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Multi-point Local Height Probabilities in the Integrable RSOS Model

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Abstract

By using the bosonization technique, we derive an integral representation for multi-point Local Hight Probabilities for the Andrews-Baxter-Forrester model in the regime III. We argue that the dynamical symmetry of the model is provided by the deformed Virasoro algebra.

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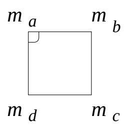
1. Introduction

We start with an extended formulation of the problem which is attacked by solving in this work.

1.1. General RSOS model

Knowing the equilibrium structure of the crystal-vapor interface is a prerequisite for understanding and controlling the growth of crystals from vapor or dilute solution. At low temperatures the interface is expected to be atomistically flat to minimize the surface energy, whereas at high temperatures it is rough and contains steps producing the configuration entropy. This change in the structure is the roughening transition of the interface. The simple model generally accepted to depict the necessary minimum of configurational degrees of freedom describing the roughening transition is so-called "Restricted solid-on-solid" (RSOS) model [1]. In this model crystal atoms are supposed to be located at the square lattice sites numerated by variables a without allowing an empty space in the bulk of the crystal. The interface is characterized by the height or number of crystal layers m_a measured from some reference level. The statistics of the model is governed by local positively defined Boltzmann weights



$$\mathbf{W}\begin{bmatrix} m_a & m_b \\ m_d & m_c \end{bmatrix} = \exp\left\{ -\frac{\varepsilon(m_a, m_b, m_c, m_d)}{k_B T} \right\} \tag{1.1}$$

assigned to every configuration (m_a, m_b, m_c, m_d) of heights round a face with the sites (a, b, c, d) ordered clockwise from the upper left. The restriction condition on the weights (1.1) expresses that no strongly fluctuating configurations are allowed. More precisely, if m's take integer values between 1 and $r-1$,

$$1 \leq m_a \leq r-1 \; , \tag{1.2}$$

with the integer $r \geq 4$, then (1.1) is non zero only if any two neighboring heights around the face differ by ± 1 :

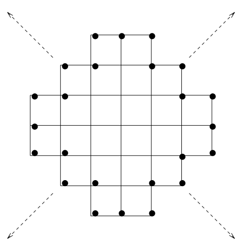
$$|m_a - m_b| = 1 \; . \tag{1.3}$$

The Boltzmann weight of a given configuration is a product of weights assigned to each face of the lattice. At low temperatures the interface tends to take on a height configuration minimizing the total surface energy

$$\mathcal{E} = \sum_{faces(a,b,c,d)} \varepsilon(m_a, m_b, m_c, m_d) \; . \tag{1.4}$$

Such configuration is usually referred as a ground state. In principle, the system may leave a room for a number of different ground states and we enumerate them by an integer l.

Consider the following lattice.



The bullets here denote boundary heights which would be fixed somehow. In the present work we assume that the boundary heights take the same values as those in one of the ground states [2]. So, there is one-to-one correspondence between these boundary conditions and the ground states and they can be labeled by the same index l. A partition function for the RSOS model with the designated l-th boundary condition is given by

$$Z_l^{(N)} = \sum_{heights\{m\}} \prod_{faces(a,b,c,d)} \mathbf{W}\begin{bmatrix} m_a & m_b \\ m_d & m_c \end{bmatrix} . \tag{1.5}$$

Here N is a number of the lattice faces. Of particular interest is the partition function $Z_l^{(N)}$ at the thermodynamic limit $N \to \infty$, when the lattice uniformly increases in all directions. The short distance interaction ensures, that

$$Z_l^{(N)} \to \kappa^N Z_l \ , \quad N \to \infty \ , \tag{1.6}$$

and κ is a partition function per face. The above mentioned boundary conditions break explicitly the translation invariance, hence Z_l in (1.6) may depend on them, in contrary to κ [1].

Other important quantities for studying are n-point Local Height Probabilities (LHP). They can be defined as follows; Let us specify n reference sites belonging the same vertical column and label them from down to up by $1, 2, ...n$. The probability that heights at these sites have the values $1 \leq k_1, ...k_n \leq r-1$ reads:

$$P^{(N)}_{k_1,...k_n}(l) = \left[Z_l^{(N)}\right]^{-1} \sum_{heights\{m\}} \prod_{j=1}^{n} \delta(k_j, m_j) \prod_{faces(a,b,c,d)} \mathbf{W}\begin{bmatrix} m_a & m_b \\ m_d & m_c \end{bmatrix} . \tag{1.7}$$

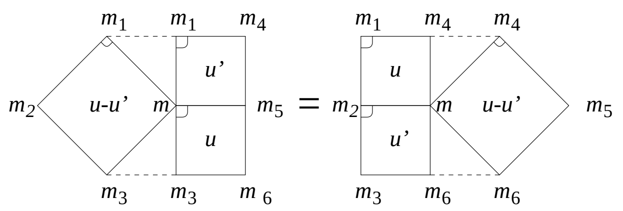
Again, of prime importance are LHP at the thermodynamic limit

$$P_{k_1,...k_n}(l) = \lim_{N\to\infty} P^{(N)}_{k_1,...k_n}(l) \ . \tag{1.8}$$

1.2. ABF model

The Boltzmann weights (1.1) are free parameters for the RSOS model, and an exact analytical calculation of LHP (1.8) for their general values does not seem to be a realizable problem nowadays. Nevertheless, there are some special cases when the model is a solvable one. As it usually is, a local integrability condition is given by the set of algebraic equations for the Boltzmann weights [2], [3]

$$\begin{gathered}
\sum_m \mathbf{W}\left[\begin{matrix} m_1 & m \\ m_2 & m_3 \end{matrix}\,\middle|\, u-u'\right] \mathbf{W}\left[\begin{matrix} m_1 & m_4 \\ m & m_5 \end{matrix}\,\middle|\, u'\right] \mathbf{W}\left[\begin{matrix} m & m_5 \\ m_3 & m_6 \end{matrix}\,\middle|\, u\right] = \\
\sum_m \mathbf{W}\left[\begin{matrix} m_1 & m_4 \\ m_2 & m \end{matrix}\,\middle|\, u\right] \mathbf{W}\left[\begin{matrix} m_2 & m \\ m_3 & m_6 \end{matrix}\,\middle|\, u'\right] \mathbf{W}\left[\begin{matrix} m_4 & m_5 \\ m & m_6 \end{matrix}\,\middle|\, u-u'\right] .
\end{gathered} \tag{1.9}$$



Here u is some variable parameterizing the Boltzmann weights. Andrews *et al.* [4] have succeeded in finding of solutions of the Yang-Baxter equation (YBE) (1.9) for any integer $r \geq 4$. In addition to the spectral

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[1] The translation invariance can be restored by a proper averaging over all possible values of l .

parameter u, the ABF solution depends on a parameter p. It gives a two parametric family in a manifold of the Boltzmann weights of the RSOS model. The integrable weights read explicitly:

$$\begin{aligned}
&\mathbf{W}\begin{bmatrix} m\pm 2 & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\ , \\
&\mathbf{W}\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\ \frac{[m\pm u]}{[1-u][m]}\ , \\
&\mathbf{W}\begin{bmatrix} m & m\pm 1 \\ m\mp 1 & m \end{bmatrix} = R\ \frac{\big([m+1][m-1]\big)^{\frac{1}{2}}}{[m]}\ \frac{[u]}{[1-u]}\ .
\end{aligned} \tag{1.10}$$

Here we use the short notation

$$[u] = h(u)/h(1)\ , \tag{1.11}$$

with

$$h(u) = 2\ |p|^{1/8}\ \sin(\pi u/r) \prod_{n=1}^{+\infty} \big(1-p^n\big)\ \big(1-2p^n \cos(2\pi u/r) + p^{2n}\big)\ .$$

In order to provide the convergence of the infinite product, the parameter p has to be restricted to the domain

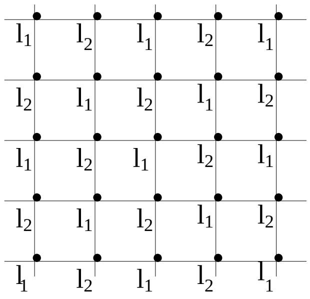
$$-1 < p < 1\ .$$

The positive factor R in (1.10) determines a normalization of the weights.

The ABF weights family consists of two distinct manifolds with $0 < u < 1$, and $-1 < u < 0$. According to [4], the manifolds are divided also into phases (regimes) by the line of critical points $p = 0$. In this work we restrict our attention to the so-called regime III:

$$0 < p < 1\ , \quad 0 < u < 1\ . \tag{1.12}$$

The parameter p measures a deviation from the criticality [2]. The value $p = 1$ can be associated with the zero temperature, since in this case the system is frozen in one of the ground states. As it follows from (1.10), each ground state in the regime III has the form:



$|l_1 - l_2| = 1$

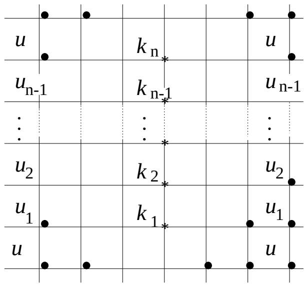
Therefore, we identify the integer number l, enumerating the ground states, with $min(l_1, l_2)$ and

$$1 \le l \le r-2\ . \tag{1.13}$$

The same number specifies also the boundary conditions, used in the definition of LHP (1.7), (1.8).

Now, let us break down the $n-1$ rows between the reference points $a = 1, 2, ..., n$ and replace the spectral parameter u in these rows to $0 < u_1, ..., u_{n-1} < 1$, respectively.

[2] In the case of the regime III, the scaling behavior in the vicinity of the critical line is described by the Quantum Field Theory and referred as the Restricted Sine-Gordon model [5].



As well as in the homogeneous case, one can introduce n-point LHP $P_{k_1,...k_n}(u_1,...,u_{n-1}|l)$ at the thermodynamic limit on the lattice with such kind of dislocations. In this work we are going to study these quantities. Our main result is an integral representation for them given by the formulas (5.11). Notice, that it would be convenient for us to use the parameters ζ and x

$$x = \exp\left[\frac{2\pi^2}{r\ln p}\right] \ , \quad \zeta = x^{2u} \ , \tag{1.14}$$

rather then u and p. Hence, the n-point LHP for the inhomogeneous lattice are denoted as

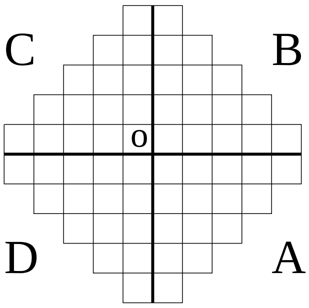
$$P_{k_1,...k_n}(\zeta_1,...,\zeta_{n-1}|l)$$

in what follows. It is worth to point out here, that due to YBE (1.9) these functions depend on the ratios $\zeta_j\zeta_1^{-1}$ only [6].

2. Corner Transfer Matrix and one-point LHP

2.1. Corner Transfer Matrix

The key calculation tool for LHP is the Corner Transfer Matrix (CTM) [7], [2], [4]. In this section we concentrate on finding the one-point function to give an idea on this important notion. Consider the square lattice with the reference site O as being the central one.



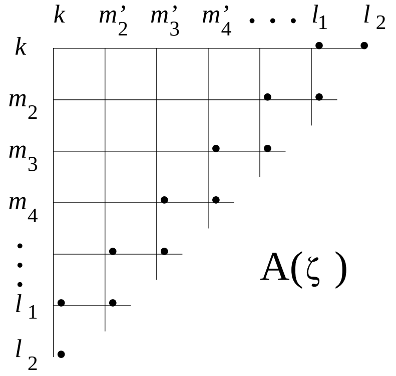
The probability $P_k(l)$ that the central height equals to k is determined by (1.7), (1.8) with $n = 1$. One can divide the whole lattice into four quadrants by the horizontal and vertical lines intersecting at O. The idea is to calculate the sum in (1.7) in two steps. First, let us perform summation over the sites inside the quadrants with given values of heights along the cuts. Doing so, one treats the resulting expressions as matrix elements of the four Corner Transfer Matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ associated with the corresponding quadrants. The second step is to find the sum of heights on the boundaries between the quadrants. More explicitly, consider the lower right quadrant with the hight configuration

$$\mathbf{m} = (m_1, m_2, ...l_1, l_2) \ , \quad m_1 = k$$

along the horizontal boundary and

$$\mathbf{m}' = (m'_1, m'_2, ...l_1, l_2) \ , \quad m'_1 = k$$

for the vertical one.



The Corner Transfer Matrix $\mathbf{A}(\zeta)$ is a matrix with elements $\big[\mathbf{A}\big]_{\mathbf{m}}^{\mathbf{m}'}$ given by the partition functions of the quadrant with the fixed boundary heights. Notice, that we specify the dependence on the spectral parameter chosen in the form (1.14). CTMs $\mathbf{B}(\zeta), \mathbf{C}(\zeta), \mathbf{D}(\zeta)$ for other quadrants are defined similarly. By the definition, CTM breaks up into $r-1$ diagonal blocks with the given value of the central height. Let $\mathbf{P}_k$ be the diagonal matrix which diagonal entries are units for the block with $m_1 = k$ and all other elements being zero. Then the remaining sum in (1.7) over the heights at the cuts consists in taking the trace:

$$P_k(l) = Tr\Big[\ \mathbf{P}_k\ \mathbf{DCBA}\ \Big]\Big/ Tr\Big[\ \mathbf{DCBA}\ \Big]\ . \tag{2.1}$$

This expression does not depend on a normalization of CTMs.

According to [7],[2], CTMs and their products would be well defined operators at the thermodynamic limit, if the Boltzmann weights are normalized in such a way that the partition function per face κ (1.6) equals to one. For the ABF weights (1.10) in the regime III this normalization means [8], [4]:

$$R = \zeta^{\frac{r-1}{2r}}\ \frac{(x^2\zeta^{-1}; x^{2r}, x^4)_\infty\ (x^{2+2r}\zeta^{-1}; x^{2r}, x^4)_\infty\ (x^4\zeta; x^{2r}, x^4)_\infty\ (x^{2r}\zeta; x^{2r}, x^4)_\infty}{(x^4\zeta^{-1}; x^{2r}, x^4)_\infty\ (x^{2r}\zeta^{-1}; x^{2r}, x^4)_\infty\ (x^2\zeta; x^{2r}, x^4)_\infty\ (x^{2+2r}\zeta; x^{2r}, x^4)_\infty}\ , \tag{2.2}$$

where the notation

$$(\zeta; q_1, q_2, ..., q_n)_\infty = \prod_{\{k_i\}=0}^{\infty} (1 - \zeta\ q_1^{k_1} q_2^{k_2} ... q_n^{k_n})$$

is used. The function $R(\zeta)$ obeys the set of relations

$$\begin{aligned} &R(\zeta)R(\zeta^{-1}) = 1\ , \\ &R(x^2\zeta^{-1}) = \big(x\zeta^{-1}\big)^{\frac{r-1}{r}}\ \frac{(\zeta; x^{2r})_\infty\ (x^{2r}\zeta^{-1}; x^{2r})_\infty}{(x^2\zeta^{-1}; x^{2r})_\infty\ (x^{2r-2}\zeta; x^{2r})_\infty}\ R(\zeta)\ . \end{aligned} \tag{2.3}$$

With this normalization, the ABF weights satisfy

- *unitarity relation*

$$\sum_{m_2} \mathbf{W}\begin{bmatrix} m_1 & m_2 \\ m_4 & m_3 \end{bmatrix}\!\!\left|\, z \right] \ \mathbf{W}\begin{bmatrix} m_1 & m'_4 \\ m_2 & m_3 \end{bmatrix}\!\!\left|\, z^{-1} \right] = \delta_{m_4, m'_4}\ , \tag{2.4}$$

- *crossing symmetry relation*

$$\mathbf{W}\begin{bmatrix} m_1 & m_2 \\ m_4 & m_3 \end{bmatrix}\!\!\left|\, x^2\zeta^{-1} \right] = \sqrt{\frac{[m_2][m_4]}{[m_1][m_3]}}\ \mathbf{W}\begin{bmatrix} m_4 & m_1 \\ m_3 & m_2 \end{bmatrix}\!\!\left|\, \zeta \right]\ , \tag{2.5}$$

- *"initial" condition*

$$\mathbf{W}\left[\begin{matrix} k_1 & k_2 \\ k_4 & k_3 \end{matrix}\Big| \zeta=1\right] = \delta_{k_2 k_4} \ . \tag{2.6}$$

From this point on we assume the normalization (2.2) holds and at the thermodynamic limit CTMs are well defined linear operators acting in the space covered by vectors $(k, m_2, m_3, ...l_1, l_2, l_1, ...)$. We denote such spaces as $\mathcal{L}_{l,k}$:

$$\mathbf{A}(\zeta): \ \mathcal{L}_{l,k} \to \mathcal{L}_{l,k} \ . \tag{2.7}$$

2.2. Spectrum of CTM in ABF model

At the thermodynamic limit the Corner Transfer Matrix for solvable statistical models has the remarkably simple form [7],[2]

$$\mathbf{A}(\zeta) = \zeta^{\mathbf{H}_C} \ , \tag{2.8}$$

where the Corner Hamiltonian $\mathbf{H}_C$ is a ζ-independent operator. Notice, that this is a common property, since it follows from YBE (1.9) and the crossing symmetry condition for the Boltzmann weights. The relation (2.5) allows one to express the CTMs $\mathbf{B}, \mathbf{C}, \mathbf{D}$ in terms of $\mathbf{A}$:

$$\mathbf{B}(\zeta) = \mathbf{S}_{l,k} \ \mathbf{A}(x^2\zeta^{-1}) \ , \quad \mathbf{C}(\zeta) = \mathbf{A}(\zeta) \ , \quad \mathbf{D}(\zeta) = \mathbf{S}_{l,k} \ \mathbf{A}(x^2\zeta^{-1}) \ . \tag{2.9}$$

The matrix $\mathbf{S}_{l,k}$ depends on the value of the central height k and the boundary condition only:

$$\mathbf{S}_{l,k} = \sqrt{[k][\hat{l}]} \ \mathbf{I} \ . \tag{2.10}$$

Here $\mathbf{I}$ is the identity matrix. An explicit form of the function $[\hat{l}]$ is inessential for the problem under consideration, since it is cancelled out in the final expressions.

Using the formulas (2.8)-(2.10), we can rewrite (2.1) as:

$$P_k(l) = [k] \ Tr_{\mathcal{L}_{l,k}}\Big[\ x^{4\mathbf{H}_C} \ \Big] \Big/ \Big\{ \sum_{m=1}^{r-1} \ [m] \ Tr_{\mathcal{L}_{l,m}}\Big[\ x^{4\mathbf{H}_C} \ \Big] \Big\} \ . \tag{2.11}$$

Therefore, the calculation is reduced to finding the spectrum of the Corner Hamiltonian $\mathbf{H}_C$ in the space $\mathcal{L}_{l,k}$. At the thermodynamic limit the spectrum is notable for the following reasons; It is

$$\begin{aligned} &- \ bounded\ from\ \ below \ , \\ &- \ discrete \ , \\ &- \ equidistant \ . \end{aligned} \tag{2.12}$$

These features seem to be rather general ones for solvable lattice models [2], [3][3]. In the ABF case, they follow from the fact that the Boltzmann weights are quasiperiodic functions under the changing $\zeta \to e^{2\pi i}\zeta$. Andrews *et al.* [4] have shown that the eigenvalues of the Corner Hamiltonian $\mathbf{H}_C$ in the regime III have the form:

$$\Delta_{l,k} - \frac{c}{24} + m \ , \quad m = 0, 1, 2, ... \ , \tag{2.13}$$

where

$$\Delta_{l,k} = \frac{\big(rl - (r-1)k\big)^2 - 1}{4r(r-1)} \ , \ c = 1 - \frac{6}{(r-1)r} \ . \tag{2.14}$$

[3] It is a starting point for so-called Angular Quantization approach for the integrable Quantum Field Theory [9]. In the scaling limit the Corner Hamiltonian coincides (up to a multiplicative normalization) with the Lorentz boost generator, acting in the Angular Quantization space.

The discreteness of the spectrum implies that it does not change discontinuously with the temperature parameter p. Hence, to find the multiplicities of eigenvalues (2.13) of the Corner Hamiltonian acting in the space $\mathcal{L}_{l,k}$ (2.13) it is enough to perform the computation for any value of p (1.12). They can be derived at low temperatures limit $p \to 1$ by combinatorial methods [4]. The amazing observation [10], [11], [12] is the spectrum of $\mathbf{H}_C$ *coincides* with one of the Virasoro algebra generator $L_0 - \frac{c}{24}$ acting in the irreducible representation with the highest weight $\Delta_{l,k}$ and the central charge c (2.14). This immediately leads to the formula for the one-point LHP [4]

$$P_k(l) = \bar{Z}_l^{-1}\ [k]\ \chi_{l,k}(x^4)\ , \tag{2.15}$$

where[4]

$$\bar{Z}_l = \sum_{m=1}^{r-1} [m]\ \chi_{l,m}(x^4)\ . \tag{2.16}$$

Here we denote the character of the irreducible module [13] by

$$\begin{aligned}\chi_{l,k}(q) = (q;q)_\infty^{-1}\ q^{-\frac{c}{24}}\ \Big\{ & q^{\Delta_{l,k}} E(-q^{rl-(r-1)k+(r-1)r}, q^{2(r-1)r}) - \\ & q^{\Delta_{-l,k}} E(-q^{-rl+(r-1)k+(r-1)r}, q^{2(r-1)r}) \Big\}\ .\end{aligned} \tag{2.17}$$

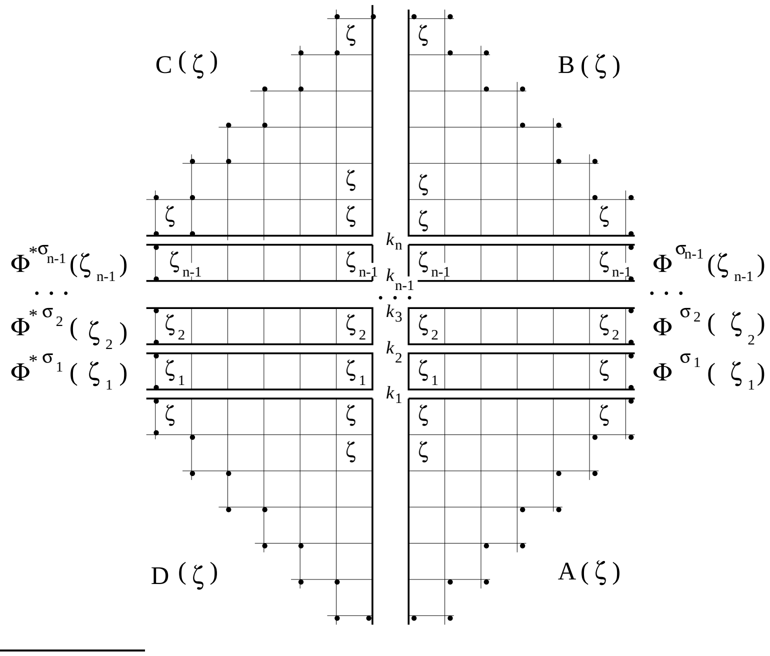
and use the notation for the elliptic function

$$E(z,q) = (q;q)_\infty (z;q)_\infty (qz^{-1};q)_\infty\ . \tag{2.18}$$

3. Vertex operators and multi-point Local Height Probabilities

3.1. Vertex operators

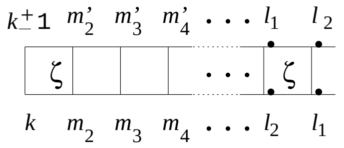
Now we begin to study the multi-point LHP. In this subsection we follow the ideas of the work [14]. Consider the inhomogeneous lattice. Our challenge is to find the probabilities that heights in n successive sites from the same vertical column take values $k_1, \dots k_n$. It is useful to divide the lattice into $2n+2$ parts, as it is shown in the picture:



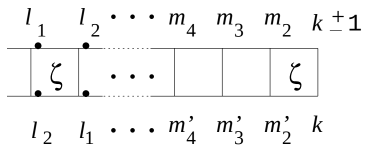
[4] Notice, that $Z_l = \hat{[l]}\ \bar{Z}_l$, where the function Z_l is defined by (1.6) and $\hat{[l]}$ is the same as in (2.10).

Again, at the first step of the calculation we specify the heights along the cuts and introduce the partition functions for the corresponding parts. The four of them are CTMs. Others are given by the products of the weights in the form of half infinite lanes:

$$[\,\Phi^{\pm}(\zeta)\,]_{m}^{m'} =$$



$$[\,\Phi^{*\pm}(\zeta)\,]_{m'}^{m} =$$



We treat such half infinite products with a given values of the spectral parameter ζ, as matrix elements of operators $\mathbf{\Phi}^{\pm}(\zeta)$, $\mathbf{\Phi}^{*\pm}(\zeta)$. The operators intertwine the spaces $\mathcal{L}_{l,k}$ and $\mathcal{L}_{l,k\pm1}$

$$\begin{aligned} &\mathbf{\Phi}^{\pm}(\zeta):\ \mathcal{L}_{l,k}\rightarrow\mathcal{L}_{l,k\pm1}\\ &\mathbf{\Phi}^{*\pm}(\zeta):\ \mathcal{L}_{l,k\pm1}\rightarrow\mathcal{L}_{l,k}\quad, \end{aligned} \tag{3.1}$$

and are referred as vertex operators (VO). The second step is to carry out the sum of heights over the cuts assuming that variables at the sites $1,...n$ equal to $k_1,...k_n$, respectively. As before, it is equivalent to taking a trace. In such a way, we obtain the expression for the multi-point LHP:

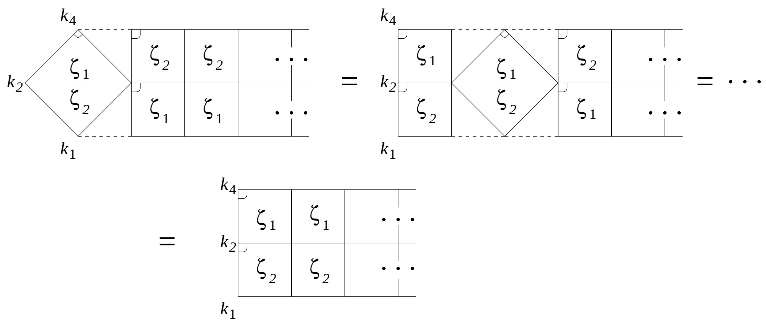
$$\begin{aligned} &P_{k_1,...k_n}(\zeta_1,...\zeta_{n-1}\ |l)=\left\{\sum_{m=1}^{r-1}\ Tr_{\mathcal{L}_{l,m}}\Big[\ \mathbf{D}(\zeta)\mathbf{C}(\zeta)\mathbf{B}(\zeta)\mathbf{A}(\zeta)\ \Big]\right\}^{-1}\times\\ &Tr_{\mathcal{L}_{l,k_1}}\Big[\ \mathbf{D}(\zeta)\mathbf{\Phi}^{*\sigma_1}(\zeta_1)..\mathbf{\Phi}^{*\sigma_{n-1}}(\zeta_{n-1})\mathbf{C}(\zeta)\mathbf{B}(\zeta)\mathbf{\Phi}^{\sigma_{n-1}}(\zeta_{n-1})..\mathbf{\Phi}^{\sigma_1}(\zeta_1)\mathbf{A}(\zeta)\ \Big]\ , \end{aligned} \tag{3.2}$$

where $\sigma_s=k_{s+1}-k_s$.

To proceed further, one need to study the properties of VO. They can be derived from the heuristic graphic arguments [14].

I. Commutation relations

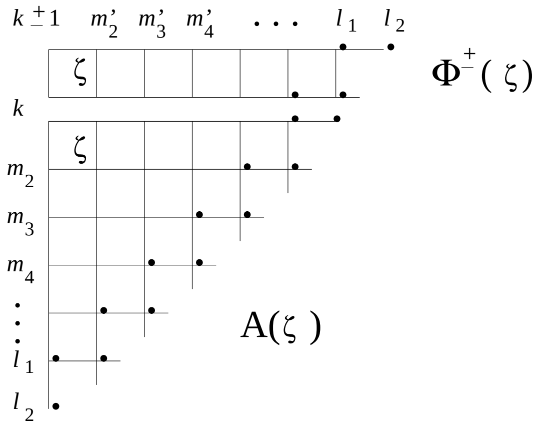
This property follows from the Yang-Baxter equation (1.9).



The face added in the first figure is pushed up to the infinity due to YBE. At the operator notations this implies that $\mathbf{\Phi}^{\pm}(\zeta)$ constitute the Zamolodchikov-Faddeev algebra in the Interaction Round a Face form:

$$\begin{aligned} &\mathbf{\Phi}^{k_4-k_2}(\zeta_1)\mathbf{\Phi}^{k_2-k_1}(\zeta_2)|_{\mathcal{L}_{l,k_1}}=\\ &\qquad=\sum_{k_3}\mathbf{W}\left[\begin{matrix}k_4 & k_3\\ k_2 & k_1\end{matrix}\,\middle|\,\zeta_1\zeta_2^{-1}\right]\ \mathbf{\Phi}^{k_4-k_3}(\zeta_2)\mathbf{\Phi}^{k_3-k_1}(\zeta_1)|_{\mathcal{L}_{l,k_1}}\ . \end{aligned} \tag{3.3}$$

II. Homogeneity condition
Consider the composition of CTM and VO:



At the thermodynamic limit this would be CTM $\mathbf{A}(\zeta)$ defined, however, in the space $\mathcal{L}_{l,k\pm1}$ rather than in $\mathcal{L}_{l,k}$. To obtain an equality, these spaces must be intertwined. It is provided by action of the operator $\mathbf{\Phi}^{\pm}(1)$ on the space $\mathcal{L}_{l,k}$, because of the "initial" condition (2.6). Therefore, we have

$$\mathbf{\Phi}^{\pm}(\zeta)\ \mathbf{A}(\zeta)|_{\mathcal{L}_{l,k}} = \mathbf{A}(\zeta)\ \mathbf{\Phi}^{\pm}(1)|_{\mathcal{L}_{l,k}}\ .$$

This leads to the equation:

$$z^{-\mathbf{H}_C}\mathbf{\Phi}^{\pm}(\zeta)\ z^{\mathbf{H}_C} = \mathbf{\Phi}^{\pm}(\zeta z^{-1})\ . \tag{3.4}$$

III. Normalization condition
To obtain the last property of VO we need to use the unitarity (2.4) and crossing symmetry (2.5) relations. They allow one to get so-called normalization condition:

$$\left\{\sqrt{[k+1]}\ \mathbf{\Phi}^{-}(x^{-2}\zeta)\mathbf{\Phi}^{+}(\zeta) + \sqrt{[k-1]}\ \mathbf{\Phi}^{+}(x^{-2}\zeta)\mathbf{\Phi}^{-}(\zeta)\right\}\Big|_{\mathcal{L}_{l,k}} = \sqrt{[k]}\ \mathbf{I}\ \Big|_{\mathcal{L}_{l,k}}\ . \tag{3.5}$$

Now, let us express CTMs $\mathbf{B},\mathbf{C},\mathbf{D}$ and VO $\mathbf{\Phi}^{*\pm}$ via $\mathbf{A}$ and $\mathbf{\Phi}^{\pm}$, respectively, by the formulas (2.9) and

$$\mathbf{\Phi}^{*\pm}(\zeta) = \mathbf{\Phi}^{\mp}(\zeta)\ . \tag{3.6}$$

The latter follows from the crossing symmetry condition (2.5). Taking (3.4) into account, it is easy to find that LHPs can be written in the form [14]:

$$\begin{aligned} P_{k_1,\ldots k_n}(\zeta_1,\ldots\zeta_{n-1}\ |l) = \bar{Z}_l^{-1}\ \sqrt{[k_1][k_n]}\times \\ Tr_{\mathcal{L}_{l,k_1}}\Big[\ x^{4\mathbf{H_C}}\ \mathbf{\Phi}^{-\sigma_1}(x^{-2}\zeta_1)\ldots\mathbf{\Phi}^{-\sigma_{n-1}}(x^{-2}\zeta_{n-1})\ \mathbf{\Phi}^{\sigma_{n-1}}(\zeta_{n-1})\ldots\mathbf{\Phi}^{\sigma_1}(\zeta_1)\ \Big]\ , \end{aligned} \tag{3.7}$$

where $\sigma_s = k_{s+1} - k_s$, and $\bar{Z}_l$ is given by (2.16).

3.2. Two-point LHP

The simplest example of LHP (3.7) with $n = 1$ was treated in the previous section. Now we would like to show that from the properties of VO and CTM we can deduce the two-point function. As well as the one-point function, the two-point LHP do not depend on the spectral parameter. Then, according to (3.7), the probabilities that heights at two adjacent sites 1 and 2 are equal to $k, k\pm1$, respectively, are given by:

$$P_{k,k\pm1}(l) = \bar{Z}_l^{-1}\ \sqrt{[k][k\pm1]}\ Tr_{\mathcal{L}_{l,k}}\Big[\ x^{4\mathbf{H_C}}\ \mathbf{\Phi}^{\mp}(x^{-2}\zeta)\ \mathbf{\Phi}^{\pm}(\zeta)\ \Big]\ . \tag{3.8}$$

It follows immediately from (3.5) that these functions obey the equation [5]

$$P_{k,k+1}(l) + P_{k,k-1}(l) = \bar{Z}_l^{-1}\ [k]\ \chi_{l,k}(x^4)\ . \tag{3.9}$$

In addition, the two-point LHP has to satisfy the evident requirements:

$$P_{k,k-1}(l) = P_{k-1,k}(l)\ , \tag{3.10}$$

$$P_{1,0}(l) = 0\ . \tag{3.11}$$

The formulae (3.9) and (3.10) specify the two-point functions uniquely:

$$P_{k+1,k}(l) = P_{k,k+1}(l) = \bar{Z}_l^{-1} \sum_{m=1}^{k} (-1)^{k-m}[m]\ \chi_{l,m}(x^4)\ . \tag{3.12}$$

This equation will lend support to the validity of our general result for the multi-point LHP .

4. Bosonization of Vertex Operator algebra

In this way, the multi-point LHP are expressed in terms of the traces of VO. From the mathematical point of view, the vertex operators and the Corner Hamiltonian constitute the algebra (3.3), (3.4), (3.5). It acts in the direct sum of the spaces $\mathcal{L}_{l,k}$. Unfortunately, our "definitions" were rather heuristic than rigorous ones. Due to Andrews *et al.* [4], we surely know that $\mathcal{L}_{l,k}$ has a structure of a $\mathbf{Z}$-graded space provided by the Corner Hamiltonian. The features (2.12) are characteristic ones for a spectrum of the grade operator in a Fock space. Our main idea is to construct a representation of (3.3), (3.4), (3.5) in a direct sum of Fock spaces. This representation is expected to be reducible. However, its restriction to an irreducible component would provide the proper multiplicities for the eigenvalues of $\mathbf{H}_C$. A similar bosonization procedure was applied for calculations of conformal blocks in the Conformal Field Theory [15] and lattice correlation functions for the XXZ Heisenberg spin chain [16], [17].

4.1. Bosonic representation of VO

It is convenient for us to perform the simple transformation of VO

$$\mathbf{\Psi}^{\pm}(\zeta)|_{\mathcal{L}_{l,k}} = i^{k-l}\ \sqrt{[k]}\ \mathbf{\Phi}^{\pm}(\zeta)|_{\mathcal{L}_{l,k}}\ . \tag{4.1}$$

The operators $\mathbf{\Psi}^{\pm}$ also constitute the associative quadratic algebra. It can be easily verified by substitution of (4.1) into (3.3). In particular, the commutation relation of $\mathbf{\Psi}^{+}(\zeta_1)$ and $\mathbf{\Psi}^{+}(\zeta_2)$ has the form:

$$\mathbf{\Psi}^{+}(\zeta_1)\mathbf{\Psi}^{+}(\zeta_2) = R(\zeta_1\zeta_2^{-1})\ \mathbf{\Psi}^{+}(\zeta_2)\mathbf{\Psi}^{+}(\zeta_1)\ , \tag{4.2}$$

where the function $R(\zeta)$ is given by (2.2). Our first task is to carry out the bosonization of $\mathbf{\Psi}^{+}(\zeta)$ satisfying this simple equation. To do it we follow the procedure developed in [9]; Let us introduce the operator valued function $\varphi(\zeta)$ obeying the commutation relation:

$$[\varphi(\zeta_1), \varphi(\zeta_2)] = -\ln R\big(\zeta_1\zeta_2^{-1}\big)\ . \tag{4.3}$$

The quasi-periodicity of $R(\zeta)$ under the substitution $\zeta \to e^{2\pi i}\zeta$ implies the following Laurent decomposition for $\varphi(\zeta)$:

$$\varphi(\zeta) = -\sqrt{\frac{r-1}{2r}}\ \big(\mathcal{Q} - i\mathcal{P}\ \ln\zeta\big) + i \sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}} \frac{\beta_m}{m}\ \zeta^{-m}\ . \tag{4.4}$$

[5] Note, that (3.9) obviously follows from the definition of the one-point function and (2.15).

In order to provide (4.3), we specify the non-vanishing commutation relations for the operators $\mathcal{P}, \mathcal{Q}, \beta_n$ as being

$$\begin{aligned} &[\beta_m, \beta_n] = m\ \frac{[m]_x[(r-1)m]_x}{[2m]_x[rm]_x}\ \delta_{m+n,0}\ , \\ &[\mathcal{P}, \mathcal{Q}] = -i\ , \end{aligned} \tag{4.5}$$

where the notation

$$[m]_x = \frac{x^m - x^{-m}}{x - x^{-1}}$$

is used [6]. With these definitions the operator $\varphi(\zeta)$ satisfies the commutation relations (4.3). It is easily checked, since the function $R(\zeta)$ (2.2) can be rewritten in the form:

$$R(\zeta) = \zeta^{\frac{r-1}{2r}}\ \exp\Big\{ \sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}} \frac{[m]_x[(r-1)m]_x}{m\ [2m]_x[rm]_x}\ \zeta^m\ \Big\}\ . \tag{4.6}$$

The Heisenberg algebra (4.5) is represented in Fock spaces. In the usual fashion, one defines the Fock space $\mathcal{F}_P$ as the module generated by action of the creation operators $\beta_{-n}, n > 0$ on the highest weight vector $\mathbf{v}_P$

$$\beta_n \mathbf{v}_P = 0\ ,\ n > 0\ ; \qquad \mathcal{P}\ \mathbf{v}_P = P\ \mathbf{v}_P\ . \tag{4.7}$$

Namely, $\mathcal{F}_P$ is covered by vectors $\beta_{-n_1}\beta_{-n_2}\dots\beta_{-n_j}\mathbf{v}_P$, $n_m > 0$. It is endowed with the structure of a $\mathbf{Z}$-graded module by the operator:

$$\mathbf{H}_C = \sum_{m>0} \frac{[2m]_x[rm]_x}{[m]_x[(r-1)m]_x}\ \beta_{-m}\beta_m + \frac{\mathcal{P}^2}{2} - \frac{1}{24}\ . \tag{4.8}$$

Its eigenvalues in $\mathcal{F}_P$ have the form (2.13) and we would like to identify (4.8) with the Corner Hamiltonian. As it follows from (2.13), only the Fock spaces

$$\mathcal{F}_{l,k} \equiv \mathcal{F}_{\frac{rl-(r-1)k}{\sqrt{2(r-1)r}}}\ , \tag{4.9}$$

with l and k being integers, are relevant in our construction.

Now, let us introduce the following bosonic representation:

$$\mathbf{\Psi}^+(\zeta) = e^{i\varphi(\zeta)}\ :\ \mathcal{F}_{l,k} \to \mathcal{F}_{l,k+1}\ . \tag{4.10}$$

Then the relation (4.2) is satisfied immediately. In order to describe the bosonic representation for the vertex operator $\mathbf{\Psi}^-(\zeta)$, we need the notations:

$$U(\zeta) = e^{i\varphi(\zeta)}\ , \qquad \bar{U}(z) = e^{-i(\varphi(zx)+\varphi(zx^{-1}))}\ . \tag{4.11}$$

Define also the operator $F(z,\mathcal{P})$ depending on the "zero" mode $\mathcal{P}$

$$F(z,\mathcal{P}) = z^{-\omega-\frac{r-1}{r}} x^{r\omega(\omega+1)-\omega}\ \frac{E\big(x^{1-2r\omega}z, x^{2r}\big)}{E\big(xz^{-1}, x^{2r}\big)}\ , \tag{4.12}$$

where $\omega = \sqrt{\frac{2r-2}{r}}\,\mathcal{P}$ and $E(z,q)$ is given by (2.18). We propose the following bosonization prescription:

$$\mathbf{\Psi}^-(\zeta) = \eta^{-1}\oint_C \frac{dz}{2\pi i z}\ U(\zeta)\bar{U}(z)\ F\big(z\zeta^{-1},\mathcal{P}\big) = \eta^{-1}\oint_C \frac{dz}{2\pi i z}\ F\big(\zeta z^{-1}, -\mathcal{P}\big)\ \bar{U}(z)U(\zeta)\ . \tag{4.13}$$

[6] It should not be confused with the symbol $[u]$ (1.11).

Here the anti-clockwise contour $\mathcal{C}$ encloses the poles $z = \zeta x^{1+2rm}$ $(m = 0, 1, 2, ..)$ of the integrand and η is some constant, which we are going to specify later. It is significant that the integrand in (4.13) is a single-valued function of z on the complex plane, so the integration contour is closed. Therefore, the action of the operator (4.13) is well defined for an arbitrary Fock space $\mathcal{F}_P$. We claim that;

The operators (4.10), (4.13)

$$\mathbf{\Psi}^{\pm}(\zeta):\ \mathcal{F}_{l,k} \to \mathcal{F}_{l,k\pm1}\ , \tag{4.14}$$

satisfy the commutation relations

$$\mathbf{\Psi}^{a}(\zeta_1)\mathbf{\Psi}^{b}(\zeta_2)|_{\mathcal{F}_{l,k}} = \sum_{c+d=a+b} \mathbf{U}\left[\begin{matrix} k+a+b & k+c \\ k+b & k \end{matrix}\middle|\zeta_1\zeta_2^{-1}\right]\ \mathbf{\Psi}^{d}(\zeta_2)\mathbf{\Psi}^{c}(\zeta_1)|_{\mathcal{F}_{l,k}}\ . \tag{4.15}$$

The coefficients $\mathbf{U}$ (4.15) are connected with the Boltzmann weights (1.10) via the simple transformation

$$\mathbf{U}\begin{bmatrix} m_1 & m_2 \\ m_4 & m_3 \end{bmatrix} = i^{m_4-m_2}\sqrt{\frac{[m_4]}{[m_2]}}\ \mathbf{W}\begin{bmatrix} m_1 & m_2 \\ m_4 & m_3 \end{bmatrix}\ . \tag{4.16}$$

It is apparent that the commutation relations (4.15) are equivalent to (3.3) if VO $\mathbf{\Psi}^{\pm}$ and $\mathbf{\Phi}^{\pm}$ are related as in (4.1). The explicit form of the matrix (4.16) is presented in Appendix A. We give the flavour of the proof (4.15) in Appendix B. It is easy to see that the operators (4.10), (4.13) obey the proper commutation relation (3.4) with the Corner Hamiltonian (4.8). One can also check that they satisfy the normalization condition:

$$\left\{\ \mathbf{\Psi}^{-}(x^{-2}\zeta)\mathbf{\Psi}^{+}(\zeta) - \mathbf{\Psi}^{+}(x^{-2}\zeta)\mathbf{\Psi}^{-}(\zeta)\right\}\Big|_{\mathcal{F}_{l,k}} = A\ (-1)^{k-l}\ [k]\ \mathbf{I}\ \Big|_{\mathcal{F}_{l,k}}\ , \tag{4.17}$$

where the numerical constant A is equal to

$$A = \eta^{-1}\ \sqrt{1-z}|_{z\to1}\ x^{\frac{1-r}{2r}}\ \frac{\sqrt{(x^2, x^{2r})_\infty (x^{2r-2}, x^{2r})_\infty}}{(x^{2r}, x^{2r})_\infty}\ . \tag{4.18}$$

The occurrence of the multiple $\sqrt{1-z}|_{z\to1}$ is connected with a fact that we do not use the normal ordered forms of the exponents (4.11). The normal ordering of $U(\zeta)$ produces only an additional multiple factor which is a non zero and finite constant for $0 < x < 1$. At the same time, the normal ordering of $\bar{U}(z)$ gives rise to the factor $\sqrt{1-z}|_{z\to1}$. It can be cancelled out by a suitable choice of η. We specify this constant in such a way that

$$A = i\ ,$$

and (4.17) would be equivalent to the normalization condition (3.5).

4.2. Felder complex

Now we should concentrate our attention on the description of $\mathcal{L}_{l,k}$ in terms of the bosonic Fock spaces. Recall, the Corner Hamiltonian was identified with the grade operator (4.8). Although the eigenvalues of (4.8) coincide with ones for the Corner Hamiltonian, their multiplicities are different. Hence, the spaces $\mathcal{F}_{l,k}$ and $\mathcal{L}_{l,k}$ cannot be identified directly. More sophisticated treatment shows that the bosonic representation of the VO algebra is reducible. To construct the proper space of states one has to extract an irreducible component by throwing out some states from the Fock spaces. Explicitly, the procedure reads as follows; Introduce the notation

$$\bar{V}(z) = e^{-i(\phi(zx)+\phi(zx^{-1}))} \tag{4.19}$$

with $\phi(\zeta)$ given by

$$\phi(\zeta) = \sqrt{\frac{r}{2(r-1)}}\ \left(\mathcal{Q} - i\mathcal{P}\ \ln\zeta\right) - i\sum_{\substack{m\in\mathbf{Z}\\ m\neq0}} \frac{\alpha_m}{m}\ \zeta^{-m} \tag{4.20}$$

and

$$[(r-1)m]_x\ \alpha_m = [rm]_x\ \beta_m\ . \tag{4.21}$$

A central place in the analysis belongs to the operator

$$X = \tilde{\eta}^{-1} \oint \frac{dz}{2\pi i z}\ \bar{V}(z)\ \tilde{F}(z,\mathcal{P})\ , \tag{4.22}$$

where

$$\tilde{F}(z,\mathcal{P}) = z^{-\tilde{\omega}-\frac{r}{r-1}}\ x^{(r-1)\tilde{\omega}(\tilde{\omega}+1)+\tilde{\omega}}\ \frac{E\big(x^{-1-2(r-1)\tilde{\omega}}z, x^{2r-2}\big)}{E\big(x^{-1}z^{-1}, x^{2r-2}\big)}\ ,$$

with $\tilde{\omega} = -\sqrt{\frac{2r}{r-1}}\mathcal{P}$ and the constant $\tilde{\eta}$ provides the regularization of $\bar{V}(z)$. The operator (4.22) defines the following maps:

$$\begin{aligned} X_{2j} = X^{l}\ :\ \mathcal{F}_{l-2j(r-1),k} \to \mathcal{F}_{-l-2j(r-1),k}\ , \\ X_{2j+1} = X^{r-1-l}\ :\ \mathcal{F}_{-l-2j(r-1),k} \to \mathcal{F}_{l-2(j+1)(r-1),k} \end{aligned} \tag{4.23}$$

for $j \in \mathbf{Z},\ 1 \le l \le r-2,\ 1 \le k \le r-1$. As a result, we can construct the infinite chain:

$$\ldots \xrightarrow{X_{-2}} \mathcal{F}_{2r-2-l,k} \xrightarrow{X_{-1}} \mathcal{F}_{l,k} \xrightarrow{X_{0}} \mathcal{F}_{-l,k} \xrightarrow{X_{1}} \ldots\ . \tag{4.24}$$

We claim that it is the Felder resolution [18]. In other words,

I. The chain of maps (4.24) is a BRST complex for any $0 < x < 1$ *, i.e.*

$$X_j X_{j-1} = 0\ ; \tag{4.25}$$

II. The cohomologies of the complex turn out to be non-trivial only for $j = 0$

$$Ker\ X_j \Big/ Im\ X_{j-1} = 0\ ,\ \text{if } j \neq 0\ . \tag{4.26}$$

The first statement can be checked by the direct calculations [19]. The arguments in favour of (4.26) are based on the fact that the operator (4.22) commute with $\mathbf{H}_C$ (4.8). So, the cohomology spaces are $\mathbf{Z}$-graded, as well as $\mathcal{F}_P$. Let us consider the finite-dimensional eigensubspaces of the operator $\mathbf{H}_C$ in $Ker\ X_j \Big/ Im\ X_{j-1}$. The restriction of (4.24) on these subspaces provides BRST complexes involving only a finite number of the maps. Dimensions of the cohomology spaces for such complexes are integers and would not depend on the continuous parameter x. Therefore, it is enough to find them for an arbitrary $0 < x < 1$. The statement (4.26) surely holds at the limit $x \to 1$, where our construction turns out to be the Feigin-Fuks-Dotsenko-Fateev bosonization [20], [15]. In addition, we conclude that the spectrum of $\mathbf{H}_C$ in the space

$$\mathcal{L}_{l,k} = Ker\ X_0 \Big/ Im\ X_{-1} = Ker_{\mathcal{F}_{l,k}}\ X^{l}\ \Big/ Im_{\mathcal{F}_{2r-2-l,k}}\ X^{r-1-l} \tag{4.27}$$

coincides with the spectrum of the grade operator $L_0 - \frac{c}{24}$ in the corresponding irreducible representation of the Virasoro algebra.

To complete the identification of the factor-space (4.27) with $\mathcal{L}_{l,k}$ one notes that VO satisfy the commutation relations:

$$\mathbf{\Psi}^{\pm}(\zeta)\ X_j = X_j\ \mathbf{\Psi}^{\pm}(\zeta)\ . \tag{4.28}$$

So they are BRST invariant operators.

The BRST properties and the structure of the complex (4.24) lead to the following computation procedure of traces over $\mathcal{L}_{l,k}$ in terms of traces over the Fock spaces [18]. Introduce the notation:

$$\mathcal{O}_n = \mathbf{\Psi}^{-\sigma_1}(x^{-2}\zeta_1)\ldots\mathbf{\Psi}^{-\sigma_{n-1}}(x^{-2}\zeta_{n-1})\ \mathbf{\Psi}^{\sigma_{n-1}}(\zeta_{n-1})\ldots\mathbf{\Psi}^{\sigma_1}(\zeta_1)\ . \tag{4.29}$$

Then, due to the commutation relations (4.28) the following diagram is a commutative one:

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{X_{-2}} & \mathcal{F}_{-l+2r-2,k} & \xrightarrow{X_{-1}} & \mathcal{F}_{l,k} & \xrightarrow{X_{0}} & \mathcal{F}_{-l,k} & \xrightarrow{X_{1}} & \ldots \\ & & \Big\downarrow x^{4\mathbf{H}_C}\mathcal{O}_n & & \Big\downarrow x^{4\mathbf{H}_C}\mathcal{O}_n & & \Big\downarrow x^{4\mathbf{H}_C}\mathcal{O}_n & & \\ \ldots & \xrightarrow[X_{-2}]{} & \mathcal{F}_{-l+2r-2,k} & \xrightarrow[X_{-1}]{} & \mathcal{F}_{l,k} & \xrightarrow[X_{0}]{} & \mathcal{F}_{-l,k} & \xrightarrow[X_{1}]{} & \ldots \end{array}\ . \tag{4.30}$$

The complex is exact excluding the $j=0$ term, hence the trace of the operator $x^{4\mathbf{H_C}}\mathcal{O}_n$ over the space (4.27) equals to the following alternating sum:

$$Tr_{\mathcal{L}_{l,k}}\Big[\,x^{4\mathbf{H_C}}\mathcal{O}_n\,\Big] = Tr_{Ker_{\mathcal{F}_{l,k}}\;X^l\big/Im_{\mathcal{F}_{2r-2-l,k}}\;X^{r-1-l}}\Big[\,x^{4\mathbf{H_C}}\mathcal{O}_n\,\Big] =$$
$$\sum_{j=-\infty}^{\infty} Tr_{\mathcal{F}_{l-2j(r-1),k}}\Big[\,x^{4\mathbf{H_C}}\mathcal{O}_n\,\Big] - \sum_{j=-\infty}^{\infty} Tr_{\mathcal{F}_{-l-2j(r-1),k}}\Big[\,x^{4\mathbf{H_C}}\mathcal{O}_n\,\Big]\ . \tag{4.31}$$

In the simplest case $\mathcal{O}_1=1$ the formula (4.31) gives

$$Tr_{\mathcal{L}_{l,k}}\Big[\,x^{4\mathbf{H_C}}\,\Big] = \sum_{j=-\infty}^{\infty} Tr_{\mathcal{F}_{l-2j(r-1),k}}\Big[\,x^{4\mathbf{H_C}}\,\Big] - \sum_{j=-\infty}^{\infty} Tr_{\mathcal{F}_{-l-2j(r-1),k}}\Big[\,x^{4\mathbf{H_C}}\,\Big] =$$
$$x^{\frac{1}{6}}\,(x^4;x^4)^{-1}_{\infty}\,\Big\{\,\sum_{j=-\infty}^{\infty} x^{\frac{(rl-(r-1)k+2(r-1)rj)^2}{(r-1)r}} - \sum_{j=-\infty}^{\infty} x^{\frac{(rl+(r-1)k+2(r-1)rj)^2}{(r-1)r}}\,\Big\}\ . \tag{4.32}$$

It coincides with the character $\chi_{l,k}(x^4)$ (2.17), due to the Jacobi identity:

$$E(z,q) = \sum_{m=-\infty}^{+\infty} (-1)^m\, q^{\frac{(m-1)m}{2}}\, z^m\ .$$

5. Integral representation for multi-point LHP

5.1. Calculation of traces

In this section we work out an integral representation for LHP by using the results of the bosonization procedure. In terms of VO $\mathbf{\Psi}^{\pm}(\zeta)$ the equation (3.7) can be rewritten as

$$P_{k_1,\dots k_n}(\zeta_1,\dots\zeta_{n-1}|l) = i^{2l-k_1-k_n}\,\bar{Z}_l^{-1}\,\prod_{s=2}^{n-1}[k_s]^{-1}(-1)^{l-k_s}\times$$
$$Tr_{\mathcal{L}_{l,k_1}}\Big[\,x^{4\mathbf{H}_C}\,\mathbf{\Psi}^{-\sigma_1}(x^{-2}\zeta_1)\dots\mathbf{\Psi}^{-\sigma_{n-1}}(x^{-2}\zeta_{n-1})\mathbf{\Psi}^{\sigma_{n-1}}(\zeta_{n-1})\dots\mathbf{\Psi}^{\sigma_1}(\zeta_1)\,\Big]\ . \tag{5.1}$$

It is obvious that in the calculation of (5.1) we can perform the traces over the zero and oscillator modes separately. The zero mode contribution is governed by the structure of the complex (4.30) and it was obtained in [18]. The trace over the oscillator part is carried out by using the Clavelli-Shapiro technique [21], [17]. More explicitly, the prescription implies the introducing together with β_n the oscillators

$$\beta^*_m:\quad [\beta_m,\beta^*_n]=0\ ,$$

satisfying the same commutation relations as (4.5). Define the following operators acting in the tensor product $\mathcal{F}[\beta]\otimes\mathcal{F}[\beta^*]$ of the Fock spaces:

$$b_m = (1-x^{4m})^{-1}\,\beta_m\otimes 1 + 1\otimes\beta^*_{-m}\ ,\ m>0\ ;$$
$$b_m = \beta_m\otimes 1 + (x^{4m}-1)^{-1}\,1\otimes\beta^*_{-m}\ ,\ m<0\ . \tag{5.2}$$

Notice, that we omit the index P since the result does not depend on the zero mode eigenvalue in this case. For any operator $\mathcal{O}[\beta]$ acting in $\mathcal{F}[\beta]$ one can construct the operator

$$\mathcal{O}[b]:\ \mathcal{F}[\beta]\otimes\mathcal{F}[\beta^*]\to\ \mathcal{F}[\beta]\otimes\mathcal{F}[\beta^*]$$

by the replacement of the modes β_m with b_m (5.2). Then the "oscillator" trace over the Fock space is expressed in terms of the vacuum expectation value $<<\mathcal{O}[b]>>\equiv<0|\mathcal{O}[b]|0>$ of the operator $\mathcal{O}[b]$ with respect to the vacuum $|0>=\mathbf{v}\otimes\mathbf{v}$. Namely,

$$Tr_{osc}\Big[\, x^{4\mathbf{H}_C}\,\mathcal{O}[\beta]\,\Big]=(x^4;x^4)^{-1}_{\infty}\ <<\mathcal{O}[b]>>\ . \tag{5.3}$$

Due to the Wick theorem, the expectation value of a product of the exponential operators (4.11) is factorized into the two point functions [7]:

$$\begin{aligned}&<<U(\zeta_2)U(\zeta_1)>>=\mathcal{R}^2\ G'(\zeta_1\zeta_2^{-1})\\&<<\bar U(\zeta_2)U(\zeta_1)>>=\eta\ \mathcal{R}\ \bar{\mathcal{R}}\ W'(\zeta_1\zeta_2^{-1})\\&<<\bar U(\zeta_2)\bar U(\zeta_1)>>=\eta^2\ \bar{\mathcal{R}}^2\ \bar G'(\zeta_1\zeta_2^{-1})\ .\end{aligned}\tag{5.4}$$

The calculations of the functions and constants in (5.4) are straightforward, and their explicit forms are written down in Appendix C.

5.2. Integral representation

We begin with the following trace:

$$\hat P_{k,k+1}(\zeta_1\zeta_2^{-1}|l)=i\ (-1)^{l-k-1}\ \bar Z_l^{-1}\ Tr_{\mathcal{L}_{l,k}}\Big[\, x^{4\mathbf{H}_C}\ \mathbf{\Psi}^-(\zeta_2)\mathbf{\Psi}^+(\zeta_1)\,\Big]\ . \tag{5.5}$$

According to our analysis, (5.5) admits the bosonic representation (4.31) with

$$\mathcal{O}_2=\eta^{-1}\oint_{|z|=|\zeta_2|}\frac{dz}{2\pi i z}\ F(\zeta_2z^{-1},-\mathcal{P})\ \bar U(z)U(\zeta_2)U(\zeta_1)\ . \tag{5.6}$$

First, let us derive the zero modes contribution into the integrand. For the $j=0$ term in the sum (4.31) one obtains:

$$\begin{aligned}&x^{2\mathcal{P}^2-\frac16}\ F(\zeta_2z^{-1},-\mathcal{P})\ e^{2i\sqrt{\frac{r-1}{r}}(Q-i\mathcal{P}\ln z)}\ e^{-i\sqrt{\frac{r-1}{r}}(Q-i\mathcal{P}\ln\zeta_2)}\ e^{-i\sqrt{\frac{r-1}{r}}(Q-i\mathcal{P}\ln\zeta_1)}\Big|_{\mathcal{F}_{l,k}}=\\&=x^{\frac{r\omega^2}{r-1}-\frac16}\ \big(\zeta_1\zeta_2{}^{-1}\big)^{\frac{r-1}{4r}-\frac{\omega}{2}}\ \big(z\zeta_2^{-1}\big)^{\omega-\frac{r-1}{r}}\ F(\zeta_2z^{-1},-P)\ ,\end{aligned}\tag{5.7}$$

where $\omega=\sqrt{\frac{2r-2}{r}}\ P=l-\frac{r-1}{r}k$. All other terms can be found by the substitution $l\to\pm l-2j(r-1)\ ,\ j\in\mathbf{Z}$ into (5.7). Then, performing the sum over j we get the zero mode contribution:

$$(x^4;x^4)_\infty\ \big(\zeta_1\zeta_2^{-1}\big)^{\frac{r-1}{4r}-\frac{\omega}{2}}\ O_{l,k}\big(\zeta_1\zeta_2z^{-2}\big)\ (z\zeta_2^{-1})^{\omega-\frac{r-1}{r}}\ F(\zeta_2z^{-1},-P)\ ,$$

where

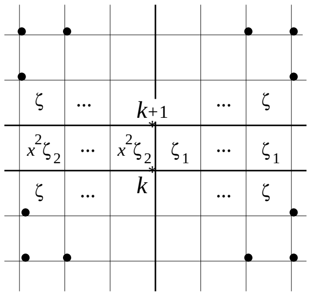
$$\begin{aligned}O_{l,k}(\zeta)=(x^4;x^4)^{-1}_\infty\ x^{-\frac16}\ \Big\{&x^{\frac{(rl-(r-1)k)^2}{(r-1)r}}\ E(-\zeta^{1-r}x^{4(rl-(r-1)k+(r-1)r)},x^{8(r-1)r})-\\&\zeta^l\ x^{\frac{(rl+(r-1)k)^2}{(r-1)r}}\ E(-\zeta^{1-r}x^{-4(rl+(r-1)k-(r-1)r)},x^{8(r-1)r})\Big\}\ .\end{aligned}\tag{5.8}$$

The trace over the oscillator modes is provided by the Wick theorem and (5.4). As a result, we derive the integral representation for (5.5):

$$\begin{aligned}&\hat P_{k,k+1}(\zeta_1\zeta_2^{-1};l)=\bar Z_l^{-1}\ i\ (-1)^{l-k-1}\ \mathcal{R}^2\ \bar{\mathcal{R}}\ G'(\zeta_1\zeta_2^{-1})\ \big(\zeta_1\zeta_2^{-1}\big)^{\frac{r-1}{4r}-\frac{\omega}{2}}\times\\&\oint_{|z|=|\zeta_2|}\frac{dz}{2\pi i z}\ O_{l,k}(\zeta_1\zeta_2z^{-2})\ W'(\zeta_1z^{-1})W'(\zeta_2z^{-1})\ (z\zeta_2^{-1})^{\omega-\frac{r-1}{r}}\ F(\zeta_2z^{-1},-P)\ .\end{aligned}\tag{5.9}$$

[7] These functions were calculated in the work [9] (see also [22]), and we preserved here the same notations. The prime should not be confused with the symbol of the derivative.

The explicit form of the functions and constants appearing here is rather complicated (see Appendix C). The function (5.9) represents LHP on the lattice with more general dislocation, than one considered before:



In the case $\zeta_1 = x^2\zeta_2$ the expression (5.9) is simplified drastically. Indeed, using the obvious relation

$$W'(\zeta x)W'(\zeta x^{-1}) = \bar{G}'^{-1}(\zeta) \ ,$$

it is possible to show that the two-point LHP (3.8) are given by:

$$\begin{aligned} P_{k,k\pm1}(l) = (-1)^{l-k} \ \bar{Z}_l^{-1} \ \frac{(x^2,x^2)^3_\infty}{E(x^{2r-2},x^{2r})} \ x^{r\omega(\omega-1)+\frac{r-1}{r}} \times \\ \oint_{|z|=x^{\mp 2}} \frac{dz}{2\pi i z} \ O_{l,k}(z^{-2}) \ \frac{E(z^{-1}x^{2rl-2(r-1)k},x^{2r})}{E(zx^2,x^2)} \ . \end{aligned} \tag{5.10}$$

In Appendix D, we prove that these functions coincide with (3.12).

In a similar manner, we have obtained the integral representation for the multi-point Local Height Probabilities:

$$\begin{aligned} & P_{k_1,\dots k_n}(\zeta_1,\dots\zeta_{n-1}|l) = (-1)^{\frac{k_n-k_1+n-1}{2}} \ [k_1] \ \bar{Z}_l^{-1} \times \\ & \prod_{s=1}^{n-1} B^{-1}_{l,k_s} \prod_{s<j} B^{-1}(\zeta_s\zeta_j^{-1}) \prod_{s=1}^{n-1} \Big\{ \oint_{\mathcal{C}_s} \frac{dz_s}{2\pi i z_s} \Big\} \ O_{l,k_1}\big(\prod_{s=1}^{n-1} \zeta_s^2 z_s^{-2} \big) \times \\ & \prod_{s<j} B^{-1}(z_s z_j^{-1} x^{2k_{s+1}-2k_s}) \prod_{j<s} B(\zeta_j z_s^{-1}) \prod_{j>s} B(\zeta_j z_s^{-1} x^{2k_s-2k_{s+1}}) \prod_{s=1}^{n-1} B_{l,k_s}(z_s\zeta_s^{-1}) \ , \end{aligned} \tag{5.11}$$

where

$$B(z) = \frac{E(z,x^{2r})}{E(z,x^2)} \ ,$$

$$B_{l,k}(z) = \frac{E(zx^{-2rl+2(r-1)k},x^{2r})}{E(z,x^2)} \ ,$$

$$B_{l,k} = \frac{E(x^{-2rl+2(r-1)k},x^{2r})}{(x^2;x^2)^3} \ ,$$

and the functions $[k]$, $\bar{Z}_l$, $E(z,q)$, $O_{l,k}(\zeta)$ are given by (1.11), (2.16), (2.18), (5.8), respectively. The integration contour $\mathcal{C}_s$, $(s = 1,\dots n-1)$ in (5.11) goes anti-clockwise and encloses the following poles:

$$z_s = \begin{cases} \zeta_j x^{2m+1-k_{s+1}+k_s} \ , \\ z_j x^{r(2m+1)+(r-2)(k_j-k_{j+1})} \ \ (j<s) \ , \quad z_j x^{r(2m+1)+(r-2)(k_{s+1}-k_s)} \ \ (j>s) \end{cases} ,$$

where $m = 0,1,2,\dots$.

6. Conclusion

In summary, we would like to make some comments on an algebraic interpretation of the presented results. Our construction leads to a natural conjecture that the space $\mathcal{L}_{l,k}$ can be treated as the irreducible representation of some infinite-dimensional algebra. Since at the limit $x \to 1$ the bosonization turns out to be the Feigin-Fuks-Dotsenko-Fateev procedure [20], [15], then we have called the algebra as the deformed Virasoro one $Vir_{c,x}$ in the work [23]. The explicit basis for $Vir_{c,x}$ was found in the works [24], [25][8]. It can be described in terms of the oscillators λ_m :

$$[(r-1)m]_x \ \alpha_m = [rm]_x \ \beta_m = (x - x^{-1})^{-1} \ m \ \lambda_m \ . \tag{6.1}$$

If one introduces the field

$$\Lambda(z) = x^{\sqrt{2r(r-1)}\mathcal{P}} \ : \exp\left(- \sum_{m \neq 0} \lambda_m z^{-m} \right) : \ , \tag{6.2}$$

then the generating function for the elements of $Vir_{c,x}$ has the form [26], [24], [25]:

$$T(z) = \Lambda(zx^{-1}) + \Lambda^{-1}(zx) \ . \tag{6.3}$$

This field commutes with the operators (4.23) and obeys the relation

$$\begin{aligned} &f(\zeta z^{-1}) \ T(z)T(\zeta) - f(z\zeta^{-1}) \ T(\zeta)T(z) = \\ &\qquad 2\pi \ (x - x^{-1}) \ [r-1]_x \ [r]_x \ \left(\ \delta(\zeta z^{-1} x^{-2}) - \delta(\zeta z^{-1} x^{2}) \ \right) \ , \end{aligned} \tag{6.4}$$

where we denote $\delta(z) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} z^m$ and

$$f(z) = (1-z)^{-1} \ \frac{(zx^{2r}; x^4)_\infty \ (zx^{2-2r}; x^4)_\infty}{(zx^{2r+2}; x^4)_\infty \ (zx^{4-2r}; x^4)_\infty} \ .$$

The limiting $x \to 1$ behavior of (6.3) is governed by the expansion:

$$T(z) = 2 + r(r-1) \ (x - x^{-1})^2 \left(\sum_{m=-\infty}^{+\infty} L_m z^{-m} - \frac{c}{24} \right) + O((x - x^{-1})^4) \ , \tag{6.5}$$

and (6.4) gives us the Virasoro algebra commutation relations for L_m with the central charge c (2.6) .

Remind, that the operators $\mathbf{\Psi}^{\pm}$ (4.10) , (4.13) intertwine the irreducible representations $\mathcal{L}_{l,k}$ with the same first index l. In the terminology of the work [16], they are the first kind VO. It would be rather natural to introduce the second kind VO, which intertwine the spaces $\mathcal{L}_{l,k}$ and $\mathcal{L}_{l\pm1,k}$. Their bosonization is easily performed in terms of the oscillators α_m (4.21). Then the "Corner" space of states [9]

$$\pi_Z \equiv \sum_{1 \leq l < k \leq r-1} \oplus \mathcal{L}_{l,k} \tag{6.6}$$

of the ABF model at the regime III can be treated as the irreducible representation of the complete algebra of both kinds VO. Notice, that we restrict the sum (6.6) by $1 \leq l < k \leq r-1$, since

$$\mathcal{L}_{l,k} \simeq \mathcal{L}_{r-1-l,r-k} \ .$$

The integrability of the ABF model implies an existence of an infinite dimensional Abelian symmetry. The deformed Virasoro algebra provides another type of the symmetry which is usually referred as the dynamical one [16] (see also [27]). The generators of such symmetry do not mutually commute, but create

[8] J. Shiraishi *et al.* [24] use the notations $p = x^{-2}$, $q = x^{-2r}$, $t = qp^{-1} = x^{2-2r}$.

[9] The lattice analogue of the Angular Quantization space [9].

the space of states of the model. Other well known examples of the dynamical symmetries were produced by the Conformal Field Theory [28]. From the algebraic point of view the symmetry corresponding to the deformed Virasoro algebra is very similar to the conformal invariance, although the simple geometrical meaning of the latter seems to be lost.

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7. Appendix A

For convenience, we collect in this Appendix two different forms of the matrix $\mathbf{U}$ (4.16). In terms of the parameters u and p, it reads:

$$\begin{aligned}
&\mathbf{U}\begin{bmatrix} m\pm 2 & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\ ,\\
&\mathbf{U}\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\,\frac{[m\pm u]}{[1-u][m]}\ ,\\
&\mathbf{U}\begin{bmatrix} m & m\pm 1 \\ m\mp 1 & m \end{bmatrix} = -R\,\frac{[u][m\mp 1]}{[1-u][m]}\ .
\end{aligned} \tag{7.1}$$

Via the variables ζ and x (1.14), the same functions are given by the formulas:

$$\begin{aligned}
&\mathbf{U}\begin{bmatrix} m\pm 2 & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\ ,\\
&\mathbf{U}\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\ \zeta^{\frac{(r-1)(\mp m-1)}{r}}\ \frac{E(x^{2r-2},x^{2r})\ E(x^{\mp 2(r-1)m}\zeta,x^{2r})}{E(x^{2r-2}\zeta,x^{2r})\ E(x^{\mp 2(r-1)m},x^{2r})}\ ,\\
&\mathbf{U}\begin{bmatrix} m\pm 2 & m\mp 1 \\ m\pm 1 & m \end{bmatrix} = -R\ x^{\mp\frac{2(r-1)m}{r}}\ \zeta^{\frac{1-r}{r}}\ \frac{E(x^{2(r-1)(\pm m+1)},x^{2r})\ E(\zeta,x^{2r})}{E(x^{2r-2}\zeta,x^{2r})\ E(x^{\mp 2(r-1)m},x^{2r})}\ .
\end{aligned} \tag{7.2}$$

The function $R=R(\zeta)$ is specified by the equation (2.2).

8. Appendix B

We show that bosonic operators (4.10), (4.13) satisfy (4.15) by working out the example of the commutation relation

$$\begin{aligned}
\mathbf{\Psi}^-(\zeta_1)\mathbf{\Psi}^+(\zeta_2)|_{\mathcal{L}_{l,k}} = {} & \mathbf{U}\left[\begin{array}{cc|c} k & k-1 & \\ k+1 & k & \zeta_1\zeta_2^{-1} \end{array}\right]\mathbf{\Psi}^+(\zeta_2)\mathbf{\Psi}^-(\zeta_1)|_{\mathcal{L}_{l,k}} + \\
& \mathbf{U}\left[\begin{array}{cc|c} k & k+1 & \\ k+1 & k & \zeta_1\zeta_2^{-1} \end{array}\right]\ \mathbf{\Psi}^-(\zeta_2)\mathbf{\Psi}^+(\zeta_1)|_{\mathcal{L}_{l,k}}\ .
\end{aligned} \tag{8.1}$$

To simplify the formulas, let us return to the parameterization (1.14). In terms of the variables p and u the function $F(\zeta,\mathcal{P})=F[u,\omega]$ (4.12) reads:

$$F[u,\omega] = \frac{[1/2+u-r\omega]}{[1/2-u]}\ , \tag{8.2}$$

where $\omega=\sqrt{\frac{2r-2}{r}}\,\mathcal{P}$ and the symbol $[u]$ is defined by (1.11). Then, the following equation

$$\begin{aligned}F[v-u_1,r\omega-r+1]=&\frac{[r-1][r\omega+u_1-u_2][1/2+v-u_1][1/2+u_2-v]}{[r-1+u_1-u_2][r\omega][1/2-v+u_1][1/2-u_2+v]}\times\\ F[v-u_2,r\omega-r+1]-&\frac{[r-1-r\omega][u_1-u_2][1/2+u_2-v]}{[\omega][r-1+u_1-u_2][1/2-u_2+v]}\;F[v-u_1,\omega]\end{aligned}\tag{8.3}$$

is a simple consequence of the famous Riemann identity:

$$\begin{aligned}[2x][2y][2z][2t]=&[x+y+z+t][x-y-z+t][x+y-z-t][x-y+z-t]+\\ &[-x+y+z+t][x-y+z+t][x+y-z+t][x+y+z-t]\;.\end{aligned}\tag{8.4}$$

By using (8.3), it is easy to obtain the relation between the exponential operators (4.11):

$$\begin{aligned}U(\zeta_1)\bar U(z)F(z\zeta_1^{-1},\mathcal{P})U(\zeta_2)|_{\mathcal{L}_{l,k}}=&\,\mathbf{U}\left[\begin{array}{cc|c}k&k+1&\\k+1&k&\zeta_1\zeta_2^{-1}\end{array}\right]U(\zeta_2)U(\zeta_1)\bar U(z)\times\\ F(z\zeta_2^{-1},\mathcal{P})|_{\mathcal{L}_{l,k}}+\mathbf{U}\left[\begin{array}{cc|c}k&k-1&\\k+1&k&\zeta_1\zeta_2^{-1}\end{array}\right]&\;U(\zeta_2)\bar U(z)F(z\zeta_1^{-1},\mathcal{P})U(\zeta_1)|_{\mathcal{L}_{l,k}}\;,\end{aligned}\tag{8.5}$$

here $\zeta_{1,2}=x^{2u_{1,2}}$, $z=x^{2v}$. Now, to derive (8.1), we should integrate both parts of this equation over the variable z.

9. Appendix C. Reference formulae for trace calculations

Here we collect the formulae for the functions and constants in (5.4):

$$\begin{aligned}G'(\zeta)=&\frac{(x^{2r+2}\zeta;x^{2r},x^4,x^4)_\infty\;(x^2\zeta;x^{2r},x^4,x^4)_\infty}{(x^{2r}\zeta;x^{2r},x^4,x^4)_\infty\;(x^4\zeta;x^{2r},x^4,x^4)_\infty}\times\\ &\frac{(x^{2r+6}\zeta^{-1};x^{2r},x^4,x^4)_\infty\;(x^6\zeta^{-1};x^{2r},x^4,x^4)_\infty}{(x^{2r+4}\zeta^{-1};x^{2r},x^4,x^4)_\infty\;(x^8\zeta^{-1};x^{2r},x^4,x^4)_\infty}\;,\end{aligned}\tag{9.1}$$

$$W'(\zeta)=\frac{(x^{2r-1}\zeta;x^{2r},x^4)_\infty\;(x^{2r+3}\zeta^{-1};x^{2r},x^4)_\infty}{(x\zeta;x^{2r},x^4)_\infty\;(x^5\zeta^{-1};x^{2r},x^4)_\infty}\;,\tag{9.2}$$

$$\bar G'(\zeta)=\frac{(\zeta;x^2)_\infty\;(x^4\zeta^{-1};x^2)_\infty}{(x^{2r-2}\zeta;x^{2r})_\infty\;(x^{2r+2}\zeta^{-1};x^{2r})_\infty}\;,\tag{9.3}$$

$$\bar{\mathcal{R}}=i\;x^{\frac{r-1}{2r}}\;\frac{(x^2;x^2)_\infty\;(x^{2r};x^{2r})_\infty}{(x^{2r-2};x^{2r})_\infty\;(x^2;x^{2r})_\infty}\;,\tag{9.4}$$

$$\mathcal{R}^2=G'(1)\;.\tag{9.5}$$

The functions satisfy the relations

$$\begin{aligned}&G'(\zeta x)G'(\zeta x^{-1})={W'}^{-1}(\zeta)\;,\\ &W'(\zeta x)W'(\zeta x^{-1})=\bar G'^{-1}(\zeta)\;,\\ &G'(\zeta)=G'(x^4\zeta^{-1})\;,\\ &W'(\zeta)=W'(x^4\zeta^{-1})\;,\\ &\bar G'(\zeta)=\bar G'(x^4\zeta^{-1})\;.\end{aligned}\tag{9.6}$$

10. Appendix D.

In this Appendix we demonstrate that the expressions (3.12), (5.10) for the two-point LHP are equivalent. The integral (5.10) is given by a sum of the residues of the poles situated inside the circle $|z| = x^{-2}$. The direct calculation leads to the formula:

$$\begin{aligned} &P_{k,k+1}(l) = (-1)^{l-k}\ \bar{Z}_l^{-1}\ \frac{x^{4\Delta_{l,k}-\frac{c}{6}}}{E(x^{2r-2},x^{2r})(x^4;x^4)_\infty}\times \\ &x^{r\omega(\omega-1)+\frac{r-1}{r}}\ \Big\{J\big(x^{2rl-2(r-1)k}\big) - x^{4rlk+2rl}\ J\big(x^{-2rl-2(r-1)k}\big)\Big\}\ . \end{aligned} \tag{10.1}$$

Here $\Delta_{l,k}$, c are defined by (2.14) and

$$J(m) = -\frac{m}{2}\sum_{s=0}^{2r-3}\ (-1)^s\ x^{s(s-1)}E(-m^2x^{4(r-1)s+4(r-1)r}, x^{8(r-1)r})E'(mx^{-2s},x^{2r})\ . \tag{10.2}$$

The prime in (10.2) denotes a derivative with respect to the first argument. Using simple properties of the elliptic function $E(z,q)$, one can show that $J(m)$ satisfies the identity:

$$J(x^{2r-2}m) = -m^{-1}E(-m^2x^{4(r-1)r}, x^{8(r-1)r})E(m,x^{2r}) + m^{-1}J(m)\ . \tag{10.3}$$

It provides the recursion relation

$$P_{k,k+1}(l) + P_{k,k-1}(l) = \bar{Z}_l^{-1}\ [k]\ \chi_{l,k}(x^4)\ , \tag{10.4}$$

where $\chi_{l,k}(x^4)$ is given by (2.17). It only remains to check that the function $P_{1,0}(l)$ obeys the condition (3.11). In terms of integrals this means:

$$\begin{aligned} &\oint_{|z|=x^2}\frac{dz}{2\pi i z}\ E\big(-z^{2r-2}x^{4rl+(4r-6)(r-1)}, x^{8(r-1)r}\big)\ \frac{E(zx^{2r-3-2rl},x^{2r})}{E(z^{-1}x^3,x^2)} = \\ &\oint_{|z|=x^2}\frac{dz}{2\pi i z}\ z^{-2l}x^{6l}E\big(-z^{2r-2}x^{-4rl+(4r-6)(r-1)}, x^{8(r-1)r}\big)\ \frac{E(zx^{2r-3-2rl},x^{2r})}{E(z^{-1}x^3,x^2)}\ . \end{aligned} \tag{10.5}$$

To prove the equality we should change the variable $z\to z^{-1}$ and then deform the integration contour at the left hand side of (10.5).

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