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A note on the deformed Virasoro algebra

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Abstract

A current of the deformed Virasoro algebra is identified with the Zamolodchikov-Faddeev operator for the basic scalar particle in the XYZ model.

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The studies of infinite dimensional algebras has turned out to be the main tendency in the recent development of the theory of two-dimensional quantum integrable systems. This approach was originated by the fundamental work of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov. Now it has become clear that the Virasoro-type symmetry is not a unique feature of Conformal Field Theory models. In the work [1] an existence of a nontrivial deformation of the Virasoro algebra was conjectured (see also [2]). It is expected to be an algebra of dynamical symmetry in the XYZ Heisenberg chain with the Hamiltonian:

$$H = -\sum_n (J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z) \ , \tag{1}$$

where

$$J_x > J_y > |J_z| \ .$$

Precisely speaking, the symmetry with respect to the deformed Virasoro algebra (DVA) arises in the thermodynamic limit only, when the number of sites in the chain goes to infinity. DVA plays the same role as $U_q(\widehat{sl(2)})$ at the level one for the anti-ferromagnetic XXZ chain $(-J_z > J_x = J_y > 0)$ [3]. The bosonization procedure of [1] allowed one to construct irreducible representations and intertwining vertex operators for DVA without using an explicit form of its commutation relations. In the recent important paper [4] the generators of DVA were constructed explicitly by the same bosonization procedure. This construction was already generalized for the WA_n algebras [5], [6].

Let us recall some basic facts on DVA in notations close to [1]. The algebra depends on two real parameters $\xi > 0$ and $0 < x < 1$. [1] Notice that the parameters are related to the constants of interaction of the XYZ chain (1) as follows:

$$\begin{aligned} \frac{J_x - J_y}{J_x + J_y} &= \prod_{n=0}^{\infty} \frac{(1 - x^{\xi + (\xi+1)n})^2 (1 - x^{1 + (\xi+1)n})^2}{(1 + x^{\xi + (\xi+1)n})^2 (1 + x^{1 + (\xi+1)n})^2} \ , \\ \frac{2J_z}{J_x + J_y} &= \frac{x^{\xi}}{4} \prod_{n=0}^{\infty} \frac{(1 + x^{(1+\xi)n})^4 (1 - x^{2\xi + (\xi+1)(2n+1)})(1 - x^{-2\xi + (\xi+1)(2n+1)})}{(1 - x^{(\xi+1)(2n+1)})^2 (1 + x^{\xi + (\xi+1)n})^2 (1 + x^{1 + (\xi+1)n})^2} \ . \end{aligned} \tag{2}$$

It is convenient to introduce the set of oscillators λ_m which satisfy the commutation relations:

$$[\lambda_m, \lambda_n] = \frac{1}{m} \, \frac{(x^{\xi m} - x^{-\xi m})(x^{(\xi+1)m} - x^{-(\xi+1)m})}{x^m + x^{-m}} \, \delta_{m+n,0} \ . \tag{3}$$

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[1] J. Shiraishi, e.a. [4] use the notations $p = x^{-2}$, $q = x^{-2(\xi+1)}$, $t = qp^{-1} = x^{-2\xi}$.

Such oscillators are "self dual" with respect to the transform $\xi \to -\xi - 1$. [2] We need to define also the " zero mode" operators P, Q, which commute with λ_m and satisfy the relation:

$$[Q, P] = i \ . \tag{4}$$

Central objects in the bosonization procedure are "screening charges" [7], [8], [9]. In terms of the oscillators (3), (4) the "screening currents" can be written in the form [1]:

$$\begin{aligned} I_+(z) &= e^{-i\sqrt{\frac{2(\xi+1)}{\xi}}Q} z^{-\sqrt{\frac{2(\xi+1)}{\xi}}P+\frac{1}{\xi}} : \exp\left(- \sum_{m\neq 0} \frac{x^m + x^{-m}}{x^{\xi m} - x^{-\xi m}} \ \lambda_m \ z^{-m} \right) : , \\ I_-(z) &= e^{i\sqrt{\frac{2\xi}{(\xi+1)}}Q} z^{\sqrt{\frac{2\xi}{(\xi+1)}}P-\frac{1}{\xi+1}} : \exp\left(\sum_{m\neq 0} \frac{x^m + x^{-m}}{x^{(\xi+1)m} - x^{-(\xi+1)m}} \ \lambda_m \ z^{-m} \right) : \ . \end{aligned} \tag{5}$$

Now introduce the field

$$\Lambda(z) = x^{\sqrt{2\xi(\xi+1)}P} \ : \exp\left(- \sum_{m\neq 0} \lambda_m z^{-m} \right) : , \tag{6}$$

then a current of DVA reads [4], [2]:

$$T(z) = \Lambda(zx^{-1}) + \Lambda^{-1}(zx) \ . \tag{7}$$

It commutes with the "screening charges"

$$[X^l_\pm, T(z)] = 0 \ , \tag{8}$$

where

$$X^l_\pm = \int_{C_1} ... \int_{C_l} \frac{dz_1}{2\pi i} ... \frac{dz_l}{2\pi i} \ I_\pm(z_1)...I_\pm(z_l) \ ,$$

if the integration contours are chosen according to Felder' s prescription [9], [1]. The field $T(z)$ generates the algebra with the basic relation [4]:

$$\begin{aligned} & f(\zeta z^{-1}) \ T(z)T(\zeta) - f(z\zeta^{-1}) \ T(\zeta)T(z) = \\ & \qquad 2\pi \ \frac{(x^\xi - x^{-\xi})(x^{(\xi+1)} - x^{-(\xi+1)})}{x - x^{-1}} \ \left(\ \delta(\zeta z^{-1} x^{-2}) - \delta(\zeta z^{-1} x^2) \ \right) , \end{aligned} \tag{9}$$

[2] The operators λ_m are connected with the oscillators b_m, b'_m from [1] in such a way:

$$b'_m \ (x^{(\xi+1)m} - x^{-(\xi+1)m}) = b_m (x^{\xi m} - x^{-\xi m}) = m \ \lambda_m \ .$$

here

$$f(z)=(1-z)^{-1}\ \prod_{n=0}^{\infty}\frac{(1-zx^{2(\xi+1)+4n})(1-zx^{-2\xi+4n})}{(1-zx^{2(\xi+1)+2(2n+1)})(1-zx^{-2\xi+2(2n+1)})}$$

and

$$\delta(z)=\frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}z^m\ .$$

This relation can be equivalently written in terms of the modes T_m: $T(z)=\sum_{m=-\infty}^{+\infty}T_m z^{-m}$

$$\sum_{l=0}^{+\infty}f_l\ (T_{n-l}T_{m+l}-T_{m-l}T_{n+l})=\frac{(x^{\xi}-x^{-\xi})(x^{(\xi+1)}-x^{-(\xi+1)})}{x-x^{-1}}(x^{2n}-x^{-2n})\ \delta_{n+m,0}\ , \quad (10)$$

where f_l :

$$f(z)=\sum_{l=0}^{+\infty}f_l\,z^l.$$

If we fix z and consider the limit $x\to 1$, then

$$T(z)=2+\xi(\xi+1)\ (x-x^{-1})^2\left(\sum_{m=-\infty}^{+\infty}L_m z^{-m}+\frac{1}{4\xi(\xi+1)}\right)+O((x-x^{-1})^4)\ . \quad (11)$$

One can check that the defining relations (10) give us the Virasoro algebra commutators for the modes L_m and the corresponding central charge is equal to

$$c=1-\frac{6}{\xi(\xi+1)}\ .$$

To explain the physical meaning of the field $T(z)$ for the XYZ model let us introduce the following parameterization

$$x^2=e^{-\pi\epsilon},z=e^{-i\epsilon\beta}\ . \quad (12)$$

After a little algebra, (9) can be rewritten in the form:

$$T(\beta')T(\beta)=S_\epsilon(\beta'-\beta)\ T(\beta)T(\beta')+C_\epsilon\big(\ \delta(\beta'-\beta-i\pi)+\delta(\beta'-\beta+i\pi)\ \big)\ , \quad (13)$$

where

$$S_\epsilon(\beta)=\frac{\theta_1(i\frac{\beta}{2\pi}-\frac{\xi}{2})\ \theta_2(i\frac{\beta}{2\pi}+\frac{\xi}{2})}{\theta_2(i\frac{\beta}{2\pi}-\frac{\xi}{2})\ \theta_1(i\frac{\beta}{2\pi}+\frac{\xi}{2})}. \quad (14)$$

Here the functions $\theta_{1,2}(u)$ are the conventional theta functions with the modular parameter

$$\tau = i\epsilon^{-1}$$

and the constant C_ϵ reads explicitly

$$C_\epsilon = \frac{2\pi}{\epsilon} \prod_{n=0}^{\infty} \frac{(1-x^{2(\xi+1)+4n})(1-x^{-2\xi+4n})}{(1-x^{2\xi+4(n+1)})(1-x^{-2(\xi+1)+4(n+1)})} \ . \tag{15}$$

It is useful to consider the limit $\epsilon \to 0$ in the commutation relation (13) . Now we will fix β. So this limit differs from the "conformal" one, when we fixed z. From the physical point of view this limit corresponds to the scaling limit of the infinite XYZ-chain (1), when it can be described by relativistic quantum field theory with the Lagrangian [10]

$$L = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2} \, (\partial_\mu \phi)^2 + \frac{m_0^2}{b^2} \cos(b\phi) \right) \tag{16}$$

where

$$b^2 = 8\pi \ \frac{\xi}{\xi+1} \ .$$

The "bare" mass m_0 and the physical mass scale m_{phys} in the Sine-Gordon model are connected with the parameters of the initial XYZ chain as

$$m_0^2 \sim \frac{J_x - J_y}{J_x} \ , \quad m^2_{phys} \sim \left(\frac{J_x - J_y}{J_x} \right)^{\xi+1} \sim e^{-\frac{2\pi}{\epsilon}} \ . \tag{17}$$

It is easy to see that in the scaling limit the operator $T(\beta)$ will generate the simple Zamolodchikov - Faddeev (ZF) algebra

$$T(\beta')T(\beta) = S(\beta'-\beta) \ T(\beta)T(\beta'), \tag{18}$$

where both β, β' are real. The function

$$S(\beta) = \frac{\sinh\beta + i\sin(\pi\xi)}{\sinh\beta - i\sin(\pi\xi)} \tag{19}$$

coincides with the two-particle S-matrix of the basic scalar particle in the Sine-Gordon model [11]. Moreover, as it follows from (13), the singular part of the operator product $T(\beta')T(\beta)$, being considered as a function of the complex variable β' for real β in the upper half plane $\Im m \ \beta' \geq 0$, contains a simple pole at the point $\beta' = \beta + i\pi$ with a numerical residue. This operator product condition together with the commutation relation (18) are the basic properties of ZF algebra acting in the space of angular quantization [12], [13]. In such a way we can identify $T(\beta)$ with the ZF operator of the basic scalar particle. Note that such particle exists only for $0 < \xi < 1$, so this interpretation is valid only within this restriction.

Now let us go back to the relation (13) and interpret it as the ZF algebra for the basic scalar excitation in the XYZ model (1) with $J_z > 0$. The function (14) is a two-particle S-matrix of this excitation. Having at hand the oscillator representation (6), (7) for the ZF operator of the basic scalar particle it is easy to get the form-factors of local operators [3], [12], [13]. I hope to return to this point in future publications.

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