/4\pi} . \tag{6.5}$$

Thus, as μ increases, ω_0 decreases, reaching an ultraviolet fixed point at $\omega_0 = 0$.

One can derive a relation which describes the behavior of Σ as one scales the dimensionful parameters ω, ω_0, L. Ordinary dimensional analysis implies

$$\Sigma(\omega, \omega_0(\mu), L, \mu) = \mu^2 \widetilde{\Sigma}\left(\frac{\omega}{\mu}, \frac{\omega_0(\mu)}{\mu}, L\mu\right), \tag{6.6}$$

where $\widetilde{\Sigma}$ is a function of dimensionless parameters. Rescaling all dimensionful parameters by a dimensionless parameter s, and using the renormalization group equation (6.3), one has

$$\mu^2 \widetilde{\Sigma}\left(\frac{s\omega}{\mu}, \frac{s\omega_0(\mu)}{\mu}, L\mu/s\right) = \mu'^2 \widetilde{\Sigma}\left(\frac{s\omega}{\mu'}, \frac{s\omega_0(\mu')}{\mu}', L\mu'/s\right). \tag{6.7}$$

Taking $\mu' = s\mu$, one obtains the scaling equation

$$\Sigma(s\omega, s\omega_0(\mu), L/s, \mu) = s^2 \Sigma(\omega, \omega_0(s\mu), L, \mu). \tag{6.8}$$

This means that at higher energies $s\omega$, the two-level splitting $\omega_0(\mu)$ is screened to $\omega_0(s\mu) = s^{-\beta^2/4\pi}\omega_0(\mu)$.

7. Comparison with the Reduced Maxwell-Bloch Theory

Two approximations commonly made in the quantum optics literature are the so-called slowly varying envelope and rotating wave approximations. In this section we compare the above results with the analagous results obtained in these approximations.

6.1 Definition of the Model

The slowly varying envelope approximation is suitable for dealing with near resonant phenomena. Consider the mode expansion of the free scalar photon field:

$$\phi(x,t) = \int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} \left(a(k)e^{-i\vec{k}\cdot\vec{x}} + a^\dagger(k)e^{i\vec{k}\cdot\vec{x}}\right), \tag{7.1}$$

where $\vec{k} = (|k|, k)$. The photon creation operators satisfy

$$[a(k), a^\dagger(k')] = \delta(k - k'), \tag{7.2}$$

which implies (2.13). We suppose that near resonant photons are most important, and let

$$k = \omega_0 + k_e, \tag{7.3}$$

where k_e denotes an 'envelope' wave vector. Letting $|k| \approx \omega_0$, one has

$$\phi(x,t) \approx e^{-i\omega_0(t-x)}\psi(x,t) + e^{i\omega_0(t-x)}\psi^\dagger(x,t), \tag{7.4}$$

where

$$\begin{aligned} \psi(x,0) &= \frac{1}{\sqrt{2\omega_0}} \int \frac{dk_e}{\sqrt{2\pi}}\, \widehat{a}(k_e) e^{ik_e x} \\ \psi^\dagger(x,0) &= \frac{1}{\sqrt{2\omega_0}} \int \frac{dk_e}{\sqrt{2\pi}}\, \widehat{a}^\dagger(k_e) e^{-ik_e x}, \end{aligned} \tag{7.5}$$

and

$$\widehat{a}(k_e) = a(\omega_0 + k_e). \tag{7.6}$$

The operators $\widehat{a}, \widehat{a}^\dagger$ satisfy the same commutations as (7.2).

In classical theory, ψ and $\psi^\dagger$ are referred to as the slowly varying envelopes if $k_e^2 \ll \omega_0^2$, which implies

$$|\partial_x \psi| \ll \omega_0 |\psi|, \qquad |\partial_t \psi| \ll \omega_0 |\psi|. \tag{7.7}$$

Note that in expanding the field as in (7.4), we are quantizing about a *right-moving* plane wave. One can also begin with envelopes of left-moving waves separately; however we will not consider interactions between the left and right moving envelopes.

Using (7.7), one finds

$$\int dx dt \frac{1}{2} \left((\partial_t \phi)^2 - (\partial_x \phi)^2 \right) \approx 2i\omega_0 \int dx dt \; \psi^\dagger (\partial_x + \partial_t) \psi. \tag{7.8}$$

Without any additional interactions with atoms, the equation of motion is $(\partial_x + \partial_t)\psi = 0$, which means that the envelope also consists of right-moving excitations only at zero coupling. The canonical commutation relation which follow from (7.8) are

$$\left[\psi(x,t), \psi^\dagger(x',t)\right] = \frac{1}{2\omega_0} \delta(x - x'), \tag{7.9}$$

and is compatible with (7.5).

The interaction (2.31) contains terms with both photon creation operators $a^\dagger$ and S^+ operators which excite the atoms. Such terms lead to vacuum fluctuations wherein both photons and atoms are simultaneously excited, and also to real processes where e.g. an incoming photon excites the atom and emerges as two photons. The rotating wave approximation sets such processes to zero. For the charge-model, the rotating wave approximation, combined with the slowly varying envelope approximation leads to

$$H_{\rm int} = -i\frac{\omega_0\beta}{2}\int dx\,\left(\psi e^{-i\omega_0(t-x)}S^+ - \psi^\dagger e^{i\omega_0(t-x)}S^-\right). \tag{7.10}$$

Note that the phases $e^{\pm i\omega_0 t}$ in (7.10) cancel the time dependence (3.6) of $S^\pm$ which comes from the hamiltonian $H_0^{\rm atom}$. Thus we can replace $e^{\mp i\omega_0(t-x)}S^\pm$ with new operators $\widehat{S}^\pm$ and set the $H_0^{\rm atom}$ piece of the hamiltonian to zero. Thus we consider the model defined by the complete hamiltonian:

$$H = -2i\omega_0\int dx\;\psi^\dagger\partial_x\psi - i\frac{\omega_0\beta}{2}\int dx\,\left(\psi S^+ - \psi^\dagger S^-\right). \tag{7.11}$$

The first term follows from (7.8) and we relabeled the $\widehat{S}^\pm$ operators back to $S^\pm$. The algebra satisfied by the S's is the same as before. In the classical context, the equations of motion for the model (7.11) is sometimes referred to as the reduced Maxwell-Bloch theory.

The same approximations as above applied to the current-model leads to the same reduced Maxwell-Bloch theory (7.10), since $\partial_t(\psi e^{-i\omega_0 t}) \approx -i\omega_0\psi e^{-i\omega_0 t}$.

6.2 Exact One-Polariton States

One can construct the one-polariton state exactly for the above model. Let us define momentum space S-operators as follows:

$$\begin{aligned}\widetilde{S}^\pm(k) &= \int \frac{dx}{\sqrt{2\pi}}\;e^{\pm ikx}S^\pm(x,0)\\ \widetilde{S}_3(k) &= \int \frac{dx}{\sqrt{2\pi}}\;e^{-ikx}S_3(x,0)\end{aligned} \tag{7.12}$$

.

The hamiltonian now reads

$$H = \int dk_e\;k_e\;\widehat{a}^\dagger(k_e)\widehat{a}(k_e) - i\frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\int dk\,\left(\widehat{a}(k)\widetilde{S}^+(k) - \widehat{a}^\dagger(k)\widetilde{S}^-(k)\right). \tag{7.13}$$

The state $|\Omega\rangle$, defined in (2.42), is now, in contrast to before, the exact ground state:

$$H|\Omega\rangle = 0. \tag{7.14}$$

By dropping $H_0^{\rm atom}$ in (7.13), we have merely shifted the ground state energy by $N\omega_0/2$.

Below, we will need $\widetilde{S}_3(k)|\Omega\rangle$. From (7.12) and (2.46), one has

$$\widetilde{S}_3(k)|\Omega\rangle = -\frac{N}{L}\sqrt{2\pi}\delta_L(k)|\Omega\rangle, \tag{7.15}$$

where δ_L is a delta function $\int dk\delta_L(k) = 1$, and

$$\lim_{L\to\infty}\frac{\delta_L(0)}{L} = \frac{1}{2\pi}. \tag{7.16}$$

In this way,

$$H_0^{\rm atom}|\Omega\rangle = \sqrt{2\pi}\frac{\omega_0}{2}\widetilde{S}_3(0)|\Omega\rangle = -\frac{N\omega_0}{2}|\Omega\rangle. \tag{7.17}$$

Understanding that the polariton quasiparticle is a combination of photon and atomic degrees of freedom, let us take as an ansatz for the one-polariton states:

$$|k_e\rangle = \left(\widehat{a}^\dagger(k_e) - i\lambda(k_e)\widetilde{S}^+(k_e)\right)|\Omega\rangle, \tag{7.18}$$

where λ is some function of k_e. One has

$$\begin{aligned}\left[H, \widehat{a}^\dagger(k_e)\right] &= k_e\ \widehat{a}^\dagger(k_e) - i\frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\widetilde{S}^+(k_e)\\ \left[H, \widetilde{S}^+(k_e)\right] &= -i\frac{\beta}{2}\sqrt{\frac{\omega_0}{4\pi}}\int dk'\ \widehat{a}^\dagger(k')\widetilde{S}_3(k'-k_e).\end{aligned} \tag{7.19}$$

This gives

$$H|k_e\rangle = \left[\left(k_e + \frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\frac{N}{L}\lambda(k_e)\right)\widehat{a}^\dagger(k_e) - i\frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\widetilde{S}^+(k_e)\right]|\Omega\rangle. \tag{7.20}$$

Thus, $|k_e\rangle$ is an exact eigenstate of H,

$$H|k_e\rangle = \omega_e(k_e)|k_e\rangle, \tag{7.21}$$

with

$$\omega_e = \frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\frac{1}{\lambda}, \tag{7.22}$$

if k_e satisfies

$$k_e = \frac{\beta}{2}\sqrt{\frac{\omega_0}{2}}\left(\frac{1}{\lambda} - \frac{N}{L}\lambda\right). \tag{7.23}$$

One can thus view λ as a spectral parameter, parameterizing the envelope energy and momentum ω_e, k_e. Eliminating λ one finds the dispersion relation

$$\omega_e^2 - \omega_e k_e = m^2, \tag{7.24}$$

where m^2 is defined in (4.25). The above dispersion relation is *exact*; thus one sees that there are no one-loop corrections of the kind computed in section 4. This also means that the reduced Maxwell-Bloch theory does not incorporate spontaneous emission effects.

The result (7.24) can now be compared with the result (5.1) obtained to lowest order in perturbation theory. Recalling that ω_e and k_e denote envelope quantities, we let $\omega = \omega_0 + \omega_e$, $k = \omega_0 + k_e$ and substitute in (5.1). One obtains

$$2\omega_0\omega_e + \omega_e^2 - 2\omega_0 k_e - k_e^2 = 4m^2\frac{\omega_0^2}{2\omega_e\omega_0 + \omega_e^2}. \tag{7.25}$$

Using the slowly varying envelope inequalities,

$$\omega_e^2 \ll \omega_0^2, \qquad k_e^2 \ll \omega_0^2, \tag{7.26}$$

one obtains precisely (7.24).

Remarkably, it is known that the model (7.11) is integrable. The Heisenberg operator equations of motion are

$$\begin{aligned} (\partial_t + \partial_x)\psi = \frac{\beta}{4}S^-, \qquad (\partial_t + \partial_x)\psi^\dagger &= \frac{\beta}{4}S^+ \\ \partial_t S_3 = -\beta\omega_0\left(\psi^\dagger S^- + \psi S^+\right)& \\ \partial_t S^+ = \frac{\beta\omega_0}{2}\psi^\dagger S_3, \qquad \partial_t S^- &= \frac{\beta\omega_0}{2}\psi S_3. \end{aligned} \tag{7.27}$$

These equations of motion have a zero-curvature representation

$$[\partial_t + A_t, \partial_x + A_x] = 0, \tag{7.28}$$

where A_x, A_t are auxiliary 2×2 matrices of quantum operators:

$$\begin{aligned} A_x &= \mu\begin{pmatrix} b_2^\dagger b_2 & S^+ \\ S^- & b_1^\dagger b_1 \end{pmatrix} - \frac{1}{\mu}\frac{\beta^2\omega_0}{16}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\beta\omega_0}{2}\begin{pmatrix} 0 & -\psi^\dagger \\ \psi & 0 \end{pmatrix} \\ A_t &= \frac{\beta\omega_0}{2}\begin{pmatrix} 0 & \psi^\dagger \\ -\psi & 0 \end{pmatrix} + \frac{1}{\mu}\frac{\beta^2\omega_0}{16}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{7.29}$$

Above, μ is an arbitrary spectral parameter, and requiring (7.28) to be valid for all μ is equivalent to (7.27).

The zero-curvature representation allows the model to be solved by the Quantum Inverse Scattering Method[5], as was carried out by Rupasov[6]. The integrability leads to a Bethe-ansatz construction of the multi-particle states that generalizes the above construction for the one-polariton states.

8. Semi-Classical Analysis

The reduced Maxwell-Bloch equations (7.27) have been extensively studied in a semi-classical approximation. See e.g. [2][4] and [7]. The nature of this semi-classical approximation is the following. Consider the expectation of the equations (7.27) in the state $|\Omega\rangle$. Let us assume that the atomic and photon correlators are approximately decoupled:

$$\langle \psi S^+ \rangle \approx \langle \psi \rangle \langle S^+ \rangle, \tag{8.1}$$

where $\langle O \rangle = \langle \Omega | O | \Omega \rangle$. In this semi-classical approximation $\langle \psi \rangle$ is now interpreted as a classical electromagnetic field. Define

$$\rho^{\pm} = S^+ \pm S^-. \tag{8.2}$$

Imposing a reality condition $\langle \psi \rangle = \langle \psi^\dagger \rangle$, one obtains the c-number equations

$$\begin{aligned} \partial_t \langle \rho^+ \rangle &= \beta\omega_0 \; \langle \psi \rangle \langle S_3 \rangle \\ \partial_t \langle S_3 \rangle &= -\beta\omega_0 \; \langle \psi \rangle \langle \rho^+ \rangle \\ \partial_t \langle \rho^- \rangle &= 0. \end{aligned} \tag{8.3}$$

Since $\langle \rho^-(t=0) \rangle = 0$, the last equation above allows us to impose $\langle \rho^- \rangle = 0$ for all times.

As a consequence of $\vec{S}^2$ being a Casimir for su(2), $\langle \vec{S} \rangle \cdot \langle \vec{S} \rangle$ is a constant of the motion. Having set $\langle \rho^- \rangle$ to zero, this implies

$$\langle S_3 \rangle^2 + \langle \rho^+ \rangle^2 = \text{constant} = \left(\frac{N}{L} \right)^2. \tag{8.4}$$

This constraint can be parameterized by introducing an angle function $\Theta(x,t)$:

$$\langle S_3(x,t) \rangle = -\frac{N}{L} \; \cos(\beta\Theta(x,t)), \qquad \langle \rho^+(x,t) \rangle = -\frac{N}{L} \; \sin(\beta\Theta(x,t)). \tag{8.5}$$

The equations (8.3) then imply

$$\partial_t \Theta = \omega_0 \langle \psi \rangle. \tag{8.6}$$

Inserting this into (7.27) one gets the sine-Gordon like equation:

$$\left(\partial_t^2 + \partial_t \partial_x\right) \Theta = -\frac{m^2}{\beta} \frac{N}{L} \sin(\beta\Theta). \tag{8.7}$$

The classical soliton solutions to this equation were observed some time ago by McCall and Hahn[2].

The sine-Gordon (SG) equation is easily seen to be consistent with the one-polariton dispersion relation obtained above in perturbation theory. Taking β to be very small, and expanding the $\sin(\beta\Theta)$ leads to the linear equation

$$\left(\partial_t^2 + \partial_t \partial_x\right) \Theta = -m^2 \ \Theta, \tag{8.8}$$

with a dispersion relation that is precisely (7.24).

The classical SG equation has a rich spectrum of solutions consisting of solitons and breathers. The lowest energy breather solution can be identified with the polariton, as (8.8) shows. The existence of these solutions suggests that the quantum Maxwell-Bloch theory may have a rich spectrum of bound states in addition to the polariton. In the next section we attempt to study this question by considering the quantum version of the SG theory.

9. Quantum Sine-Gordon as an Effective Theory

In the last section we saw how the classical sine-Gordon equation emerged from the reduced Maxwell-Bloch theory in a semi-classical approximation wherein the electromagnetic field was treated classically. Suppose one attempts to re-quantize the semi-classical treatment by quantizing the sine-Gordon theory in the canonical manner. What such a quantum theory has to do with the fully quantum Maxwell-Bloch theory is a delicate question. One can hope that an intricate manifestation of the correspondence principle in the end will save the day. In this section we explore these issues and conclude that the quantum sine-Gordon theory has some validity.

In order to make use of the standard quantization of sine-Gordon, let us make a change of variables

$$\widetilde{x} = 2x - t, \qquad \widetilde{t} = t, \tag{9.1}$$

such that

$$\partial_t^2 + \partial_t \partial_x = \partial_{\tilde{t}}^2 - \partial_{\tilde{x}}^2. \tag{9.2}$$

The action which leads to the equations of motion (8.7) is

$$S = \gamma \int d\tilde{t} d\tilde{x} \, \left[\frac{1}{2} \left(\partial_{\tilde{t}} \Theta \partial_{\tilde{t}} \Theta - \partial_{\tilde{x}} \Theta \partial_{\tilde{x}} \Theta \right) + \frac{m^2}{\beta^2} \cos(\beta \Theta) \right], \tag{9.3}$$

where γ is an arbitrary constant. In the classical theory the constant γ is irrelevant, i.e. the classical equations of motion are independent of γ. In the quantum theory however, γ determines the fundamental commutation relations:

$$\left[\Theta(\tilde{x}, \tilde{t}), \partial_{\tilde{t}} \Theta(\tilde{x}', \tilde{t}) \right] = \frac{i}{\gamma} \delta(\tilde{x} - \tilde{x}'), \tag{9.4}$$

and is thus meaningful.

To promote Θ to an operator and impose the commutation relation (9.4) is potentially perilous given the origin of Θ, i.e. as a way of solving the c-number constraint (8.4). Let us try and interpret the quantization of Θ by replacing (8.5), (8.6) with operator equations:

$$S_3 = -\frac{N}{L} \cos(\beta \Theta), \qquad \rho^+ = -\frac{N}{L} \sin(\beta \Theta) \tag{9.5}$$

$$\partial_t \Theta = \omega_0 \, \psi. \tag{9.6}$$

The S-commutation relations (2.29), after setting $\rho^- = 0$, imply $[S_3(x,t), \rho^+(x',t')] = 0$. This is consistent since $[\Theta(x,t), \Theta(x',t')] = 0$.

In [8], we argued that to connect with the quantum Maxwell-Bloch theory, one must take $\gamma = 1$[1]. Let us repeat a version of this argument here. First, note that the operator equation (9.6) combined with the commutation relation (7.9) does not by itself give a commutation relation of the kind (9.4). In a sense, one must go to next to leading order in the slowly varying envelope approximation in order to obtain (9.4), as follows. Recalling the reality condition imposed in the previous section $\langle \psi \rangle = \langle \psi^\dagger \rangle$, let us impose this classically. Then $\phi = 2\psi \cos(\omega_0(t-x))$. Since $\partial_t \phi \approx -2\omega_0 \psi \sin(\omega_0(t-x))$, we have $\partial_t \phi \approx -2\partial_t \Theta \sin(\omega_0(t-x))$. The term in the action (2.12) for ϕ that determines the commutation relations is

$$\int dx dt \, \frac{1}{2} \partial_t \phi \partial_t \phi \approx 4 \int dx dt \, \partial_t \Theta \partial_t \Theta \sin^2(\omega_0(t-x)). \tag{9.7}$$

[1] The discrepancy of 4π between equation 3.4 of [8] and equation (2.17) is because Heaviside-Lorentz units were not used throughout in [8]; in particular the 4π in (2.8) was omitted in [8], which amounts to a redefinition of e. All formulas in this paper are in the esu system of units.

Averaging over the rapid oscillations let us replace $\sin^2(\omega_0(t-x))$ by 1/2. Then the commutation relations which follow from (9.7) are

$$[\Theta(x,t), \partial_t \Theta(x',t)] = \frac{i}{2}\delta(x-x'). \tag{9.8}$$

Since $\delta(x-x') = 2\delta(\widetilde{x}-\widetilde{x}')$ at equal time, (9.8) is precisely (9.4) with $\gamma = 1$.

Further evidence for the relevance of the quantum SG theory defined by (9.3) with $\gamma = 1$ comes from the perturbative computations in section 6, in particular the beta-function (6.4). From the 2-point function

$$\langle 0|\Theta(\widetilde{x},\widetilde{t})\Theta(0)|0\rangle_{m=0} = -\frac{1}{4\pi}\log(\widetilde{t}^2 - \widetilde{x}^2), \tag{9.9}$$

one has

$$\langle 0|e^{i\beta\Theta(\widetilde{x},\widetilde{t})}e^{-i\beta\Theta(0)}|0\rangle_{m=0} = \frac{1}{(\widetilde{t}^2-\widetilde{x}^2)^{\beta^2/4\pi}}. \tag{9.10}$$

This implies that the $\cos(\beta\Theta)$ operator in (9.3) has anomalous mass dimension $\beta^2/4\pi$. Since the action S is dimensionless, m^2 has a dimension of $2-\beta^2/4\pi$. Since $m^2 \propto \omega_0$, then this precisely corresponds to the beta-function (6.4).

One does not expect of course that the quantum SG theory precisely reproduces the quantum corrections computed in section 4 for the fully quantum Maxwell-Bloch theory. One can study this explicitly by computing one loop corrections in the SG model. Expanding out the $\cos(\beta\Theta)$, the lagrangian is

$$\mathcal{L} = \frac{1}{2}\widetilde{\partial}_\mu\Theta\widetilde{\partial}^\mu\Theta - \frac{m^2}{2}\Theta^2 + \frac{m^2\beta^2}{24}\Theta^4 + \ldots \tag{9.11}$$

The one loop contribution to the photon self-energy can be computed using standard perturbation theory. In the conventional coordinates $\widetilde{x}, \widetilde{t}$, the frequency and wave-vector $\widetilde{\omega}, \widetilde{k}$ are related to the envelope quantities by $\widetilde{\omega} = \omega_e - k_e/2$, $\widetilde{k} = k/2$, so that $\widetilde{\omega}^2 - \widetilde{k}^2 = \omega_e^2 - \omega_e k_e$. Introducing a u.v. cuttoff $\widetilde{\mu}$ as in section 4, one finds the dispersion relation:

$$\omega_e^2 - \omega_e k_e - m^2\left[1 - \frac{\beta^2}{8\pi}\left(\log\left(\frac{\widetilde{\mu}}{m} + \sqrt{\frac{\widetilde{\mu}^2}{m^2}-1}\right)^2 - i\pi\right)\right] = 0. \tag{9.12}$$

This should be compared to the dispersion relation $\omega^2 - k^2 - \Sigma = 0$ for the current-model near resonance, i.e. with $\omega = \omega_0 + \omega_e$, $k = \omega_0 + \omega_e$. Using the slowly varying inequalities (7.26), and the expression (4.31), one obtains:

$$\omega_e^2 - \omega_e k_e - m^2\left[1 + \frac{\beta^2}{4\pi} - \frac{\beta^2}{8\pi}\frac{\omega_0}{\omega_e}\left(\log\left(\frac{\mu^2}{\omega_0^2} - 1\right) - i\pi\right)\right] = 0. \tag{9.13}$$

The expressions (9.13) and (9.12), including the imaginary parts, agree when both cutoffs are large and when $\omega_e \approx \omega_0$; however this contradicts the slowly varying inequalities.

The conclusion of the above analysis is that the quantum SG theory captures some aspects of the fully quantum Maxwell-Bloch theory, in particular the current-model defined in section 2, and does incorporate spontaneous emission, but is not equivalent to it even in the slowly varying envelope approximation.

As a step toward understanding the spectrum of the quantum Maxwell-Bloch theory, one can assume the approximate validity of the quantum SG description. The quantum SG spectrum is known to consist of breathers, the lowest mass breather being the particle associated with the SG field Θ itself, and a pair of solitons. See e.g. [9]. The mass of the n-th breather is given by

$$m_n = 2m_s \sin\left(\frac{n\xi}{16}\right), \qquad n = 1, 2, ... < \frac{8\pi}{\xi}, \quad \xi = \frac{\beta^2}{1 - \beta^2/8\pi}. \tag{9.14}$$

where m_s is the mass of the soliton. As $\beta \to 0$, the mass of the lowest breather m_1 approaches m thus the polariton is identified as the lowest breather. The higher breathers are polariton bound states. As $\beta \to 0$, $m_s \approx 8m/\beta^2$, thus for very small β, the mass of the soliton can be very large compared with the polaritons. In the quantum theory, the polariton can actually be viewed as a bound state of two solitons.

It would be very interesting to understand whether aspects of this spectrum can be seen in the models defined in section 2.

10. Conclusions

We have defined some models which describe quantized radiation in interaction with a medium of two-level atoms arranged in a fiber geometry. Our main computational results are the photon self-energy (4.23) and (4.31), which determine the first quantum corrections to the polariton dispersion relation. We also compared our results with known semi-classical results in the slowly-varying envelope and rotating-wave approximations, and argued for the approximate validity of the quantum sine-Gordon theory. We found that the model which follows from an interaction hamiltonian $-\vec{d} \cdot \vec{E}$ (current-model) is better behaved than the model which follows from the minimal coupling $\vec{p} \to \vec{p} - e\vec{A}$, the latter suffering from infrared divergences. In the current model we derived a renormalization group equation for the energy splitting of the two-level atoms which follows from the beta function (6.4), and shows that the splitting is screened at higher energies.

Though the quantum corrections are generically small, we hope that the trend toward fabricating smaller optical devices will eventually lead to the observation of these quantum effects.

The models defined in section 2 deserve further theoretical study, in particular it would be interesting to determine whether they have a bound state spectrum that resembles the spectrum of the quantum sine-Gordon theory.

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