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Enhanced Gauge Symmetries and Calabi–Yau Threefolds

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ABSTRACT

We consider the general case of a type IIA string compactified on a Calabi–Yau manifold which has a heterotic dual description. It is shown that the nonabelian gauge symmetries which can appear nonperturbatively in the type II string but which are understood perturbatively in the heterotic string are purely a result of string-string duality in six dimensions. We illustrate this with some examples.

1 Introduction

Our understanding of the dynamics of $N=2$ Yang-Mills theories in four dimensions has greatly improved recently due to the work of Seiberg and Witten [1, 2]. The moduli of such theories appear in two kinds of supermultiplets, i.e., hypermultiplets and vector multiplets. Our attention will focus on the vector multiplets in this letter. Consider a theory with n vector multiplets. This will correspond to a Yang-Mills theory with a gauge group of rank n.

Roughly speaking, the vector moduli live in the Cartan subalgebra of the gauge group and will break this group down to the elements that commute with the value of the moduli. Thus, the effective gauge group, G, will always be of rank n and will generically be $U(1)^n$. At particular points and subspaces within the moduli space the gauge group will become nonabelian. This is the classical picture but quantum effects modify this behaviour [1] (see also [3, 4]). Instead of enhancement of the gauge group at special points, the gauge group remains $U(1)^n$ but hypermultiplets which are massive at generic points in the moduli space can become massless. We want to consider the classical limit in which the gauge group becomes enhanced but in the context of string theory. Having found such points in the moduli space we can assume that Seiberg-Witten theory will take over in the case of nonzero coupling.

Until recent developments in string duality it was generally believed that a type II superstring compactified on a Calabi-Yau manifold would not exhibit a nonabelian gauge symmetry. When it was realized that the type IIA string compactified on a K3 surface could naturally be identified as the dual of the heterotic string on a four-torus [5, 6, 7] then it followed that, since the heterotic string can have nonabelian gauge symmetries, the same must be true for the type IIA string on a K3 surface. The way in which this was possible was discussed in [6] and was further developed in [8, 9, 10].

A similar duality in four dimensions has been conjectured [11, 12] in which the type II string compactified on a Calabi-Yau threefold is considered equivalent by some kind of duality to a heterotic string compactified on K3$\times T^2$ (or some variant thereof). Since this heterotic string can again lead to nonabelian gauge groups the same must be true of type II strings on a Calabi-Yau threefold. Based on these conjectured dualities, points in the moduli space for specific examples of compactification of type II strings on Calabi-Yau manifolds where the gauge group should become enhanced were identified. In these cases the reduction to Seiberg-Witten theory for nonzero coupling has been explicitly shown in [13].

Our goal here will be to make general statements about the appearance of enhanced gauge groups in type IIA compactifications. The basic idea is actually very simple thanks to the results of [14]. We will restrict our attention (until some comments at the end) to gauge groups which are visible perturbatively in the heterotic string picture. The dual picture to this is a type IIA string theory compactified on X which must be a K3 fibration and the gauge group arises from properties of the generic fibre. It was shown that the dilaton modulus in the heterotic string corresponds to the size of the base $\mathbb{P}^1$ of the fibration and that the weak-coupling limit corresponds to the size of this base space being taken to infinity. Clearly if we look at a generic fibre and the base space becomes infinitely large then we have effectively decompactified the 4-dimensional picture to 6 dimensions and we have reduced the question the that studied in [6, 8].

In a recent paper [10] some questions regarding enhanced gauge groups were studied using D-branes, particularly in the type IIB context. Our results here concern type IIA strings but clearly have some overlap with the conjectures and examples of [10]. The D-brane approach is probably the more powerful tool for answering questions regarding type II string compactifications but the purpose of this letter is to demonstrate how simply many of the properties of gauge groups on Calabi-Yau manifolds can be reduced to the simpler picture of 6-dimensional duality.

In section 2 we will review the machinery we need to apply our picture to examples and in section 3 we give some examples. Finally we present some concluding remarks in section 4.

2 K3 Fibrations

We will be concerned with the case that X can be written as a fibration over $\mathbb{P}^1$ with generic fibre a K3 surface. The importance of such a class in the context of string duality was first realized in [15, 16]. Let us consider the general case of a dual pair consisting of a type IIA string compactified on a Calabi-Yau manifold, X, and a heterotic string such that the weakly-coupled heterotic string is identified with X at some kind of large radius limit. It was shown in [14] that X must be a K3-fibration in this case.

For a generic dual pair, the gauge group is $U(1)^{r+1}$. This comes from r vector multiplets and the graviphoton. In the type IIA picture, the vector multiplets come from $H^{1,1}(X)$ and $r = h^{1,1}(X)$. The group $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is known as the Picard group of X. The rank of the Picard group is the Picard number. In our case we will assume that $h^{2,0}(X) = 0$ and so the Picard group of X is simply $H^2(X, \mathbb{Z})$. That is, the generators of the Picard group of X can be considered to form a basis of the space of vector multiplets.

The Picard group of X when X is a K3 fibration essentially has three different contributions. We work in terms of the dual group $H_4(X)$. The generic K3 fibre itself gives one contribution. Another source is from elements of the Picard group of the K3 fibre. In this case a curve in the fibre is transported over the whole base $\mathbb{P}^1$ to build up a 4-cycle. Note that such curves may have monodromy under such a transport. One can see that monodromy-invariant elements of the Picard group of the generic K3 fibre contribute to the Picard group of X. Note that the Picard group of a K3 surface is a more subtle object than the Picard group of a Calabi-Yau threefold. The Picard group of a K3 surface can depend upon the complex structure of the K3 surface since $h^{2,0} \neq 0$. Lastly, there will be degenerate fibres over a finite number of points in the base $\mathbb{P}^1$. In some cases such fibres can also contribute to the Picard group of X.

In [14] these three sources for elements of the Picard group of X were given different interpretations in terms of the dual heterotic string. The generic fibre element was identified with the dilaton, which lies in a vector multiplet. The elements coming from the Picard group of the fibre were matched with the rest of the gauge group that was visible perturbatively in the heterotic string. Lastly, the contributions from the degenerate fibres were expected to have some nonperturbative origin in the heterotic string.

We are interested in the gauge group that is perturbatively visible in the heterotic string — that is the part associated to the Picard group of the K3 fibres. We would like to know if we can obtain nonabelian groups by varying the moduli in these vector multiplets in the weakly-coupled limit. The weakly coupled limit corresponds to the base $\mathbb{P}^1$ becoming very large. Clearly in this limit, any question about the generic fibres can be treated purely in terms of the type IIA string compactified on a K3 surface along the lines of [6, 8].

This means that the analysis is actually very simple — the question of enhanced groups for $N = 2$ theories in four dimensions is actually completely reducible to the case of six-dimensions, at least as far as the perturbative heterotic string is concerned.

To simplify our discussion we will make an assumption concerning the way in which X is written as a K3 fibration. We will assert that the Picard lattice of a generic fibre is invariant, i.e., does not undergo any monodromy transformation, as we move about the base space. One can certainly find examples which will not obey our assumption but simple examples, such as the ones we discuss later, do not have monodromy. It should not be difficult to extend the analysis here to nontrivial monodromy.

We are thus concerned with the enhanced gauge groups that can appear on a K3 surface as we vary its Kähler form. The fibration of X will restrict the K3 fibre to be of a particular type. Indeed, the generic fibre will be an algebraic K3 surface which may be considered to be embedded in some higher-dimensional projective space. An abstract K3 surface has variations in its complex structure and Kähler structure which naturally fill out a space of 80 real dimensions in string theory [17]. In the case of a specific algebraic K3 surface however only some of these deformations are allowed. In particular, the Kähler form is only allowed to vary within the space spanned by the Picard group.

For the general case, the stringy moduli space of K3 surfaces is given locally by the Grassmanian of space-like 4-planes in $\mathbb{R}^{4,20}$. The space $\mathbb{R}^{4,20}$ may be viewed as the space of total real cohomology $H^*(\mathrm{K3},\mathbb{R})$ which contains the lattice $H^*(\mathrm{K3},\mathbb{Z})$ which has intersection form $E_8 \oplus E_8 \oplus H \oplus H \oplus H \oplus H$ [18]. Here E_8 denotes *minus* the Cartan matrix of the Lie group E_8 and H is the hyperbolic plane. The global form of the moduli space is obtained by dividing this Grassmanian by the group of isometries of this lattice.

In the case of an algebraic K3 surface, this moduli space is restricted as follows.[1] Divide

[1] This analysis rests heavily on work done in collaboration with D. Morrison [19]. Aspects have also been discussed in [20, 21].

the lattice as

$$H^*(\mathrm{K3},\mathbb{Z}) \supseteq \Lambda_K \oplus \Lambda_c. \tag{1}$$

That is, take Λ_K to be a sublattice of the integral cohomology of the K3 surface and Λ_c to be its orthogonal complement. We demand that the sum of the ranks of these lattices is equal to the rank of $H^*(\mathrm{K3},\mathbb{Z})$. That is, the lattice $\Lambda_K \oplus \Lambda_c$ is equal to $H^*(\mathrm{K3},\mathbb{Z})$ or is a sublattice of finite index.

Let $V_K = \Lambda_K \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector spanned by the generators of Λ_K with V_c similarly defined. Now we may consider the Grassmanian of space-like 2-planes in V_K to be our restricted moduli space of complexified Kähler forms and the Grassmanian of space-like 2-planes in V_c to be the restricted moduli space of complex structures. Clearly the product of these two moduli spaces is a subspace of the total stringy moduli space of K3 surfaces locally. Globally some of the isometries of $H^*(\mathrm{K3},\mathbb{Z})$ will descend to isometries of Λ_K and Λ_c to give identifications within these Grassmanians.

By restricting to a special class of K3 surfaces we have thus managed to locally factorize the moduli space into deformations of complex structure and deformations of Kähler form. If we want to make contact with classical geometry then we must insist that the moduli space of Kähler forms contains the large radius limit. In this case

$$\Lambda_K = \mathrm{Pic} \oplus H, \tag{2}$$

where Pic is the Picard lattice of the algebraic K3 surface. The H factor is then identified with $H^0 \oplus H^4$.

Let us denote the Picard number of the fibre by r. Since the signature of the Picard lattice is $(1, r-1)$, we see immediately from (2) that the moduli space of Kähler forms within the fibre is given by

$$\frac{O(2,r)}{O(2)\times O(r)}, \tag{3}$$

divided by the group of isometries of Λ_K. Thus we give a geometrical interpretation to the result of [22].

Note that the mirror map exchanges the rôles of Λ_K and Λ_c. Thus the mirror of one algebraic K3 surface is generally a different algebraic K3 surface and the Picard numbers of these two surfaces will add up to 20.

Now recall how, using string-string duality in six dimensions, we find the points in the moduli space of type IIA strings on a K3 surface where we have enhanced gauge groups [6, 8]. Let Π be the space-like 4-plane in $H^*(\mathrm{K3},\mathbb{R})$. The set of vectors

$$\{\alpha \in H^*(\mathrm{K3},\mathbb{Z}); \alpha^2 = -2 \text{ and } \alpha \in \Pi^\perp\}, \tag{4}$$

give the roots of the semi-simple part of the gauge group G.

For the case of interest to us, we are only concerned with the slice of the moduli space given by deformations of the Kähler form on the K3 surface. Given the decomposition (1) we see that our gauge group now has roots

$$\{\alpha \in \Lambda_K; \alpha^2 = -2 \text{ and } \alpha \in \mho^{\perp}\}, \tag{5}$$

where $\mho$ is the space-like 2-plane[2] in V_K.

This essentially gives the full description of how the generic fibres can enhance the gauge group in the limit that the base space becomes infinitely large, i.e., when the dual heterotic string becomes weakly-coupled. First find the Picard group of the generic fibre, then build Λ_K from (2). The gauge group is then given by (5).

3 Examples

Let us clarify the discussion of the last section by giving two examples. The first example we take from [11]. Let X_0 be the Calabi-Yau hypersurface of degree 24 in the weighted projective space $\mathbb{P}^4_{\{1,1,2,8,12\}}$. The first two homogeneous coordinates may be used as the homogeneous coordinates of the base $\mathbb{P}^1$ to form a K3-fibration. The generic fibre may then be written as a degree 12 hypersurface in $\mathbb{P}^3_{\{1,1,4,6\}}$ which is indeed a K3 surface which we denote F_0.

The space $\mathbb{P}^3_{\{1,1,4,6\}}$ contains a curve of $\mathbb{Z}_2$-quotient singularities which F_0 intersects once at a point. We blow-up the quotient singularities in X_0 to obtain X and this induces a blow-up of F_0 which we call F. The latter is locally a blow-up of a $\mathbb{Z}_2$-quotient singularity and so gives us a rational curve of self-intersection -2 within F. Let us call the homology class of this curve C. Clearly C lies in the Picard group of F. The other contribution to the Picard group comes from hyperplanes of $\mathbb{P}^3_{\{1,1,4,6\}}$ slicing F_0. Let us denote the resulting curve A. Representatives of A pass through the quotient singularity of F_0. When F_0 is blown-up such self-intersections are removed and thus A has self-intersection 0 in F. Since A passed through this singularity in F_0, the intersection number between A and C is equal to 1.

We claim then that F has Picard number 2 with intersection lattice

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \tag{6}$$

Clearly by replacing C with the cycle $B = A + C$ we obtain an intersection form between A and B equal to H, the hyperbolic plane.

Thus we obtain

$$\Lambda_K \cong H \oplus H \tag{7}$$

[2] The reason for this notation comes from [23] as explained in [18].

and the part of the moduli space coming from the fibre is given by

$$\frac{O(2,2)}{O(2)\times O(2)} \tag{8}$$

divided by $O(2,2;\mathbb{Z})$ in the usual language. This is, of course, the Narain moduli space for a string on the 2-torus. If the conjecture in [11] is true then this is no accident as explained in [16]. This part of the moduli space would arise from deformations of the T^2 of the K3$\times T^2$ on which the dual heterotic string is compactified.

Now we can look for places in the moduli space where the gauge group is enhanced. The easiest case is when the K3 fibre can be taken to be at large radius. Thus corresponds to a direction in $\mho$ becoming almost light-like being close to a null vector in the H factor in (2) [18]. Consistent with such a limit be may take $\mho$ to be perpendicular to C. This means that the Kähler form or B-field when integrated over the curve C is zero. That is, we have blown down F back to F_0. The roots corresponding to $\pm C$ give the root lattice of $SU(2)$. Thus we expect an $SU(2)$ gauge symmetry on the space X_0 (for suitable B-field).

X_0 contains a curve of $\mathbb{Z}_2$-quotient singularities. Note that this is easy to generalize — when we can enhance the gauge symmetry by blowing down a curve in the K3 generic fibre then every fibre will have a singular point meaning that X will contain a curve of singularities. The fact that curves of singularities can be associated with enhanced gauge symmetries in a type IIB context has been discussed in [10].

We can also embed the root lattice of $SU(2)\times SU(2)$ or $SU(3)$ into Λ_K for the case at hand. In both cases the plane $\mho$ is fixed and so these are isolated points in the moduli space. Note also that in both cases we cannot be near the large radius limit of the K3 fibre. These further enhancements of the gauge group thus correspond to effects of quantum geometry within the K3 fibre when the volume of the fibre will be of the order of $(\alpha')^2$.

We can make closer contact with the conjecture of [11] by going to the mirror picture of a type IIB string compactified on Y, the mirror of X. In this case, Y is an orbifold of the hypersurface

$$x_1^{24}+x_2^{24}+x_3^{12}+x_4^3+x_5^2+a_0x_1x_2x_3x_4x_5+a_1x_1^6x_2^6x_3^6+a_2x_1^{12}x_2^{12}=0 \tag{9}$$

in $\mathbb{P}^4_{\{1,1,2,8,12\}}$. Application of the monomial-divisor mirror map in the Calabi-Yau phase as described in [24] immediately tells us that, to leading order, the size of the base $\mathbb{P}^1$ is given by $\log(a_2^2)$; the size of the blown-up curve, C, within the K3 fibre is $\log(a_1^2/a_2)$; and the size of the fibre itself is given by $\log(a_0^6/a_1)$. Thus, the weak-coupling limit of the heterotic string is given by $a_2\to\infty$ from [14] in agreement with the conjecture of [11].

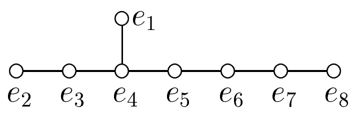
Keeping the K3 fibre big, by keeping a_0^6/a_1 big, we can blow down C by decreasing a_1^2/a_2. In the limit that a_1^2/a_2 becomes zero we reach the conformal field theory orbifold. This is not what we want however. In order to get the enhanced gauge symmetry we need the component of B along the blow-up to be zero as explained in [8], whereas the conformal field theory orbifold gives a value of $\frac{1}{2}$. Fortunately for this blow-up, the B-field value for the mirror map has been explicitly worked out in section 5.5 of [24]. To obtain a zero-sized exceptional divisor and zero B-field we require $a_1^2/a_2 = 4$. Thus we expect an $SU(2)$ enhanced gauge group here again in agreement with [11].

Determination of the $SU(2) \times SU(2)$ or $SU(3)$ points of enhanced gauged symmetry is harder to analyze in this direct manner and we will not pursue it here. It is clear that we should again reproduce the results of [11] however.

As a second example we turn to one of the conjectured dual pairs of [25] which has also been analyzed in [26]. Consider the Calabi-Yau hypersurface of degree 84 in $\mathbb{P}^4_{\{1,1,12,28,42\}}$. This is a K3 fibration where the generic fibre is a hypersurface of degree 42 in $\mathbb{P}^3_{\{1,6,14,21\}}$. This K3 surface has Picard number 10 and the Picard lattice has intersection form $E_8 \oplus H$. This is most easily seen following the methods explained in [20]. Thus we have

$$\Lambda_K \cong E_8 \oplus H \oplus H. \tag{10}$$

It is easy then to see that there will be dual heterotic strings with an E_8 factor in the gauge group in agreement with [25]. We can also say in more detail how to obtain this gauge group. Let us label the simple roots of E_8 as follows:



(11)

Each of these roots is associated with a rational curve in the K3 fibre. All the roots, with the exception of e_4, come from blowing-up the quotient singularities of the ambient $\mathbb{P}^4_{\{1,1,12,28,42\}}$. Thus we can blow these down to get a gauge group $SU(2) \times SU(3) \times SU(5)$ while keeping the fibre at the large radius limit. The curve corresponding to e_4 comes from the hyperplane section from the ambient projective space however. To blow this down we have to shrink down the whole K3 fibre to take us into the realms of quantum geometry. Thus in order to obtain the full E_8 symmetry the fibre must be shrunk down to volume of order $(\alpha')^2$.

Actually one can enhance the gauge group beyond this using the H factors in Λ_K. For example $E_8 \times SU(3)$ or $SU(10)$ should appear in the moduli space.

One can see that we can reproduce all of the gauge groups for type IIA strings compactified on hypersurfaces listed in [25]. Note also that any enhanced gauge groups appearing in the dual pairs studied in [12, 27] can also be recovered by the same methods.

4 Comments

We have seen how enhanced gauge groups appearing on a type IIA string compactified on a Calabi-Yau manifold that can be seen perturbatively in the dual heterotic string can be understood purely in terms of string-string duality in six-dimensions. We used this fact to reproduce all the currently known results about gauge groups from conjectured dual pairs.

We should emphasize that we have not completed the problem of understanding the appearance of nonabelian groups for type II strings on Calabi-Yau manifolds however. Firstly there can be parts of the gauge group which cannot be understood perturbatively from either the type II or the heterotic string point of view. Such groups have been analyzed recently in six dimensions by using nonperturbative methods for a heterotic string compactified on a K3 surface [28]. Similar effects must be expected for the case considered in this letter. It is tempting to conjecture how they will appear. We know from [14] that the vector multiplets that cannot be seen perturbatively in the heterotic string arise from contributions to the Picard group of X from the degenerate fibres. We also have seen above that curves of quotient singularities lead to nonabelian groups in the type IIA string when this curve can be fibred over the base $\mathbb{P}^1$. If we suppose that the appearance of nonabelian groups is a result purely of singular curves and not whether they fibre properly over the base $\mathbb{P}^1$ then we can consider the case that we pick up a curve of singularities within the degenerate fibre. Thus, if we have a type IIA string compactified on X and X is a K3 fibration with curves of quotient singularities within the degenerate fibres, then the dual heterotic string will have nonperturbatively enhanced gauge groups as in [28]. We should add however that one may have to worry about what one means by the "weak-coupling limit" in which one actually sees these nonabelian gauge groups. This should be investigated further.

Another aspect which we ignored above concerns hypermultiplets. We have explored questions involving vector multiplets only. Hypermultiplets can become massless when the gauge group gets enhanced. This is essential for analysis of phase transitions as discussed in [29, 12, 27]. On a related point, gauge groups can also become enhanced as we move about in the moduli space of hypermultiplets. The D-brane picture should be of help here. We may also need to worry about discrete R-R degrees of freedom since precisely these issues concerning hypermultiplets were sensitive to such effects in the example of [12]. Clearly we must address these problems before the subject of enhanced gauge symmetries on Calabi-Yau manifolds is completely understood.

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