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Radiative Corrections to the Muonium Hyperfine Structure. I. The $\alpha^2(Z\alpha)$ Correction

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Abstract

This is the first of a series of papers on a systematic application of the NRQED bound state theory of Caswell and Lepage to higher-order radiative corrections to the hyperfine structure of the muonium ground state. This paper describes the calculation of the $\alpha^2(Z\alpha)$ radiative correction. Our result for the complete $\alpha^2(Z\alpha)$ correction is 0.424(4) kHz, which reduces the theoretical uncertainty significantly. The remaining uncertainty is dominated by that of the numerical evaluation of the nonlogarithmic part of the $\alpha(Z\alpha)^2$ term and logarithmic terms of order α^4. These terms will be treated in the subsequent papers.

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I. INTRODUCTION

The hyperfine structure of hydrogenic atoms is one of the well-understood problems both experimentally and theoretically. Especially, the muonium has played an important role in the precision test of QED because its radiative corrections have been calculated to high orders and its hyperfine splitting has been measured very precisely [1]:

$$\Delta\nu(\mathrm{exp}) \;=\; 4\ 463\ 302.88\ (16)\ \mathrm{kHz} \qquad (0.036\ \mathrm{ppm}). \tag{1}$$

Furthermore, a new experiment is in progress to improve the precision of $\Delta\nu(\mathrm{exp})$ to about 0.007 ppm [2]. To match this experimental accuracy, it is necessary to improve the theory of the $\alpha^2(Z\alpha)$ and $\alpha(Z\alpha)^2$ non-recoil radiative corrections as well as the leading $\ln(Z\alpha)$ terms of order $\alpha^{4-n}(Z\alpha)^n$, n = 1, 2, 3, and some relativistic corrections. This paper presents details of the calculation of the $\alpha^2(Z\alpha)$ radiative correction. A preliminary report of this work has been published [3].

As is well known, the bulk of the hyperfine splitting can be explained by the nonrelativistic quantum mechanics and is given by the Fermi formula [4]

$$E_F = \frac{16}{3}\alpha^2 c R_\infty \frac{m}{M}\left[1+\frac{m}{M}\right]^{-3}, \tag{2}$$

where R_∞ is the Rydberg constant for infinite nuclear mass, and m and M are the electron and muon masses, respectively.

Many correction terms have been calculated over several decades since the pioneering work of Fermi. Unfortunately, different terms were often evaluated by different methods making comparison of the results nontrivial in some cases. This causes a particularly difficult problem in identifying and evaluating higher order correction terms. Recently, however, Lepage and his collaborators have developed an approach, called NRQED, to deal with the nonrelativistic and weakly coupled bound systems consistently, starting from quantum electrodynamics (QED) [5–7]. This provides a solid framework for evaluating higher order radiative corrections systematically and unambiguously. This series of papers deal with a treatment of radiative corrections of the muonium hyperfine structure within the framework of NRQED.

Before describing our calculation, let us summarize the previous results on the muonium hyperfine splitting $\Delta\nu$. It is customary to classify the QED corrections to $\Delta\nu$ into three types: radiative non-recoil correction, pure recoil correction, and radiative-recoil correction. We use the convention such that electron charge is e and the charge of the positive muon is $-Ze$. Of course $Z = 1$ for the muon, but it is kept in the formula in order to identify the origin of corrections. Note that each radiative photon on the electron-line contributes a factor α, that on the muon line a factor $Z^2\alpha$, and one jumping from electron to muon a factor $Z\alpha$. This factor also arises from the effect of binding on the velocity distribution of atomic electrons. In addition, there are small corrections due to the hadronic vacuum polarization and weak interaction effects. Thus one may write

$$\begin{aligned}\Delta\nu(\text{theory}) = {} & \Delta\nu(\text{rad}) + \Delta\nu(\text{recoil}) + \Delta\nu(\text{rad-recoil}) \\ & + \Delta\nu(\text{hadron}) + \Delta\nu(\text{weak}).\end{aligned} \tag{3}$$

Purely radiative terms of orders $\alpha(Z\alpha)$ and $\alpha(Z\alpha)^2$ have been known for some time [8]:

$$\begin{aligned}\Delta\nu(\text{rad}) = (1 + a_\mu) \Big(1 & + \frac{3}{2}(Z\alpha)^2 + a_e + \alpha(Z\alpha)(\ln 2 - \frac{5}{2}) \\ & - \frac{8\alpha(Z\alpha)^2}{3\pi} \ln(Z\alpha) \left[\ln(Z\alpha) - \ln 4 + \frac{281}{480}\right] \\ & + \frac{\alpha(Z\alpha)^2}{\pi}(14.88 \pm 0.29)\Big) E_F.\end{aligned} \tag{4}$$

Here a_e and a_μ are the anomalous magnetic moments of the electron and muon, respectively. The appearance of the factor $(1 + a_\mu)$ in (4) is in accord with our definition of E_F in (2). Note that the number 14.88 in the $\alpha(Z\alpha)^2$ correction is different from 15.39 reported in Ref. [8]. This is due to the recent discovery of two mistakes in the literature. The first error is in the calculation of the $\alpha(Z\alpha)^2$ correction due to the vacuum polarization insertion in the transverse photon in Ref. [9]. Recently several people independently found [10–12] that this contribution is $E_F\alpha(Z\alpha)^2/\pi(-4/5)$, not $E_F\alpha(Z\alpha)^2/\pi(-2/3)$ given in Ref. [9]. The second error was caused by omission of a part of the contribution due to the vacuum polarization insertion in the Coulomb photon, $E_F\alpha(Z\alpha)^2/\pi(-8/15)\ln 2$, when it was combined with the radiative photon contribution $E_F\alpha(Z\alpha)^2/\pi(15.10 \pm 0.29)$ [11].

The known recoil corrections add up to [8]

$$\Delta\nu(\text{recoil}) = \Big(-\frac{3Z\alpha}{\pi}\frac{mM}{M^2-m^2}\ln\frac{M}{m} + \frac{\gamma^2}{mM}\left[2\ln\frac{m_r}{2\gamma} - 6\ln 2 + \frac{65}{18}\right]\Big) E_F, \tag{5}$$

where $\gamma \equiv Z\alpha m_r$, $m_r = mM/(m+M)$. The radiative-recoil contributions, which arise from both electron and muon lines and from vacuum polarizations, are given by [1]

$$\begin{aligned}\Delta\nu(\text{rad-recoil}) = \frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\Big(&-2\ln^2\frac{M}{m} + \frac{13}{12}\ln\frac{M}{m} + 6\zeta(3) + \zeta(2) - \frac{71}{72} + 3\pi^2\ln 2 \\ &+ Z^2\left[\frac{9}{2}\zeta(3) + \frac{39}{8} - 3\pi^2\ln 2\right] \\ &+ \frac{\alpha}{\pi}\left[-\frac{4}{3}\ln^3\frac{M}{m} + \frac{4}{3}\ln^2\frac{M}{m} + \mathcal{O}\left(\ln\frac{M}{m}\right)\right]\Big) E_F\ . \end{aligned} \tag{6}$$

The $\alpha(Z\alpha)(m/M)$ and $Z^2\alpha(Z\alpha)(m/M)$ terms are known exactly [8,13]. The $\ln^3$ and $\ln^2$ parts of the $\alpha^2(Z\alpha)$ term were evaluated by Eides *et al.* [14].

The hadronic vacuum polarization contributes [15]

$$\begin{aligned}\Delta\nu(\text{hadron}) &= \frac{\alpha(Z\alpha)}{\pi^2}\frac{mM}{m_\pi^2}(3.75 \pm 0.24)E_F \\ &= 0.250(16)\ \text{kHz}\ ,\end{aligned} \tag{7}$$

where m_π is the charged pion mass.

Finally there is a small contribution due to the Z^0 exchange. Our re-evaluation of the standard-model estimate [16,17] gives [2]

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[1]Eq. (6) of Ref. [3] is valid only for $Z = 1$ although it does not affect the muonium. We thank B. N. Taylor and P. Mohr for pointing out this oversight.

[2]This is in agreement with the corrected value given in Ref. [17] and has a sign opposite to that of Ref. [3]. The same result was also obtained by J. R. Sapirstein and by M. I. Eides. We thank B. N. Taylor and P. Mohr for calling a possible problem of sign to our attention.

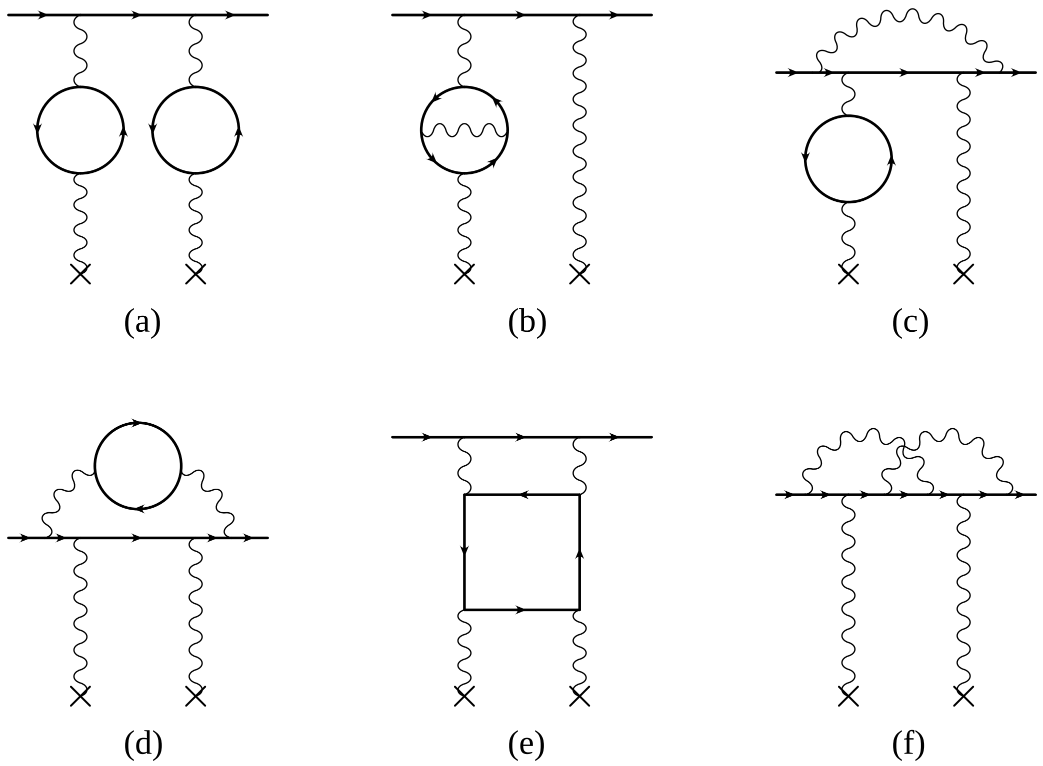


FIG. 1. Representative diagrams contributing to the $\alpha^2(Z\alpha)$ radiative corrections to the muonium hyperfine structure in which two virtual photons are exchanged between e^- and μ^+. The muon is represented by $\times$.

$$\begin{aligned}\Delta\nu(\text{weak}) &= -G_F \frac{3\sqrt{2}mM}{8\alpha\pi} E_F \\ &\simeq -0.065 \text{ kHz}.\end{aligned} \tag{8}$$

As is clear from these results one must know the $\alpha^2(Z\alpha)$ pure radiative correction in order to improve the theoretical prediction further. Fig. 1 shows typical diagrams contributing to this order. Recently, terms represented by the diagrams (a) - (e) of Fig. 1 have been evaluated by Eides *et al.* [18]. Their results are as follows:

$$\begin{aligned}\Delta\nu(\text{Fig.1(a)}) &= \frac{36}{35}\frac{\alpha^2(Z\alpha)}{\pi}E_F \\ &= 0.567 \text{ kHz},\end{aligned} \tag{9}$$

$$\Delta\nu(\text{Fig.1(b)}) = \left(\frac{224}{15}\ln 2 - \frac{38}{15}\pi - \frac{118}{225}\right)\frac{\alpha^2(Z\alpha)}{\pi}E_F$$

$$= 1.030 \text{ kHz}, \tag{10}$$

$$\Delta\nu(\text{Fig.1(c)}) = \left(-\frac{4}{3}z^2 - \frac{20\sqrt{5}}{9}z - \frac{64}{45}\ln 2 + \frac{\pi^2}{9} + \frac{1043}{675} + \frac{3}{8}\right)\frac{\alpha^2(Z\alpha)}{\pi}E_F$$
$$= -0.369 \text{ kHz}, \tag{11}$$

$$\Delta\nu(\text{Fig.1(d)}) = -0.310\ 742\cdots\ \frac{\alpha^2(Z\alpha)}{\pi}E_F$$
$$= -0.171 \text{ kHz}, \tag{12}$$

where $z = \ln((1+\sqrt{5})/2)$. The results (9), (10) and (11) are analytic, while (12) was evaluated numerically after reducing the integral to one dimension. We confirmed these results by an independent numerical calculation. However, our purely numerical evaluation of Fig. 1(e):

$$\Delta\nu(\text{Fig.1(e)}) = -0.472\ 48\ (9)\frac{\alpha^2(Z\alpha)}{\pi}E_F$$
$$= -0.261 \text{ kHz} \tag{13}$$

disagreed with the semi-analytic result of Ref. [19]. With our help, Eides [20] found an error in the Table after Eq. (23) of Ref. [19]. Their corrected value is in good agreement with (13).

Fig. 2 shows the complete set of Feynman diagrams of the type (f) of Fig. 1, which have not been evaluated before our work [3]. The preliminary result of our calculation for all diagrams of Fig. 2 was

$$\Delta\nu(\text{Fig.1(f)}) = -0.63\ (4)\frac{\alpha^2(Z\alpha)}{\pi}E_F$$
$$= -0.347\ (22) \text{ kHz}, \tag{14}$$

where the error is mainly due to the uncertainty in extrapolating the integral to zero infrared cutoff. The main purpose of this paper is to report a further improvement of this result:

$$\Delta\nu(\text{Fig.1(f)}) = -0.675\ 7(81)\frac{\alpha^2(Z\alpha)}{\pi}E_F$$
$$= -0.373\ (4) \text{ kHz}. \tag{15}$$

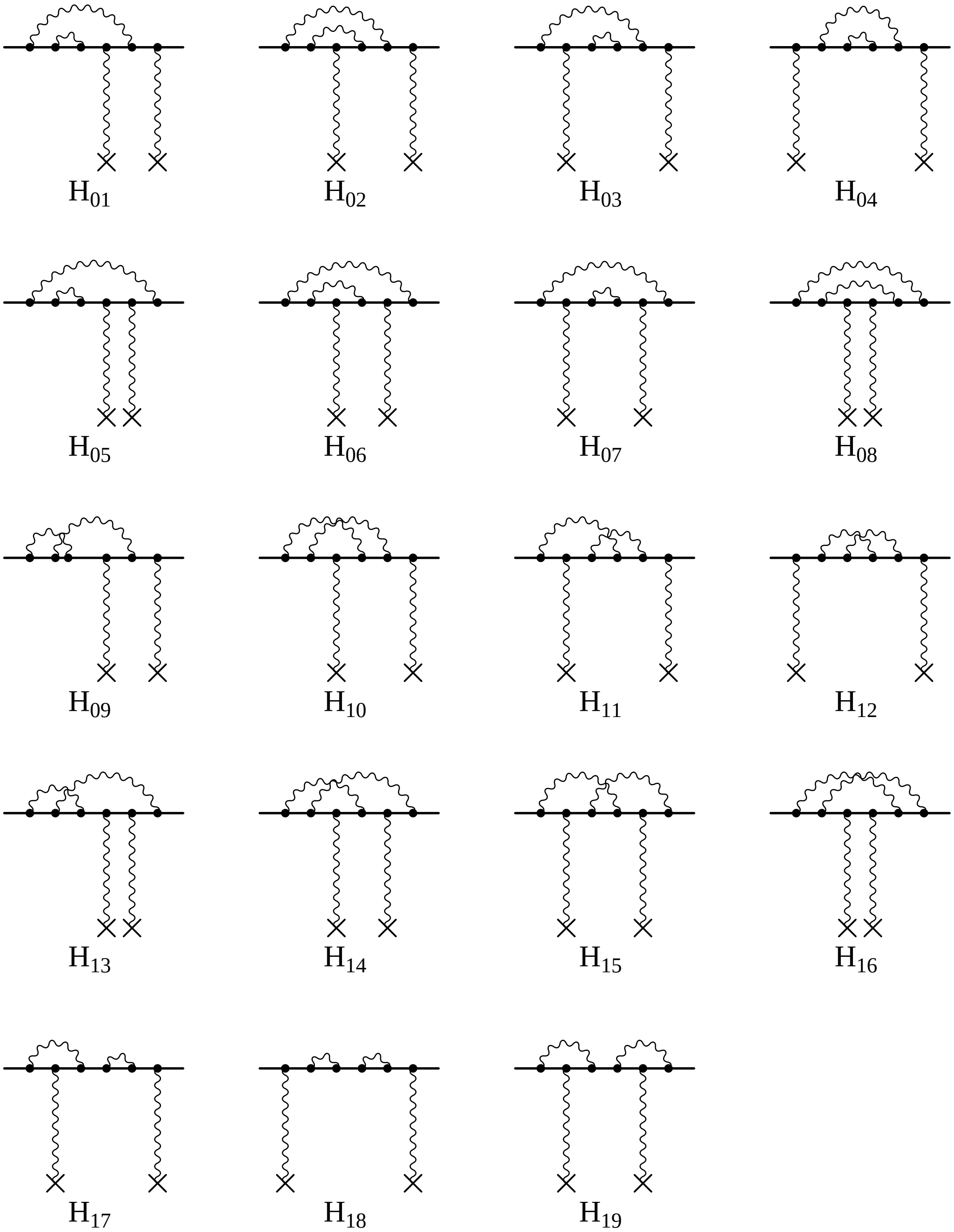


FIG. 2. Two-photon exchange diagrams with fourth-order radiative corrections on the electron-line. Diagrams which are related to these diagrams by time reversal are not shown explicitly. The muon is represented by $\times$.

As a consequence of this result, the total contribution of the $\alpha^2(Z\alpha)$ correction to the muonium hyperfine splitting becomes

$$\begin{aligned}\Delta\nu(\text{Fig.1}) &= 0.768\ 6\ (81)\frac{\alpha^2(Z\alpha)}{\pi}E_F \\ &= 0.424\ (4)\ \text{kHz}.\end{aligned} \tag{16}$$

This removes the dominant theoretical uncertainty in $\Delta\nu$(theory).

In Sec. II we outline the NRQED treatment of two-body bound system. It serves as the theoretical basis for the calculation of the $\alpha^2(Z\alpha)$ correction as well as the calculation of the $\alpha(Z\alpha)^2$ and higher order corrections discussed in the subsequent papers. In Sec. III we illustrate the general procedure of NRQED choosing the well-known $\alpha(Z\alpha)$ non-recoil radiative correction as an example. In Sec. IV we present our calculation of the $\alpha^2(Z\alpha)$ purely radiative non-recoil correction to the muonium hyperfine structure. Some problems encountered in the numerical work are also discussed there. Sec. V is devoted to the discussion of our results.

II. NRQED

A. Why NRQED ?

The Lorentz invariance has been one of the most important guiding principles for the development of quantum field theory. However, relativistic quantum field theory is often very cumbersome to apply to nonrelativistic bound systems. Such a calculation tends to be very complicated and requires an enormous effort, while the result reflects mostly the nonrelativistic feature of the system. For such a system an approach that incorporates most of the bound state effects from the beginning would minimize the amount of computation necessary to achieve the desired precision. For the case of electromagnetic interaction, this has been realized by a theory called nonrelativistic quantum electrodynamics, or NRQED. The NRQED enables us to avoid some, if not all, of the problems encountered in the usual treatment based on the Bethe-Salpeter equation.

The NRQED, formulated by Caswell and Lepage [5], is a rigorous adaptation of QED to bound systems. This theory enables us to take a consistent and systematic approach to loosely bound nonrelativistic systems. Compared with the conventional bound state theories, it allows easier power counting, more transparent cancellation of UV and IR divergences, and is manifestly gauge invariant. In spite of its superiority, however, the details of the theory has not yet been fully worked out. In this series of papers, we present an explicit construction of the NRQED Hamiltonian and develop a bound state perturbation theory based on it.

As for the computation of the $\alpha(Z\alpha)$ and $\alpha^2(Z\alpha)$ corrections, NRQED or any other relativistic bound state formalism gives the same simple recipe: calculate the forward scattering amplitude in QED and multiply it with $|\phi(0)|^2$, where $\phi(0)$ is the nonrelativistic wave function at the origin. In NRQED this recipe can be directly justified by inspection of relevant diagrams and power counting. In other bound state formalisms, the corresponding procedure may be less straightforward. The latter approach becomes very complicated for higher-order corrections such as the $\alpha(Z\alpha)^2$ and $\alpha(Z\alpha)^3$ corrections. Difficulty in achieving high numerical precision by this method is one of the sources of theoretical uncertainty at present [8,21].

The approach adopted by the NRQED, however, loses its effectiveness for the high Z system. In such a case it is desiable to avoid expanding in $v \sim Z\alpha$. Recently, an attempt has been made to calculate the order α term without exapanding in $Z\alpha$ [22]. However, this aproach may have difficulty in providing a good precision for $Z = 1$. This is primarily because, for low Z systems, the bound electron is almost on-the-mass-shell and causes the near infrared divergence. As a consequence the convergence of numerical integration deteriorates as Z decreases. As is shown in the subsequent papers, the NRQED method enables us to deal with the near infrared divergence problem order by order in a systematic expansion in $Z\alpha$, and allows us to calculate the expansion coefficients with high precision. This is why the NRQED method is a powerful tool for low Z systems.

B. Outline of NRQED

In the NRQED approach to the bound state problem, one first derives the NRQED Lagrangian from the QED, and then uses it to determine the correction to the energies and wave functions by a systematic application of the Rayleigh-Schrödinger perturbation theory.

The NRQED Lagrangian consists of all possible local interactions satisfying the required symmetries, such as gauge invariance, parity invariance, time reversal, galileian invariance, hermiticity, and locality. We use the same photon Lagrangian $(-1/4)F_{\mu\nu}F^{\mu\nu}$ as that of QED. In addition, new photon interaction terms are introduced to represent the insertion of the fermion loop, such as vacuum polarization and light-by-light scattering.

In order to define the NRQED Lagrangian precisely, we must regularize the interaction terms of NRQED, e.g., by cutting off contributions of large momenta. Since this theory is meant to apply to nonrelativistic systems, the cut-off Λ may be chosen as the typical mass scale of the system, e.g., the rest mass of an electron. With the cut-off Λ thus fixed, the theory becomes well-defined, even though the interaction terms are strongly dependent on the cut-off parameter. In the following the cut-off is understood implicitly, and will be exhibited only when it is necessary. The choice of the momentum cut-off used for the NRQED scattering amplitudes is arbitrary but the physical quantity computed should be independent of any particular choice. In other words, the NRQED theory must have reparametrization invariance with respect to the choice of cut-off. This is analogous to the existence of the renormalization group in the renormalizable relativistic field theory. It is important to note that the NRQED is fully equivalent to the QED. The only difference is that it is better adapted to low energy bound systems.

The NRQED rule for determining the operators which appear in its Lagrangian and their coefficients is simple and straightforward: Each term of the *scattering* amplitude calculated in the NRQED must coincide with the corresponding *scattering* amplitude of the original QED at some given momentum scale, e.g., at the threshold of the external on-shell particles. The center of mass frame is used for both bound state and scattering state calculations. Since the same argument about reparametrization invariance holds for the momentum scale chosen for comparison of QED and NRQED scattering amplitudes, the at-threshold condition is just for convenience. However, the on-shell condition for the external fermion is more than a matter of convenience. In order to regulate the IR singularity it is convenient to introduce the photon mass λ in the calculation of scattering amplitude of both QED and NRQED. This finite photon mass together with the on-shell condition ensures that the NRQED scattering theory has a pole in the region of the complex energy plane of the external fermion in which the scattering theory can be analytically continued to the off-shell bound state theory.

We use the normalization $u^\dagger u = 1$ for the external 4-component spinors in the QED calculation instead of the conventional relativistic normalization $\bar{u}u = 1$ so that both QED and NRQED S-matrix have the same normalization [23]. This ensures that physical quantities, such as decay rate and cross section, calculated in both theories are the same.

Note that the scattering amplitude of QED is fully renormalized, namely, it is finite and completely determined within QED. This enables us to fix the NRQED "renormalization" constants without ambiguity. This also means that the coupling constant α and fermion masses in the QED are the renormalized ones determined on-shell, and these α and fermion masses are used as the "bare" coupling constant and "bare" masses of NRQED.

It is convenient to write the NRQED Lagrangian in two parts: $L_{\rm main}$ and $L_{\rm contact}$. The $L_{\rm main}$ part consists of the fermion bilinear operators. Fermions in NRQED are expressed by the Pauli two component spinor field $\psi(t,\vec{x})$ (instead of the Dirac spinor). If one takes into account the required symmetries of the theory, the main part of NRQED Lagrangian $L_{\rm main}$ must have the general form [5,6]

$$\begin{aligned}
L_{\rm main} = \psi^\dagger \{ & iD_t + \frac{\vec{D}^2}{2m} + \frac{\vec{D}^4}{8m^3} \\
& + c_F \frac{e\vec{\sigma}\cdot\vec{B}}{2m} + c_D \frac{e(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})}{8m^2} \\
& + c_S \frac{ie\vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})}{8m^2} + c_{W1}\frac{e\{\vec{D}^2, \vec{\sigma}\cdot\vec{B}\}}{8m^3} \\
& + c_{W2}\frac{-e\vec{D}^i\vec{\sigma}\cdot\vec{B}\vec{D}^i}{4m^3} + c_{p'p}\frac{e(\vec{\sigma}\cdot\vec{D}\vec{B}\cdot\vec{D} + \vec{D}\cdot\vec{B}\vec{\sigma}\cdot\vec{D})}{8m^3} + \ldots \}\psi\ .
\end{aligned} \tag{17}$$

where $D_t = \partial_t + ieA^0$ and $\vec{D} = \vec{\partial} - ie\vec{A}$. (We put $c = 1$ and $\hbar = 1$ henceforth.) The positron part can be written down in a similar way. The particle-antiparticle mixed interaction is not present in L_{main}. The first three terms are related to the kinetic term of the QED Lagrangian. The second and third terms are derived from the expansion

$$E = \sqrt{\vec{p}^{\,2} + m^2} = m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} + \ \ldots \quad . \tag{18}$$

These three terms of (17) have coefficients unaffected by the radiative correction as a consequence of the renormalizability of QED, while the coefficients c_i of other terms are modified by the QED interaction and can be expressed as a power series in the coupling constants α

$$c_i = c_i^{(0)} + c_i^{(1)}\alpha + c_i^{(2)}\alpha^2 + \ldots \qquad . \tag{19}$$

Some of the operators in (17) can be generated by the Foldy-Wouthuysen-Tani transformation of the Dirac Lagrangian. These operators have the coefficient $c_i^{(0)} = 1$ while other operators have $c_i^{(0)} = 0$. Note that c_i's do not have coefficients involving $Z\alpha$ caused by the binding effect because they are determined solely by comparison of the NRQED and QED *scattering* amplitudes without referring to the bound states.

Eq. (17) has an infinite number of terms. Not all of them, of course, are needed in a practical calculation. The operators necessary to carry out a particular calculation are determined by the power counting rule of NRQED for the bound state. We will show this process explicitly in the next subsections where the NRQED Hamiltonian is constructed.

The L_{main} of NRQED alone is not sufficient to produce the same physical quantities as those from QED. To make NRQED equivalent to QED, we must add another term to the NRQED Lagrangian. It consists of terms of contact interaction type:

$$\begin{aligned} L_{\text{contact}} = {} & d_1 \frac{1}{mM} (\psi^\dagger \vec{\sigma} \psi) \cdot (\chi^\dagger \vec{\sigma} \chi) + d_2 \frac{1}{mM} (\psi^\dagger \psi)(\chi^\dagger \chi) \\ & + d_3 \frac{1}{mM} (\psi^\dagger \vec{\sigma} \chi) \cdot (\chi^\dagger \vec{\sigma} \psi) + d_4 \frac{1}{mM} (\psi^\dagger \chi)(\chi^\dagger \psi) \\ & + d_5 \frac{1}{m^3 M} (\psi^\dagger \vec{D}^2 \vec{\sigma} \psi) \cdot (\chi^\dagger \vec{\sigma} \chi) + \ \cdots \quad , \end{aligned} \tag{20}$$

where χ represents a fermion field (of mass M) such as a muon or a positron. The third and fourth terms in (20) are needed only when both electron and positron are present. This is because, from the viewpoint of NRQED, the electron-positron annihilation is a high energy process and can only be represented as a contact interaction term. For the muonium, only the first and second terms are relevant. The fifth term is an example of contact terms including derivative interactions, which are of higher order in $< \vec{p}^{\,2}/m^2 >\sim (Z\alpha)^2$.

The coefficients d_i are choosen such that these contact interactions make up the difference between the QED electron-muon scattering amplitude and the corresponding NRQED scattering amplitude derived from the Lagrangian $L_{\rm main}$. This procedure enebles us to determine the coefficients d_i completely.

As is clear from the above discussion, these NRQED "renormalization" constants c_i and d_i have the parameter dependence

$$\begin{aligned} c_i &= c_i(\alpha, \Lambda, \lambda, m), \\ d_i &= d_i(\alpha, Z\alpha, Z^2\alpha, \Lambda, \lambda, m, M), \end{aligned} \tag{21}$$

where λ is the photon mass introduced as the infrared (IR) cut-off. Of course, the experimentally observable result of calculation must be independent of both cut-offs Λ and λ, and gauge invariant. This is realized by a systematic application of the nonrelativistic Rayleigh-Schrödinger perturbation theory to the bound states.

Just as the actual execution of renormalization program of QED must rely on the covariant perturbation theory, a comprehensive formulation of NRQED can be realized explicitly only within the framework of the nonrelativistic Rayleigh-Schrödinger perturbation theory. This means that we have to choose an appropriate part of the Hamiltonian as the unperturbed term and treat the rest as perturbation.

To deal with the muonium we find it generally convenient to define the unperturbed system in terms of the ground state solution of the nonrelativistic Schrödinger equation:

$$\left(\frac{\vec{p}^{\,2}}{2m_r} - \frac{Z\alpha}{r}\right)\phi = E^0\phi \ , \tag{22}$$

where m_r is the reduced mass, $E^0 = -\gamma^2/(2m_r)$ is the ground state binding energy, $\gamma \equiv (Z\alpha)m_r$ being a typical momentum scale of the Coulomb bound state. The solution of this equation is

$$\phi(\vec{p}) = \frac{8\sqrt{\pi\gamma^5}}{(\vec{p}^2+\gamma^2)^2} \ . \tag{23}$$

The unperturbed electron field $\psi(\vec{p})$ is thus expressed by this wave function $\phi(\vec{p})$ times the Pauli spin factor. Using the remaining interaction terms in the NRQED Lagrangian together with the photon Lagrangian, we can construct the effective potentials. These potentials are to be treated as perturbation.

The nonrelativistic Rayleigh-Schrödinger perturbation theory gives

$$\begin{aligned}
\Delta E_n &= \psi_n^\dagger V \psi_n [1 + \psi_n^\dagger \left(\frac{\partial}{\partial E} V\right) \psi_n]_{E=E_n^0} \\
&+ \psi_n^\dagger V (\tilde{G}_0 - \frac{\psi_n\psi_n^\dagger}{E-E_n^0}) V \psi_n \ |_{E=E_n^0} \\
&+ \psi_n^\dagger V (\tilde{G}_0 - \frac{\psi_n\psi_n^\dagger}{E-E_n^0}) V (\tilde{G}_0 - \frac{\psi_n\psi_n^\dagger}{E-E_n^0}) V \psi_n \ |_{E=E_n^0} \\
&+ \psi_n^\dagger V \psi_n \psi_n^\dagger \left(\frac{\partial}{\partial E}\left(V(\tilde{G}_0 - \frac{\psi_n\psi_n^\dagger}{E-E_n^0})V\right)\right) \psi_n \ |_{E=E_n^0} \\
&+ \dots \ .
\end{aligned} \tag{24}$$

The Green function $\tilde{G}_0(\vec{k},\vec{q}: E_n^0)$ appearing here is known in a closed form for the nonrelativistic Coulomb potential [24]. For the ground state $n=1$, we find

$$\begin{aligned}
\lim_{E\to E_{n=1}^0} (\tilde{G}_0 - \frac{\psi_{n=1}\psi_{n=1}^\dagger}{E-E_{n=1}^0}) &= \frac{-2m_r}{\vec{k}^2+\gamma^2}(2\pi)^3\delta^3(\vec{k}-\vec{q}) \\
&+ \frac{-2m_r}{(\vec{k}^{\,2}+\gamma^2)}\frac{-Ze^2}{|\vec{k}-\vec{q}|^2}\frac{-2m_r}{(\vec{q}^{\,2}+\gamma^2)} \\
&- \frac{64\pi}{Z\alpha\gamma^4}\tilde{R}(\vec{k},\vec{q}) \ ,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\tilde{R}(\vec{k},\vec{q}) = \frac{\gamma^8}{(\vec{k}^{\,2}+\gamma^2)^2(\vec{q}^{\,2}+\gamma^2)^2} [\ & \frac{5}{2} - 4\frac{\gamma^2}{\vec{k}^{\,2}+\gamma^2} - 4\frac{\gamma^2}{\vec{q}^{\,2}+\gamma^2} \\
& + \frac{1}{2}\log A + \frac{2A-1}{(4A-1)^{1/2}} \tan^{-1}(4A-1)^{1/2} \ ] \ ,
\end{aligned} \tag{26}$$

and

$$A = \frac{(\vec{k}^{\,2}+\gamma^2)(\vec{q}^{\,2}+\gamma^2)}{4\gamma^2|\vec{k}-\vec{q}|^2} \ . \tag{27}$$

The first, second, and third terms of the expression (25) can be understood as corresponding to zero, one, and two or more Coulomb-photon exchanges.

In order to determine which terms of the Hamiltonian are needed to obtain the desired precision it is useful to know the expectation values of various operators with respect to appropriate wave functions [23]. For a nonrelativistic Coulombic bound system, one finds

$$
\begin{aligned}
&< \vec{\partial} >\sim m(v/c), \quad < \partial_t >\sim m(v/c)^2, \quad < eA^0 >\sim m(v/c)^2, \\
&< e\vec{A} >\sim m(v/c)^3, \quad < e\vec{E} >\sim m^2(v/c)^3, \quad < e\vec{B} >\sim m^2(v/c)^4,
\end{aligned} \tag{28}
$$

where m is the electron mass, v is the typical velocity of a bound electron, and $c(=1)$ is the velocity of light. Thus, in Eq. (17), the first two terms, next four terms, and the remaining terms correspond to the interactions which start at orders v^2, v^4, and v^6, respectively. Radiative corrections, which alter the values of the coefficients c_i's and d_i's, will keep the estimate (28) intact. The information (28) can be used to terminate the series of interaction terms at the desired precision. In this sense, the NRQED Lagrangian is an expansion in both the coupling constant α and the velocity v.

The NRQED "renormalization" coefficients play important roles in restoring gauge invariance which might have been broken by regularization. The explicit form of these coefficients depends on the regularization method. Gauge invariant regularization is desirable but not necessary. If one proceeds carefully, even a simple momentum cut-off method may be used [26]. (This is only true for an Abelian gauge theory such as NRQED.) In a calculation of the would-be divergent quantity in NRQED, we put the UV cut-off Λ not in the fermion momentum but in the photon momentum [27].

Because of the way NRQED is constructed, gauge invariance of the NRQED amplitude with its complete set of "renormalization" constants is automatically guaranteed by the gauge invariance of the corresponding QED amplitude. Since QED and NRQED are separately gauge invariant, we may choose different gauges for QED and NRQED. We will use the Feynman gauge for QED calculation, and we use the Coulomb gauge for NRQED. The Feynman gauge minimizes the amount of work for numerical computation, and the Coulomb gauge is more suitable for describing the nonrelativistic behavior of the electron.

The dominant contribution to the hyperfine splitting between the spin J=1 and J=0 states originates from the interaction between the electron spin and the muon spin mediated by a transverse photon of momentum $\vec{k}$. This leads to a potential of the form

$$V_F = \frac{ie}{2m}(\psi^\dagger \vec{k} \times \vec{\sigma_e}\psi) \cdot \frac{-iZe}{2M}(\chi^\dagger(-\vec{k}) \times \vec{\sigma_\mu}\chi)\frac{-1}{\vec{k}^{\,2}} \tag{29}$$

in the momentum space representation, where M is the muon mass. Using this Fermi potential V_F in the first order perturbation theory and taking the difference between J=1 and J=0 states, we obtain the hyperfine Fermi splitting

$$\begin{aligned} E_F &= \langle n=1|V_F|n=1\rangle|^{J=1}_{J=0} \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\frac{(8\sqrt{\pi\gamma^5})^2}{(\vec{p}^{\,2}+\gamma^2)^2(|\vec{p}+\vec{k}|^2+\gamma^2)^2}\cdot \\ &\quad \frac{ie}{2m}\langle(\vec{k}\times\vec{\sigma_e})\cdot\frac{-iZe}{2M}((-\vec{k})\times\vec{\sigma_\mu})\rangle\frac{-1}{\vec{k}^{\,2}} \\ &= \frac{2(Z\alpha)\gamma^3}{3mM}\langle\vec{\sigma_e}\cdot\vec{\sigma_\mu}\rangle|^{J=1}_{J=0} \\ &= \frac{8(Z\alpha)\gamma^3}{3\ mM}\ . \end{aligned} \tag{30}$$

Needless to say, the Fermi potential is of order $v^4(m/M)m \sim (Z\alpha)^4(m/M)m$.

Various interaction terms and propagators are represented by the NRQED "Feynman" diagrams shown in Fig. 3. It is convenient and useful to express higher-order amplitudes representing scattering states or bound states by corresponding diagrams.

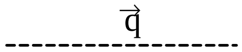
We classify the diagrams according to the number of external photons in the QED Feynman diagrams. In the following subsections we shall show step by step how the corresponding NRQED Lagrangian $L_{\rm main}$ (or Hamiltonian $H_{\rm main}$) is determined.

C. Scattering by a static external potential

We have already found the general form of the main part of the NRQED Lagrangian given by Eq. (17) using the required symmetries for the theory and the power counting rules.

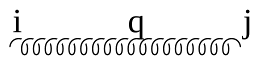
Propagators

Coulomb photon propagator



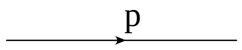
$$\frac{1}{\vec{q}^{\,2}+\lambda^2}$$

Transverse photon propagator



$$\frac{\delta^{ij}-\frac{q^i q^j}{\vec{q}^{\,2}+\lambda^2}}{(q^0)^2-\vec{q}^{\,2}-\lambda^2+i\epsilon}$$

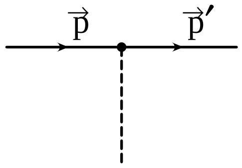
Fermion propagator



$$\frac{1}{E-\frac{\vec{p}^{\,2}}{2m}+i\epsilon}$$

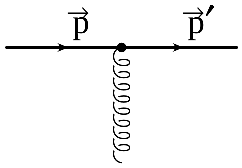
Vertices

Coulomb vertex



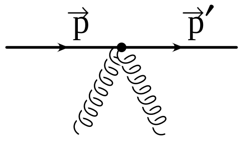
$$e$$

Dipole vertex



$$-e\frac{\vec{p}\,'+\vec{p}}{2m}$$

$\vec{A}\cdot\vec{A}$ vertex



$$\frac{e^2\delta^{ij}}{2m}$$

FIG. 3. NRQED "Feynman" rules for vertices and propagators. They can be used for both scattering and bound state calculations. E in the fermion propagator represents the fermion's bound state energy: $E=0$ for scattering and $E=-\gamma^2/(2m_r)$ for the ground state muonium. The photon mass λ is set to zero in the bound state calculation.