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The fine structure constant

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1 Introduction

The fine structure constant α is a dimensionless number defined by

$$\alpha = \frac{\mu_0 c e^2}{2h}, \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7} H/m$ is the permeability of the vacuum, c is the speed of light in vacuum, e is the electric charge of the electron, and h is the Planck constant. It is one of the most fundamental constants of physics, important not only to the theoretical structure of physics but also to metrology.

The name “fine structure constant” reflects the historical fact that it was originally introduced by Sommerfeld as a measure of the relative order of magnitude ($\sim 1/137$) of the splitting (called fine structure) of degenerate levels of the Bohr atom due to the relativistic effect [1]. As is obvious from (1), it is closely related to the elementary charge e. In fact, if h were known precisely (μ_0 and c are exact by definition), α could be regarded as a way of expressing the charge e in a dimensionless manner. This means that α is a relevant parameter quite apart from the spin-orbit splitting effect. Although its original meaning was more or less forgotten, the name has somehow stuck and its importance to physics has remained unchanged. It is a fundamental constant that plays a crucial role whenever the electromagnetic interaction is involved, namely the whole range of physics including elementary particle, atomic, mesoscopic, and macroscopic systems. This in turn implies that α can be measured by a variety of independent and competitive physical methods.

For many years atomic physics was the only source for an accurate value of α. Unfortunately even the best atomic methods, based on the precise measurements of the $2P_{3/2} - 2P_{1/2}$ fine structure splitting of the hydrogen atom [2] and the hyperfine structure of the hydrogen ground state [3], were unable to go beyond a precision of several ppm (1 ppm $= 10^{-6}$) due to the fact that the corresponding theories required information on the structure of the proton, which was and still is poorly known. Another method based on the precise measurement of the helium fine structure [4] was able to determine α to about 1 ppm. But further progress has been hindered because of the complexity of theoretical calculations for two-electron atoms [5].

The situation was changed radically by the discovery in 1967 [6] that α can be determined more precisely based on the ac Josephson effect, a macroscopic quantum phenomenon. Another macroscopic quantum phenomenon, the quantum Hall effect, joined the field of high precision measurements of α in 1980 [7]. Meanwhile, the electron anomalous magnetic moment has evolved into a very powerful tool for determining α as a consequence of extremely accurate measurements by the University of Washington group [8] combined with increasingly accurate theoretical work [9, 10, 11].

Recently the field has been broadened further by a very precise determination of α based on the measurement of the de Broglie wavelength of slow neutrons [12]. Furthermore, in the last few years a couple of very promising methods have been proposed and are being actively pursued. One is based on the measurement of the ratio of the Planck constant and the mass of an atom [13], and the other is based on single-electron tunneling devices [14]. And, to complete the circle, atomic physics determinations of α itself show strong signs of revival.

This review will concentrate on determinations of α which have reached a precision of $\sim 10^{-7}$ or better. Such determinations may be classified into two categories:

(a) *Direct method.* The quantum Hall effect and anomalous magnetic moment of the electron belong to this category.
(b) *Indirect method.* This approach utilizes combinations of accurately measured quantities such as e/h, m/h, and the Rydberg constant R_∞.

Of course, all of these methods are driven by the development of better experimental techniques in recent years, in particular, by the remarkable progress in laser spectroscopy and condensed matter physics. It is important to note, however, that accurate measurements by themselves will not necessarily lead to accurate determinations of α. For this to happen the theory must be developed to a comparable precision. Recent progress in theory has occurred mostly in higher order QED calculations. Of course, this is due to the lucky circumstance that the smallness of α enables us to solve problems of quantum electrodynamics (QED) systematically in terms of a perturbative expansion in α. To the extent that QED has no known limitation, the precision of its prediction is limited only by our meager ability to work out the numerical details.

At some point, however, QED by itself is no longer sufficient, and one must take account of other forces, such as the strong interaction, as was alluded to earlier. For instance, the pure QED prediction of the hyperfine structure of the ground state of the hydrogen atom [15]

$$\Delta\nu(\mathrm{QED}) = 1\ 420.451\ 99\ (14)\ \mathrm{MHz}, \tag{2}$$

disagrees with the extremely accurate measurement [3]

$$\Delta\nu(\mathrm{exp}) = 1\ 420.405\ 751\ 766\ 7\ (9)\ \mathrm{MHz} \tag{3}$$

by about 33 ppm. This discrepancy has been attributed to the proton structure correction, which is usually written as a sum of three terms

$$\delta\nu_p(\mathrm{finite-size}) + \delta\nu_p(\mathrm{recoil}) + \delta\nu_p(\mathrm{polarizability}). \tag{4}$$

These terms have been found to contribute $-38.72(56)$ ppm, 5.68 ppm, and less than 4 ppm to $\Delta\nu$, respectively. The contribution $\delta\nu_p$(polarizability) was particularly difficult to evaluate, and was the major obstacle to deriving α from the hydrogen hyperfine structure, in spite of the extremely precise measurement available. (See Sec. 2.6 for further discussion.)

Another example of difficulty concerns the $2S_{1/2} - 2P_{1/2}$ Lamb shift L_{2S} of the hydrogen atom. The theoretical calculation of this shift requires information on the charge radius of the proton. Depending on which of the two radii available in the literature was used [16, 17], early theoretical work gave [15]

$$\begin{aligned} L_{2S} &= 1\ 057.853\ (9)\ (10)\mathrm{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.805\ (11)\ \mathrm{fm}, \\ L_{2S} &= 1\ 057.871\ (9)\ (10)\mathrm{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.862\ (11)\ \mathrm{fm}, \end{aligned} \tag{5}$$

where the first error refers to the uncertainty in the proton radius and the value of the function G_{SE} (of which (52) is the updated result), and the second is an estimate of the uncalculated terms. The difference between these results highlights the difficulty arising from poorly known hadronic structure in precision tests of QED.

In recent years, however, it has become possible to obtain very precise information on some selected energy intervals because of progress in laser spectroscopy. In particular, the so-called Doppler-free two-photon spectroscopy eliminates measurement errors due to the first order Doppler effect and makes it possible to determine with great precision the energy intervals between various narrow metastable states.

In Ref. [18] the frequency of the hydrogen $1S - 2S$ transition was measured by comparison with an accurately known reference frequency. In Ref. [19] the $1S$ ground state Lamb shift in hydrogen was measured by direct comparison of the frequencies of the $1S-2S$ and $2S-4S$ two-photon transitions, and by taking into account the theoretical calculation of the $2S$ Lamb shift. The method adopted in Ref. [20] is to relate the frequency of the $2S-8S/D$ two-photon transitions in hydrogen to two optical standards. Another precise measurement of the $1S$ state Lamb shift has been achieved recently by comparison of the frequencies of the $1S_{1/2} - 3S_{1/2}$ and $2S_{1/2} - 6S_{1/2}$ or $2S_{1/2} - 6D_{5/2}$ two-photon transitions [21].

A byproduct of these works, particularly important for the determination of α, is the very precise determination of the Rydberg constant:

$$R_\infty = \frac{m_e c}{2h}\alpha^2. \tag{6}$$

Recall that c is by definition an exact number 299 792 458 m/s. Thus a very precise R_∞ means a very precise α once m_e/h is determined accurately by some means.

Recent measurements of the hydrogen $1S - 2S$ transition lead to

$$\begin{array}{lllll}
R_\infty & = & 10\ 973\ 731.568\ 41\ (42)\mathrm{m}^{-1} & (0.038\mathrm{ppb}), & [18] \\
R_\infty & = & 10\ 973\ 731.568\ 49\ (30)\mathrm{m}^{-1} & (0.028\mathrm{ppb}), & [19] \\
R_\infty & = & 10\ 973\ 731.568\ 34\ (24)\mathrm{m}^{-1} & (0.022\mathrm{ppb}), & [20] \\
R_\infty & = & 10\ 973\ 731.568\ 44\ (31)\mathrm{m}^{-1} & (0.028\mathrm{ppb}). & [22]
\end{array} \tag{7}$$

The agreement between these results is very encouraging in view of the fact that they were obtained by very different techniques.

In order to determine R_∞ as accurately as in (7), it is necessary to know the theoretical energy levels of the hydrogen atom to high precision. This requires determination of these levels including various radiative and recoil corrections using the full power of QED [15]. The status of the theory behind this measurement is described in Appendix A.

The phrase "non-QED determination of α" is sometimes used to refer to the determinations by the quantum Hall effect, ac Josephson effect, etc. It is important to keep in mind, however, that these non-QED methods make sense only if they are based ultimately on a renormalizable theory, namely QED. Otherwise, we would not be able to identify the parameters of these theories m, e, etc. with measured physical quantities. In this sense all determinations of α are tests of the validity of renormalized QED, or, more precisely, the standard model of electroweak and strong interactions. In other words "non-QED" means only that it does not depend directly on detailed calculations of higher-order corrections of QED.

The central question concerning high precision determinations of α is "What is their relevance to physics in general ?" To answer this question, it is necessary to scrutinize the theoretical basis of various measurements of α. Sec. II gives an overview of various currently or imminently available methods for the high precision determination of α, the comparison of these α's, and the prospects for further improvement in the near future. In Sec. III possible impacts of very high precision determinations of α on the underlying theories are discussed. A concluding remark is presented in Sec. IV. Some theoretical background is provided in the Appendices.

In view of the fact that this paper covers very diverse disciplines, a comprehensive reference to all relevant papers would be very bulky. I restrict myself to more recent references, older papers being quoted only when they are specifically needed. Some of the references to earlier literature may be found in the quoted papers and reviews.

2 Various determinations of α

2.1 Neutron h/m_n measurement

One of the most accurate measurements of α comes from a precise measurement of h/m_n, where m_n is the mass of the neutron [12]. The singularly appealing feature of this measurement is that it depends very little on theoretical considerations, being based on just the Rydberg constant (6) and the de Broglie relation of elementary quantum theory:

$$\lambda = \frac{h}{m_n v} \tag{8}$$

between the velocity v and the wavelength λ of the neutron.

In this approach, α is calculated from the equation

$$\alpha^2 = \frac{2R_\infty}{c}\frac{m_n}{m_e}\frac{h}{m_n}, \tag{9}$$

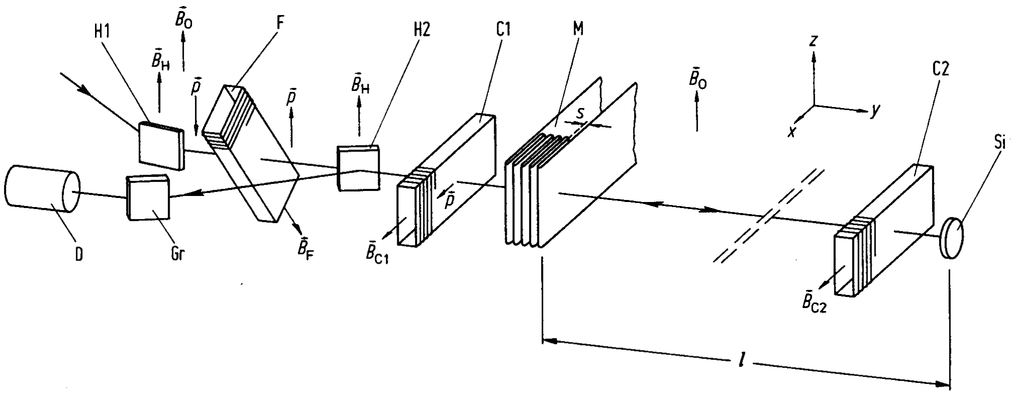

Figure 1: Arrangement for measuring h/m_n. H1, H2: Heusler crystals; F: flipping coil; C1, C2: $\pi/2$ coils; M: meander coil; Si: silicon crystal; Gr: oriented graphite; D: detector; $\vec{B}_0$: magnetic induction of the guide field; $\vec{B}_H$: magnetic induction magnetizing a Heusler crystal; $\vec{p}$: polarization vector. (Reproduced from Figure 1 of Ref. [12] with authors' permission.)

which is a simple rearrangement of the Rydberg formula (6). On the right-hand-side of (9), R_∞/c is very precisely known as is seen from (7). The ratio m_n/m_e is equal to M_n/M_e where M_n and M_e are relative atomic masses of neutron and electron. The latest values of M_n, M_e are [23, 24]

$$\begin{aligned} M_n &= 1.008\ 664\ 923\ 43\ (221), \\ M_e &= 0.000\ 548\ 579\ 911\ (12), \end{aligned} \tag{10}$$

respectively. (The tremendous precision of these measurements attests to the remarkable achievement of nuclear spectroscopy.) Thus α can be determined very accurately if h/m_n is measured precisely. In Ref. [12] this is achieved by very precise measurements of the velocity v and de Broglie wavelength λ of a slow neutron beam.

The neutron beam experiment was carried out at the high-flux reactor of the Institut Laue-Langevin in Grenoble, France. The arrangement for measuring v is shown schematically in Fig. 1. Note that the method of velocity measurement adopted in this experiment is analogous to the early terrestial measurement of the speed of light by Fizeau in 1849. Here the meander coil (denoted M in Fig. 1), which is designed to select the orientation of the neutron spin, plays the role of a rotating toothed wheel which chops up the beam of light into a succession of wave trains of limited length.

In this experiment the velocity of a monochromatic beam of polarized neutrons is measured in the following way. The beam is modulated by periodically changing the direction of the polarization vector of the neutron, the modulation frequency being about 750 kHz. After a path of several meters the neutrons are back-reflected by a silicon crystal (Bragg angle 90°). They return to the modulator which again modulates the direction of the polarization vector. The two modulations are superposed. The resulting total modulation is analyzed and measured as a function of the distance between the modulator and the silicon crystal. This function is periodic. The velocity of the neutrons follows from the length of the period (about 1 mm) and the modulation frequency.

The wave length λ of the neutron is measured using the Bragg reflection by the silicon crystal. The difficulty in measuring λ precisely, which requires an extremely pure sample of silicon crystal, is the largest source of experimental uncertainty.

In this way, the authors of Ref. [12] were able to determine h/m_n as

$$\frac{h}{m_n} = 3.956\ 033\ 320\ (302) \times 10^{-7}\mathrm{m}^2\mathrm{s}^{-1}, \tag{11}$$

with a fractional uncertainty of 7.7×10^{-8}. From these data and Eq. (9) they obtained

$$\alpha^{-1}(\mathrm{h/m_n}) = 137.036\ 011\ 28\ (524) \quad (0.039\ \mathrm{ppm}). \tag{12}$$

Although this method seems to require no theoretical sophistication beyond elementary quantum theory, it should be kept in mind, as was noted already, that its precision is dependent on the validity of the Rydberg formula (6) including all radiative corrections.

2.2 ac Josephson effect

The ac Josephson effect is a phenomenon in which a dc voltage V across the partitioning barrier of a superconducting junction gives rise to an alternating current of frequency[25]

$$\nu = \frac{2eV}{h}. \tag{13}$$

This equation tells us that one can determine $2e/h$ by measurement of ν and V. Studies of the relation (13) for a large variety of superconducting junctions have shown that it is independent of materials to within 2 parts in 10^{16} [26]. Unfortunately such a measurement is not enough to determine the fine structure constant α since it also requires a knowledge of the Planck constant h or the elementary charge e.

At present the best string of auxiliary precision measurements which are needed to relate $2e/h$ to α is given by the formula

$$\alpha^{-2} = \frac{c}{4R_\infty \gamma_{p'}} \frac{\mu_{p'}}{\mu_B} \frac{2e}{h}, \tag{14}$$

where μ_B is the Bohr magneton, $\mu_{p'}$ is the magnetic moment of the proton (in water), and $\gamma_{p'}$ is the corresponding gyromagnetic ratio. Note that (14) is nothing but the Rydberg formula (6) in disguise.

The most accurate result obtained by this method is [27]

$$\alpha^{-1}(\mathrm{acJ}) = 137.035\ 977\ 0\ (77) \quad (0.056\ \mathrm{ppm}). \tag{15}$$

The largest source of error on this measurement is the gyromagnetic ratio of the proton $\gamma_{p'}$, whose precision is 0.21 ppm. The uncertainty in $2e/h$ itself is only 0.03 ppm.

The question of possible uncertainties in the theory of superconductivity is discussed in Sec. 3.

2.3 Quantum Hall effect

The quantum Hall effect [7] measures e^2/h and hence α directly. Measurement of the Hall resistance R_H has thus far been carried out mostly for Si-MOSFET and modulation-doped GaAs-(AlGa)As heterojunctions. Measured values are identical within experimental errors [28], and suggest very strongly that the relation

$$R_H^{(n)} = \frac{h}{e^2 n}, \qquad n = 1, 2, \ldots, \tag{16}$$

holds for any 2-dimensional electron gas under a strong perpendicular magnetic field and at sufficiently low temperature, independent of size, shape, material, and random impurities. In this approach, α is obtained from

$$\alpha^{-1} = \frac{2nR_H^{(n)}}{\mu_0 c}, \tag{17}$$

for some chosen value of n. One of the most precise results obtained by this method is [29]

$$\alpha^{-1}(\text{q.Hall}) = 137.035\ 997\ 9(32) \quad (0.024\ \text{ppm}). \tag{18}$$

The error of this measurement is dominated by the uncertainty in the calibration of the standard ohm.

2.4 The anomalous magnetic moment of the electron

From the theoretical point of view the simplest method for determining α is by comparison of theory and measurement of the anomalous magnetic moment of the electron a_e. Instead of reviewing the history of its measurement, which can be found elsewhere [30], let me quote the latest result from Penning trap measurements of the electron (e^-) and positron (e^+) anomalies [8]:

$$\begin{aligned} a_{e^-} &= 1\ 159\ 652\ 188.4\ (4.3) \times 10^{-12}, \\ a_{e^+} &= 1\ 159\ 652\ 187.9\ (4.3) \times 10^{-12}. \end{aligned} \tag{19}$$

The Penning trap is a device which constrains the z component of the position of an electron in a uniform magnetic field set up in the $+z$ direction by means of a hyperboloid cavity whose rotational symmetry axis is aligned along the z axis.

At present the best theoretical result for a_e is given by

$$\begin{aligned} a_e(\text{theory}) &= 0.5\left(\frac{\alpha}{\pi}\right) \\ &- 0.328\ 478\ 965\ldots\left(\frac{\alpha}{\pi}\right)^2 \\ &+ 1.181\ 241\ 456\ldots\left(\frac{\alpha}{\pi}\right)^3 \end{aligned}$$

$$
\begin{aligned}
& - \quad 1.557\ (70) \left(\frac{\alpha}{\pi}\right)^4 \\
& + \quad 4.695\ (42) \times 10^{-12},
\end{aligned} \tag{20}
$$

where the α^3 term has now been evaluated fully analytically [11] and the last line is an updated value of the sum of contributions of the muon vacuum polarization, tau vacuum polarization, hadronic vacuum polarization, and the weak interaction effect. (See Appendix B for more details.)

In order to compare the theory with the measurement (19), it is necessary to know the value of the fine structure constant α. As is seen from the preceding subsections, there are currently three measurements of α whose precision exceeds 0.1 ppm, namely those based on the quantum Hall effect [29], the ac Josephson effect [27], and the measurement of the ratio of the Planck constant h and the neutron mass m_n [12]:

$$
\alpha^{-1}(\mathrm{q.Hall}) = 137.035\ 997\ 9(32) \quad (0.024\ \mathrm{ppm}), \tag{21}
$$

$$
\alpha^{-1}(\mathrm{acJ}) = 137.035\ 977\ 0(77) \quad (0.056\ \mathrm{ppm}), \tag{22}
$$

$$
\alpha^{-1}(\mathrm{h/m_n}) = 137.036\ 010\ 82(524) \quad (0.039\ \mathrm{ppm}). \tag{23}
$$

Substitution of these α's in (20) leads to the theoretical predictions

$$
\begin{aligned}
a_e(\mathrm{q.Hall}) &= 1\ 159\ 652\ 201.4\ (2.1)\ (27.1) \times 10^{-12}, \\
a_e(\mathrm{acJ}) &= 1\ 159\ 652\ 378.0\ (2.1)\ (65.3) \times 10^{-12}, \\
a_e(\mathrm{h/m_n}) &= 1\ 159\ 652\ 092.2\ (2.1)\ (44.4) \times 10^{-12},
\end{aligned} \tag{24}
$$

where the numerals enclosed in parentheses on each line are, respectively, the uncertainty in the numerical integration of the eighth-order term and that of α used in the evaluation. The values in (24) are about $-$ 0.5, + 2.9, and $-$ 2.2 standard deviations away from the measurement (19).

Note that the intrinsic theoretical uncertainty of a_e(theory) is already quite small and thus the overall uncertainty in Eq. (24) is dominated by the measurement uncertainty of α given in (21), (22), or (23). This means, in turn, that comparison of theory and measurement of a_e will give us a more precise value of α. The value of α determined from the average of a_{e^-} and a_{e^+} of (19) and the theory (20) is

$$
\begin{aligned}
\alpha^{-1}(\mathrm{a_e}) &= 137.035\ 999\ 47\ (25)\ (51) \\
&= 137.035\ 999\ 47\ (57) \quad (0.004\ 1\ \mathrm{ppm}),
\end{aligned} \tag{25}
$$

where the errors on the first line are those in the numerical integration of the eighth-order term and the experimental uncertainty in the measurement of a_e, respectively. The error on the second line is the statistical combination of errors on the first line, which is also expressed as a ppm error. This is the most accurate of all determinations of α available at present. See Appendix B for further discussion of the current status of theory.

The experimental uncertainties in (19) are dominated by a cavity shift effect of $\pm 4 \times 10^{-12}$, which arises from a lack of control over the resonant interaction of an electron with the electromagnetic modes of the surrounding microwave cavity of the Penning trap. Thus far two approaches have been proposed to reduce this uncertainty. One is to use a hyperboloid cavity whose resonances do not couple strongly with the electron motion. Following this idea Van Dyck *et al.* have built a new Penning trap that minimizes the cavity shift problem. Their preliminary result [31]

$$a_e = 1\ 159\ 652\ 185.5\ (4.0) \times 10^{-12} \tag{26}$$

agrees with (19). They expect to reach a precision of 0.5 ppb eventually. Another method, pursued by Gabrielse [32], is to utilize a cylindrical cavity whose resonance structure has been studied theoretically [33]. This approach is expected to give results in a few years.

2.5 Muonium hyperfine structure

The hyperfine splitting of the muonium ground state has been measured very precisely [34]:

$$\Delta\nu(\text{exp}) \ = \ 4\ 463\ 302.88\ (16)\ \text{kHz} \qquad (0.036\ \text{ppm}). \tag{27}$$

Currently a new experiment is in progress to improve the measurement of $\Delta\nu(\text{exp})$ and muon mass by a factor of five [35].

The challenge of the muonium hyperfine structure and its relevance to our discussion of α comes from the fact that the measurement (27) is much more accurate than the current theory and thus provides an excellent source of α if higher-order correction terms are calculated to a comparable precision. The hyperfine structure of the muonium ground state is particularly suited for a detailed theoretical study of the bound state, being relatively free from the effects of hadronic interaction.

Including the results of new calculations [36, 37] the current status of theoretical prediction is as follows. Using the value of α from (21), R_∞ from (7) (any value listed there may be used), and m_μ/m_e from (68), we find

$$\Delta\nu(\text{theory}) \;=\; 4\ 463\ 302.51\ (1.34)\ (0.21)\ (0.16)\ (0.06)\ \text{kHz}\ , \tag{28}$$

where the first and second errors reflect the uncertainties in the measurements of m_μ and α^{-1}, listed in (68) and (21). The third error comes from the uncertainty in the numerical evaluation of the $\alpha(Z\alpha)^2$ correction term [38]. The fourth error is an estimate of theoretical uncertainty arising from higher order terms. More details of the theoretical calculation are described in Appendix C.

The agreement between $\Delta\nu(\text{theory})$ and $\Delta\nu(\text{exp})$ is excellent, the difference being

$$-\ 0.37\ (0.16)\ (1.34)\ (0.21)\ (0.16)\ (0.06)\ \text{kHz}, \tag{29}$$

where the first error is from the experiment (27) and the rest are carried over from (28).

The result (28) was obtained using the value of α from (21), which is currently the best non-QED value. If one uses $\alpha(\mathrm{a_e})$ of (25) instead, one finds

$$\Delta\nu(\text{theory})' \;=\; 4\ 463\ 302.41\ (1.34)\ (0.04)\ (0.16)\ (0.06)\ \text{kHz}\ . \tag{30}$$

At present it is not possible to determine α very accurately from the theory and experiment of the muonium hyperfine structure, mainly because of the uncertainty in the mass ratio m_e/m_μ given in (68). This is clearly visible from the value of α determined by comparing (28) and (27):

$$\alpha^{-1}(\mu\text{hfs}) = 137.035\ 994\ 0\ (182)\ (23)\ (22) \quad (0.134\ \text{ppm}), \tag{31}$$

where the errors are due to the muon mass, theory, and the measurement (27), respectively. The new Los Alamos experiment [35] will improve the muon mass as well as $\Delta\nu$ by about a factor of 5. The precision of $\alpha^{-1}(\mu\text{hfs})$ will be improved to about 0.03 ppm, comparable to that of the quantum Hall value (21).

2.6 Hydrogen hyperfine structure

The hyperfine structure of the hydrogen ground state is potentially a very good source of α. As was noted in (3), it is measured with an astonishing precision of better than 10^{-12}. The QED part of the theory is in a better shape than that of the muonium hyperfine structure because the mass ratio m_e/m_p is known very precisely. The problem arises from the difficulty in evaluating the proton structure correction as was noted in (4). The best previous evaluation of this correction, excluding the proton polarizability contribution, led to [15]

$$\frac{\nu(\text{theory}) - \nu(\text{exp})}{\nu_F} = (-0.47 \pm 0.56 \pm \text{unknown})\ \text{ppm}. \tag{32}$$

The error of 0.56 ppm contains a small contribution from the uncertainty in α, but it arises mainly from the uncertainty in the proton elastic form factors. The part of the error labeled "unknown" represents all remaining uncertainties in ν(theory) except for the proton polarizability.

The proton polarizability $\delta\nu_p$(polarizability) is the largest source of theoretical uncertainty. De Rafael [39] and Gnädig and Kuti [40] have shown how to use data from inelastic electron scattering with polarized beam and target to put a bound on $\delta\nu_p$(polarizability). Based on the available experimental data Hughes and Kuti gave the bound [41]

$$|\delta\nu_p(\text{polarizability})| < 4\text{ppm}. \tag{33}$$

Clearly it is impossible to obtain an accurate α from the hydrogen hyperfine structure measurement until $\delta\nu_p$(polarizability) is determined more precisely. I should like to emphasize here that, with the recent progress in the *ab initio* calculation of lattice QCD [42], there is now hope that this quantity can be calculated with a good precision in the near future. Together with further improvements on other proton structure corrections, it may become possible to determine α from the hydrogen hyperfine structure with a precision competitive with others. Remember that the measured value (3) is challenging the theorists with six more significant digits!

2.7 Helium atom

Before 1967 spectroscopy of the hydrogen and helium atoms was the main source of precise values of α. In fact, the fine structure splitting of the

2^3P states was measured very precisely [43]. The difficulty in obtaining α from the helium atom measurement was due to the enormous complexity in calculating the energy levels of the two-electron atom [5].

Recent progress in helium spectroscopy using laser excitation of an atomic beam has renewed interest in using the $2^3S - 2^3P$ fine structure to determine a precise value of α. The current result is [44]

$$\alpha^{-1}(\text{He} - \text{fs}) = 137.035\ 853\ (3)\ (7)\ (99) \quad (0.73\ \text{ppm}), \tag{34}$$

where the three values of standard deviation are from the uncertainty in the measured atomic frequency interval, the total experimental error (including the frequency calibration), and the theory of the helium atom [5], respectively.

Clearly, most errors in (34) come from theory. This has stimulated renewed interest in more precise calculations of the 2^3P fine structure [45]. Confirmation and re-evaluation of all previously considered terms appears to be nearing completion, with uncertainties below 3 ppb. Higher order terms are also being studied with the goal of reducing their theoretical uncertainty to this level [46]. When these calculations are completed, the error in α(He-fs) will be reduced to the level of 0.03 ppm.

The new and highly accurate measurements [47] of the ground state Lamb shift and the $1^1S - 2^1P$ transition isotope shift are in good agreement with the theoretical values [48], providing another test of QED.

2.8 h/m_A measurement for atom A

One new and innovative approach to the determination of α utilizes the atomic recoil from absorption of photons. Such a recoil of an atom was first observed spectroscopically in the doubling of certain spectral peaks in saturation spectroscopy [49]. This paper pointed out that the frequency splitting of these peaks is proportional to $\hbar/m_A$, m_A being the mass of the atom A. This opens up the possibility of a precise measurement of $\hbar/m_A$ by means of an "atom interferometer."

This is one of the new breed of interferometers in which the conventional roles of matter and wave are reversed so that the incident wave is made of atoms and the slits are made of light.

The basic physics of the atomic recoil velocity measurement does not depend on atomic interference. Suppose an atom moving along the $+z$ direction absorbs a photon of momentum $\hbar\vec{k}$ from a π pulse of laser beam propagating in the $+x$ direction tuned on resonance. To conserve momentum the atom's velocity changes by $\hbar\vec{k}/m$. The atom is then deexcited by a π pulse of leftward propagating (i. e., propagating in the $-x$ direction) laser beam by the stimulated emission of a photon of momentum $-\hbar\vec{k}$. The excitation and deexcitation frequencies are separated by $\Delta\omega = 2\hbar\vec{k}^2/m$ so that measurements of $\Delta\omega$ and the photon wavelength determine $\hbar/m$.

This experiment is too simple to give high precision results because it forgoes the advantages of Ramsey spectroscopy, which include increased resolution, larger signal, and reduced systematic shifts. The first improvement adopted in Ref. [13] is to replace each π pulse with two spatially separated $\pi/2$ pulses, forming a double atom interferometer sequence aligned along the z axis. Each $\pi/2$ pulse puts the atom in a superposition of two trajectories. Thus the first three $\pi/2$ pulses generate eight coherent trajectories, two pairs of which interfere at the final $\pi/2$ pulse. The crucial difference between the two interfering pairs is that their velocities differ by two photon recoils, so that the center frequencies of the sets of Ramsey fringes they produce are displaced by $2\hbar\vec{k}^2/m$. Further improvement in the sensitivity of the atom interferometer can be achieved by adding several π pulses with alternating propagation directions, sandwiched between the middle two $\pi/2$ pulses. Each π pulse adds an ultraviolet photon recoil to each atomic path, so that for N π pulses the separation between the two sets of interference fringes is multiplied by $N + 1$.

In the method developed in Ref. [13], the excitation and deexcitation is achieved by stimulated Raman transition between two hyperfine levels of the ground state of the cesium atom. They were able to separate the two interferometer end points by over 4 mm, using up to 15 π pulses.

The experiment described above transfers about 32 photon momenta on the average with a transfer efficiency of 85% per pair of photons. More recently, they reported a new approach based on the coherent transfer of 140 photon momenta to cesium atoms with an efficiency of 95% per exchanged photon pair [50]. This method utilizes the population transfer based on adiabatic passage.

2.9 Single electron tunneling

This method of measuring α is based on the rapidly developing technology of Coulomb blockade devices made from ultra-small tunnel junctions [51]. With the advent of such devices, it is now possible to transfer a countable number of electrons onto, for instance, a cryogenic capacitor. Given a definite number of electrons accumulated on the capacitor, it is possible to obtain a value of α by measuring the voltage across the capacitor in terms of the Josephson effect and the capacitance in terms of the *calculable capacitor* [52].

The early turnstile and pump devices [53], which allow the passage of a single electron with each oscillation of a gate bias, have demonstrated that a countable numer of electrons can be transferred onto a cryogenic capacitor. The electrons were counted with an experimental uncertainty of about 0.1%. In a more recent version [14], a counting error of less than 1 ppm was achieved by an electron pump, made from a linear array of five ultrasmall tunnel junctions and four capacitative gates. This demonstrates the feasibility of using the device for metrological applications. At present, however, the error on the measurements is significantly higher than the error predicted by the theory. The errors seem to arise from photon-assisted tunneling, where the photon energy is supplied by noise from the environment. This method will be an excellent source of α if these problems can be controlled by a greater understanding of the various electron-tunneling processes.

2.10 Comparison of α determined by various means

To facilitate a comparison of best α's determined by various means, let us collect them here:

$$\begin{aligned}
\alpha^{-1}(\mathrm{h/m_n}) &= 137.036\,011\,28\,(524) \quad (0.039\ \mathrm{ppm}),\\
\alpha^{-1}(\mathrm{acJ}) &= 137.035\,977\,0\,(77) \quad (0.056\ \mathrm{ppm}),\\
\alpha^{-1}(\mathrm{q.Hall}) &= 137.035\,997\,9\,(32) \quad (0.024\ \mathrm{ppm}),\\
\alpha^{-1}(\mathrm{a_e}) &= 137.035\,999\,47\,(57) \quad (0.004\,1\ \mathrm{ppm}),\\
\alpha^{-1}(\mu\mathrm{hfs}) &= 137.035\,994\,0\,(184) \quad (0.134\ \mathrm{ppm}).
\end{aligned} \tag{35}$$

I also give the CODATA 1986 value for α [54]:

$$\alpha^{-1}(\mathrm{CODATA\ 1986}) = 137.035\,989\,5\,(61) \quad (0.045\ \mathrm{ppm}). \tag{36}$$

They are shown together in Fig. 2. They are clearly in agreement with each other to 7 digits. This is an encouraging start but it also means that we may not learn anything new at this level. The eighth digit seems to show signs of disagreement, although it is not very serious at present. This situation can be clarified only by further improvements in measurement.

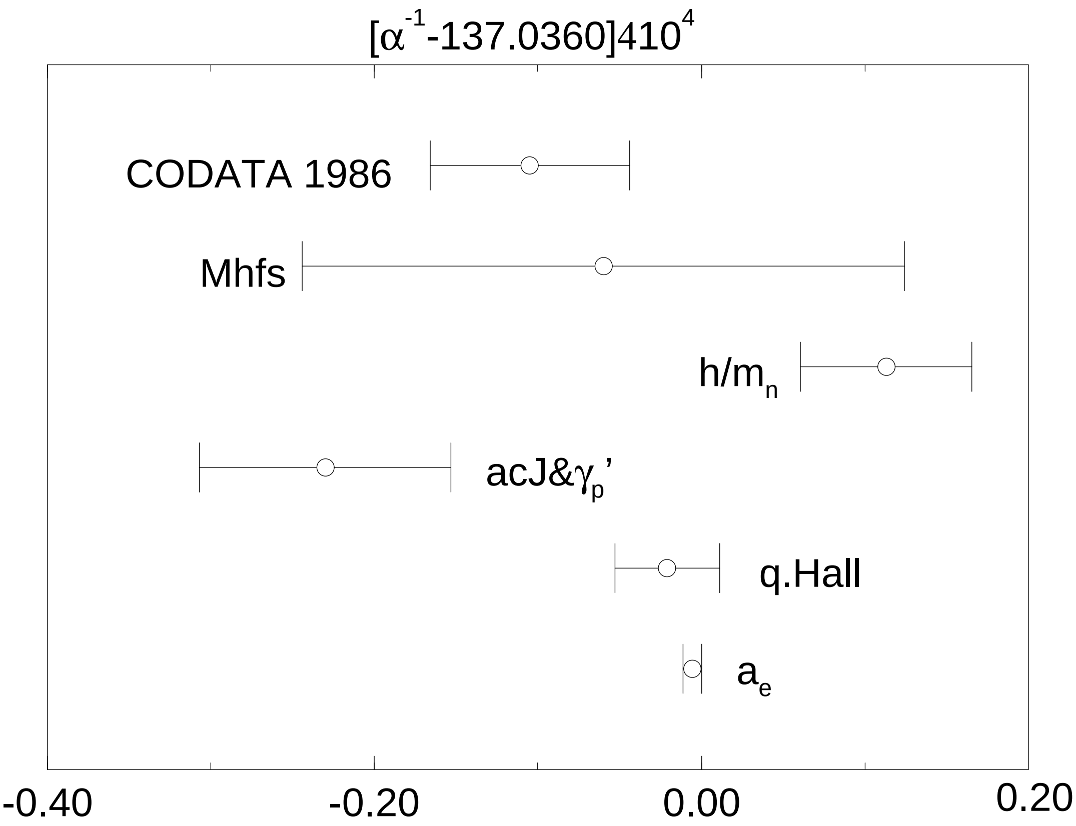

Figure 2: Fine structure constants determined by various means.

Of the values listed in (35), $\alpha(\mu\text{hfs})$ will be improved to about 0.03 ppm shortly [35]. Some improvement of other α's may be achieved by improving the standard of the ohm. It must be kept in mind that the methods based on the h/m_A measurement and the single electron tunneling have no intrinsic reason which prevents them from achieving a precision of ~ 1 ppb [13, 14]. If successful, these methods may also provide a better calibration of the standard ohm and hence improve the measurements of α(q.Hall) and α(acJ). The value of $\alpha(a_e)$ will be determined to 1 ppb once the new measurements of a_e succeed [31, 32]. Thus, in the next few years, we might have at our disposal several independent determinations of α whose precision is better than 10^{-8}, with some possibly pushing 10^{-9}.

3 Possible implications to physics

It is a commonly held view that the disagreement among various α's is a sign of the breakdown of QED. However, this is not necessarily the case, because all theories used in extracting α from experiments are ultimately based on the same theory, namely QED. To put it differently, all measurements of α discussed above must produce identical results irrespective of the correctness of QED (assuming of course that the QED calculations are carried out correctly). This means that, if any disagreement is found, not only the higher-order calculation of QED but also all theories that are used to deduce α from particular experiments must be scrutinized very carefully. The specific question about any of these theories is "Is it accurate enough to predict α at the level of 10^{-9} ?" This question has been examined partially in the past but not in its full ramifications.

Before proceeding further, let us recall that the fine structure constant α being considered here is actually a specific value of the running coupling constant $\alpha(Q^2)$ at $Q^2 = -q^2 = 0$, where q is the momentum transfer relevant for measurement of α. The Q^2 dependence of $\alpha(Q^2)$ is controlled by the renormalization group equation. Its dependence on Q^2 is hardly visible for small Q^2 and becomes noticeable only at very large Q^2, namely at very short distances. For instance, at $Q^2 = m_W^2$ where m_W is the weak boson mass ($m_W = 80.22$ GeV/c^2), α takes the value $\simeq 1/128$. Even if QED breaks down at very short distances, it will not have any noticeable impact on the value of $\alpha(0)$. In other words, none of the determinations of α discussed here is sensitive to possible breakdown of QED at short distances. In spite of the Q^2 dependence, the value $\alpha(0)$ has a specific significance since it can be measured extremely accurately and provides the normalization point to the running quantity $\alpha(Q^2)$.

Thus, if disagreement develops among various determinations of α, we must seek the cause not in the breakdown of QED but in something else. In this connection we must keep in mind that the errors of $\alpha(\mathrm{acJ})$ in (15) and $\alpha(\mathrm{q.Hall})$ in (18) do not include any consideration for the possibly approximate nature of the underlying theory. I shall come back to this question later.

Of all the determinations of α, those by a_e and h/m_n (or h/m_A) seem to be the least complicated from the theoretical point of view. Thus, for instance, if any discrepancy is observed between $\alpha(h/m_n)$ and $\alpha(h/m_A)$, it must be of experimental origin since they are based on an identical theory. Equally straightforward is the implication of an inequality of $\alpha(\mathrm{a_e})$ and $\alpha(\mathrm{h/m_n})$.

For the sake of argument, let us suppose that the current values of $\alpha(\mathrm{a_e})$ and $\alpha(\mathrm{h/m_n})$ given in (35) remain unchanged after further measurements have reduced their uncertainties substantially. In this case, the only cause of discrepancy will be a computational error or an inaccurate error estimate in the theoretical calculation. Note that the effects of the hadronic and weak interaction contributions are small and known with sufficient precision, as is seen from Eqs. (69) – (71) of Appendix B. This leaves the calculation of the eighth-order coefficient $A_1^{(8)}$ (see (64)) as a possible source of discrepancy. An additional source is the tenth-order coefficient $A_1^{(10)}$, which is not yet calculated.

If one wants to explain the discrepancy between $\alpha(\mathrm{a_e})$ and $\alpha(\mathrm{h/m_n})$ entirely in terms of $A_1^{(8)}$, $A_1^{(8)}$ must be increased from the current value $-\ 1.557$ to $\sim +1.9$. Although the real error in $A_1^{(8)}$ might be 2 or even 3 times larger than the value ± 0.070 given in (64) (which is not inconceivable because of insufficient Monte Carlo sampling of the integrand in the quoted calculations) the probability that the result of a VEGAS integration is off by as much as 50 standard deviations is infinitesimal.

This would force us to examine the tenth-order coefficient $A_1^{(10)}$ as a possible source of discrepancy. If one assumes that this is the case, the comparison of $\alpha(\mathrm{a_e})$ and $\alpha(\mathrm{h/m_n})$ will lead to the prediction

$$A_1^{(10)} \sim 1500, \tag{37}$$

which is ridiculously large. It is very unlikely that $|A_1^{(10)}|$ is much larger than ~ 10 in view of the experiences from the calculation of lower-order terms. In other words, the contribution of the tenth-order term to a_e will be at most of order 0.7×10^{-12}.

Of course, the strange situation (37) is the consequence of the dubious assumption that future measurements leave the values of $\alpha(\mathrm{a_e})$ and $\alpha(\mathrm{h/m_n})$ unchanged. In any case, it will be useful to obtain a theoretical estimate of $A_1^{(10)}$, however crude that might be.

The values of $\alpha(\mathrm{acJ})$, $\alpha(\mathrm{qH})$, and $\alpha(\mathrm{s.e.t})$, where "s.e.t" means single electron tunneling discussed in Sec. 2.9, must agree with each other since they are all based on the same theory of condensed matter physics. For instance, the cause of the current discrepancy between $\alpha(\mathrm{qH})$ and $\alpha(\mathrm{acJ})$ must be purely experimental.

When the dust settles and the equality of $\alpha(\mathrm{qH})$, $\alpha(\mathrm{acJ})$, and $\alpha(\mathrm{s.e.t})$ is established experimentally, there still remains the possibility that they disagree with $\alpha(\mathrm{a_e})$ and/or $\alpha(\mathrm{h/m_n})$. If this happens, it will become necessary to examine other possible sources of error. Here I want to call attention to the fact that the errors quoted for $\alpha(\mathrm{qH})$ or $\alpha(\mathrm{acJ})$ do not take into account the possibility that the theory of condensed matter physics might require some modification to remedy the nonrelativistic and unrenormalizable nature of the basic Hamiltonian of these theories.

The following discussion on this question is borrowed from Ref. [55] with minor modifications.

In considering theoretical errors which might be hidden in the theory of condensed matter physics itself, it is important to distinguish clearly which level of theory one is referring to. Let us define the lowest level (*level* 0) to be a fully relativistic quantum theory which, for our purpose, may be taken to be quantum electrodynamics. The next level (*level* 1) is characterized by a Hamiltonian which describes electrons and nuclei interacting with each other (nonrelativistically) by the Coulomb force and also with a transverse electromagnetic field. Solid state physics is described at an even higher level (*level* 2) involving an effective mass approximation and quasiparticle description.

The theory of Josephson junctions has been developed in *level* 2 or higher, and possible theoretical errors have also been discussed mostly within this level. An example is the frequency pulling effect in a cavity which, as it turns out, leaves the relation (13) unchanged [56]. Studies of the effects of thermal noise [57] and nonequilibrium noise [58] are also made at this or a higher level. These effects are found to be completely negligible compared with the present experimental uncertainties.

Bloch's work [59] may be regarded as an attempt to connect *level* 2 to *level* 1 or even *level* 0. However, he implicitly assumes that the Hilbert space for *level* 1 is identical with that of *level* 2. In particular, in constructing the free energy of the system, he ignores the possibility that excited states may not share the periodicity of the ground state. Errors arising from this assumption have not been estimated to my knowledge.

The only work I am aware of which might belong to *level* 0 is that of Nordtvedt [60] who showed that the electric charge in a metal differs from the free electron value by an amount of order $10^{-10}e$ because of the effect of virtual electron-positron pairs. However, it has been argued [61] that this is irrelevant to our question since the Josephson relation (13) actually measures the chemical potential difference eV, not e itself, and this is not affected by Nordtvedt's argument.

The observed material independence of the relation (13) suggests strongly that it is an exact consequence of general properties of superconducting junctions. In fact Bloch [59] attempted to show that it is a general consequence of the single-valuedness of wave functions and gauge invariance. As was noted above, however, his proof is incomplete, being restricted to *level* 2. The more serious deficiency is that his Hamiltonian is not renormalizable and hence cannot maintain gauge invariance beyond the formal level.

Similarly, since the discovery of the quantum Hall effect, attempts have been made to give it a theoretical foundation. An insightful understanding of the phenomenology of the integer and fractional quantum Hall effects is achieved in terms of the composite boson and composite fermion models [62]. An argument was given at the *level* 2 that this effect is a consequence of gauge invariance for a strictly 2-dimensional system with mobility gap [63]. Here again the assumption of gauge invariance is not warranted because of UV divergences of the theory. Thus the remaining problem is how to relate the *level* 2 to lower levels, which is analogous to that of the ac Josephson effect. Thus both cases may be solved by the same method.

There has been no serious attempt thus far to construct a theory of many body systems starting from first principles. It is important to note, however, that there is a possibility to develop such a theory starting from the NRQED method proposed by Caswell and Lepage [64], which enables us to calculate higher-order terms systematically starting with the nonrelativistic Schrödinger equation as the zeroth-order approximation. This method is fully equivalent to the relativistic QED. In particular it is fully renormalizable. Thus it provides us with a rigorous theoretical basis for the study of bound systems. There seems to be no serious obstacle which might prevent us from extending this theory from two-body to many-body systems and constructing explicitly a renormalized Hamiltonian for many-body bound systems [65]. An outline of the NRQED method is presented in Appendix D.

4 Concluding remark

Those of us who work in high precision physics are often asked: "Why do you work so hard to extract just an extra digit?" Behind this question is the widely-held view that QED is so well established by now that there is nothing new to learn by pushing it further. Of course, this is not the first time that such a sentiment was expressed. For instance, in 1894, A. A. Michelson expressed a fear of possible intellectual death of physics, invoking a nightmarishly sterile picture of future generations of physicists working only to add decimal points [66]. As was quoted in [67], he wrote:

> "While it is never safe to affirm that the future of Physical Science has no marvels in store even more astonishing than those of the past, it seems probable that most of the grand underlying principles have been firmly established and that further advances are to be sought chiefly in the rigorous application of these principles to all the phenomena which come under our notice."
>
> "It is here that the science of measurement shows its importance — where quantitative results are more to be desired than qualitative work. An eminent physicist has remarked that the future truths of Physical Science are to be looked for in the sixth place of decimals."

One hundred years later his picture is remembered as something of a joke, coming as it did just at the threshold of the tremendous discoveries of twentieth century physics. Both bold new ideas and painstakingly precise measurements, enabled by the rapidly developing technologies, played crucial roles in these discoveries. The horizon of physics has been broadened so much that many areas of physics, which have opened up since then or currently under vigorous development, are beyond the wildest dream of Michelson or his contemporaries.

Nevertheless, there is one area of modern physics in which his view might have some relevance, namely, QED, the theory of the interaction of the quantized radiation field with charged particles. Particularly in this area, where physical systems of extreme simplicity can be studied with a theoretical formalism of great predictive power, Michelson's nightmare of aimlessly pushing back the decimal points might become a reality some day. What I have tried to show in this review is that the sterility one might expect is not yet present; on the contrary, the research involved has led to fruitful links with several other fields of physics, most notably condensed matter physics, laser spectroscopy, and even high energy particle physics.

As I have emphasized, QED (or its generalization, the standard model) is the fundamental theory which unifies all theories of low energy physics. In particular, all determinations of the fine structure constant α discussed in this review must lead to an identical result. This is irrespective of the fact that QED (or the standard model), which depends on external parameters such as e and m that cannot be generated from within the theory, is not yet close to the ultimate theory.

If the complete consistency of α is not achieved in spite of further theoretical and experimental efforts, we may have to conclude that it is not interpretable within the framework of the known physics. Whether this leads to a breakdown of physics as we now know it, and possibly indicates the emergence of new physics, we are not able to say at present. One can only maintain that such a breakdown, if it occurs, cannot be attributed to the failure of QED at short distances. Indeed, not just QED, but also the standard model of the electroweak and strong interactions, or even the string theory, will be powerless to resolve the issue in such a case.

Meanwhile, we are still pushing back additional decimal points, hoping to uncover some wrinkles in the marvelous fabric of physics. It remains to be seen whether the confidence in the current theoretical structure of physics is as unfounded as that of 102 years ago.

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A Energy levels of the hydrogen atom

For a comprehensive review of earlier work on hydrogen-like atoms see, for instance, Ref. [15]. The purpose of this Appendix is to add references to more recent works. We restrict ourselves to very light atoms ($Z\alpha \ll 1$) since we are primarily interested in providing the theoretical background for the determination of the Rydberg constant R_∞.

The gross feature of the energy levels is given by the exact solutions of the Dirac equation in a static Coulomb potential. These must be modified by relativistic corrections, radiative corrections, nuclear size effect, and so on, whose treatment requires a comprehensive application of QED.

We use the convention where the electron charge is e and the charge of the positive "nucleon" is $-Ze$. Of course, $Z = 1$ for the proton or the muon, but it is kept in order to identify the origin of correction terms. Note that each radiative photon on the electron line contributes a factor α, each on the nucleon line a factor $Z^2\alpha$, and each jumping from electron to nucleon a factor $Z\alpha$. This factor also arises from the effect of binding on the velocity distribution of atomic electrons.

A.1 Lowest-order energy levels

We start from the energy levels for the Dirac-Coulomb problem without recoil and radiative corrections. These are given by

$$E_{nj} = m_e c^2 f(n, j), \tag{38}$$

where

$$f(n, j) = \left[1 + \frac{(Z\alpha)^2}{(n-\beta)^2}\right]^{-1/2}. \tag{39}$$

Here n is the principal quantum number,

$$\beta = j + \frac{1}{2} - \sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2}, \tag{40}$$

and j is the total angular momentum quantum number. The energy levels are independent of the orbital angular momentum given by l, which is related to j by $l = j \pm \frac{1}{2}$. For small Z we may expand f in powers of $(Z\alpha)^2$:

$$\begin{aligned} f(n,j) &= 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) \\ &- \frac{(Z\alpha)^6}{8n^3}\left(\frac{1}{(j+1/2)^3} + \frac{3}{n(j+1/2)^2} + \frac{5}{2n^3} - \frac{6}{n^2(j+1/2)}\right) \\ &\qquad + \dots . \end{aligned} \tag{41}$$

The effect of nuclear recoil modifies this equation. Actually the first term has the correct mass dependence corresponding to the rest mass of the electron. The second term must be modified by a factor of $m_r/m_e = 1/(1 + m_e/m_N)$ to take into account the dominant non-relativistic behavior embodied in the Bohr energy levels.

For a proper treatment of the recoil effect, it is necessary to start from the quantum field theory of a two-body bound system. Usually, such a calculation starts from the Bethe-Salpeter equation, which enables us to obtain much of the correct mass dependence of the first three terms of (41), with the remaining mass dependence through that order coming from one-photon exchange kernels. An expression for the total energy in which terms of order $(Z\alpha)^6 m_e^3/m_N^2$ have been neglected and perturbation kernels through order $(Z\alpha)^4$ have been incorporated is [68]

$$\begin{aligned} E = (m_e + m_N)c^2 &+ m_r c^2[f(n,j) - 1] - \frac{m_r^2 c^2}{2(m_e + m_N)}[f(n,j) - 1]^2 \\ &+ \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2}\left[\frac{1}{j+1/2} - \frac{1}{l+1/2}\right](1 - \delta_{l0}). \end{aligned} \tag{42}$$

A.2 Relativistic recoil effect

We now turn to the correction of order $(Z\alpha)^5 m_r^3/(m_e^2 m_N)$ derived by Salpeter [69], which is often called the Salpeter correction. This term arises mainly from the exchange of a transverse photon between the electron and the nucleus with any number of Coulomb interactions occurring between emission and absorption. It involves both nonrelativistic and relativistic momentum regions. In addition there are Coulomb-Coulomb and transverse-transverse contributions to the high energy region. The following expression is valid for arbitrary masses:

$$\begin{aligned}
\Delta E_n(\text{Salpeter}) &= \frac{m_r^3}{m_e^2 m_N}\frac{(Z\alpha)^5}{\pi n^3} m_e c^2 \left(\frac{3}{2}\delta_{l0}\ln\left(\frac{1}{Z\alpha}\right)\right. \\
&- \frac{8}{3}\ln k_0(n,l) - \frac{1}{9}\delta_{l0} - \frac{7}{3}a_n \\
&- \left.\frac{2}{m_N^2 - m_e^2}\delta_{l0}\left[m_N^2\ln\left(\frac{m_e}{m_r}\right) - m_e^2\ln\left(\frac{m_N}{m_r}\right)\right]\right). \quad (43)
\end{aligned}$$

where $k_0(n,l)$ is the Bethe logarithm. Some $k_0(n,l)$ of low order are given by [70]

$$\begin{aligned}
\ln k_0(1,0) &= 2.984\ 128\ 555\ldots, \\
\ln k_0(2,0) &= 2.811\ 769\ 893\ldots, \\
\ln k_0(4,0) &= 2.749\ 811\ 840\ldots, \\
\ln k_0(2,1) &= -0.030\ 016\ 708\ldots. \quad (44)
\end{aligned}$$

The state-dependent constants a_n, which are associated with the relativistic region, are given by the formula [71]

$$a_n = -2\left(\ln\left(\frac{2}{n}\right) + \sum_1^n \frac{1}{i} + 1 - \frac{1}{2n}\right)\delta_{l0} + \frac{1-\delta_{l0}}{l(l+1)(2l+1)}. \quad (45)$$

The next-order correction has also been calculated [72, 73, 74, 75]:

$$\Delta E_R = \frac{m_e}{m_N}\frac{(Z\alpha)^6}{n^3} m_e c^2 D_{60} \quad (46)$$

where

$$D_{60} = \begin{cases} 4\ln 2 - \frac{7}{2} & \text{n } S_{1/2} \\ \frac{1}{3} & 2\ P_{1/2} \\ \frac{1}{3} & 2\ P_{3/2} \end{cases} \quad (47)$$

In particular, the corrections to the $1S$, $2S$, and $2P_{1/2}$ states of the hydrogen atom are − 7.4 kHz, − 0.77 kHz, and + 0.58 kHz, respectively. These results were confirmed further in a recent numerical evaluation of the relativistic recoil correction to all orders in $Z\alpha$ [76].

A.3 Order α radiative corrections

In the following we use the terminology "Lamb shift" to represent any deviation of the energy levels from the first two terms $(m_e + m_N)c^2 + m_r c^2[f-1]$ of Eq. (42). Where appropriate, we refer to the $2S_{1/2} - 2P_{1/2}$ splitting as the "classic" Lamb shift. If one ignores nuclear motion temporarily, the bulk of the Lamb shift due to one photon radiative corrections to the electron and vacuum polarization corrections to the potential can be expressed by the formula

$$\Delta E_n(\mathrm{one-loop}) \equiv \frac{\alpha(Z\alpha)^4}{\pi n^3} m_e c^2 F_n(Z\alpha). \tag{48}$$

It is customary to write F_n as follows:

$$\begin{aligned} F_n(Z\alpha) &= A_{40}(n) + A_{41}(n)\ln[\sigma(Z\alpha)^{-2}] + (Z\alpha)A_{50}(n) \\ &+ (Z\alpha)^2\Big(G(n,Z\alpha) + A_{61}(n)\ln[\sigma(Z\alpha)^{-2}] \\ &+ A_{62}(n)\ln^2[\sigma(Z\alpha)^{-2}]\Big), \end{aligned} \tag{49}$$

where $\sigma = m_e/m_r$, p in $A_{pq}(n)$ refers to the overall power of $Z\alpha$ in ΔE_n and q refers to the power of logarithmic factors. The results of calculations are [15]

$$\begin{aligned}
A_{40}(nS) &= \left[\frac{10}{9} - \frac{4}{15} - \frac{4}{3}\ln k_0(n,S)\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{40}(n, l\neq 0) &= \left[\frac{C_{lj}}{2(2l+1)}\right]\left(\frac{m_r}{m_e}\right)^2 + \left[-\frac{4}{3}\ln k_0(n, l\neq 0)\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{41}(nS) &= \left[\frac{4}{3}\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{50}(nS) &= 4\pi\left[\frac{139}{128} + \frac{5}{192} - \frac{1}{2}\ln 2\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{61}(1S) &= \left[\frac{28}{3}\ln 2 - \frac{21}{20} - \frac{2}{15}\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{61}(2S) &= \left[\frac{16}{3}\ln 2 + \frac{67}{30} - \frac{2}{15}\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{61}(4S) &= \left[\frac{4}{3}\ln 2 + \frac{391}{80} - \frac{2}{15}\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{61}(2P_{1/2}) &= \left[\frac{103}{180}\right]\left(\frac{m_r}{m_e}\right)^3
\end{aligned}$$

$$
\begin{aligned}
A_{61}(2P_{3/2}) &= \left[\frac{29}{90}\right]\left(\frac{m_r}{m_e}\right)^3 \\
A_{62}(nS) &= -[1]\left(\frac{m_r}{m_e}\right)^3
\end{aligned}
\tag{50}
$$

where $C_{lj} = 2(j-l)/(j+1/2)$. The constants A_{41}, A_{50}, and A_{62} vanish for P states. The above expressions are sums of the self-energy contribution (including the anomalous magnetic moment) and vacuum polarization contribution (the $-4/15$ term of A_{40}, the $5/192$ term of A_{50}, and the $-2/15$ term of $A_{61}(nS)$). The reduced mass effect is expressed by the $(m_r/m_e)^3$ factor in all terms of (50) except for the first term of $A_{40}(n, l \neq 0)$ which has the factor $(m_r/m_e)^2$.

The function $G(n, Z\alpha)$ in (49) represents non-logarithmic terms of order $m_e c^2 \alpha (Z\alpha)^6$ plus all terms of higher order in $Z\alpha$. It can be written as a sum of two parts:

$$
G(n, Z\alpha) = [G_{SE}(n, Z\alpha) + G_{VP}(n, Z\alpha)]\left(\frac{m_r}{m_e}\right)^3 , \tag{51}
$$

where G_{SE} comes from the electron self-energy contribution and G_{VP} comes from diagrams containing vacuum polarization loops. For $Z = 1$ some values of G_{SE} are given by

$$
\begin{aligned}
G_{SE}(1S) &= \begin{cases} -30.31\ (4) & [77] \\ -30.268\ (13) & [78] \end{cases} \\
G_{SE}(2S) &= \begin{cases} -31.23\ (4) & [77] \\ -31.149\ (13) & [78] \end{cases} \\
G_{SE}(4s) &= -30.9\ (2) \qquad [77] \\
G_{SE}(2P_{1/2}) &= -0.98\ (4) \qquad [77] \\
G_{SE}(2P_{3/2}) &= -0.49\ (2) \qquad [77]
\end{aligned}
\tag{52}
$$

where the values from Ref. [77] were obtained by extrapolation to $Z = 1$ of numerical values calcul ated at high Z, while the values from Ref. [78] are the combinations of direct evaluation of the $\alpha(Z\alpha)^6$ coefficient at $Z = 1$ and the low Z extrapolation. Further refinement of the above results can be found in [79].

G_{VP} is known including higher order corrections in $Z\alpha$. Some G_{VP} for $Z = 1$ can be found in Ref. [77]:

$$\begin{aligned}
G_{VP}(1S) &= \left(\frac{4}{15}\ln 2 - \frac{1289}{1575}\right) + \left(\frac{19}{45} - \frac{\pi^2}{27}\right) + \mathcal{O}(Z\alpha) \\
&= -0.562\ 01\cdots \\
G_{VP}(2S) &= -\frac{743}{900} + \left(\frac{19}{45} - \frac{\pi^2}{27}\right) + \mathcal{O}(Z\alpha) \\
&= -0.752\ 21\cdots \\
G_{VP}(4S) &= -0.806\ 6\cdots + \left(\frac{19}{45} - \frac{\pi^2}{27}\right) + \mathcal{O}(Z\alpha) \\
&= -0.749\ 91\cdots \\
G_{VP}(2P_{1/2}) &= -\frac{9}{140} + \mathcal{O}(Z\alpha) \\
&= -0.064\ 0\cdots \\
G_{VP}(2P_{3/2}) &= -\frac{1}{70} + \mathcal{O}(Z\alpha) \\
&= -0.014\ 1\cdots.
\end{aligned} \tag{53}$$

The first term of G_{VP} arises from the Uehling potential [80]. The second part $(19/45 - \pi^2/27)$ is the $\alpha(Z\alpha)^6$ contribution proportional to the cube of the external potential [81].

There is another set of single-loop non-recoil terms to be taken into account. These are vacuum polarization corrections from heavier mass particles. We will not include them here since they are very small, being proportional to $1/\text{mass}^2$. For example, the muon loop contributes about 0.6 kHz to the classic Lamb shift in hydrogen.

A.4 Radiative recoil effect

Nuclear motion effects in the radiative correction that cannot be incorporated in the reduced mass dependence considered in the last subsection are known as radiative-recoil contributions. Those of order $\alpha(Z\alpha)^5 m_e^2/m_N$ have been calculated by Bhatt and Grotch [82]. Their result consists of terms called pole correction, double-Coulomb correction, and seagull correction. Another term, called vacuum-polarization correction, has also been evaluated [83]. Recently all these contributions have been evaluated by Pachucki

[84]. Unfortunately their calculations disagree on the pole correction and seagull correction and, as of now, the difference has not been resolved. Since this effect is relatively small (on which both agree), this term will not be included in our numerical values of the Lamb shifts.

A.5 Order α^2 radiative corrections

We turn now to two-loop radiative corrections. This is where most recent works have been carried out [37, 85, 86, 87, 88, 89, 90, 91, 92]. If one writes the two-loop radiative corrections in the form [80]

$$\Delta E_n(\mathrm{two-loop}) \equiv \frac{\alpha^2(Z\alpha)^4}{\pi^2 n^3} m_e c^2 [B_{40} + (Z\alpha)B_{50}], \tag{54}$$

the results of these calculations can be summarized as

$$\begin{aligned} B_{40} &= \left[-\frac{6131}{1296} - \frac{49}{108}\pi^2 + 2\pi^2 \ln 2 - 3\zeta(3)\right]\delta_{l0} \\ &+ \left[-\frac{197}{144} - \frac{1}{12}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3)\right]\left(\frac{1}{j+1/2} - \frac{1}{l+1/2}\right) \\ B_{50} &= -4\pi(1.715\,5\ (3))\delta_{l0}. \end{aligned} \tag{55}$$

The factor 4π in B_{50} is what one would obtain by an order of magnitude estimate. The factor 1.715 5 shows that this simple estimate is approximately correct. Estimates of higher-order terms have also been made [93, 94, 95, 96].

A.6 Nuclear size effect

We turn now to the effect of nuclear size. The finite distribution of charge in the nucleus affects the binding of S state and for nonrelativistic systems leads to the well-known shift

$$\Delta E_n(\mathrm{finite-size}) = \frac{2}{3n^3}(Z\alpha)^4 \left(\frac{R}{\alpha a_0}\right)^2 \delta_{l0} m_e c^2, \tag{56}$$

where R is the root-mean-square charge radius of the nucleus, and a_0 is the Bohr radius. As was discussed already, this contribution amounts to 0.145 MHz or 0.126 MHz in the classic Lamb shift depending on which value of the proton radius, Ref. [17] or Ref. [16], one uses. This is the cause of the largest uncertainty in the theory of the classic Lamb shift.

A.7 Nuclear polarizability

The nuclear polarizability correction arises from interactions between the electron and the nucleus in which excited states of the nucleus are included in the intermediate states. It is negligibly small in hydrogen fine structure at present, but may contribute significantly to the hyperfine structure of hydrogen ground state. Indeed, it is the largest source of theoretical uncertainty in the hydrogen hyperfine structure as was discussed in Sec. 2.6.

A.8 Comparison with experiments

Theoretical values of the Lamb shifts can be calculated putting together the results (42) − (56) and using α from (36), but leaving out terms which have not yet been confirmed or are known to be very small. For the classic Lamb shift $L_{2S} = E(2S_{1/2}) - E(2P_{1/2})$ in hydrogen one finds

$$\begin{aligned} L_{2S} &= 1\ 057.821\ (6)\ \text{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.805\ (11)\ \text{fm}, \\ L_{2S} &= 1\ 057.840\ (6)\ \text{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.862\ (11)\ \text{fm}. \end{aligned} \tag{57}$$

Recent measurements of the classic Lamb shift are as follows:

$$\begin{aligned} L_{2S} &= 1\ 057.862\ (20)\ \text{MHz}, && [97] \\ L_{2S} &= 1\ 057.851\ 4\ (19)\ \text{MHz}, && [98] \\ L_{2S} &= 1\ 057.845\ (9)\ \text{MHz}, && [99] \\ L_{2S} &= 1\ 057.841\ (12)\ \text{MHz} && [100]. \end{aligned} \tag{58}$$

Another well-studied Lamb shift is that of the ground state hydrogen atom. If one uses G_{SE} of Ref. [78], the theoretical value is given by

$$\begin{aligned} L_{1S} &= 8\ 172.659\ (50)\ \text{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.805\ (11)\ \text{fm}, \\ L_{1S} &= 8\ 172.808\ (50)\ \text{MHz} \qquad \text{for} \quad < r^2 >^{1/2} = 0.862\ (11)\ \text{fm}. \end{aligned} \tag{59}$$

A somewhat more precise value is given in Ref. [79]. The recent measurements are

$$\begin{aligned} L_{1S} &= 8\ 172.874\ (60)\ \text{MHz}, && [19] \\ L_{1S} &= 8\ 172.815\ (70)\ \text{MHz}, && [20] \\ L_{1S} &= 8\ 172.827\ (51)\ \text{MHz}, && [101] \\ L_{1S} &= 8\ 172.798\ (46)\ \text{MHz}. && [21] \end{aligned} \tag{60}$$

In Ref. [19] the energy difference $E(1S)-E(2S)$ in hydrogen is measured by direct frequency comparison of the $1S-2S$ and $2S-4S/4D$ two-photon transitions. The method adopted in Ref. [20] is to relate the frequency of the $2S-8S/D$ two-photon transitions in hydrogen to two optical standards. Ref. [101] measures the combination $(L_{1S}-L_{2S})-4(L_{2S}-L_{4P})$. Ref. [21] relies on comparison of the frequencies of the $1S_{1/2}-3S_{1/2}$ and $2S_{1/2}-6S_{1/2}$ or $2S_{1/2}-6D_{5/2}$ two-photon transitions.

The $1S$ Lamb shift is determined by combining the experimental value of L_{1S} with the experimental value of the $2S$ Lamb shift and theoretical values of $2P_{1/2}$, and various excited S, P, and D states.

The measurements of L_{1S} and L_{2S} seem to favor the larger proton radius. But a new measurement of the proton charge radius is urgently needed to decide on this issue. An *ab initio* calculation of the proton charge radius in lattice QCD will be very useful here [42].

Very accurate values of R_∞ quoted in (7) are obtained by combining the information on the Lamb shifts with the precisely known frequency of the hydrogen $1S-2S$ transition [18]

$$\nu(2S)-\nu(1S)=2\ 466\ 061\ 413.182\ (45)\ \mathrm{M}Hz, \tag{61}$$

which was measured by comparison with an accurately known reference frequency.

B Theory of the electron $g-2$

The QED part of the anomalous magnetic moment of the electron can be written in the form

$$a_e^{QED}=A_1+A_2(m_e/m_\mu)+A_2(m_e/m_\tau)+A_3(m_e/m_\mu,m_e/m_\tau), \tag{62}$$

where m_e, m_μ, and m_τ are the masses of the electron, muon, and tau, respectively.

The terms A_1, A_2, and A_3 can be expanded as a power series in α/π:

$$A_i=A_i^{(2)}\left(\frac{\alpha}{\pi}\right)+A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2+A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3+\ldots,\quad i=1,2,3. \tag{63}$$

The renormalizability of QED guarantees that the coefficients $A_i^{(n)}$ are finite and calculable for all i and n. Thus far A_1 has been evaluated to order α^4. The result available at the end of 1995 was

$$
\begin{aligned}
A_1^{(2)} &= 0.5, \\
A_1^{(4)} &= -0.328\ 478\ 965\ldots, \\
A_1^{(6)} &= 1.181\ 259\ (40), \\
A_1^{(8)} &= -1.557\ (70).
\end{aligned} \tag{64}
$$

$A_1^{(2)}$ and $A_1^{(4)}$ are known exactly [102]. $A_1^{(6)}$ is composed of contributions from 72 Feynman diagrams. All 72 have been evaluated by numerical integration [9] using an iterative-adaptive Monte-Carlo routine called VEGAS [103]. Over many years 67 of the 72 have also been evaluated analytically [104, 105]. The value of $A_1^{(6)}$ given in (64) has been evaluated by combining the best analytic and numerical results. Recent improved numerical works on $A_1^{(6)}$ have revealed that earlier evaluations suffered from a slight mismatch in combining analytic and numerical contributions. This caused a small but significant shift in its value in earlier evaluations. This is corrected in (64) [10]. Now, at last, the remaining five have been evaluated analytically [11], completing the analytic evaluation of the entire α^3 term. The result

$$A_1^{(6)} = 1.181\ 241\ 456\ldots \tag{65}$$

is in good agreement with the numerical result in (64), reinforcing the confidence in the reliability of VEGAS integration. $A_1^{(8)}$ is composed of contributions from 891 Feynman diagrams. Although some of them are known analytically [106] the majority of them have been evaluated by numerical integration only [107].

If one uses the value of α from (21), one finds from (64) that

$$A_1 = 1\ 159\ 652\ 196.7\ (2.1)\ \ (27.1) \times 10^{-12}. \tag{66}$$

where the first error comes from the theoretical uncertainty in the eighth-order term, while the second comes from α of (21). To this one must add [108]

$$
\begin{aligned}
A_2^{(4)}(m_e/m_\mu)(\alpha/\pi)^2 &= 2.804 \times 10^{-12}, \\
A_2^{(4)}(m_e/m_\tau)(\alpha/\pi)^2 &= 0.010 \times 10^{-12},
\end{aligned} \tag{67}
$$

where $m_e c^2 = 0.510\ 999\ 06\ (15)$ MeV, $m_\mu c^2 = 105.658\ 389\ (34)$ MeV, and $m_\tau c^2 = 1776.9\ (+0.4/\text{-}0.5)\ (0.2)$ MeV. Note also that [34]

$$\frac{m_\mu}{m_e} = 206.768\ 259\ (62), \quad (0.15\ \text{ppm}). \tag{68}$$

The results (67) are known analytically [109]. Sixth-order results such as $A_2^{(6)}(m_e/m_\mu)$ are also known analytically [110]. These and other QED terms are too small to be significant at present.

The non-QED contribution to the electron anomaly a_e arises from the effects of hadronic vacuum polarization and the weak interaction. The latest evaluation of the hadronic vacuum-polarization effect of order α^2 gives [111]

$$a_e^{\text{had.vp}} = 1.884\ 7\ (165)\ (375) \times 10^{-12}, \tag{69}$$

where the first uncertainty is statistical and the second is systematic. A rough estimate of higher-order hadronic corrections can be obtained by scaling down the muon $g-2$ values $a_\mu(\text{had1b})$, $a_\mu(\text{had1c})$, and $a_\mu(\text{had1d})$ given by Eq. (1.6) of Ref. [112] by $(m_e/m_\mu)^2$, $(m_e/m_\mu)^2$, and $(m_e/m_\mu)^4$, respectively, and the hadronic light-by-light contribution given in Ref. [113] by $(m_e/m_\mu)^2$. The sum of these contributions is

$$a_e^{\text{higher.had}} = -0.034 \times 10^{-12}, \tag{70}$$

with an uncertainty of about 20 percent. Although this is not a correct procedure strictly speaking, the error caused by this approximation is very small and can be ignored for the moment. The contribution of the weak interaction can be obtained by adapting one-loop [114] and two-loop electroweak corrections [115] of the muon moment to the electron:

$$a_e^{\text{weak}} = 0.030 \times 10^{-12}. \tag{71}$$

The two-loop term amounts to $-\ 35\%$ of the one-loop term and hence is relatively more important than the case of the muon anomalous magnetic moment. Summing up (67), (69), (70) and (71), one obtains the last line of (20).

Currently, the only possible improvement to $A_1^{(8)}$ is through numerical integration. Numerical work in progress will cut down the uncertainty by at least one-half. Although no attempt has been made to obtain an estimate of $A_1^{(10)}$ or higher-order terms, it is highly unlikely that they will be numerically significant at this moment. A crude estimate of large-order terms of perturbation series tends to confirm that the perturbative expansion of a_e is at best asymptotic [116], but again it will not be numerically significant at this level of precision.

Together with the forthcoming experiments [31, 32], further theoretical work will reduce the uncertainty in $\alpha(a_e)$ to about 1 ppb.

C Muonium hyperfine structure

I present here a sketch of new calculations of higher order terms and also correct errors in previous literature.

As is well known, the bulk of the hyperfine splitting is given by the Fermi formula

$$E_F = \frac{16}{3}\alpha^2 c R_\infty \frac{m_e}{m_\mu}\left[1+\frac{m_e}{m_\mu}\right]^{-3}, \tag{72}$$

where R_∞ is the Rydberg constant for infinite nuclear mass, and m_e and m_μ are the electron and muon masses, respectively.

Many correction terms of binding and radiative types have been calculated over the last 40 years. It is customary to classify them into three types: radiative non-recoil correction, pure recoil correction, and radiative recoil correction. In addition there are contributions of hadronic vacuum-polarization and a small weak interaction effect. Thus one may write

$$\begin{aligned}\Delta\nu(\text{theory}) \;&=\; \Delta\nu(\text{rad}) + \Delta\nu(\text{recoil}) + \Delta\nu(\text{rad-recoil}) \\ &\quad + \Delta\nu(\text{hadron}) + \Delta\nu(\text{weak}).\end{aligned} \tag{73}$$

Purely radiative terms of orders $\alpha(Z\alpha)$ and $\alpha(Z\alpha)^2$ have been known for some time [15]:

$$\begin{aligned}\Delta\nu(\text{rad}) \;=\; & (1+a_\mu)\Big(1+\frac{3}{2}(Z\alpha)^2 + a_e + \alpha(Z\alpha)(\ln 2 - \frac{5}{2}) \\ & -\frac{8\alpha(Z\alpha)^2}{3\pi}\ln(Z\alpha)\left[\ln(Z\alpha) - \ln 4 + \frac{281}{480}\right] \\ & +\frac{\alpha(Z\alpha)^2}{\pi}(14.88 \pm 0.29)\Big) E_F.\end{aligned} \tag{74}$$

Here a_e and a_μ are the anomalous magnetic moments of the electron and muon, respectively. The appearance of the factor $(1+a_\mu)$ in (74) is in accord with our definition of E_F in (72). Note that the number 14.88 in the $\alpha(Z\alpha)^2$ correction is different from 15.39 reported in Ref. [15]. This is due to the recent discovery of two mistakes in the literature. For a detailed discussion see Ref. [117]. The 0.29 uncertainty is one standard deviation estimated by VEGAS [103].

The known recoil corrections add up to [15]

$$\begin{aligned}\Delta\nu(\text{recoil}) &= \left(-\frac{3Z\alpha}{\pi}\frac{m_e m_\mu}{m_\mu^2 - m_e^2}\ln\frac{m_\mu}{m_e}\right.\\ &\left.+\frac{\gamma^2}{m_e m_\mu}\left[2\ln\frac{m_r}{2\gamma} - 6\ln 2 + \frac{65}{18}\right]\right) E_F, \end{aligned} \quad (75)$$

where $\gamma \equiv Z\alpha m_r,\ m_r = m_e m_\mu/(m_e + m_\mu)$. The radiative-recoil contributions, which arise from both lepton lines and vacuum polarizations, are given by

$$\begin{aligned}\Delta\nu(\text{rad-recoil}) &= \frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e}{m_\mu}\left(-2\ln^2\frac{m_\mu}{m_e} + \frac{13}{12}\ln\frac{m_\mu}{m_e}\right.\\ &+6\zeta(3) + \zeta(2) - \frac{71}{72} + 3\pi^2\ln 2\\ &+Z^2\left[\frac{9}{2}\zeta(3) + \frac{39}{8} - 3\pi^2\ln 2\right]\\ &\left.+\frac{\alpha}{\pi}\left[-\frac{4}{3}\ln^3\frac{m_\mu}{m_e} + \frac{4}{3}\ln^2\frac{m_\mu}{m_e} + \mathcal{O}\left(\ln\frac{m_\mu}{m_e}\right)\right]\right) E_F, \end{aligned} \quad (76)$$

The $\alpha(Z\alpha)$ term is known exactly [15, 118]. The $\ln^3$ and $\ln^2$ parts of the $\alpha^2(Z\alpha)$ term were evaluated by Eides *et al.* [119].

The hadronic vacuum polarization contributes [120].

$$\begin{aligned}\Delta\nu(\text{hadron}) &= \frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e m_\mu}{m_\pi^2}(3.75 \pm 0.24)E_F\\ &= 0.250(16)\text{kHz}, \end{aligned} \quad (77)$$

where m_π is the charged pion mass.

Finally, there is a small contribution due to Z^0 exchange. There has been some confusion about the sign of the standard-model estimate [121, 122]. It is now agreed that the correct value is [123]

$$\begin{aligned}\Delta\nu(\text{weak}) \quad &= -G_F \frac{3\sqrt{2}mM}{8\alpha\pi} E_F \\ &\simeq -0.065 \text{ kHz}.\end{aligned} \tag{78}$$

In order to catch up with new experiments, various higher-order corrections to $\Delta\nu$ must be evaluated. The lowest order correction to be calculated is of relative order $\alpha^2(Z\alpha)$. It consists of 42 Feynman diagrams containing second-order and fourth-order vacuum polarization diagrams, 6 Feynman diagrams each containing a light-by-light scattering subdiagram, and 60 Feynman diagrams which are two-photon exchange diagrams with fourth-order radiative corrections on the electron line. The first group was evaluated mostly analytically in Ref. [124] and confirmed by numerical integration [36]. The second group was evaluated semi-analytically [125], which, after revision, is in good agreement with the result of a purely numerical integration [36]. The third group, which is the most difficult one, was evaluated numerically in Ref. [36] in the Feynman gauge, and semi-analytically in Ref. [37] in the Fried-Yennie gauge. The results are in good agreement with each other. Collecting the contributions of the three groups one finds the total $\alpha^2(Z\alpha)$ contribution to be

$$\begin{aligned}\delta_1\Delta\nu &= 0.767\ 9\ (79)\ \frac{\alpha^2(Z\alpha)}{\pi} E_F \\ &= 0.423\ 5\ (44) \text{ kHz},\end{aligned} \tag{79}$$

which is an improvement [117] of the preliminary value reported in Ref. [36], and

$$\begin{aligned}\delta_1\Delta\nu &= 0.771\ 7\ (4)\ \frac{\alpha^2(Z\alpha)}{\pi} E_F \\ &= 0.425\ 6\ (2) \text{ kHz}\end{aligned} \tag{80}$$

is from Ref. [37].

It turns out that the leading logarithmic corrections of order $\alpha^n(Z\alpha)^{4-n}$, n = 1, 2, 3, $\alpha^3(m/M)$, and $\alpha^2(m/M)^2$ contribute to the muonium hyperfine structure as much as the $\alpha^2(Z\alpha)$ contribution (79). Some of these correction terms have been evaluated in Ref. [94]:

$$\begin{aligned}
\delta_2 \Delta\nu &= -\frac{8}{3}\left(\frac{\alpha}{2\pi} - 2\frac{m}{M} + \frac{Z}{4}\frac{m}{M}\right)\frac{\alpha}{\pi}(Z\alpha)^2 \ln^2(Z\alpha)E_F \\
&+ \frac{17}{8}(Z\alpha)^4 E_F \\
&= 0.287 \text{ kHz}.
\end{aligned} \tag{81}$$

Preliminary values of correction terms of order $\alpha^2(Z\alpha)^2$ and $(Z\alpha)^3(m/M)$ evaluated by the NRQED method are given by [126, 127]:

$$\begin{aligned}
\delta_3 \Delta\nu &= -0.110 \text{ kHz}, \\
\delta_4 \Delta\nu &= -0.361 \text{ kHz}.
\end{aligned} \tag{82}$$

For further improvement of theory it is necesssary to reduce the uncertainty in the previous evaluation of the $\alpha(Z\alpha)^2$ contribution, which was carried out starting from the Bethe-Salpeter equation [15]. Actually, that calculation includes contributions of powers of $Z\alpha$ beyond $(Z\alpha)^2$. In the NRQED approach, which is a systematic expansion in α and $Z\alpha$, it is possible to identify terms of definite power in $Z\alpha$. The calculation carried out thus far is of exact orders $\alpha(Z\alpha)^2$ and $\alpha(Z\alpha)^3$. Although this calculation is still preliminary, the sum of these two terms is in agreement with the result of Ref. [38], with smaller error bars. When this calculation is completed, the uncertainty 0.16 in (28) will be reduced substantially.

D The NRQED method

The relativistic quantum field theory is often very cumbersome to apply to nonrelativistic bound systems. Such a calculation tends to be very complicated and requires an enormous effort, while the result reflects mostly the nonrelativistic features of the system, which are well described by the nonrelativistic Schrödinger theory. The Schrödinger quantum mechanics, of course, belongs to *level* 1 or even *level* 2, as was defined in Sec. 3. The problem we face is how we relate the *level* 1 (2) theory to *level* 0 theory, namely, relativistic quantum field theory. For the case of the electromagnetic interaction, this has been realized by a theory called nonrelativistic quantum electrodynamics, or NRQED.

The NRQED, formulated by Caswell and Lepage [64], is a rigorous adaptation of QED to bound systems. It is an example of an effective theory with finite cut-off which is *completely* equivalent to QED as long as it is applied to a system with typical momentum less than the chosen cut-off. The cut-off dependence disappears from the physical quantities computed.

Thus far the NRQED has been applied to the loosely bound atomic systems, muonium and positronium, and enabled us to take a consistent and systematic approach to these systems. Compared with the conventional relativistic bound state theories, such as the Bethe-Salpeter equation, it is manifestly gauge invariant and allows easier power counting and more transparent cancellation of ultraviolet (UV) and infrared (IR) divergences. In spite of its superiority, however, the details of the theory were not fully worked out until recently. In this Appendix, I present only an outline of the NRQED and its application to an atomic bound state perturbation theory. Details may be found in Refs. [117, 126, 127].

In the NRQED approach to the bound state problem, one first derives the NRQED Lagrangian from QED, and then uses it to determine the correction to the energies and wave functions by a systematic application of the Rayleigh-Schrödinger perturbation theory.

The NRQED Lagrangian consists of all possible local interactions satisfying the required symmetries, such as gauge invariance, parity invariance, time reversal, galileian invariance, hermiticity, and locality. We use the same photon Lagrangian $(-1/4)F_{\mu\nu}F^{\mu\nu}$ as that of QED. In addition, new photon interaction terms, such as vacuum polarization and light-by-light scattering, are introduced to represent the insertion of the fermion loop.

In order to define the NRQED Lagrangian precisely, we must regularize the interaction terms of NRQED, e.g., by cutting off contributions of large momenta. Since this theory is meant to apply to nonrelativistic systems, the cut-off Λ may be chosen as the typical mass scale of the system, e.g., the rest mass of an electron. With the cut-off Λ thus fixed, the theory becomes well-defined, even though the interaction terms are strongly dependent on the cut-off parameter. In the following the cut-off is understood implicitly and will be exhibited only when it is necessary. The choice of the momentum cut-off used for the NRQED scattering amplitudes is arbitrary but the physical quantity computed should be independent of any particular choice. In other words, the NRQED theory must have reparametrization invariance with respect to the choice of cut-off. This is analogous to the existence of the renormalization group in the renormalizable relativistic field theory. It is important to note that the NRQED is fully equivalent to the QED. The only difference is that it is better adapted to low energy bound systems.

The NRQED rule for determining the operators which appear in its Lagrangian and their coefficients is simple and straightforward: Each term of the *scattering* amplitude calculated in the NRQED must coincide with the corresponding *scattering* amplitude of the original QED at some given momentum scale, e.g., at the threshold of the external on-shell particles. The center-of-mass frame is used for both bound state and scattering state calculations. Since the same argument about reparametrization invariance holds for the momentum scale chosen for comparison of QED and NRQED scattering amplitudes, the at-threshold condition is just for convenience. However, the on-shell condition for the external fermion is more than a matter of convenience. In order to regulate the IR singularity it is convenient to introduce the photon mass λ in the calculation of scattering amplitude for both QED and NRQED. This finite photon mass together with the on-shell condition ensures that the NRQED scattering theory has a pole in the region of the complex energy plane of the external fermion in which the scattering theory can be analytically continued to the off-shell bound state theory.

Note that the scattering amplitude of QED is fully renormalized, namely, it is finite and completely determined within QED. This enables us to fix the NRQED "renormalization" constants without ambiguity. This also means that the coupling constant α and fermion masses in QED are the renormalized ones determined on-shell, and these α and fermion masses are used as the "bare" coupling constant and "bare" masses of NRQED.

It is convenient to write the NRQED Lagrangian in two parts: L_{main} and L_{contact}. The L_{main} part consists of the fermion bilinear operators. Fermions in NRQED are expressed by the Pauli two component spinor field $\psi(t, \vec{x})$ (instead of the Dirac spinor). If one takes into account the required symmetries of the theory, the main part of NRQED Lagrangian L_{main} must have the general form [64, 65]

$$\begin{aligned} L_{\text{main}} &= \psi^\dagger \{ iD_t + \frac{\vec{D}^2}{2m} + \frac{\vec{D}^4}{8m^3} \\ &+ c_F \frac{e\vec{\sigma} \cdot \vec{B}}{2m} + c_D \frac{e(\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D})}{8m^2} \end{aligned}$$

$$
\begin{aligned}
&+ \quad c_S \frac{ie\vec{\sigma}\cdot(\vec{D}\times\vec{E}-\vec{E}\times\vec{D})}{8m^2} + c_{W1}\frac{e\{\vec{D}^2,\vec{\sigma}\cdot\vec{B}\}}{8m^3} \\
&+ \quad c_{W2}\frac{-e\vec{D}^i\vec{\sigma}\cdot\vec{B}\vec{D}^i}{4m^3} + c_{p'p}\frac{e(\vec{\sigma}\cdot\vec{D}\vec{B}\cdot\vec{D}+\vec{D}\cdot\vec{B}\vec{\sigma}\cdot\vec{D})}{8m^3} \\
&+ \quad \ldots\}\psi \,. \qquad (83)
\end{aligned}
$$

where $D_t = \partial_t + ieA^0$ and $\vec{D} = \vec{\partial} - ie\vec{A}$. (We define $c = 1$ and $\hbar = 1$ henceforth.) The positron part can be written down in a similar way. The particle-antiparticle mixed interaction is not present in L_{main}. The first three terms are related to the kinetic term of the QED Lagrangian. The second and third terms are derived from the expansion

$$
E = \sqrt{\vec{p}^{\,2}+m^2} = m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} + \ \ldots \ . \qquad (84)
$$

These three terms of (83) have coefficients unaffected by the radiative corrections as a consequence of the renormalizability of QED, while the coefficients c_i of other terms are modified by the QED interaction and can be expressed as a power series in the coupling constants α

$$
c_i = c_i^{(0)} + c_i^{(1)}\alpha + c_i^{(2)}\alpha^2 + \ldots \quad . \qquad (85)
$$

Some of the operators in (83) can be generated by the Foldy-Wouthuysen-Tani transformation of the Dirac Lagrangian. These operators have the coefficient $c_i^{(0)} = 1$ while other operators have $c_i^{(0)} = 0$. Note that c_i's do not have coefficients involving $Z\alpha$ caused by the binding effect because they are determined solely by comparison of the NRQED and QED *scattering* amplitudes without referring to the bound states.

Eq. (83) has an infinite number of terms. Not all of them, of course, are needed in a practical calculation. The operators necessary to carry out a particular calculation are determined by the power counting rule of NRQED for the bound state.

The L_{main} of NRQED alone is not sufficient to produce the same physical quantities as those from QED. To make NRQED equivalent to QED, we must add another term to the NRQED Lagrangian. It consists of terms of the contact interaction type:

$$
L_{\mathrm{contact}} \;=\; d_1\frac{1}{mM}(\psi^\dagger\vec{\sigma}\psi)\cdot(\chi^\dagger\vec{\sigma}\chi) + d_2\frac{1}{mM}(\psi^\dagger\psi)(\chi^\dagger\chi)
$$

$$
\begin{aligned}
&+ \quad d_3 \frac{1}{mM} (\psi^\dagger \vec{\sigma} \chi) \cdot (\chi^\dagger \vec{\sigma} \psi) + d_4 \frac{1}{mM} (\psi^\dagger \chi)(\chi^\dagger \psi) \\
&+ \quad d_5 \frac{1}{m^3 M} (\psi^\dagger \vec{D}^2 \vec{\sigma} \psi) \cdot (\chi^\dagger \vec{\sigma} \chi) + \cdots \quad , \qquad (86)
\end{aligned}
$$

where χ represents a fermion field (of mass M) such as a muon or a positron. The third and fourth terms in (86) are needed only when both electron and positron are present. This is because, from the viewpoint of NRQED, the electron-positron annihilation is a high energy process and can only be represented as a contact interaction term. For the muonium, only the first and second terms are relevant. The fifth term is an example of contact terms including derivative interactions, which are of higher order in $< \vec{p}^{\,2}/m^2 > \sim (Z\alpha)^2$.

The coefficients d_i are chosen such that these contact interactions make up the difference between the QED electron-muon scattering amplitude and the corresponding NRQED scattering amplitude derived from the Lagrangian $L_{\rm main}$. This procedure enables us to determine the coefficients d_i completely.

The result of this construction to order α is the NRQED Hamiltonian of the form

$$
\begin{aligned}
H^\Lambda_{\rm main} = \psi^\dagger(\vec{p}\,') \quad & \Big[\frac{\vec{p}^{\,2}}{2m} + eA^0 - \frac{(\vec{p}^{\,2})^2}{8m^3} - \frac{e}{2m}(\vec{p}\,' + \vec{p}) \cdot \vec{A} + \frac{e^2}{2m} \vec{A} \cdot \vec{A} \\
- & \quad \frac{ie}{2m} c_F \vec{\sigma} \cdot (\vec{q} \times \vec{A}) - \frac{e}{8m^2} c_D \vec{q}^{\,2} A^0 \\
+ & \quad \frac{ie}{4m^2} c_S \vec{\sigma} \cdot (\vec{p}\,' \times \vec{p}) A^0 - \frac{ie^2}{4m^2} c_S \vec{\sigma} \cdot (\vec{q_1} \times \vec{A}(q_1)) A^0(q_2) \\
+ & \quad \frac{ie}{8m^2} c_S q^0 \vec{\sigma} \cdot ((\vec{p}\,' + \vec{p}) \times \vec{A}) \\
+ & \quad \frac{ie}{8m^3} c_W (\vec{p}\,'^2 + \vec{p}^{\,2}) \vec{\sigma} \cdot (\vec{q} \times \vec{A}) + \frac{ie}{8m^3} c_{q^2} \vec{q}^{\,2} \vec{\sigma} \cdot (\vec{q} \times \vec{A}) \\
+ & \quad \frac{ie}{8m^3} c_{p'p} \{ \vec{p} \cdot (\vec{q} \times \vec{A})(\vec{\sigma} \cdot \vec{p}\,') + \vec{p}\,' \cdot (\vec{q} \times \vec{A})(\vec{\sigma} \cdot \vec{p}) \} \\
+ & \quad \ldots \ \Big] \psi(\vec{p}) \\
+ & \quad c_{\rm vp} A^i(q) \frac{\vec{q}^{\,4}}{m^2} A^j(q) (\delta^{ij} - \frac{q^i q^j}{\vec{q}^{\,2}}) \\
+ & \quad c_{\rm vp} A^0(\vec{q}) \frac{-\vec{q}^{\,4}}{m^2} A^0(\vec{q}) \ , \qquad (87)
\end{aligned}
$$

where $\vec{p}'$ and $\vec{p}$ are the outgoing and incoming fermion momenta, respectively, and $q = (q^0, \vec{q})$ is the incoming photon momentum. In the seagull vertex, $\vec{q_1}$ is the incoming momentum of the vector potential $\vec{A}$. The superscript Λ indicates that the Hamiltonian is regularized with the UV cut-off Λ. The "renormalization" coefficients are

$$\begin{aligned}
c_F &= 1 + a_e \ , \\
c_D &= 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{2\Lambda}\right) - \frac{3}{8} + \frac{5}{6}\right] + 2a_e \ , \\
c_S &= 1 + 2a_e \ , \\
c_W &= 1 \ , \\
c_{q^2} &= \frac{\alpha}{\pi}\frac{4}{3}\left[\ln\left(\frac{m}{2\Lambda}\right) - \frac{3}{8} + \frac{5}{6} + \frac{1}{4}\right] + \frac{a_e}{2} \ , \\
c_{p'p} &= a_e \ , \\
c_{\rm vp} &= \frac{\alpha}{15\pi} \ .
\end{aligned} \tag{88}$$

The contact term $H_{\rm contact}$ of the NRQED Hamiltonian is also obtained by comparison of the electron-muon scattering amplitudes in QED and NRQED. Some low order terms of $H_{\rm contact}$ are given by

$$\begin{aligned}
H^{\Lambda}_{\rm contact} &= \left[\alpha(Z\alpha)^2\pi\frac{2}{3}\left(\ln 2 - \frac{13}{4} + \frac{3}{4}\right)\right. \\
&+ \left.\alpha^2(Z\alpha)^2\frac{2}{3}(0.767\,9(79)) + \cdots\right]\frac{1}{mM}\psi^\dagger\vec{\sigma}_e\psi\cdot\chi^\dagger\vec{\sigma}_\mu\chi \\
&+ \left[\alpha(Z\alpha)^2 4\pi\left(\frac{139}{128} + \frac{5}{192} - \frac{1}{2}\ln 2\right) + \cdots\right]\frac{1}{m^2}\psi^\dagger\psi\chi^\dagger\chi \\
&+ \cdots .
\end{aligned} \tag{89}$$

This Hamiltonian contributes to the $\alpha(Z\alpha)$ and $\alpha^2(Z\alpha)$ corrections to the muonium hyperfine structure and the $\alpha(Z\alpha)^5$ term of the Lamb shift. Note that the NRQED Hamiltonian given here, although it is derived for the calculation of atomic binding energies, can be readily generalized to any nonrelativistic system.

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