SPHERICAL ABERRATIONS-FREE WIGGLER*

A.Mikhailichenko#, Cornell University, LEPP, Ithaca, NY 14853, U.S.A

Abstract

We represented analyses of a wiggler with a linear piecewise longitudinal field dependence. This type of field distribution eliminates spherical aberrations in the wiggler. This wiggler can be recommended for usage in cooler rings including ILC one.

INTRODUCTION

In our previous publications we introduced the concept of Ideal Wiggler [1]-[3], [7]. Here we analyze this device for the absence of aberrations. Representation of the wiggler as a series of quadrupoles, installed 90^o towards the beam trajectory, Fig.1, allows immediate recommendations on how to improve the quality of the field. Basic idea behind this was very simple: if the magnets are the quadrupoles- let make the field distribution as close to the quadrupole type (linear) as possible. In some sense this representation answers the question about the best longitudinal field distribution. In many publications the question about this distribution type risen, but we think the right answer revealed either this must be a rectangular, trapezoidal or sin-like distribution in a favor of piece-wise linear one.

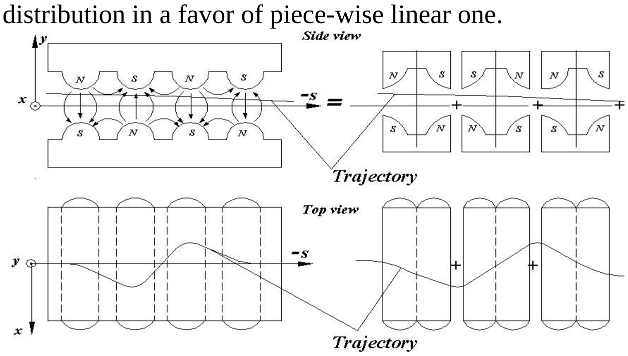

Figure 1: Representation of wiggler as series of quadrupoles with *transverse* orientation.

Other question which arises in association with the field shape linked to the spectral properties of radiation from such wiggler–does not considered in this publication.

FIELDS IN A WIGGLER

Presentation of fields in a wiggler is a rather tricky process. Usually for description of particle motion in a circular machine one uses natural coordinates, in which the fields are represented as expansions from the reference (central) trajectory calculated for the particle nominated as a reference one, running with zero transverse coordinates and reference energy, so this trajectory runs through the centers of quadrupoles.

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#mikhail@lepp.cornell.edu

Trajectory could be treated as a geodesic one with respect to these coordinates. That is why it allows broadly used graphical presentation of longitudinal coordinate as a straight line while one describes the transverse motion.

Other presentation of fields, generated by some device (magnet), uses natural coordinates associated with this magnet. Usually for quadrupoles, sextupoles, etc (or speaking more generally, for any field configuration having zero dipole field at center) –these coordinates coincide. The difference emerges in bending magnets, which introduce curved coordinates. Curved means here with respect to outer Lab coordinates. Some appropriate Canonical transformation links these two presentations. All that matters now is the complexity of these transformations. Really, as the particle wiggles in these simplified coordinates with angle $\alpha \cong K/\gamma$ with respect to the s axis ($K = eB_0\lambda_w/(2\pi mc^2)$, λ_w stands for longitudinal period of wiggler), the complicated field structure introduces linear focusing field $\sim \alpha B_0$ as a result of entrance the dipole field with angle and effective octupole $\sim \alpha B'(s)$, where $B(s)$ is longitudinal dependence of wiggler field. The last is a result of entrance the sextupole with angle. Sextupole is inevitable component of the field change in vertical direction as a result of its change in *longitudinal* direction. All this is a sequence of a theorem proven in [4] which states that "A passage with an angle through the edge field of a multipole, acting to the particle as the *next* order multipole with the power, proportional to the tangent of the entrance angle multiplied by original multipole value at the center and reduced by the order of original multipole".

As the presentation of wiggler field in natural coordinates is rather complicated, the fields usually represented in Cartesian coordinates so the trajectory wiggles. Typically in mostly computer codes in vicinity of wiggler the coordinates jump from natural to Cartesian ones and back again after the wiggler passed. These jumps from natural coordinates of damping ring to Cartesian and back done in places where the bending field is absent.

Presentation of fields as series

In numerous publications the field in a wiggler represented as series something like

$$B_x = -\sum (B_0 k_x/k_y) Sin(k_x x) Sinh(k_y y) Cos(k_s s + \varphi_s)$$
$$B_y = \sum B_0 Cos(k_x x) Cosh(k_y y) Cos(k_s s + \varphi_s) \qquad (1)$$
$$B_s = -\sum (B_0 k_s/k_y) Cos(k_x x) Sinh(k_y y) Sin(k_s s + \varphi_s)$$

$k_s = 2\pi/\lambda_w$, and with restrain $k_y^2 = k_x^2 + k_s^2$. Summing is going over all possible wave numbers k_s, k_x and k_y.

This presentation is contradictory from the formal point of view, however. Really, equation $div\vec{B}=0$ requires that restrain for $k_{x,y,s}$ applied to all sum of terms, not to each term separately, so the relation between k_s, k_x and k_y is less restrictive. Other circumstances associated with the choice of k_s, k_x and k_y itself. Of course it was natural to choose for longitudinal wavenumber $k_s = 2p / l_w$, as the wiggler period is well known and namely it determines particle dynamics. But from the point of formal mathematics, however, one needs to choose the coordinate box {x_{max}, y_{max}, s_{max}} and represent the field as Fourier harmonics with the wave numbers $\{k_x, k_y, k_s\}$ as sum of terms having {$2pm/x_{max}$, $2pn/y_{max}$, $2pl/s_{max}$} for every m,n,l = 0,1,2,… Only with this presentation, (1) will be mathematically correct. Usually for adequate presentation of fields the number of terms required in (1) easily reaches few tens or even a hundred. Basically this is a sequence of necessity for proper presentation of field dependence in transverse direction. As each term describes dependence along all three coordinates, so some redundancy is present here.

We developed *another approach* to the field presentation, which requires knowledge of few (typically two-three) functions, which could be obtained either from measurements or from calculations. With this presentation it is clear what terms are responsible for the aberrations, and some recommendations for its elimination could be given at once.

Presentation of fields with Generating functions

First, it is useful to combine transverse coordinates x and y in one complex variable $z = x + iy$. The longitudinal coordinate s runs straight. Magnetic field in 3D could be represented in these coordinates as [6]

$$\overline{B} = B_x - iB_y = \frac{\partial W}{\partial z} + \frac{\partial W}{\partial \bar{z}} = 2\frac{\partial}{\partial z}\mathrm{Re}W(z,\bar{z},s,t)$$

$$B_s = Re\int\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)W(z,\bar{z},s,t)ds \qquad (2)$$

where function of complex variable W satisfy equation

$$4\frac{\partial^2 W}{\partial z \partial \bar{z}} + \left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)W(z,\bar{z},s,t) = 0 \cdot \qquad (3)$$

Here *by definition* $\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and $\bar{z} = x - iy$, c–stands for the speed of light. General solution of (3) could be represented as [8]

$$W = (-i)\sum_{m=1}^{\infty}\frac{z^m}{m}\left\{G_{m-1}(s,t) - \frac{|z|^2}{4(m+1)}\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_{m-1}(s,t) + \right.$$
$$\left. + \frac{|z|^4}{32(m+1)(m+2)}\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^2 G_{m-1}(s,t) - \cdots\right\} \qquad (4)$$

where $|z|^2 = z \cdot \bar{z}$, t stands for time variable, and complex functions (generating functions) $G_{m-1}(s,t) = a_{m-1}(s,t) + ib_{m-1}(s,t)$ describe multipoles as functions of longitudinal coordinate and time. If all b_{m-1}=0, then the closest to the axis x in the first quadrant where the *south* pole is located. As function W contains terms $(z \cdot \bar{z})^{2k}$, k=1,2,.. it does not satisfy Cauchy-Riemann differential equations in all space and it is analytical only at z=0. This is explained by escaping field lines from one complex plane z to another one in 3D. So first terms in brackets (3) $G_{m-1}z^m / m$, describe field distribution in 2D case, far from edges of the magnet. One interesting sequence of presentation (3) is that if $G_{m-1}(s,t)$ satisfy wave equation, then all terms in brackets, except the first one, are vanished. So, in the situation, when $G_{m-1}(s,t)$ demonstrates the wave-like dependence, the field is a two-dimensional one (plane wave). So the presentation of laser field as a kind of wiggler demonstrates appearance of aberrations only at the entrance and at the exit of the laser bunch. For the purposes of focusing system design, namely this opens a possibility to make a focusing system time dependent, so it does not manifest these terms, and hence is aberration free.

As far as the wiggler business, usually fields are steady, so time derivatives are zero. If zero x lies in medial plane sy, y=0 which is the plane of up-down anti-symmetry, see Fig.1, then analytical representation for the wiggler fields defined by (2) and (4) is the following [2], [5]

$$B_x(x,y,s) = -\frac{xy}{4}B''(s) + 2S(s)xy + \frac{x^3y + xy^3}{48}B^{(IV)} + \ldots \qquad (5)$$

$$B_y = B(s) + S(s)(x^2 - y^2) - \frac{x^2 + 3y^2}{8}B''(s) + D(s)\cdot(x^4 + y^4 - 6x^2y^2) + \ldots$$

$$B_s = y\cdot B'(s) - \frac{x^2y + y^3}{8}B'''(s) + \frac{3x^2 - y^3}{3}S'(s) + D'(s)\frac{5x^4 - 10x^2y^2 + y^5}{5} + \ldots$$

where it was introduced $B(s)$=G_0, $S(s)$= G_2, $D(s)$=G_4 which stand for generating functions for dipole, sextupole, decapole, … field accordingly. One can see from (5), that for transverse dependence, say proportional to x^2, are responsible both terms- the pure sextupole one $\sim S(s)x^2$ and the one arising from the second derivative of magnetic field dependence in longitudinal direction -+-+i~$\frac{1}{8}x^2(\partial^2 B(s)/\partial s^2)$. So for wide poles of wiggler, these terms cancel each other and sextupole-type field dependence manifests itself only in the vertical direction. That is why the octupole type dependence appeared as a result of wiggling in the field of sextupole, mentioned above.

Now let us show how these functions could be found. Function $B(s)$ can be recognized simply by measurements along s –axis, as in this case x=y=0 by definition. The same could be done by calculations with appropriate 3D code. After that, all necessary derivatives along s appeared in (5), can be obtained numerically with necessary accuracy. So now the difference between calculated (or measured) transverse field variations as functions of s and found from (1) can be treated as higher harmonics, with sextupole $S(s)$ as a lowest one. Or in other words, if one has transverse field roll-off, say at field maximum, calculated or measured, sextupole component *S*(*s*), having dependence~x^2 can be identified *after subtracting* the terms, associated with derivatives. As the second derivative proportional to the curvature of the graph, one can conclude, that biggest input these terms introduce at points with extremum. Other harmonics, such as *D*(*s*) can be found in the same manner, taking in consideration dependence ~x^4 and so on. When functions *B*, *S*, *D* restored in points along *s* it is possible to represent them as Fourier series with Fast Fourier Transformation, so the accuracy will be limited only by the number of points along *s*. This gives a *simple and powerful recipe* for representation of wiggler field for the purposes of calculation of nonlinearities and calculations of dynamic aperture associated with these nonlinearities.

Now one can conclude from (5), that if dependence along *s* is linear, all terms with derivatives are vanished and the only terms remain which responsible for the principal harmonics, as in pure 2D case [2], [3]. So the field dependence desirable looks like represented in Fig.2.

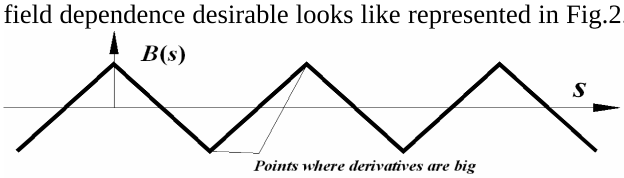

Figure 2: Longitudinal profile of magnetic field with linear dependence between extremes.

As the second derivative is proportional to the local curvature, terms with derivatives responsible for aberrations might be big at these points. Fortunately the vertical force is acting on the particle is proportional to the transverse angle, $a \cong dx/ds$, the force becomes zero at points of extreme. The power of *s* which defines the angle dependence around zero is at least a next order one, in comparison with the linear power, so resulting action of force at these extreme points is zero. This is valid for realistic field dependence in these extreme points, as the derivative proportional $B'(s) \cong B_{\max}/a^2$, where *a* stands for characteristic dimension, which might be an aperture as the angle is always proportional to the integral of magnetic field, so it power (as function of *s*) is always higher.

FIELD GENERATION

The field shape as it is represented in Fig. 2, could be generated if poles of wiggler equipped with quadrupole-type shape of poles, Fig.3, [2]. So basically any wiggler could be transformed into the one generating linear piece-wise longitudinal distribution with adding curved prism-like poles. Sides of the prisms must have a hyperbolic shape as it is required for quadrupole. Maximal value of field remains the same practically under this attachment. Of cause there is a limitation in realization of such dependence namely at the pole tips. One can make theses very tips with highly saturated magnetic material (such as Permendur, for example). Of course some nonlinearities (and aberrations) will remain, but with a much reduced level. Calculated field distribution is shown in Fig. 4.

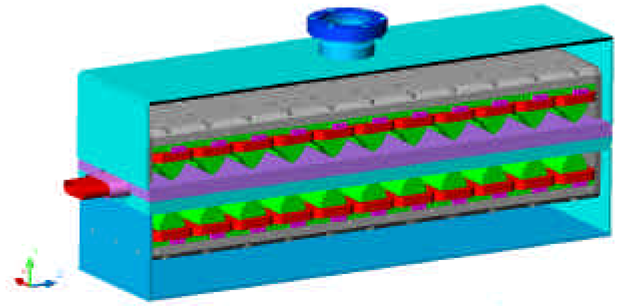

Figure 3: 11-pole wiggler cold mass. Two side walls removed. Room temperature vacuum chamber and 70°K shield are seen also.

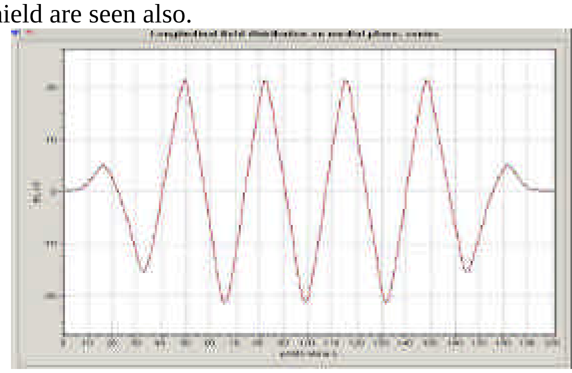

Figure 4: Calculated field distribution (in kGauss) in 11-pole wiggler with quadrupole-type poles. Right point #200 on this graph located at *s*=150 *cm*.

Tracking in a field from Fig.4 demonstrates extended linearity within vertical aperture ±1.5cm, while the closest gap between poles in Fig.3 is ±2.7 cm, period is 25 cm. [1].

CONCLUSIONS

With the approach described above one can make wiggler period rather short and by doing so, one can reduce equilibrium radial emittance achievable in wiggler dominated ring ~ l_w^2 and increase dynamic aperture of the cooling ring, which might be important for ILC.

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