CLNS 95/1380

Asymmetric Orbifolds and Wilson Lines

Zurab Kakushadze* and S.-H. Henry Tye
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853-5001, USA
(December 19, 1995)

Abstract

We generalize the rules for the free fermionic string construction to include other asymmetric orbifolds. Examples are given to illustrate the use of these rules.

Typeset using REVTeX

I. INTRODUCTION

The construction of string models has a long history. The number of consistent string models is clearly very large (One may consider various string models as different classical string vacua of a single theory; in this case, we are talking about the construction of classical vacua). The best understood string models are probably those obtained via toroidal compactification, and also their orbifolds [1]. However, classification of such orbifold string models is still largely unexplored. This is in part due to the lack of simple rules for constructing such models, in particular, for asymmetric orbifolds [2].

The first class of asymmetric orbifold string models are the free fermionic string models [3]. Although this class of models is rather restrictive (allowing only multiple $\mathbf{Z}_2$ twists), the rules for such model building are quite simple. As a result, rather complicated models can be readily constructed [4], sometimes with the help of computers.

A general framework for asymmetric orbifolds was given in [2], and it provides a setting for all orbifold models, since some symmetric orbifolds at fixed radii may be considered as special cases of asymmetric orbifolds. However, this general approach is not easy to use in actual practice; as a result, asymmetric orbifold models are not well explored. In this paper, we simplify the construction of asymmetric orbifold models by presenting explicit and rather simple rules for their model buildings.

The rules for the asymmetric orbifold construction are quite similar to those for the free fermionic string models. Here we shall follow the notations of Ref. [5]. We shall impose the consistency requirements on the string one-loop partition function in the light-cone gauge: (*i*) one-loop modular invariance; (*ii*) world-sheet supersymmetry (if present), which insures space-time Lorentz invariance in the covariant gauge; and (*iii*) physically sensible projection; this means the contribution of a space-time fermionic (bosonic) degree of freedom to the partition function counts as minus (plus) one. In all cases that can be checked, this condition plus the one-loop modular invariance and factorization imply multi-loop modular invariance.

The building blocks for any specific string model partition function are the (appropriate) characters for world-sheet fermions and bosons. The fermion characters are the same as the ones used in the free fermionic string models. The characters for bosons are combined from types: those for twisted chiral bosons and those for chiral lattices. A key to obtaining simple rules is the choice of basis for the chiral lattices. They are chosen so that, up to phases, all the chiral lattice characters are permuted under any modular transformation, as is the case for the chiral fermion characters. Our discussion shall focus on heterotic strings compactified to four spacetime dimensions; the generalization to other dimensions and to Type II strings is straightforward.

As in the free fermionic string model constructions, the rules may be used to build new models without direct reference to their original partition functions and/or the characters in. This turns out to be useful because the partition function can get rather complicated. In this paper, we consider models with Wilson lines and only one twist. The rules given here can be used as a basis for further generalization to the non-abelian orbifold case, which will be discussed elsewhere. For a general lattice, the sublattice invariant under the twist may be difficult to identify. Sometimes, it is easier to start with a lattice whose invariant sublattice is obvious, and then introduce background fields, in particular Wilson lines, that commute with the twist. In fact, this approach is very useful in the symmetric orbifold construction

[6]. Here, our work can be considered as a generalization of their work to the asymmetric orbifold case.

In section II, we briefly review the fermion and the boson characters that we shall use later. In section III, we derive the rules for model building. To be specific, we shall consider only heterotic strings compactified to four spacetime dimensions. Also, we shall confine ourselves to the elementary particle sectors, where the conformal field theory description (as given by the characters mentioned above) is sufficient. The rules for model building are summarized. In section IV, some examples of asymmetric orbifold models, with and without Wilson lines, are explicitly constructed to illustrate the rules. In section V, we discuss a model with higher-level gauge group. For the sake of completeness, we present a couple of symmetric orbifolds as well, with and without Wilson lines, in section VI. Section VII contains the discussion and remarks. Some of the details are relegated to the appendices.

II. PRELIMINARIES

A. Framework

In this subsection we set up the framework for the remainder of this paper. To be specific, we consider heterotic strings compactified to four space-time dimensions. In the light-cone gauge, which we adopt, we have the following world-sheet degrees of freedom: One complex boson ϕ^0 (corresponding to two transverse space-time coordinates); three right-moving complex bosons ϕ^ℓ_R, $\ell = 1, 2, 3$ (corresponding to six internal coordinates); four right-moving complex fermions ψ^r, $r = 0, 1, 2, 3$ (ψ^0 is the world-sheet superpartner of the right-moving component of ϕ^0, whereas ψ^ℓ are the world-sheet superpartners of ϕ^ℓ_R, $\ell = 1, 2, 3$); eleven left-moving complex bosons ϕ^ℓ_L, $\ell = 4, 5, ..., 14$ (corresponding to twenty-two internal coordinates). Before orbifolding, the corresponding string model has $N = 4$ space-time supersymmetry and the internal momenta span an even self-dual Lorentzian lattice $\Gamma^{6,22}$.

It is convenient to organize the string states into sectors labeled by the monodromies of the string degrees of freedom. Thus, consider the sector where

$$\begin{aligned} &\psi^r(\overline{z}e^{-2\pi i}) = \exp(-2\pi i V_i^r)\psi^r(\overline{z}) \ , \\ &\phi^\ell_R(\overline{z}e^{-2\pi i}) = \exp(-2\pi i W_i^\ell)\phi^\ell_R(\overline{z}) - U_i^\ell \ , \quad \ell = 1, 2, 3 \ , \\ &\phi^\ell_L(ze^{2\pi i}) = \exp(-2\pi i W_i^\ell)\phi^\ell_L(z) + U_i^\ell \ , \quad \ell = 4, ..., 14 \end{aligned} \tag{2.1}$$

(Note that $\phi^0(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = \phi^0(z, \overline{z})$ since ϕ^0 corresponds to space-time coordinates). These monodromies can be combined into a single vector

$$V_i = (V_i^0(V_i^1 \ \ (W_i^1, U_i^1))(V_i^2 \ \ (W_i^2, U_i^2))(V_i^3 \ \ (W_i^3, U_i^3))||(W_i^4, U_i^4)...(W_i^{14}, U_i^{14})) \ . \tag{2.2}$$

The double vertical line separates the right- and left-movers. Without loss of generality we can restrict the values of V_i^r and W_i^ℓ as follows: $-\frac{1}{2} \leq V_i^r < \frac{1}{2}$; $0 \leq W_i^\ell < 1$ (A complex boson (fermion) with boundary condition W_i^ℓ (V_i^r) $= 0$ or $\frac{1}{2}$ can be split into two real bosons (fermions)). The shifts U_i^ℓ can be combined into a real $(6, 22)$ dimensional Lorentzian vector $\vec{U}_i$ defined up to the identification $\vec{U}_i \sim \vec{U}_i + \vec{P}$, where $\vec{P}$ is an arbitrary vector of $\Gamma^{6,22}$.

The monodromies (2.1) can be viewed as fields Φ (where Φ is a collective notation for the fields ψ^r, ϕ^ℓ_R and ϕ^ℓ_L) being periodic $\Phi(ze^{2\pi i},\overline{z}e^{-2\pi i})=\Phi(z,\overline{z})$ up to the identification $\Phi\sim g(V_i)\Phi g^{-1}(V_i)$, where $g(V_i)$ is an element of the *orbifold* group G. In this paper we will only consider the cases where G is an *abelian* group. For two elements $g(V_i)$ and $g(V_j)$ to commute, we must have $U^\ell_i=0$ if $W^\ell_j\neq 0$, and $U^\ell_j=0$ if $W^\ell_i\neq 0$.

This leads us to a simpler form of V_i where instead of having a double entry (W^ℓ_i,U^ℓ_i) for each complex boson we will specify either W^ℓ_i (whenever $W^\ell_i\neq 0$, in which case $U^\ell_i=0$), or U^ℓ_i (whenever $U^\ell_i\neq 0$, in which case $W^\ell_i=0$). To keep track of whether a given entry corresponds to a twist or a shift, it is convenient to introduce an *auxiliary* vector

$$W=(0(0\ W^1)(0\ W^2)(0\ W^2)||W^4\ ...\ W^{14})\ . \tag{2.3}$$

The entries W^ℓ are defined as follows: $W^\ell=\frac{1}{2}$ if in at least one sector (labeled by, say, V_i) of the model the corresponding boson has twisted boundary conditions (*i.e.*, $W^\ell_i\neq 0$); $W^\ell=0$, otherwise. For example, if

$$W=(0(0\ \frac{1}{2})^3||0^{11})\ , \tag{2.4}$$

then

$$V_i=(V^0_i(V^1_i\ W^1_i)(V^2_i\ W^2_i)(V^3_i\ W^3_i)||U^4_i\ ...\ U^{14}_i) \tag{2.5}$$

is *a priori* compatible with W. Here W^1_i, W^2_i and W^3_i correspond to the twists, U^4_i,...,U^{14}_i correspond to the shifts, and V^r_i, $r=0,1,2,3$, specify the fermionic spin structures.

The notation we have introduced proves convenient in describing the sectors of a given string model based on the orbifold group G. For G to be a finite discrete group, the element $g(V_i)$ must have a finite order $m_i\in\mathbf{N}$, *i.e.* $g^{m_i}(V_i)=1$. This implies that V^r_i and W^ℓ_i must be rational numbers, and the shift vector $\vec{U}_i$ must be a rational multiple of a vector in $\Gamma^{6,22}$; that is, $m_iV^r_i, m_iW^\ell_i\in\mathbf{Z}$, and $m_i\vec{U}_i\in\Gamma^{6,22}$. To describe all the elements of the group G, it is convenient to introduce the set of generating vectors $\{V_i\}$ such that $\overline{\alpha V}=\mathbf{0}$ if and only if $\alpha_i\equiv 0$. Here $\mathbf{0}$ is the null vector:

$$\mathbf{0}=(0(0\ 0)^3||0^{11})\ . \tag{2.6}$$

Also, $\alpha V\equiv\sum_i\alpha_iV_i$ (The summation is defined as $(V_i{+}V_j)^\ell=V^\ell_i{+}V^\ell_j$), α_i being integers that take values from 0 to m_i-1. The overbar notation is defined as follows: $\overline{\alpha V}\equiv\alpha V-\Delta(\alpha)$, and the components of $\overline{\alpha V}$ satisfy $-\frac{1}{2}\leq\overline{\alpha V}^r<\frac{1}{2}$, $0\leq\overline{\alpha W}^\ell<1$; here $\Delta^r(\alpha),\Delta^\ell(\alpha)\in\mathbf{Z}$. So the elements of the group G are in one-to-one correspondence with the vectors $\overline{\alpha V}$ and will be denoted by $g(\overline{\alpha V})$. It is precisely the abelian nature of G that allows this correspondence (by simply taking all possible linear combinations of the generating vectors V_i).

Now we can identify the sectors of the model. They are labeled by the vectors $\overline{\alpha V}$, and in a given sector $\overline{\alpha V}$ the monodromies of the string degrees of freedom are given by $\Phi(ze^{2\pi i},\overline{z}e^{-2\pi i})=g(\overline{\alpha V})\Phi(z,\overline{z})g^{-1}(\overline{\alpha V})$.

G is a symmetry of the Hilbert space of the original string model with $N=4$ supersymmetry compatible with the operator algebra of the underlying (super)conformal field theory. If $|\chi\rangle$ is a state in the original Hilbert space, $g(\overline{\alpha V})|\chi\rangle$ (where there must exist a representation of $g(\overline{\alpha V})$ via the vertex operators of the theory) also belongs to the same Hilbert space.

One consequence of this requirement is that G must commute with the Virasoro algebra, which is indeed the case for the class of abelian asymmetric orbifolds considered in this paper. We must also require that G (anti)commutes with the right-moving super-Virasoro algebra (which ensures space-time Lorentz invariance in the covariant gauge). This implies the following *supercurrent* constraint

$$V_i^\ell + W_i^\ell = V_i^0 \equiv s_i \ (\mathrm{mod}\ 1)\ , \quad \ell = 1,2,3\ . \tag{2.7}$$

Here s_i is the monodromy of the supercurrent $\overline{S}(\overline{z}e^{-2\pi i}) = \exp(2\pi i s_i)\overline{S}(\overline{z})$, which must satisfy $s_i \in \frac{1}{2}\mathbf{Z}$. Then the sectors with $\overline{\alpha V}^0 = 0$ give rise to space-time bosons, while the sectors with $\overline{\alpha V}^0 = -\frac{1}{2}$ give rise to space-time fermions.

B. Fermion and Boson Characters

Let us confine our attention to the orbifolds with a single twist of prime order, generated by the V_1 vector (The order of this twist is defined as the smallest positive integer t_1, such that $\forall \ell\ t_1 W_1^\ell \in \mathbf{Z}$; note that t_1 is a divisor of m_1).

In a given sector $\overline{\alpha V}$, the right- and left-moving Hamiltonians are given by the corresponding sums of the Hamiltonians for individual string degrees of freedom. The Hilbert space in the $\overline{\alpha V}$ sector is given by the momentum states $|\vec{P}_{\overline{\alpha V}} + \alpha\vec{U}\rangle$, and also the states obtained from these states by acting with the fermion and boson creation operators (oscillator excitations). In the untwisted sectors, that is, sectors $\overline{\alpha V}$ with $\alpha_1 = 0$, we have $\vec{P}_{\overline{\alpha V}} \in \Gamma^{6,22}$. In the twisted sectors $\overline{\alpha V}$ with $\alpha_1 \neq 0$, we have $\vec{P}_{\overline{\alpha V}} \in \tilde{I}$, where $\tilde{I}$ is the lattice dual to the lattice I, which in turn is the sublattice of $\Gamma^{6,22}$ invariant under the action of the twist part of the group element $g(V_1)$. This lattice must have a prime N_I, where N_I is the smallest positive integer such that for all vectors $\vec{P} \in \tilde{I}$, $N_I \vec{P}^2 \in 2\mathbf{Z}$; moreover, for the corresponding characters to have the correct modular transformation properties, it must be the case that either $N_I = 1$ (in which case I is an even self-dual lattice), or $N_I = t_1$ (in which case I is even but not self-dual).

Now we turn to expressing the group elements $g(\overline{\beta V})$ (in a given sector $\overline{\alpha V}$) in terms of the generators of twists $J^\ell_{\overline{\alpha V}}$, shifts $\vec{P}_{\overline{\alpha V}}$, and $U(1)$ rotations of the right-moving complex fermions $-N^\ell_{\overline{\alpha V}}$ (see Appendix A):

$$g(\overline{\beta V}) = \exp(2\pi i \beta V \cdot \mathcal{N}_{\overline{\alpha V}} + \frac{1}{2}\nu(\alpha_1,\beta_1)\vec{P}^2_{\overline{\alpha V}})\ . \tag{2.8}$$

Here $\nu(\alpha_1,\beta_1)$ is an integer taking value between 0 and $N_I - 1$ defined as

$$\alpha_1 \nu(\alpha_1,\beta_1) = \beta_1 \ (\mathrm{mod}\ N_I)\ , \quad \alpha_1 \neq 0\ , \tag{2.9}$$

and $\nu(0,\beta_1) \equiv 0$. $\mathcal{N}_{\overline{\alpha V}} = (N^r_{\overline{\alpha V}}, J^\ell_{\overline{\alpha V}}, \vec{P}_{\overline{\alpha V}})$, and the dot product is understood with respect to the following signature:

$$\beta V \cdot \mathcal{N}_{\overline{\alpha V}} \equiv \beta\vec{U} \cdot \vec{P}_{\overline{\alpha V}} + \sum_r (\beta V)^r N^r_{\overline{\alpha V}} \epsilon^r + \sum_\ell (\beta W)^\ell J^\ell_{\overline{\alpha V}} \epsilon^\ell\ . \tag{2.10}$$

The dot product of the vectors $\beta\vec{U}$ and $\vec{P}_{\overline{\alpha V}}$ is understood with respect to the metric Lorentzian metric diag$((-)^6,(+)^{22})$. The signature ϵ^r for fermions equals $+1$ for left-moving complex fermions, and -1 for right-moving complex fermions, respectively. The signature ϵ^ℓ for bosons equals -1 for left-moving complex bosons, and $+1$ for right-moving complex bosons, respectively.

In section III we express the one-loop modular invariant partition function for an orbifold model as a linear combination of the following characters:

$$\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \equiv \mathrm{Tr}(q^{H^L_{\overline{\alpha V}}}\ \overline{q}^{H^R_{\overline{\alpha V}}} g^{-1}(\overline{\beta V})) \ . \tag{2.11}$$

Here $H^L_{\overline{\alpha V}}$ and $H^R_{\overline{\alpha V}}$ are the left- and right-moving Hamiltonians, respectively. The trace is taken over the states in the Hilbert space corresponding to the sector $\overline{\alpha V}$. These characters can be computed as products of building blocks, or contributions of individual string degrees of freedom, which are reviewed in Appendix A. The result can be written as a product of the corresponding fermion and boson characters:

$$\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} = \overline{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \mathcal{Y}^{\overline{\alpha V}}_{\overline{\beta V}} \ . \tag{2.12}$$

The fermion characters $\overline{Z}^{\overline{\alpha V}}_{\overline{\beta V}}$ read:

$$\overline{Z}^{\overline{\alpha V}}_{\overline{\beta V}} = \prod_r \overline{Z}^{\overline{\alpha V}^r}_{\overline{\beta V}^r} \tag{2.13}$$

(The characters $\overline{Z}^v_u$ for a right-moving fermion are complex conjugates of the characters Z^v_u for a left-moving fermion given by (8.6)).

The boson characters $\mathcal{Y}^{\overline{\alpha V}}_{\overline{\beta V}}$ read:

$$\mathcal{Y}^{\overline{\alpha V}}_{\overline{\beta V}} = Y^{\alpha\vec{U}}_{\beta\vec{U}} \ , \quad \alpha_1 = \beta_1 = 0 \ , \tag{2.14}$$

$$\mathcal{Y}^{\overline{\alpha V}}_{\overline{\beta V}} = \xi(\alpha_1) Y^{\alpha_1,\alpha\vec{U}}_{\beta_1,\beta\vec{U}} \prod_{\ell=1}^{3} \overline{X}^{\overline{\alpha W}^\ell}_{\overline{\beta W}^\ell} \prod_{\ell=4}^{14} X^{\overline{\alpha W}^\ell}_{\overline{\beta W}^\ell} \ , \quad \alpha_1+\beta_1 \neq 0 \tag{2.15}$$

(The characters $\overline{X}^v_u$ for a right-moving boson are complex conjugates of the characters X^v_u for a left-moving boson given by (8.15)). The product over ℓ does *not* include terms with $\overline{\alpha W}^\ell = \overline{\beta W}^\ell = 0$.

$Y^{\alpha\vec{U}}_{\beta\vec{U}}$ are the characters for the even self-dual lattice $\Gamma^{6,22}$, whereas $Y^{\alpha_1,\alpha\vec{U}}_{\beta_1,\beta\vec{U}}$ are the characters for the lattice I (If I is an even self-dual lattice then instead of $Y^{\alpha_1,\alpha\vec{U}}_{\beta_1,\beta\vec{U}}$ we would have to use the characters similar to $Y^{\alpha\vec{U}}_{\beta\vec{U}}$ but defined for the lattice $I \subset \Gamma^{6,22}$):

$$Y^{\alpha\vec{U}}_{\beta\vec{U}} = \frac{1}{\eta^{22}(q)\overline{\eta}^6(\overline{q})} \sum_{\vec{P}\in\Gamma^{6,22}} q^{\frac{1}{2}(\vec{P}^L+\alpha\vec{U}^L)^2} \overline{q}^{\frac{1}{2}(\vec{P}^R+\alpha\vec{U}^R)^2} \exp(-2\pi i \beta\vec{U}\cdot\vec{P}) \ , \tag{2.16}$$

$$Y^{\alpha_1,\alpha\vec{U}}_{\beta_1,\beta\vec{U}} = \frac{1}{\eta^d(q)\overline{\eta}^{d'}} \sum_{\vec{P}\in\tilde{I}} q^{\frac{1}{2}(\vec{P}^L+\alpha\vec{U}^L)^2} \overline{q}^{\frac{1}{2}(\vec{P}^R+\alpha\vec{U}^R)^2} \exp(-2\pi i(\beta\vec{U}\cdot\vec{P} + \frac{1}{2}\nu(\alpha_1,\beta_1)\vec{P}^2)) \ . \tag{2.17}$$

Here I has the Lorentzian metric $((-)^{d'},(+)^{d})$. $\vec{P}^L$, $\vec{P}^R$, and $\vec{U}^L$, $\vec{U}^R$, are the left- and right-moving parts of the momentum and shift vectors, respectively.

The integers $\xi(\alpha_1)$ are nothing but the number of fixed points in the twisted sectors (M_I is the determinant of the metric of I) [2]:

$$\xi(\alpha_1) = M_I^{-\frac{1}{2}} \prod_{\ell} 2\sin(\pi\overline{\alpha W}^{\ell}) \ , \quad \alpha_1 \neq 0 \ , \tag{2.18}$$

and $\xi(0) = 1$ (The product over ℓ does *not* include terms with $\overline{\alpha W}^{\ell} = 0$).

Under the S- and T-modular transformations the characters $\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}}$ transform as follows:

$$\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \xrightarrow{S} \exp(2\pi i\overline{\alpha V}\cdot\overline{\beta V})\mathcal{Z}^{\overline{\beta V}}_{\overline{-\alpha V}} \ , \quad \alpha_1\beta_1 = 0 \ , \tag{2.19}$$

$$\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \xrightarrow{S} \exp(2\pi i(\overline{\alpha V}-W)\cdot(\overline{\beta V}-W)+\chi(\alpha_1,\beta_1))\mathcal{Z}^{\overline{\beta V}}_{\overline{-\alpha V}} \ , \quad \alpha_1\beta_1 \neq 0 \ , \tag{2.20}$$

$$\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \xrightarrow{T} \exp(2\pi i(\frac{1}{2}\overline{\alpha V}\cdot\overline{\alpha V}-\overline{\alpha V}\cdot W+\frac{1}{2})\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V-\alpha V+V_0}} \ . \tag{2.21}$$

Here V_0 is the vector with $-1/2$ entry for each world -sheet fermion and zero otherwise:

$$V_0 = (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^{11}) \ . \tag{2.22}$$

According to the above modular transformation properties of $\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}}$, V_0 is always among the generating vectors V_i (The sector corresponding to V_0 is the Ramond sector of the original heterotic string). The dot product of two vectors $\overline{\alpha V}$ and $\overline{\beta V}$ is defined as in (2.10). For example,

$$V_i\cdot V_j = \vec{U}_i\cdot\vec{U}_j + \sum_r V_i^r V_j^r \epsilon^r + \sum_{\ell} W_i^{\ell} W_j^{\ell} \epsilon^{\ell} \ . \tag{2.23}$$

III. ORBIFOLD RULES

In this section we derive the rules for constructing consistent string models in the framework discussed in section II. The contribution to the orbifold one-loop partition function is a linear combination of the characters $\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}}$:

$$\mathcal{Z} = \frac{1}{\prod_i m_i}\sum_{\alpha,\beta} C^{\overline{\alpha V}}_{\overline{\beta V}}\mathcal{Z}^{\overline{\alpha V}}_{\overline{\beta V}} \ . \tag{3.1}$$

The coefficients $C^{\overline{\alpha V}}_{\overline{\beta V}}$ must be such that (3.1) is modular invariant. Taking into account the modular transformation properties (2.19), we have the following constraints on the coefficients $C^{\overline{\alpha V}}_{\overline{\beta V}}$ coming from the requirement of modular invariance of (3.1):

$$S: \quad C^{\overline{\alpha V}}_{\overline{\beta V}}\exp(2\pi i\overline{\alpha V}\cdot\overline{\beta V}) = C^{\overline{\beta V}}_{\overline{-\alpha V}} \ , \quad \alpha_1\beta_1 = 0 \ , \tag{3.2}$$

$$C^{\overline{\alpha V}}_{\overline{\beta V}}\exp(2\pi i(\overline{\alpha V}-W)\cdot(\overline{\beta V}-W)+\chi(\alpha_1,\beta_1)) = C^{\overline{\beta V}}_{\overline{-\alpha V}} \ , \quad \alpha_1\beta_1 \neq 0 \ , \tag{3.3}$$

$$T: \quad C^{\overline{\alpha V}}_{\overline{\beta V}}\exp(2\pi i(\frac{1}{2}\overline{\alpha V}\cdot\overline{\alpha V}-\overline{\alpha V}\cdot W+\frac{1}{2}) = C^{\overline{\alpha V}}_{\overline{\beta V-\alpha V+V_0}} \ . \tag{3.4}$$

In addition to (3.2) we require that for any physical sector labeled by $\overline{\alpha V}$ the sum over β's in (3.1) form a proper projection with eigenvalues 0 or $\xi(\alpha_1)$. Specifically, this means that

$$\frac{1}{\prod_i m_i} \sum_\beta C^{\overline{\alpha V}}_{\overline{\beta V}} g^{-1}(\overline{\beta V}) = e^{2\pi i \alpha s} \eta(\overline{\alpha V}, \mathcal{N}_{\overline{\alpha V}}, \vec{P}^2_{\overline{\alpha V}}) \ , \tag{3.5}$$

where $\eta(\overline{\alpha V}, \mathcal{N}_{\overline{\alpha V}}, \vec{P}^2_{\overline{\alpha V}})$ takes values 0 or 1 depending on the values of α_i, $\mathcal{N}_{\overline{\alpha V}}$ and $\vec{P}^2_{\overline{\alpha V}}$. As a consequence, we have in this case

$$\mathcal{Z} = \mathrm{Tr}(q^{H^L_{\overline{\alpha V}}} \ \overline{q}^{H^R_{\overline{\alpha V}}} \xi(\alpha_1) e^{2\pi i \alpha s} \eta(\overline{\alpha V}, \mathcal{N}_{\overline{\alpha V}}, \vec{P}^2_{\overline{\alpha V}})) \ . \tag{3.6}$$

This is precisely the physically sensible projection; space-time bosons contribute into the partition function with the weight plus one, whereas space-time fermions contribute wiht the weight minus one (Each with degeneracy $\xi(\alpha_1)$ due to fixed points in the twisted sectors).

The formal solution to (3.5) is given by

$$C^{\overline{\alpha V}}_{\overline{\beta V}} = \exp(2\pi i [\beta \phi(\overline{\alpha V}) + \alpha s]) \ . \tag{3.7}$$

The phases $\phi_i(\overline{\alpha V})$ are constrained due to (3.2):

$$S: \quad \beta\phi(\overline{\alpha V}) + \alpha\phi(\overline{\beta V}) + \alpha s + \beta s + \overline{\alpha V} \cdot \overline{\beta V} = 0 \ (\mathrm{mod}\ 1) \ , \quad \alpha_1\beta_1 = 0 \ , \tag{3.8}$$

$$\begin{aligned} &\beta\phi(\overline{\alpha V}) + \alpha\phi(\overline{\beta V}) + \alpha s + \beta s + \\ &\qquad (\overline{\alpha V} - W) \cdot (\overline{\beta V} - W) + \chi(\alpha_1, \beta_1) = 0 \ (\mathrm{mod}\ 1) \ , \quad \alpha_1\beta_1 \neq 0 \ , \end{aligned} \tag{3.9}$$

$$T: \quad \alpha\phi(\overline{\alpha V}) + \phi_0(\overline{\alpha V}) + \frac{1}{2}\overline{\alpha V} \cdot \overline{\alpha V} - \overline{\alpha V} \cdot W + \frac{1}{2} = 0 \ (\mathrm{mod}\ 1) \ . \tag{3.10}$$

Provided that $2V_1 \cdot W \in \mathbf{Z}$, we have:

$$\chi(\alpha_1, \beta_1) + W \cdot W - \overline{\alpha V} \cdot W - \overline{\beta V} \cdot W \equiv 0 \ (\mathrm{mod}\ 1) \ , \quad \alpha_1\beta_1 \neq 0 \ , \tag{3.11}$$

and the solution to the system of equations (3.8), (3.9) and (3.10) is given by:

$$\phi_i(\overline{\alpha V}) = \sum_j k_{ij}\alpha_j + s_i - V_i \cdot \overline{\alpha V} \ (\mathrm{mod}\ 1) \ . \tag{3.12}$$

The structure constants k_{ij} must satisfy the following constraints:

$$k_{ij} + k_{ji} = V_i \cdot V_j \ (\mathrm{mod}\ 1) \ , \tag{3.13}$$

$$k_{ij} m_j = 0 \ (\mathrm{mod}\ 1) \ , \tag{3.14}$$

$$k_{ii} + k_{i0} + s_i + V_i \cdot W - \frac{1}{2} V_i \cdot V_i = 0 \ (\mathrm{mod}\ 1) \tag{3.15}$$

(Note that there is *no* summation over repeated indices).

All the states are projected out of the sum in (3.1) except those satisfying

$$V_i \cdot \mathcal{N}_{\overline{\alpha V}} = \phi_i(\overline{\alpha V}) \ (\mathrm{mod}\ 1) \ , \quad i \neq 1 \text{ or } \alpha_1 = 0 \ , \tag{3.16}$$

$$\alpha_1 V_1 \cdot \mathcal{N}_{\overline{\alpha V}} + \frac{1}{2}\vec{P}^2_{\overline{\alpha V}} = \alpha_1 \phi_1(\overline{\alpha V}) \ (\mathrm{mod}\ 1) \ , \quad \alpha_1 \neq 0 \ . \tag{3.17}$$

This is the spectrum generating formula (Note that in the twisted sectors ($\alpha_1 \neq 0$) all the states appear with the multiplicity $\xi(\alpha_1)$).

The states that satisfy the spectrum generating formula include both on- and off-shell states. The on-shell states must satisfy the additional constraint that the left- and right-moving energies are equal. In the $\overline{\alpha V}$ sector they are given by:

$$
\begin{aligned}
E^L_{\overline{\alpha V}} = -\frac{1}{2} + \sum_{\ell:\ \text{left}} \{\frac{1}{2}\overline{\alpha W}^\ell(1-\overline{\alpha W}^\ell) + \sum_{q=1}^\infty [(q+\overline{\alpha W}^\ell - 1)n^\ell_q + (q-\overline{\alpha W}^\ell)\overline{n}^\ell_q]\} + \\
\sum_{q=1}^\infty q(n^0_q + \overline{n}^0_q) + \frac{1}{2}(\vec{P}^L_{\overline{\alpha V}} + \vec{U}^L)^2 \ , \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
E^R_{\overline{\alpha V}} = -1 + \sum_{\ell:\ \text{right}} \{\frac{1}{2}\overline{\alpha W}^\ell(1-\overline{\alpha W}^\ell) + \sum_{q=1}^\infty [(q+\overline{\alpha W}^\ell - 1)m^\ell_q + (q-\overline{\alpha W}^\ell)\overline{m}^\ell_q]\} + \\
\sum_{q=1}^\infty q(m^0_q + \overline{m}^0_q) + \frac{1}{2}(\vec{P}^R_{\overline{\alpha V}} + \vec{U}^R)^2 + \\
\sum_r \{\frac{1}{2}(\overline{\alpha V}^r)^2 + \sum_{q=1}^\infty [(q+\overline{\alpha V}^r - \frac{1}{2})k^r_q + (q-\overline{\alpha V}^r - \frac{1}{2})\overline{k}^r_q]\} \ . \qquad (3.19)
\end{aligned}
$$

Here n^ℓ_q and $\overline{n}^\ell_q$ are occupation numbers for the left-moving bosons ϕ^ℓ_L, whereas m^ℓ_q and $\overline{m}^\ell_q$ are those for the right-moving bosons ϕ^ℓ_R. These take non-negative integer values. k^r_q and $\overline{k}^r_q$ are the occupation mumbers for the right-moving fermions, and they take only two values: 0 and 1. The occupation numbers are directly related to the boson and fermion number operators. For example, $N^r_{\overline{\alpha V}} = \sum_{q=1}^\infty (k^r_q - \overline{k}^r_q)$.

We conclude this section by summarizing the rules. To construct a consistent orbifold model, start with an $N=4$ space-time supersymmetric four dimensional heterotic string model with the internal momenta spanning an even self-dual lattice $\Gamma^{6,22}$ that possesses a $\mathbf{Z}_k$ symmetry (k is a prime). The invariant sublattice I must be such that $N_I = 1$ or k. Now one can introduce a set of vectors V_i (which includes V_0) that correspond to a particular embedding of the orbifold group Z_k. A given embedding is acceptable if and only if the set $\{V_i\}$ satisfies (2.7), and the set of constraints (3.13), (3.14) and (3.15) for some choices of the structure constants k_{ij}. Then, a particular choice of the set $\{V_i, k_{ij}\}$ defines a consistent string model. The complete spectrum (on- and off-shell states) of the model is given by the spectrum generating formula (3.16), which together with the left/right energy formula (3.18) determines the on-shell physical spectrum. In the next three sections we will illustrate the rules with some examples.

IV. ASYMMETRIC ORBIFOLDS AND WILSON LINES

Consider an even self-dual Lorentzian lattice $\Gamma^{6,22} = \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^8 \otimes \Gamma^8$. Here we take $\Gamma^{2,2}$ to be the even self-dual Lorentzian lattice spanned by the vectors $(\overline{p}, p)$ such that $\overline{p}, p \in \tilde{\Gamma}^2$ ($SU(3)$ weight lattice), and $p - \overline{p} \in \Gamma^2$; Γ^2 ($SU(3)$ root lattice). Γ^8 is the E_8 root lattice. The lattice $\Gamma^{6,22}$ has a $\mathbf{Z}_3$ symmetry under 120° rotations of the right-moving momenta while the left-moving momenta are untouched. Here we discuss asymmetric orbifolds obtained via modding out by this discrete symmetry.

Consider the following set of generating vectors

$$W = (0(0\ \frac{1}{2})^3||0^3|0^8|0^8)\ , \tag{4.1}$$

$$V_0 = (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^3|0^8|0^8)\ , \tag{4.2}$$

$$V_1 = (0(-\frac{1}{3}\ \frac{1}{3})^3||0^3|v^I)\ , \tag{4.3}$$

$$V_2 = (0(0\ 0)^3||w\ 0^2|A^I)\ , \tag{4.4}$$

$$V_3 = (0(0\ 0)^3||0\ (\frac{1}{2}\zeta)^2|\tilde{A}^I) \tag{4.5}$$

(Here we have chosen the basis where all the right-moving fermions, as well as the bosons corresponding to the $\Gamma^{2,2}\otimes\Gamma^{2,2}\otimes\Gamma^{2,2}$ sublattice, are complex; the first single vertical line separates the three complex and sixteen real left-moving bosons, the latter corresponding to the $\Gamma^8\otimes\Gamma^8$ sublattice; the second single vertical line separates the real bosons corresponding to the first and second Γ^8 sublattices, respectively). Note that $V_1\cdot W=\frac{1}{2}$. Let us choose $3v^I, 3A^I, 2\tilde{A}^I\in\Gamma^8\otimes\Gamma^8$; $w\in\tilde{\Gamma}^2$, $w^2=\frac{2}{3}$; $\zeta\in\Gamma^2$, $\zeta^2=2$. Then $m_1=t_1=m_2=3$, $m_3=2$. The matrix of the dot products $V_i\cdot V_j$ reads (for simplicity we choose $\tilde{A}^I v^I=\tilde{A}^I A^I=0$):

$$V_i\cdot V_j = \begin{pmatrix} -1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & (v^I)^2 & v^I A^I & 0 \\ 0 & v^I A^I & (A^I)^2+\frac{2}{3} & 0 \\ 0 & 0 & 0 & (\tilde{A}^I)^2+1 \end{pmatrix}\ . \tag{4.6}$$

The structure constants k_{ij} are given by:

$$k_{ij} = \begin{pmatrix} k_{00} & 0 & 0 & k_{30} \\ \frac{1}{2} & \frac{1}{2}(v^I)^2 & -k_{21}+v^I A^I & 0 \\ 0 & k_{21} & \frac{1}{2}(A^I)^2+\frac{1}{3} & 0 \\ k_{30} & 0 & 0 & k_{30}+\frac{1}{2}(\tilde{A}^I)^2+\frac{1}{2} \end{pmatrix}\ . \tag{4.7}$$

To satisfy the constraints (3.14) we must have:

$$3v^I A^I, (\tilde{A}^I)^2\in\mathbf{Z}\ ,\quad 3(v^I)^2, 3(A^I)^2\in 2\mathbf{Z}\ . \tag{4.8}$$

The invariant sublattice (*i.e.*, the sublattice of $\Gamma^{6,22}$ invariant under the twist part of V_1) is $I=\Gamma^2\otimes\Gamma^2\otimes\Gamma^2\otimes\Gamma^8\otimes\Gamma^8$. Note that its dual lattice is $\tilde{I}=\tilde{\Gamma}^2\otimes\tilde{\Gamma}^2\otimes\tilde{\Gamma}^2\otimes\Gamma^8\otimes\Gamma^8$, and $N_I=3(=t_1)$. The determinant of the metric of I is $M_I=3^3=27$, and

$$\xi(\alpha_1) = M_I^{-\frac{1}{2}}[2\sin(\frac{\alpha_1\pi}{3})]^3 = 1\ ,\quad \alpha_1=1,2\ . \tag{4.9}$$

Therefore, the number of fixed points in each of the twisted sectors is one.

Before discussing the orbifold models generated by these vectors, we note that the model, which we will refer to as $N0$, generated by the set $\{V_0\}$ (which contains only V_0) is an $N=4$ space-time supersymmetric Narain model [7]. Its massless spectrum consists of the $N=4$ supergravity multiplet (graviton, four gravitinos, six vector bosons, four spin-$\frac{1}{2}$ fermions, and one complex scalar (dilaton plus axion)), and also the $N=4$ super-Yang-Mills multiplet (gauge bosons, four spin-$\frac{1}{2}$ fermions, and six real scalars) transforming in the adjoint of the gauge group $E_8\otimes E_8\otimes SU(3)\otimes SU(3)\otimes SU(3)$. The bosons come from the $\mathbf{0}$ sector, whereas their superpartners come from the V_0 sector.

A. Asymmetric Orbifolds without Wilson Lines

Next, consider the model, which we will refer to as $A1$, generated by the set $\{V_0, V_1\}$, with $v^I = (\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ 0^5|0^8)$. This is an asymmetric orbifold model without Wilson lines. It possesses $N=1$ space-time supersymmetry. The sectors $\overline{\alpha V}$, $\alpha_0 = 0$, give rise to bosons, whereas their superpartners come from the sectors $\overline{\alpha V}$, $\alpha_0 = 1$.

First consider the untwisted sectors $\overline{\alpha V} = \alpha_0 V_0$ ($\alpha_1 = 0$). In these sectors the momenta $\vec{P}_{\overline{\alpha V}} = (\overline{p}_1, \overline{p}_2, \overline{p}_3 || p_1, p_2, p_3 | p^I)$ ($\overline{p}_\ell, p_\ell \in \tilde{\Gamma}^2$, $p_\ell - \overline{p}_\ell \in \Gamma^2$, $\ell = 1,2,3$; $p^I \in \Gamma^8 \otimes \Gamma^8$) span $\Gamma^{6,22}$.

The spectrum generating formula in the untwisted sectors reads:

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}\sum_{r=0}^{3} N^r_{\overline{\alpha V}} = k_{00}\alpha_0 + \frac{1}{2}\ (\mathrm{mod}\ 1)\ , \tag{4.10}$$

$$V_1 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{3}(\sum_{r=1}^{3} N^r_{\overline{\alpha V}} + \sum_{\ell=1}^{3} J^\ell_{\overline{\alpha V}}) + v^I p^I = 0\ (\mathrm{mod}\ 1)\ . \tag{4.11}$$

Thus, the untwisted sectors $\mathbf{0}$ ($\alpha_0 = 0$) and V_0 ($\alpha_0 = 1$) give rise to the following massless states: (i) The graviton $N=1$ supermultiplet (graviton and one gravitino); (ii) the dilaton $N=1$ supermultiplet (one spin-$\frac{1}{2}$ fermion and one complex scalar (dilaton plus axion)); (iii) $N=1$ Yang-Mills supermultiplet (gauge bosons and one spin-$\frac{1}{2}$ fermion) transforming in the adjoint of $SU(3) \otimes E_6 \otimes E_8 \otimes (SU(3))^3$ (These states satisfy $v^I p^I \in \mathbf{Z}$; the first $SU(3)$ subgroup arises in the breaking $E_8 \supset SU(3) \otimes E_6$ due to the shift v^I; the other three $SU(3)$ subgroups come from the $\Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^{2,2}$ lattice); (iv) three $N=1$ chiral supermultiplets transforming in the representation $(\mathbf{3}, \mathbf{27}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ of the gauge group (These states have $v^I p^I \in \pm\frac{1}{3} + \mathbf{Z}$ for those transforming in $\mathbf{3}$ and $\overline{\mathbf{3}}$, respectively). The chirality of these states depends on the choice of k_{00}: For $k_{00} = 1/2$ they are left-movers, whereas for $k_{00} = 0$ they are right-movers. For definiteness in the following we choose $k_{00} = 1/2$.

Next, consider the twisted sectors $\overline{\alpha V} = \overline{\alpha_0 V_0 + \alpha_1 V_1}$ ($\alpha_1 = 1, 2$). In these sectors the momenta $\vec{P}_{\overline{\alpha V}} = (0,0,0 || p_1, p_2, p_3 | p^I)$ ($p_\ell \in \tilde{\Gamma}^2$, $\ell = 1,2,3$; $p^I \in \Gamma^8 \otimes \Gamma^8$) span $\tilde{I}$.

The spectrum generating formula in the twisted sectors reads ($(v^I)^2 = 2/3$)

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}\sum_{r=0}^{3} N^r_{\overline{\alpha V}} = \alpha_0(k_{00} + \frac{1}{2}\alpha_1)\ (\mathrm{mod}\ 1)\ , \tag{4.12}$$

$$\alpha_1 V_1 \cdot \mathcal{N}_{\overline{\alpha V}} + \frac{1}{2}\vec{P}^2_{\overline{\alpha V}} = \frac{\alpha_1}{3}(\sum_{r=1}^{3} N^r_{\overline{\alpha V}} + \sum_{\ell=1}^{3} J^\ell_{\overline{\alpha V}}) + v^I p^I + \frac{1}{2}\sum_{\ell=1}^{3} p_\ell^2 = -\frac{1}{3}\alpha_1^2\ (\mathrm{mod}\ 1)\ . \tag{4.13}$$

The twisted sectors give rise to one left-moving chiral $N=1$ supermultiplet transforming in the following representations of the gauge group: (i) $(\mathbf{3}, \overline{\mathbf{27}}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ (these have $v^I p^I \in -\frac{1}{3} + \mathbf{Z}$); ($ii$) $(\mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$, where one of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other two are $\mathbf{1}$ (these have $v^I p^I \in \frac{1}{3} + \mathbf{Z}$); ($iii$) $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$, where two of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $v^I p^I \in \mathbf{Z}$).

The $A1$ model is the original asymmetric orbifold model of [2].

B. Turning on Wilson Lines

Wilson lines are incorporated via vectors V_i that contain only lattice shifts. Thus, consider the model, which we will refer to as $N2$, generated by the vectors V_0 and V_2, with

$A^I = (\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ 0^5|(-\frac{1}{3})\ (-\frac{1}{3})\ (-\frac{2}{3})\ 0^5)$, and w being a weight vector corresponding to the $\mathbf{3}$ irrep of $SU(3)$. This model is an $N=4$ space-time supersymmetric Narain model. The sectors with $\alpha_0 = 0$ give rise to bosons, whereas their supperpartners are supplied by the sectors with $\alpha_0 = 1$.

The spectrum generating formula reads $((A^I)^2 = 4/3)$:

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}\sum_{r=0}^{3} N^r_{\overline{\alpha V}} = k_{00}\alpha_0 + \frac{1}{2}\ (\mathrm{mod}\ 1)\ , \tag{4.14}$$

$$V_2 \cdot \mathcal{N}_{\overline{\alpha V}} = wp_1 + A^I p^I = 0\ (\mathrm{mod}\ 1)\ . \tag{4.15}$$

The momenta that survive this projection are given by:

$$\vec{P} = (\overline{q} + 3A^I p^I w, \overline{p}_2, \overline{p}_3 || q + (3A^I p^I + \alpha_2) w, p_2, p_3 | p^I + \alpha_2 A^I)\ , \tag{4.16}$$

where $\overline{q}, q \in \Gamma^2$.

Thus, the "unshifted" sectors ($\alpha_2 = 0$) give rise to the following massless states: (i) The $N=4$ supergravity multiplet; (ii) $N=4$ vector supermultiplet transforming in the adjoint of $SU(3)\otimes E_6 \otimes SU(3) \otimes E_6 \otimes SU(3) \otimes (SU(3))^2$.

The "shifted" sectors ($\alpha_2 = 1$ and 2) give rise to the $N=4$ vector supermultiplets transforming in the representations $(\overline{\mathbf{3}},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$ and $(\mathbf{3},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$, respectively. These states combine with those in the "unshifted" sectors and give rise to the $N=4$ Yang-Mills supermultiplet transforming in the adjoint of the resulting gauge group $E_6 \otimes E_6 \otimes E_6 \otimes SU(3) \otimes SU(3)$.

C. Asymmetric Orbifolds with Wilson Lines

Now we turn to asymmetric orbifold models with Wilson lines. Consider the model, which we will refer to as $A2$, generated by the vectors V_0, V_1 and V_2. This model has $N=1$ space-time supersymmetry.

First consider the untwisted sectors $\overline{\alpha V} = \alpha_0 V_0 + \alpha_2 V_2$. The spectrum generating formula reads:

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}\sum_{r=0}^{3} N^r_{\overline{\alpha V}} = k_{00}\alpha_0 + \frac{1}{2}\ (\mathrm{mod}\ 1)\ , \tag{4.17}$$

$$V_1 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{3}(\sum_{r=1}^{3} N^r_{\overline{\alpha V}} + \sum_{\ell=1}^{3} J^\ell_{\overline{\alpha V}}) + v^I p^I = -k_{21}\alpha_2\ (\mathrm{mod}\ 1)\ , \tag{4.18}$$

$$V_2 \cdot \mathcal{N}_{\overline{\alpha V}} = wp_1 + A^I p^I = 0\ (\mathrm{mod}\ 1)\ . \tag{4.19}$$

If $k_{21} = 0$, then the gauge group of the $A2$ model is the same as that of $N2$. If $k_{21} \neq 0$, then the gauge group is broken down to $(E_6)^2 \otimes (SU(3))^5$. For definiteness we will choose $k_{21} = 0$. Then the untwisted sectors contribute the following massless states: (i) The graviton $N=1$ supermultiplet; (ii) the dilaton $N=1$ supermultiplet; (iii) $N=1$ Yang-Mills supermultiplet transforming in the adjoint of the gauge group $E_6 \otimes E_6 \otimes E_6 \otimes (SU(3))^2$ (these states satisfy $v^I p^I \in \mathbf{Z}$).

Next, consider the twisted sectors $\overline{\alpha V} = \overline{\alpha_0 V_0 + \alpha_1 V_1}$ ($\alpha_1 = 1, 2$). The spectrum generating formula in the twisted sectors reads ($(v^I)^2 = 2/3$):

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2} \sum_{r=0}^{3} N^r_{\overline{\alpha V}} = \alpha_0 (k_{00} + \frac{1}{2} \alpha_1) \text{ (mod 1)} , \tag{4.20}$$

$$\alpha_1 V_1 \cdot \mathcal{N}_{\overline{\alpha V}} + \frac{1}{2} \vec{P}^2_{\overline{\alpha V}} = \frac{\alpha_1}{3} (\sum_{r=1}^{3} N^r_{\overline{\alpha V}} + \sum_{\ell=1}^{3} J^\ell_{\overline{\alpha V}}) + v^I p^I + \frac{1}{2} \sum_{\ell=1}^{3} p_\ell^2 = -\frac{1}{3} \alpha_1^2 \text{ (mod 1)} , \tag{4.21}$$

$$V_2 \cdot \mathcal{N}_{\overline{\alpha V}} = w p_1 + A^I p^I = -\frac{2}{3} \alpha_1 \text{ (mod 1)} . \tag{4.22}$$

The twisted sectors with $\alpha_2 = 0$ give rise to one left-moving chiral $N = 1$ supermultiplet transforming in the following representations of the group $SU(3) \otimes E_6 \otimes SU(3) \otimes E_6 \otimes SU(3) \otimes (SU(3))^2$: (*i*) $(\mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $A^I p^I \in \frac{1}{3} + \mathbf{Z}$); (*ii*) $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \overline{\mathbf{3}}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $A^I p^I \in \mathbf{Z}$).

The twisted sectors with $\alpha_2 = 2$ give rise to one left-moving chiral $N = 1$ supermultiplet transforming in the following representations of the group $SU(3) \otimes E_6 \otimes SU(3) \otimes E_6 \otimes SU(3) \otimes (SU(3))^2$: (*i*) $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $A^I p^I \in -\frac{1}{3} + \mathbf{Z}$); (*ii*) $(\mathbf{1}, \mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{3}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $A^I p^I \in \mathbf{Z}$).

The twisted sectors with $\alpha_2 = 1$ give rise to one left-moving chiral $N = 1$ supermultiplet transforming in the following representations of the group $SU(3) \otimes E_6 \otimes SU(3) \otimes E_6 \otimes SU(3) \otimes (SU(3))^2$: $(\mathbf{3}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$ (these have $A^I p^I \in \mathbf{Z}$).

Thus, the states from the twisted sectors combine into the following representations of the resulting gauge group $E_6 \otimes E_6 \otimes E_6 \otimes SU(3) \otimes SU(3)$: There is one left-moving chiral $N = 1$ supermultiplet in the representations $(\mathbf{27}, \mathbf{1}, \mathbf{1}, \mathbf{x}, \mathbf{y})$, $(\mathbf{1}, \mathbf{27}, \mathbf{1}, \mathbf{x}, \mathbf{y})$, $(\mathbf{1}, \mathbf{1}, \mathbf{27}, \mathbf{x}, \mathbf{y})$, where one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{3}$ or $\overline{\mathbf{3}}$, and the other one is $\mathbf{1}$.

Finally, we briefly discuss the model, which we refer to as $A3$, generated by the vectors V_0, V_1, V_2 and V_3, with $\tilde{A}^I = (0^7\ 1 | 0^7\ 1)$. Note that if $k_{30} = 1/2$, the supersymmetry is broken to $N = 0$. For definiteness we will choose $k_{30} = 0$. Then the model possesses $N = 1$ space-time supersymmetry. The sectors $\overline{\alpha V}$ with $\alpha_3 = 1$ contribute massive string states only. However, some of the states from the other sectors are projected out due to the presence of the Wilson line generated by V_3. Thus, the gauge symmetry is broken down to $E_6 \otimes (SO(10) \otimes U(1))^2 \otimes (SU(2) \otimes U(1))^2$. In the twisted sectors we have chiral $N = 1$ supermultiplets in the following representations of $E_6 \otimes SO(10) \otimes SO(10) \otimes SU(2) \otimes SU(2)$ (Here we drop the $U(1)$ charges for the sake of simplicity; they are straightforward to work out, however): Four fields (with different $U(1)$ charges) in $(\mathbf{27}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$; one field in each of $(\mathbf{1}, \mathbf{16}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\mathbf{1}, \mathbf{16}, \mathbf{1}, \mathbf{1}, \mathbf{2})$, $(\mathbf{1}, \mathbf{1}, \mathbf{16}, \mathbf{2}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1}, \mathbf{16}, \mathbf{1}, \mathbf{2})$. We also have Higgs $N = 1$ supermultiplets in the following representations: Four fields (with different $U(1)$ charges) in each of $(\mathbf{1}, \mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1}, \mathbf{10}, \mathbf{1}, \mathbf{1})$. There are also four singlets in the twisted sectors.

The examples we have considered indicate that incorporating Wilson lines into asymmetric orbifolds may be useful for controlling the gauge symmetry and the number of chiral generatitions in a given model. As we discuss in the next section, asymmetric orbifolds are also very handy in constructing models with reduced rank.

V. RANK REDUCTION

We start an even self-dual Lorentzian lattice $\Gamma^{6,22} = \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^8 \otimes \Gamma^8$. Here we take $\Gamma^{2,2}$ to be the even self-dual Lorentzian lattice spanned by the vectors $(\overline{p}, p)$ such that $\overline{p}, p \in \tilde{\Gamma}^2$ ($SU(3)$ weight lattice), and $p - \overline{p} \in \Gamma^2$ ($SU(3)$ root lattice). Γ^8 is the E_8 root lattice.

Consider the following set of generating vectors (Here we choose $\zeta \in \Gamma^2$, $\zeta^2 = 2$).

$$W = (0(0\ \frac{1}{2})^2(0\ 0)||(\frac{1}{2})^2\ 0|(\frac{1}{2})^4|0^4)\ , \tag{5.1}$$

$$V_0 = (-\frac{1}{2}(-\frac{1}{2}\ 0)^3|0^3|0^4|0^4)\ , \tag{5.2}$$

$$V_1 = (0(-\frac{1}{2}\ \frac{1}{2})^2(0\ 0)|(\frac{1}{2})^2(\frac{1}{2}\xi)|(\frac{1}{2})^4|0^4)\ . \tag{5.3}$$

Here all the bosons are complexified. The complexification for the sixteen $E_8 \otimes E_8$ real bosons φ_1^I and φ_2^I, $I = 1, ..., 8$ is chosen as $\phi^7 = \frac{1}{2}(\varphi_1^1 - \varphi_2^1 + i(\varphi_1^2 - \varphi_2^2)), ..., \phi^{10} = \frac{1}{2}(\varphi_1^7 - \varphi_2^7 + i(\varphi_1^8 - \varphi_2^8))$; $\phi^{11} = \frac{1}{2}(\varphi_1^1 + \varphi_2^1 + i(\varphi_1^2 + \varphi_2^2)), ..., \phi^{14} = \frac{1}{2}(\varphi_1^7 + \varphi_2^7 + i(\varphi_1^8 + \varphi_2^8))$. The second single vertical line in V_i separates ϕ^ℓ, $\ell = 7, ..., 10$, from ϕ^ℓ, $\ell = 11, ..., 14$.

The twist V_1 has the following action on ϕ^ℓ, $\ell = 7, ..., 14$: $\phi^\ell \to -\phi^\ell$, $\ell = 7, ..., 10$, and $\phi^\ell \to \phi^\ell$, $\ell = 11, ..., 14$. This corresponds to moding out by the permutational symmetry $\phi_1^I \leftrightarrow \phi_2^I$, that is, the outer automorphism of the $\Gamma^8 \otimes \Gamma^8$ lattice.

The matrix of the dot products $V_i \cdot V_j$ and structure constants k_{ij} for this model are given by

$$V_i \cdot V_j = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}\ , \quad k_{ij} = \begin{pmatrix} k_{00} & k_{10} + \frac{1}{2} \\ k_{10} & k_{10} + \frac{1}{2} \end{pmatrix}\ . \tag{5.4}$$

The invariant sublattice is $I = \Gamma^{2,2} \otimes \Delta^8$, where $\Delta^8 \equiv \{\sqrt{2} p^I | p^I \in \Gamma^8\}$. The dual lattice is $\tilde{I} = \Gamma^{2,2} \otimes \tilde{\Delta}^8$, where $\tilde{\Delta}^8 \equiv \{\frac{1}{\sqrt{2}} p^I | p^I \in \Gamma^8\}$ is the lattice dual to Δ^8. Note that $N_I = 2 (= t_1)$. The determinant of the metric of I is $M_I = 2^8$, and in the twisted sectors ($\alpha_1 = 1$) we have

$$\xi(\alpha_1) = M_I^{-\frac{1}{2}}[2\sin(\frac{\alpha_1 \pi}{2})]^8 = 16\ . \tag{5.5}$$

The model generated by the set $\{V_0, V_1\}$ has $N = 2$ space-time supersymmetry. The bosons come from the sectors with $\alpha_0 = 0$, whereas the fermions arise in the sectors with $\alpha_0 = 1$.

First consider the untwisted sectors $\overline{\alpha V} = \alpha_0 V_0$ ($\alpha_1 = 0$). In these sectors the momenta $\vec{P}_{\overline{\alpha V}} = (\overline{p}_1, \overline{p}_2, \overline{p}_3||p_1, p_2, p_3|q)$ ($\overline{p}_\ell, p_\ell \in \tilde{\Gamma}^2$, $p_\ell - \overline{p}_\ell \in \Gamma^2$, $\ell = 1, 2, 3$; $q \in \Gamma^8 \otimes \Gamma^8$) span $\Gamma^{6,22}$.

The spectum generating formula in the untwisted sectors reads

$$V_0 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}\sum_{r=0}^{3} N^r_{\overline{\alpha V}} = k_{00}\alpha_0 + \frac{1}{2}\ (\mathrm{mod}\ 1)\ , \tag{5.6}$$

$$V_1 \cdot \mathcal{N}_{\overline{\alpha V}} = \frac{1}{2}(N^1_{\overline{\alpha V}} + N^2_{\overline{\alpha V}} + J^1_{\overline{\alpha V}} + J^2_{\overline{\alpha V}} - J^4_{\overline{\alpha V}} - J^5_{\overline{\alpha V}} - \\ - \sum_{\ell=7}^{10} J^\ell_{\overline{\alpha V}} + \zeta p_3) = (k_{10} + \frac{1}{2})\alpha_0\ (\mathrm{mod}\ 1)\ . \tag{5.7}$$

The untwisted sectors $\mathbf{0}$ and V_0 give rise to the following massless states: (i) The $N=2$ supergravity multiplet; (ii) $N=2$ Yang-Mills supermultiplet transforming in the adjoint of the gauge group $E_8\otimes SU(2)\otimes SU(2)\otimes(SU(2)\otimes U(1))$ gauge group; (iii) two $N=2$ scalar supermultiplets in each of the representations $(\mathbf{248},\mathbf{1},\mathbf{1},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{5},\mathbf{1},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{5},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(3)$ and $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(-3)$ of $E_8\otimes SU(2)\otimes SU(2)\otimes(SU(2)\otimes U(1))$ (The $U(1)$ charge is given in regular font in the parentheses).

In the twisted sectors $\overline{\alpha V}=\overline{\alpha_0 V_0+V_1}$ the momenta $\vec{P}_{\overline{\alpha V}}=(0,0,\overline{p}_3||0,0,p_3|0^4|Q)$ ($\overline{p}_3,p_3\in\tilde{\Gamma}^2$, $p_3-\overline{p}_3\in\Gamma^2$; $Q\in\tilde{\Delta}^8$) span $\tilde{I}$.

the spectrum generating formula in the twisted sectors reads ($\alpha_1=1$):

$$V_0\cdot\mathcal{N}_{\overline{\alpha V}}=\frac{1}{2}\sum_{r=0}^{3}N^r_{\overline{\alpha V}}=k_{00}\alpha_0+k_{10}+\frac{1}{2}\ (\mathrm{mod}\ 1)\ , \tag{5.8}$$

$$\begin{aligned}V_1\cdot\mathcal{N}_{\overline{\alpha V}}+\frac{1}{2}\vec{P}^2_{\overline{\alpha V}}=\frac{1}{2}(N^1_{\overline{\alpha V}}+N^2_{\overline{\alpha V}}+J^1_{\overline{\alpha V}}+J^2_{\overline{\alpha V}}-J^4_{\overline{\alpha V}}-J^5_{\overline{\alpha V}}-\\ -\sum_{\ell=7}^{10}J^\ell_{\overline{\alpha V}}+\zeta p_3+Q^2)=(k_{10}+\frac{1}{2})(\alpha_0+1)\ (\mathrm{mod}\ 1)\ .\end{aligned} \tag{5.9}$$

Thus, the twisted sectors give rise to the following massless states: Four $N=2$ scalar supermultiplets transforming in the representation $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})(0)$ of the gauge group.

Note the rank reduction of the gauge group from twenty-two twelve. This indicates that the gauge group is realized via a higher level Kac-Moody algebra. Also note appearance of massless states in $\mathbf{5}$ irrep of $SU(2)$. This is too a sign of a higher level Kac-Moody algebra realization. A careful analysis of the underlying conformal field theory unambiguously determines the levels of each subgroup: E_8 is realized at level two (It arises in the breaking $(E_8)_1\otimes(E_8)_1\supset(E_8)_2\times$ (Ising Model); the central charge of $(E_8)_2$ is $c=15/2$, whereas an Ising model has $c=1/2$); the first two $SU(2)$ subgroups are realized at level four (as a result of a special breaking $SU(3)_1\supset SU(2)_4$; note that under this breaking $\mathbf{8}=\mathbf{3}+\mathbf{5}$, and the central charge of both $SU(3)_1$ and $SU(2)_4$ is $c=2$); the last $SU(2)$ is realized at level one (It arises in a regular breaking $SU(3)_1\supset SU(2)_1\otimes U(1)$).

VI. OTHER EXAMPLES

For completeness, in this section we briefly discuss symmetric orbifolds with and without Wilson lines. We construct these examples (familiar from the previous developments [1,6]) using the rules given in section III. This is to further clarify the rules and notation.

A. Symmetric Orbifolds without Wilson Lines

We start from the lattice $\Gamma^{6,22}=\Gamma^{2,2}\otimes\Gamma^{2,2}\otimes\Gamma^{2,2}\otimes\Gamma^8\otimes\Gamma^8$ considered in the previous section. Consider the model generated by the following set of vectors

$$W=(0(0\ \frac{1}{2})^3||(\frac{1}{2})^3|0^8|0^8)\ , \tag{6.1}$$

$$V_0 = (-\frac{1}{2}(-\frac{1}{2}\ 0)^3||0^3|0^8|0^8)\ , \tag{6.2}$$

$$V_1 = (0(-\frac{1}{3}\ \frac{1}{3})^3||(\frac{1}{3})^3|v^I)\ , \tag{6.3}$$

where the order of the shift v^I is three, *i.e.*, $3v^I \in \Gamma^8 \otimes \Gamma^8$. The matrix of the dot prodocts $V_i \cdot V_j$ and structure constants k_{ij} are given by

$$V_i \cdot V_j = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & (v^I)^2 - \frac{1}{3} \end{pmatrix}\ , \quad k_{ij} = \begin{pmatrix} k_{00} & 0 \\ \frac{1}{2} & \frac{1}{2}(v^I)^2 + \frac{1}{3} \end{pmatrix}\ . \tag{6.4}$$

Due to the constraint (3.14) we have $m_1 k_{11} \in \mathbf{Z}$ ($m_1 = t_1 = 3$), and the shift v^I must satisfy the condition

$$3(v^I)^2 \in 2\mathbf{Z}\ . \tag{6.5}$$

The sublattice $I \subset \Gamma^{6,22}$ invariant under the twist part of V_1 is given by $I = \Gamma^8 \otimes \Gamma^8$, and $N_I = M_I = 1$ (since it is an even self-dual lattice). Therefore, the multiplicity of states $\xi(\alpha)$ in the twisted sectors ($\alpha = 1, 2$) is given by

$$\xi(\alpha_1) = M_I^{-\frac{1}{2}}[2\sin(\frac{\alpha_1 \pi}{3})]^6 = 27\ . \tag{6.6}$$

If we take $v^I = (\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ 0^5|0^8)$, then we obtain the original symmetric orbifold model of [1]. The gauge group of this model is $E_6 \otimes SU(3) \otimes E_8 \otimes (U(1))^6$ (The six $U(1)$'s survive because before orbifolding the $\Gamma^{2,2}$ sublattices were at the special radius of enhanced gauge symmetry; after orbifolding the original $SU(3)$ gauge group undergoes a regular breaking $SU(3) \supset U(1) \otimes U(1)$). The twisted sectors give rise to $\xi(\alpha_1) = 27$ chiral matter fields (Which is the number of fixed points of the original Z-orbifold).

B. Symmetric Orbifolds with Wilson Lines

Now let us start from the lattice $\Gamma^{6,22}$ of the $N2$ Narain model discussed in subsection IVB. Consider the model generated by the same vectors as in the previous subsection (The structure constants are then the same as for the previous model).

The invariant sublattice is given by $I = \{p^I | p^I A^I \in \mathbf{Z}, p^I \in \Gamma^8 \otimes \Gamma^8\}$ (Here $A^I = (\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ 0^5|(-\frac{1}{3})\ (-\frac{1}{3})\ (-\frac{2}{3})\ 0^5)$ is the Wilson line). The dual lattice is $\tilde{I} = \{p^I + A^I | p^I \in \Gamma^8 \otimes \Gamma^8\}$, and $N_I = 3 (= t_1)$. The determinant of the metric of I is $M_I = 9$, and

$$\xi(\alpha_1) = M_I^{-\frac{1}{2}}[2\sin(\frac{\alpha_1 \pi}{3})]^6 = 9\ , \quad \alpha_1 = 1, 2\ . \tag{6.7}$$

This model is one of the symmetric orbifold models with one Wilson line [6] at the special radius. The model has $N = 1$ space-time supersymmetry, and the gauge group is $E_6 \otimes E_6 \otimes (SU(6) \otimes U(1)) \otimes (U(1))^4$ (Recall that the $N2$ model has the gauge group $(E_6)^3 \otimes SU(3)^2$; after orbifolding one of the E_6 subgroups undergoes a regular breaking $E_6 \supset SU(6) \otimes U(1)$, and each of the $SU(3)$'s breaks to $U(1)^2$). The twisted sectors contribute chiral matter fields with the multiplicity $\xi(\alpha_1) = 9$.

VII. DISCUSSION AND REMARKS

We have seen that the rules for asymmetric orbifolds are almost as easy to use as those for the free fermionic string models. The rules for asymmetric orbifold model-building are summarized at the end of section III. These rules are for models with only one twist. Can the rules given in this paper be generalized to models with multi-twists? In cases where the twists do not overlap, this generalization is straightforward. In more general situations, the twist (plus shift) operators typically do not commute with each other. Can the rules given in this paper be further generalized to include such non-abelian orbifolds?

Recall the rules for the free fermionic string models [3]. It is well-known that many of the free fermionic string models, in particular, the ones with reduced rank gauge groups, can be constructed in the bosonic language as non-abelian orbifolds. However, it is also known that some of the abelian (*e.g.*, $\mathbf{Z}_2$) orbifold models easily obtainable in the bosonic language cannot be constructed with the free fermionic string model rules.

Consider the example of a single boson. By the free fermionic string construction rules, we must start with a complex fermion, i.e., a boson compactified at radius one. Using the free fermionic string rules, it is easy to change the radius to any rational value. It is also easy to construct its $\mathbf{Z}_2$ orbifold at radius one, using the real fermion basis. However, the rules [5] do not allow us to construct the $\mathbf{Z}_2$ orbifold version at any other radius. This is because this particular model involves a subgroup of the space group, which is non-abelian. Fortunately, it turns out that the free fermionic string construction rules can be generalized to include such non-abelian orbifolds. Such a generalization can also be applied to the rules given in this paper as well. This generalization will be discussed elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation.

VIII. APPENDIX A

A. Free Fermions

Consider a single free left-moving complex fermion with the monodromy

$$\psi_v(ze^{2\pi i}) = e^{-2\pi i v}\psi_v(z) \ , \quad -\frac{1}{2} \leq v < \frac{1}{2} \ . \tag{8.1}$$

The field $\psi_v(z)$ has the following mode expansion

$$\psi_v(z) = \sum_{n=1}^{\infty} \{ b_{n+v-1/2} z^{-(n+v)} + d^{\dagger}_{n-v-1/2} z^{n-v-1} \} \ . \tag{8.2}$$

Here $b_r^\dagger$ and $d_s^\dagger$ are creation operators, and b_r and d_s are annihilation operators. The quantization conditions read

$$\{b_r^\dagger, b_{r'}\} = \delta_{rr'}, \quad \{d_s^\dagger, d_{s'}\} = \delta_{ss'}, \quad \text{others vanish.} \tag{8.3}$$

The Hamiltonian H_v and fermion number operator N_v are given by

$$H_v = \sum_{n=1}^{\infty}\{(n+v-\frac{1}{2})b^\dagger_{n+v-1/2}b_{n+v-1/2} + (n-v-\frac{1}{2})d^\dagger_{n-v-1/2}d_{n-v-1/2}\} + \frac{v^2}{2} - \frac{1}{24}\ , \tag{8.4}$$

$$N_v = \sum_{n=1}^{\infty}\{b^\dagger_{n+v-1/2}b_{n+v-1/2} - d^\dagger_{n-v-1/2}d_{n-v-1/2}\}\ . \tag{8.5}$$

Note that the vacuum energy is $\frac{v^2}{2} - \frac{1}{24}$. Also note that for a Ramond fermion ($v = -1/2$) the vacuum is degenerate: There are two ground states $|0\rangle$ and $b_0^\dagger|0\rangle$.

The operator $-N_v$ is the generator of $U(1)$ rotations. The corresponding characters read

$$Z_u^v = \mathrm{Tr}(q^{H_v}g^{-1}(u)) = \mathrm{Tr}(q^{H_v}\exp(-2\pi i u N_v)) = \\ q^{\frac{v^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{n+v-1/2}e^{-2\pi i u})(1+q^{n-v-1/2}e^{2\pi i u})\ . \tag{8.6}$$

Under the generators of modular transformations ($q = \exp(2\pi i\tau)$)

$$S : \tau \to -1/\tau\ , \quad T : \tau \to \tau + 1\ , \tag{8.7}$$

the characters (8.6) transform as

$$Z_u^v \xrightarrow{S} e^{2\pi i v u} Z^u_{-v}\ , \tag{8.8}$$

$$Z_u^v \xrightarrow{T} e^{2\pi i(\frac{v^2}{2}-\frac{1}{24})} Z^v_{u-v-1/2}\ . \tag{8.9}$$

In the cases where $v = -1/2$ (Ramond fermion) or $v = 0$ (Neveu-Schwarz fermion) in (8.1) the complex fermion $\psi_v(z)$ can be represented in terms of a linear combination of two real fermions. The corresponding characters for real fermions then are square roots of the characters Z_u^v for the complex fermions (v and u being $-1/2$ or 0). A more detailed discussion of the real fermion characters is given in [5].

B. Twisted Bosons

Consider a single free left-moving complex boson with the monodromy

$$\partial\phi_v(ze^{2\pi i}) = e^{-2\pi i v}\partial\phi_v(z)\ , \quad 0 \le v < 1\ . \tag{8.10}$$

The field $\partial\phi_v(z)$ has the following mode expansion

$$i\partial\phi_v(z) = \delta_{v,0}pz^{-1} + (1-\delta_{v,0})\sqrt{v}\,b_v z^{-v-1} + \\ \sum_{n=1}^{\infty}\{\sqrt{n+v}\,b_{n+v}z^{-n-v-1} + \sqrt{n-v}\,d^\dagger_{n-v}z^{n-v-1}\}\ . \tag{8.11}$$

Here $b_r^\dagger$ and $d_s^\dagger$ are creation operators, and b_r and d_s are annihilation operators. The quantization conditions read

$$[b_r, b_{r'}^\dagger] = \delta_{rr'}, \quad [d_s, d_{s'}^\dagger] = \delta_{ss'}, \quad [x^\dagger, p] = [x, p^\dagger] = i, \quad \text{others vanish.} \tag{8.12}$$

The Hamiltonian H_v and angular momentum operator J_v are given by

$$H_v = \delta_{v,0} p p^\dagger + (1-\delta_{v,0}) v b_v^\dagger b_v + \sum_{n=1}^{\infty} \{(n+v) b_{n+v}^\dagger b_{n+v} + (n-v) d_{n-v}^\dagger d_{n-v}\} + \frac{v(1-v)}{2} - \frac{1}{12} , \tag{8.13}$$

$$J_v = \delta_{v,0} i(xp^\dagger - x^\dagger p) - (1-\delta_{v,0}) b_v^\dagger b_v - \sum_{n=1}^{\infty} \{b_{n+v}^\dagger b_{n+v} - d_{n-v}^\dagger d_{n-v}\} . \tag{8.14}$$

Note that the vacuum energy is $\frac{v(1-v)}{2} - \frac{1}{24}$.

The operator J_v is the generator of $U(1)$ rotations. The corresponding characters read $(v+u \neq 0)$:

$$X_u^v = \mathrm{Tr}(q^{H_v} g^{-1}(u)) = \mathrm{Tr}(q^{H_v} \exp(2\pi i u J_v)) = q^{\frac{v(1-v)}{2} - \frac{1}{12}} (1 - (1-\delta_{v,0}) q^v e^{-2\pi i u})^{-1} \prod_{n=1}^{\infty} (1 - q^{n+v} e^{-2\pi i u})^{-1} (1 - q^{n-v} e^{2\pi i u})^{-1} . \tag{8.15}$$

Under the generators of modular transformations the characters (8.15) transform as

$$X_u^v \xrightarrow{S} (2\sin(\pi u)\delta_{v,0} + [2\sin(\pi v)]^{-1}\delta_{u,0} + (1-\delta_{vu,0}) e^{-2\pi i(v-1/2)(u-1/2)}) X_{-v}^u , \tag{8.16}$$

$$X_u^v \xrightarrow{T} e^{2\pi i (\frac{v(1-v)}{2} - \frac{1}{12})} X_{u-v}^v . \tag{8.17}$$

In the cases where $v = -1/2$ or $v = 0$ in (8.10), the complex boson $\phi_v(z)$ can be represented in terms of a linear combination of two real bosons. The corresponding characters for real bosons then are square roots of the characters X_u^v for the complex bosons (v and u being $-1/2$ or 0). The twisted boson characters with a different overall normalization were discussed in Ref. [8].

C. Chiral Lattices

Consider d free left-moving real bosons with the monodromy

$$\phi_v^I(z e^{2\pi i}) = \phi_v^I(z) + v^I , \tag{8.18}$$

where $I = 1, 2, ..., d$, and v^I is the I^{th} component of the shift vector v. The field $\phi_v^I(z)$ has the following mode expansion:

$$i\phi_v^I(z) = ix^I + (p^I + v^I)\ln(z) - \sum_{n \neq 0} \frac{1}{\sqrt{n}} a_n^I z^{-n} . \tag{8.19}$$

Here a_n^I, $n > 0$, are annihilation operators, and a_n^I, $n < 0$, are creation operators. In the following the eigenvalues of the momentum operator p^I are taken to span an even lattice Γ^d. The quantization conditions read

$$[a_n^I, a_{n'}^J] = \delta^{IJ}\delta_{nn'}, \quad [x^I, p^J] = i\delta^{IJ}, \quad \text{others vanish.} \tag{8.20}$$

The Hamiltonian operator is given by

$$H_v = \frac{(p+v)^2}{2} + \sum_{n=1}^{\infty} n a_{-n}^I a_n^I - \frac{d}{24} \ . \tag{8.21}$$

The momentum operator p is the generator of translations. Thus, the action of the operator

$$g(u) \equiv \exp(2\pi i p u) \ , \quad pu \equiv p^I u^I \ , \tag{8.22}$$

on the field $\phi_v^I(z)$ is given by

$$g(u)\phi_v^I(z)g^{-1}(u) = \phi_v^I(z) + u^I \ . \tag{8.23}$$

The corresponding characters read

$$\begin{aligned} Y_u^v = \mathrm{Tr}(q^{H_v} g^{-1}(u)) = \mathrm{Tr}(q^{H_v}\exp(-2\pi i p u)) = \\ \frac{1}{\eta^d(q)} \sum_{p\in\Gamma^d} q^{\frac{1}{2}(p+v)^2} \exp(-2\pi i p u) \ . \end{aligned} \tag{8.24}$$

Let $w_a \in \tilde{\Gamma}^d$, $a = 1, ..., M-1$, be a set of vectors such that $\Gamma_0^d \oplus \Gamma_1^d \oplus ... \oplus \Gamma_{M-1}^d = \tilde{\Gamma}^d$, where w_0 is the null vector ($w_0^I \equiv 0$), and $\Gamma_a^d \equiv \{w_a + p | p \in \Gamma^d\}$, $a = 0, 1, ..., M-1$ (Thus, $w_a \notin \Gamma^d$ for $a \neq 0$; also note that $M = \det(g_{ij})$). Consider the set of characters $Y_u^{v+w_a}$:

$$Y_u^{v+w_a} \xrightarrow{T} \exp(2\pi i(\frac{1}{2}(w_a+v)^2 - \frac{d}{24}))Y_{u-v}^{v+w_a} \ , \tag{8.25}$$

$$Y_u^{v+w_a} \xrightarrow{S} \sum_{b=0}^{M-1} S_{ab}(v,u) Y_{-v}^{u+w_b} \ , \tag{8.26}$$

where

$$S_{ab}(v,u) = M^{-\frac{1}{2}} \exp(2\pi i(w_a+v)(w_b+u)) \ . \tag{8.27}$$

Let N the smallest positive integer such that $\forall a$ $Nw_a^2 \in 2\mathbf{Z}$. If $N = 1$ (in which case Γ^d is an even self-dual lattice with $M = \det(g_{ij}) = 1$), we will use the characters Y_u^v defined in (8.24) whose modular transformations are particularly simple for $N = 1$:

$$Y_u^v \xrightarrow{T} \exp(2\pi i(\frac{1}{2}v^2 - \frac{d}{24}))Y_{u-v}^v \ , \tag{8.28}$$

$$Y_u^v \xrightarrow{S} \exp(2\pi i v u) Y_{-v}^u \ . \tag{8.29}$$

If $N > 1$, the set of characters $Y_u^{v+w_a}$ is such that the T-transformation is diagonal (with respect to a), whereas the S-transformation is not. There exists a basis such that both S-

and T-transformations act as permutations. In particular, consider the case where N is a prime. In the discussion of asymmetric orbifolds we will use the set of characters

$$Y^{0,v}_{\sigma,u} \equiv Y^v_u \ , \tag{8.30}$$

$$Y^{\lambda,v}_{\sigma,u} \equiv \sum_{a=0}^{M-1} \exp(-2\pi i\lambda(uw_a + \frac{1}{2}\sigma w_a^2))Y^{v+\lambda w_a}_u \ , \quad \lambda \neq 0 \ , \tag{8.31}$$

where λ and σ are integers taking values between 0 and $N-1$, such that $\lambda + \sigma \neq 0$. The modular transformation properties of $Y^{\lambda,v}_{\sigma,u}$ read

$$Y^{\lambda,v}_{\sigma,u} \xrightarrow{T} \exp(2\pi i(\frac{1}{2}v^2 - \frac{d}{24}))Y^{\lambda,v}_{\sigma-\lambda,u-v} \ , \tag{8.32}$$

$$Y^{\lambda,v}_{\sigma,u} \xrightarrow{S} \{M^{-\frac{1}{2}}\delta_{\lambda,0} + M^{\frac{1}{2}}\delta_{\sigma,0} + (1-\delta_{\lambda\sigma,0})\exp(2\pi i\chi(\lambda,\sigma))\}\exp(2\pi ivu)Y^{\sigma,u}_{-\lambda,-v} \ , \tag{8.33}$$

where $Y^{\lambda,v}_{\sigma,u} \equiv Y^{\lambda+N,v}_{\sigma,u} \equiv Y^{\lambda,v}_{\sigma+N,u}$, and

$$\exp(2\pi i\chi(\lambda,\sigma)) \equiv M^{-\frac{1}{2}} \sum_{a=0}^{M-1} \exp(-2\pi i\frac{1}{2}\lambda\sigma w_a^2) \ , \quad \lambda\sigma \neq 0 \ . \tag{8.34}$$

Note that $\chi(\lambda,\sigma)$ are real numbers, and $\chi(\lambda,\lambda) \equiv -d/8$.

To illustrate the above discussion we note that the root lattices of simply-laced Lie groups are even. The groups that have prime N are the following: (*i*) $SU(n)$, n is an odd prime, and $N=n$; (*ii*) E_6, $N=3$; (*iii*) SO(8n), $N=2$; (*iv*) E_8, $N=1$.

Similar considerations apply to right-moving chiral lattices, and also Lorentzian lattices. In the latter case all the scalar products of vectors are understood with respect to the Lorentzian signature.

IX. APPENDIX B

Since the $N2$ model of subsection IVB has $N=4$ space-time supersymmetry, it must correspond to one of the four-dimensional $N=4$ supersymmetric models classified by Narain [7], *i.e.*, there must be a choice of the constant background fields such that the corresponding even self-dual Lorentzian lattice is spanned by momenta (4.16). Here we briefly construct such a lattice.

Consider a general lattice $\Gamma^{2,18}$ spanned by the vectors

$$P = k^i m_i + k_i n^i + k^J p^J \ , \quad m_i, n^i \in \mathbf{Z} \ , \quad p^J \in \Gamma^{16} \ , \tag{9.1}$$

where $(i=1,2)$

$$k^i = (\frac{1}{2}\ell^{*i}; \frac{1}{2}\ell^{*i}; \mathbf{0}) \ , \tag{9.2}$$

$$k_i = (-(B_{ji} + \frac{1}{4}A^I_j A^I_i)\ell^{*j} - \ell_i; -(B_{ji} + \frac{1}{4}A^I_j A^I_i)\ell^{*j} + \ell_i; A^I_i) \ , \tag{9.3}$$

$$k^J = (-\frac{1}{2}\ell^{*i}A^J_i; -\frac{1}{2}\ell^{*i}A^J_i; \delta^{IJ}) \tag{9.4}$$

are Lorentzian vectors of signature $((-)^2, (+)^{18})$, and

$$\ell^{*i} \cdot \ell_j = \delta^i_j \ , \quad \ell_i \cdot \ell_j = g_{ij} \ , \quad \ell^{*i} \cdot \ell^{*j} = g^{ij} \ , \tag{9.5}$$

g_{ij}, B_{ij} and A^I_i being constant background symmetric, anti-symmetric and gauge (Wilson lines) fields, respectively. Suppose that $\ell^{*i} = 2\zeta^i$ and $\ell_j = \frac{1}{2}\zeta^*_i$, where $\{\zeta^i m_i\} = \Gamma^2$ ($SU(3)$ root lattice), and $\{\zeta^*_i n^i\} = \tilde{\Gamma}^2$ ($SU(3)$ weight lattice), and in the following we will use the convention where $\zeta^1 \cdot \zeta^1 = \zeta^2 \cdot \zeta^2 = -2\zeta^1 \cdot \zeta^2 = 2$. Then, provided that $A^I_1 = -A^I_2 \equiv A^I$, $2B_{12} = -2B_{21} \in \mathbf{Z} + \frac{1}{2}$, and $\frac{1}{2}(A^I)^2 \in \mathbf{Z} + \frac{2}{3}$, the momenta P can be expressed as

$$P = (\overline{p}; p; p^I + \alpha_2 A^I) \ , \tag{9.6}$$

where $\overline{p} \in 3A^I p^I w + \Gamma^2$, $p \in (3A^I p^I + \alpha_2)w + \Gamma^2$ and $p^I \in \Gamma^{16}$ (Here $\alpha_2 \equiv n^1 - n^2$, and $w \equiv \zeta^*_2 - \zeta^*_1$).

The lattice $\Gamma^{2,2} \otimes \Gamma^{2,2} \otimes \Gamma^{2,18}$ has exactly the momentum spectrum of the model $N2$. This proves that the latter does describe an $N = 4$ space-time supersymmetric heterotic string model with a Wilson line.

REFERENCES

* E-mail address: zurab@hepth.cornell.edu.

[1] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B261** (1985) 678; **B274** (1986) 285.

[2] K.S. Narain, M.H. Sarmadi and C. Vafa, *Nucl. Phys.* **B288** (1987) 551.

[3] H. Kawai, D.C. Lewellen, and S.-H. H. Tye, *Phys. Rev. Lett.* **57** (1986) 1832,**58** (1987) 429E; *Nucl. Phys.* **B288** (1987) 1; I. Antoniadis, C. P. Bachas, and C. Kounnas, *Nucl. Phys.* **B289** (1987) 87.

[4] For a partial list, see I. Antoniadis, J. Ellis, J. Hagelin and D.V. Nanopoulos, *Phys. Lett.* **B194** (1987) 231;**B208** (1988) 209; I. Antoniadis, G.K. Leontaris and J. Rizos, *Phys. Lett.* **B245** (1990) 161; J. Lopez, D.V. Nanopoulos and K. Yuan, *Nucl. Phys.* **B399** (1993) 654. D. Lewellen, *Nucl. Phys.* **B337** (1990) 61; S. Chaudhuri, S.-W. Chung, G. Hockney and J.D. Lykken, preprint hep-th/9409151; preprint hep-ph/9501361. S. Chaudhuri, G. Hockney and J.D. Lykken, *Phys. Rev. Lett.* **75** (1995) 2264; preprint hep-th/9510241; K. Dienes and A. Faraggi, *Phys. Rev. Lett.* **75** (1995) 2646; preprint hep-th/9505046; G.B. Cleaver, preprint hep-th/9505080; preprint hep-th/9506006. D. Finnel, SLAC preprint SLAC-PUB-95-6986, hep-th/9508073.

[5] H. Kawai, D.C. Lewellen, J.A. Schwartz and S.-H. H. Tye, *Nucl. Phys.* **B299** (1988) 431.

[6] L.E. Ibáñez, H.P. Nilles and F.D. Quevedo, *Phys. Lett.* **B187** (1987) 25.

[7] K. Narain, *Phys. Lett.* **B169** (1986) 41; K. Narain, M.H. Sarmadi and E. Witten, *Nucl. Phys.* **B279** (1986) 369.

[8] A.H. Chamseddine, J.-P. Derendinger and M. Quirós, *Nucl. Phys.* **B326** (1989) 497.