CBN 99-2

Modes of Oscillation of Trains of Bunches

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Jan. 26, 1999

1. Introduction

In storage rings which operate with trains of bunches at high currents the observation of the motion of these bunches is quite important. An important reason is that the storage ring is often attempting to operate at or above the instability threshold for coupled bunch instabilities of the beam(s) and these observations are necessary to help to identify the cause of the instability and its possible cures. With single bunches or equally spaced bunches the observation of beam position signals in the frequency domain has been analyzed and gives fairly simple and understandable results. With trains of bunches the frequency spectrum is more complicated, but is still capable of providing useful information. This paper will present a basis for understanding measurements of the frequency spectrum of beam signals for trains of bunches.

Since we are typically interested in observing the beam while it is stable or just at the onset of an instability, we will assume the amplitude of the oscillating bunches is small compared to the scale distance at which nonlinearities become important. This allows us to treat the motion as linear, small amplitude motion. This analysis equally well describes motion in the longitudinal or transverse directions for dipole, quadrupole, etc. oscillations, however to simplify the expressions in this paper we will treat the case of coupled bunch dipole betatron motion in the horizontal direction. The results of this paper may be trivially applied to any of the other cases.

2. Description of the Motion of the Ensemble of Bunches

Before dealing with the signals from a position monitor, we must establish a consistent way of describing the actual beam motion. To describe the motion of the bunches we will choose a reference location in the ring which will be our measurement location. When the first bunch in the fill pattern passes this location, its phase space coordinates will be (x_0,x_0'). At the same instant the second bunch will be some distance away from the measurement point having its own independent phase space coordinates at its location. To remove the effects of differing betas around the accelerator, we will spatially project the motion of this second bunch forward to our measurement location using standard Twiss parameters. These projected phase space coordinates are then called (x_1,x_1'). The position vector Λ_i which contains these same projected phase space coordinates for all the bunches in the ring when the first bunch reaches the measurement location on the i-th turn is then

$$\Lambda_i = \begin{pmatrix} x_{0i} \\ x'_{0i} \\ x_{1i} \\ x'_{1i} \\ \circ \\ \circ \\ \circ \end{pmatrix}$$

For later analyses it is convenient to use complex variables to describe the oscillation of each bunch. In this case the real part of the expressions will describe the actual motion. For the phase space coordinates undergoing betatron oscillations we have the following form

$$\begin{pmatrix} x_{0i} \\ x'_{0i} \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} \beta^{\frac{1}{2}} & 0 \\ -\alpha\,\beta^{-\frac{1}{2}} & \beta^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \hat{x}_{0i}\; e^{j\omega t + j\phi_0} \\ -\,j\;\hat{x}_{0i}\; e^{j\omega t + j\phi_0} \end{pmatrix} \right\}$$

$$= \mathrm{Re}\left\{ \begin{pmatrix} 1 \\ \dfrac{\sqrt{1+\alpha^2}}{\beta}\; e^{j\delta_0 + j\pi} \end{pmatrix} \beta^{\frac{1}{2}}\, \hat{x}_{0i} e^{j\omega t + j\phi_0} \right\}$$

$$\tan\,\delta_0 = \frac{1}{\alpha}$$

Thus λ_i may be defined to be the full complex expression for Λ_i,

$$\Lambda_i = \mathrm{Re}\{\,\lambda_i\,\}$$

The time development of the freely propagating position vector from the i-th turn to the next turn can be described by

$$\lambda_{i+1} = (\,T + Z\,)\,\lambda_i$$

where T is the well known one turn transfer matrix and Z is a real matrix which contains the effect of the wakefields from each bunch producing deflections for subsequent bunches. The matrix T will be block diagonal, while generally the matrix Z will contain non-zero elements off the diagonal. From the form of this equation there will be a set of eigen vectors and eigen values each of which will describe simple repetitive motion of the ensemble of bunches. Each one of these eigen vectors will evolve from turn to turn in a very simple way,

$$\lambda_{i+1} = e^{j\omega T_r} \lambda_i$$

where T_r is the revolution period of the ring and $e^{j\omega T_r}$ is the eigen value corresponding to this eigen vector. The full eigen equation is then

$$(T + Z) \lambda_i = e^{j\omega T_r} \lambda_i$$

Solving this equation yields a set of eigen values and corresponding eigen vectors which describe the normal modes of the bunches.

From the form of the eigen equation several things are clear. Because (T+Z) is a real matrix, it will have solutions which will break into pairs of eigen values which are complex conjugates of each other. This gives pairs of angular frequencies which are equal in magnitude and opposite in sign. Since the elements of λ_i i pairs represent the phase space coordinates of each bunch, the equal and opposite values of each pair of ω's simply reflects each bunch's oscillation being visible in both the x and x' coordinates. If the wakefield effects are ignored, Z is zero, the eigen values are degenerate with ω being the betatron oscillation angular frequency and the eigen vectors can be any set which spans the space of all possible bunch motion.

If the wakefield effects are important, then the eigen values and eigen vectors will depend on the details of the ring's impedance, the bunch length and the exact distribution of current in each bunch. In general the eigen values will differ from each other in their real and imaginary parts. The change in the real part when compared to the zero current case gives the tune shift and the imaginary part yields the change in damping rate of the eigen mode. Depending on the ring's impedance and the distribution of current, the growth rate of the most unstable mode and its normal motion can vary dramatically. In order to measure the beam(s) stability, locating and observing coherent tune shifts and damping rates for the most unstable mode(s) are of great interest.

Making use of the results of the example in Appendix 1., for a single bunch we see that the signal processed from a position monitor gives a line spectrum of upper and lower betatron sidebands with a fairly flat envelope extending from zero frequency up to a frequency $f_c = 1/\Delta T$ determined from the sampling time, $2\ \Delta T$. When there are many bunches in the ring, the pattern of amplitudes and half widths of sidebands is not in general uniform from sideband to sideband as it is for a single bunch, but there is a characteristic frequency for which the sideband pattern does repeat. This frequency is determined by the RF accelerating system frequency or one of its subharmonics. In the case of CESR when 14 ns (7 RF wavelength) spaced bunches are filled, the characteristic frequency is 71.4 MHz or 1/7 of the RF frequency. When bunches fill relatively few of the possible bunch locations for this characteristic frequency, it should not be necessary to observe all sidebands over the same range of frequencies. One would, rather, expect that it should be sufficient to make measurements at a subset of sidebands whose number equals the number of bunches filled, i.e. the degrees of freedom of the ensemble of bunches. If the matrix Z is known exactly, then the set of eigen values and eigen vectors would determine for each mode the actual pattern of bunch oscillations, the envelope of the frequency spectrum of the motion and thus the most appropriate sideband for observing each normal mode. If (as is generally true) Z is not known in detail, but if there is a periodic or almost periodic pattern to the bunches which are filled, we should expect to find some related pattern in the sideband spectrum. For each set of bunch patterns which is filled, the goal is then to find some basis set of position vectors, b_k, (and their corresponding frequency spectrum), which span all the possible motion of the bunches. When this is done, the eigen vectors of (T+Z) can then be written in terms of these basis vectors,

$$\lambda_i = \sum_{k=1}^{\text{number of bunches}} c_k\ b_k$$

where c_k are coefficients, and the spectrum of sidebands for a given eigen mode will be the superposition of all the basis vectors' sideband spectra.

3. A Basis for Oscillations of Equally Spaced Trains of Bunches

The are many possible choices for basis vectors for the oscillation of trains of bunches. In order to choose a reasonable set of basis vectors it is useful to examine a simple case where the normal modes of oscillation are known, equal current bunches which are spaced uniformly around the ring. Uniformly spaced bunches may be also thought of as uniformly spaced trains with a single bunch per train. Although this is a semantic distinction, it will be less confusing for notation later if we adopt this view. For the case of N_t trains of one bunch there is a important symmetry which is visible in the voltage of a position monitor; the signal is periodic at with a period of T_r/N_t. If bunches are undergoing an oscillation for which all bunches are moving in phase with each other, the signal at the position monitor will appear to oscillate at the betatron frequency. For an oscillation where the bunches move with a phase shift of $2\pi/N_t$ from bunch to bunch the oscillation frequency will be the upper sideband of the first rotation harmonic. This sequence will continue by increasing the phase shift from bunch to bunch by $2\pi/N_t$ to give a total of N_t independent modes of oscillation.

With equally spaced single bunch trains the position monitor voltage signals for each bunch v(t) will be identical in shape but delayed in time if all the bunches have the same current. The signal v(t) must be understood to repeat with a period of T_r. So we can write the voltage signal from all of the single bunch trains, which are undergoing an oscillation with a phase shift $2\pi\ m_t/N_t$ from train to train, as

$$v_{tr}^{m_t}(t) = \sum_{n_t=0}^{N_t-1} v\left(t + \frac{n_t}{N_t} T_r\right) \exp\left[j\omega_\beta \frac{n_t}{N_t} T_r + 2\pi\, j\, \frac{n_t\ m_t}{N_t} \right]$$

where m_t, the coupled train mode number, ranges from 0 to N_t-1. The frequency spectrum of the train voltage signal may computed to be

$$v_{tr}^{m_t}(\omega) = \sum_{n_t=0}^{N_t-1} v(\omega)\ \exp\left[-j\omega \frac{n_t}{N_t} T_r + j\omega_\beta \frac{n_t}{N_t} T_r + 2\pi\, j\, \frac{n_t\ m_t}{N_t} \right]$$

$$= v(\omega) \left\{ \frac{1 - \exp[-j\omega T_r + j\omega_\beta T_r + 2\pi\, j\, m_t]}{1 - \exp\left[-j\omega \frac{1}{N_t} T_r + j\omega_\beta \frac{1}{N_t} T_r + 2\pi\, j\, \frac{m_t}{N_t} \right]} \right\}$$

$$= v(\omega) \left\{ \frac{1 - \exp[- j(\omega - \omega_\beta - m_t\omega_r)\, T_r]}{1 - \exp\left[- j(\omega - \omega_\beta - m_t\omega_r)\, \frac{T_r}{N_t} \right]} \right\}$$

where $v(\omega)$ will have the spectral envelope of the periodic position monitor signal. $v(\omega)$ may be written as an envelope function $v(\omega)$ and the betatron line spectrum. Substituting

$$v(\omega) \equiv \hat{v}(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\}) \,,$$

and examining

$$\sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\}) \left\{ \frac{1 - \exp[- j(\omega - \omega_\beta - m_t\omega_r)\, T_r]}{1 - \exp\left[- j(\omega - \omega_\beta - m_t\omega_r)\, \frac{T_r}{N_t} \right]} \right\}$$

$$= N_t \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_\beta - \{ nN_t + m_t \}\, \omega_r)$$

since $\{ \ \} = 0$ unless $n = m_t + N_t\, k$ (where k is an integer).

Making use of this substitution, the frequency spectrum of the train voltage signal may written more concisely as

$$v_{tr}^{m_t}(\omega) = \hat{v}(\omega)\, N_t \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_\beta - \{ nN_t + m_t \}\, \omega_r)$$

This result gives a line spectrum of upper betatron sidebands of the rotation harmonics where the set of lines which correspond to a phase shift of $2\pi\ m_t/N_t$ from train to train occur at the betatron sideband above the m_t-th rotation harmonic and every N_t sidebands thereafter. For a beam excited at one of betatron sidebands the response will be N_t times the response of a single bunch since all the bunches are superposing their signals coherently.

Having described the case of equally spaced single bunch trains, we shall now investigate the case of a single train of N_b (equally spaced) bunches. Making use of the simplicity of the multiple train case which has bunches oscillating with a phase shift of $2\pi\ m_t/N_t$ from train to train, we will choose oscillation patterns which have phase shifts of $2\pi\ m_b/N_b$ from bunch to bunch within the train. Therefore, the position monitor signal is

$$v_B^{m_b}(t) = \sum_{n_b=0}^{N_b-1} v(t + n_b T_{bb}) \ \exp\left[j\omega_\beta n_b T_{bb} + 2\pi\, j\, \frac{n_b\ m_b}{N_b} \right]$$

where N_b is the number of bunches,
T_{bb} is the spacing between bunches &
$m_b = 0, .., N_b-1$ is the coupled bunch mode number

This signal may be Fourier transformed to give its frequency spectrum which is

$$v_B^{m_b}(\omega) = \sum_{n_b=0}^{N_b-1} v(\omega) \ \exp\left[- j\omega n_b T_{bb} + j\omega_\beta n_b T_{bb} + 2\pi\, j \frac{n_b\ m_b}{N_b} \right]$$

$$= v(\omega) \left\{ \frac{1 - \exp[- j\omega N_b T_{bb} + j\omega_\beta N_b T_{bb} + 2\pi\, j\, m_b]}{1 - \exp\left[- j\omega T_{bb} + j\omega_\beta T_{bb} + 2\pi\, j\, \dfrac{m_b}{N_b} \right]} \right\}$$

$$= v(\omega) \left\{ \frac{\sin\frac{1}{2}[(\omega - \omega_\beta) N_b T_{bb} - 2\pi\ m_b]}{\sin\frac{1}{2}\left[(\omega - \omega_\beta) T_{bb} - 2\pi\, \dfrac{m_b}{N_b}\right]} \right\} \quad *$$

$$\exp\left\{ -\frac{j}{2}\left[(\omega - \omega_\beta)\ (N_b-1)\ T_{bb} - 2\pi\, \frac{m_b(N_b-1)}{N_b} \right]\right\}$$

This result gives a spectral envelope which is an interference pattern times the signal's spectral envelope. For coupled bunch mode 0, this interference pattern has a peak at ω equal to ω_β which is N_b times the amplitude of a

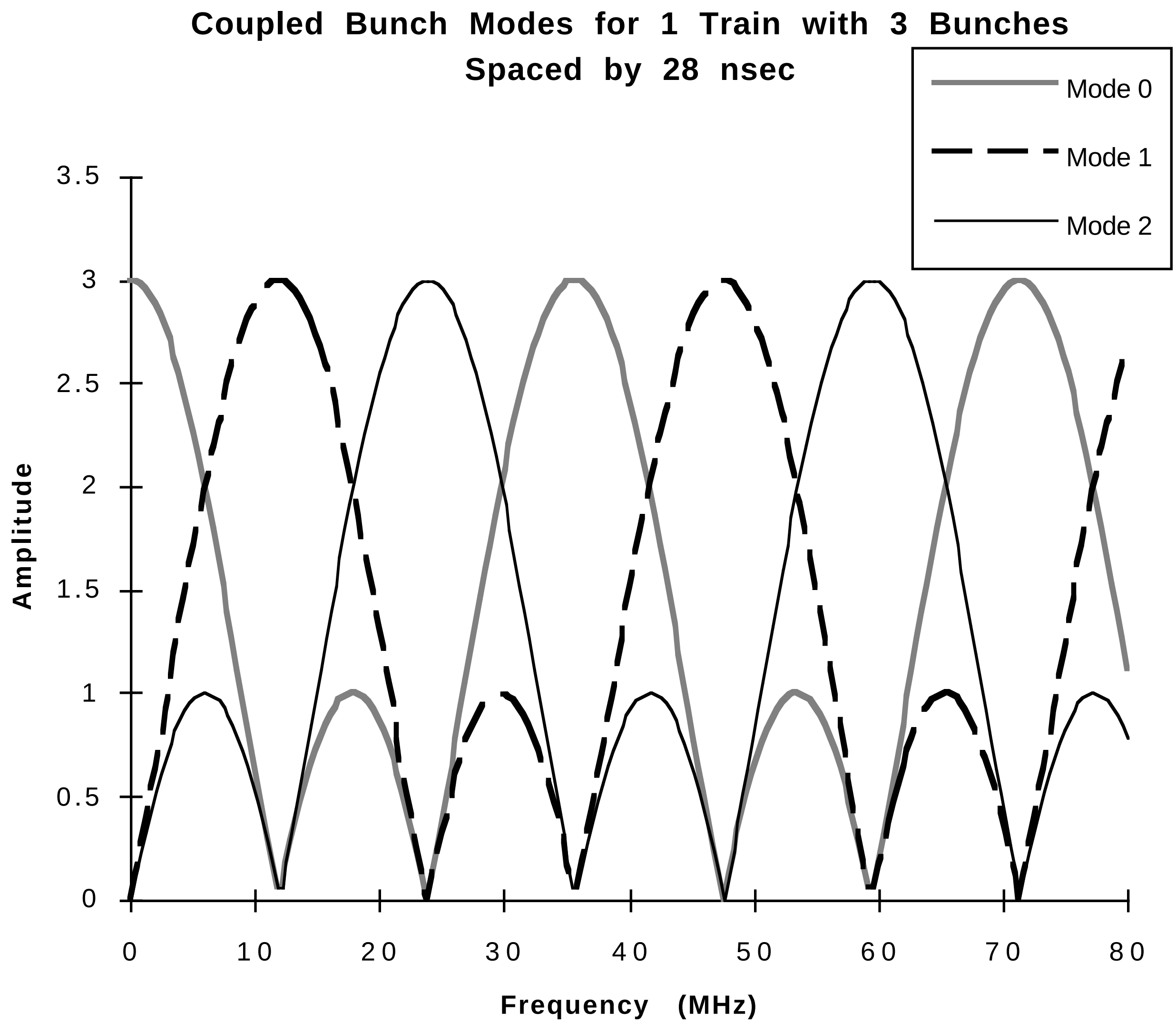


Figure 1. Envelope functions for three 28 nsec spaced bunches in a single train oscillating in each of the three coupled bunch mode patterns.

single bunch. Generally the envelope will have other smaller peaks and will repeat with a periodicity in angular frequency of $2\pi/T_{bb}$. The overall envelope will have the shape similar to the angular dependence of the interference pattern of monochromatic light passing through multiple slits. An example of the three envelope functions for three bunches spaced by 28 nsec in a single train may be found in Figure 1. Notice in Figure 1 that the highest spectral peaks occur at frequencies of m_b/T_{bb}, where the other two envelope functions have zeroes. This suggests that betatron sidebands near these spectral peaks have the largest fraction of their signal coming from this particular mode of oscillation and are the best frequencies at which these modes may be observed. Notice also that as the spacing between bunches T_{bb} increases, the peaks in the spectral envelope will crowd together until $T_{bb} = {}^{1}/_{3}\ T_r$ (or the bunches are equally spaced around the ring) at which point the peaks in the envelope function will lie on top of every third betatron sideband. This is the same result as was found for three equally spaced trains.

We are now in a position to combine the last two results to give the signal for N_t trains of N_b bunches at a position monitor,

$$v_{Tr\ .\ B}^{(m_t\ .\ m_b)}(t) = \sum_{n_t=0}^{N_t-1} \sum_{n_b=0}^{N_b-1} v\left(t + \frac{n_t}{N_t} T_r + n_b T_{bb}\right) *$$

$$\exp\left[j\omega_\beta\left(\frac{n_t}{N_t} T_r + n_b T_{bb}\right) + 2\pi\, j\, \frac{n_t\ m_t}{N_t} + 2\pi\, j\, \frac{n_b\ m_b}{N_b}\right]$$

where N_t is the number of trains, N_b is the number of bunches,
T_{bb} is the spacing between bunches within the trains
$m_t = 0, .., N_t - 1$; $m_b = 0, .., N_b - 1$
$(m_t\ .\ m_b)$ is the coupled bunch mode number

Each of the different oscillation patterns may be labeled by the pair of numbers (m_t, m_b). The frequency spectrum of the bunch train signal is then given by

$$v_{Tr\ ,\ B}^{(m_t\ ,\ m_b)}(\omega) =$$

$$= \hat{v}(\omega)\ N_t \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_\beta - \{nN_t + m_t\}\ \omega_r) \left\{ \frac{\sin\frac{1}{2}[(\omega - \omega_\beta)N_b T_{bb} - 2\pi\ m_b]}{\sin\frac{1}{2}\left[(\omega - \omega_\beta)T_{bb} - 2\pi\ \frac{m_b}{N_b}\right]} \right\}$$

$$*\quad \exp\left\{ -\frac{j}{2}\left[(\omega - \omega_\beta)\ (N_b - 1)\ T_{bb} - 2\pi\ \frac{m_b(N_b - 1)}{N_b} \right]\right\}$$

where $\hat{v}(\omega)$ is the envelope of $v(\omega)$.

This is, of course, a spectrum for which every N_t-th betatron sideband corresponds to the same train mode oscillation pattern and the overall spectral interference envelope is determined by the bunch mode oscillation pattern. From this we can now see that there are more favorable betatron sidebands which will be useful for observing each of the different basis modes of oscillation. Upper sidebands of the rotation harmonics near m_b'/T_{bb} will contain the largest fraction of their signals from the mode patterns (m_t', m_b') where $m_t' = 1, \ldots, N_t-1$. These correspond to basis vectors $\lambda_{m_t', m_b'}$ in which the bunches oscillate with phase shifts of $2\pi\ m_t'/N_t$ from train to train, and $2\pi\ m_b'/N_b$ from bunch to bunch within each train.

The set of basis modes λ_{m_t, m_b} is a complete set of independent modes so that any oscillation of bunches in the beam may be written as a superposition of elements in this set. For any arbitrary impedance in the accelerator there will be eigen modes for the beam and these, therefore, may be written as the superposition of the set λ_{m_t, m_b}. As an example of this superposition, we will measure the coherent damping rate of the least stable eigen mode. To accomplish this we will select the betatron sideband at ω corresponding to a particular λ_{m_t, m_b} which has a significant projection from the eigen mode's spectrum. Since the set of basis modes is complete there will always be a mode (m_t', m_b') which has the largest projection from the given eigen mode. To measure the damping rate of the given eigen mode, we excite the beam at the frequency which has the largest component coming from $\lambda_{m_t', m_b'}$. This means that on the n-th turn there will be an excitation $\delta\lambda_n$ driven by some type of kicker in the ring. This will produce a position on the n+1-th turn given by

$$\lambda_{n+1} = (T + Z) \{ \lambda_n + \delta\lambda_n \}$$

If we make use of a periodic excitation,

$$\delta\lambda_n = \lambda_e\, e^{j\, n\omega T_r}$$

for a total of N turns beginning initially with $\lambda_0 = 0$, we may formally solve for λ_N, the beam's position after the excitation has ceased, and obtain

$$\lambda_N = \sum_{n=0}^{N-1} (T + Z)^n\ \delta\lambda_n = \sum_{n=0}^{N-1} (T + Z)^n\ \lambda_e\, e^{j\, n\omega T_r}$$

$$= [\ (T + Z)\, e^{j\,\omega T_r} - I\ ]^{-1}\, [\ (T + Z)^N\, e^{j\,N\,\omega T_r} - I\ ]\ \ \lambda_e$$

$$\text{when} \quad \det [\ (T + Z)\, e^{j\,\omega T_r} - I\ ] \neq 0$$

Having solved for λ_N we may then project it into its different eigen vector components. Therefore, after turning off the excitation, we will, in general, see a decay in the amplitude which is made of several different signals (coming from different eigen modes which have projections to this

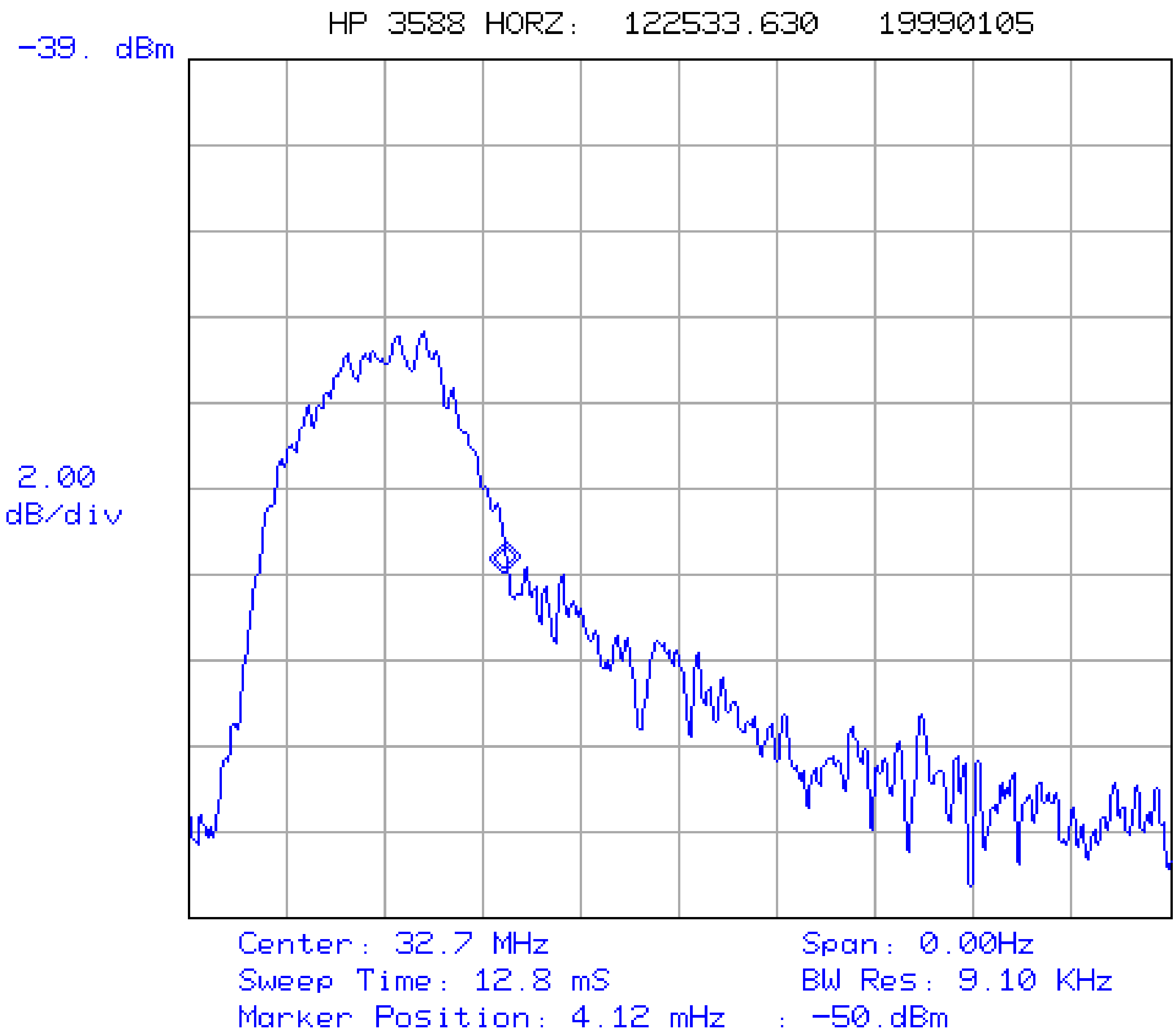


Figure 2. Temporal response of the spectral amplitude of a synchrotron sideband at 32.7 MHz due to the excitation of a multiple bunch train beam at this sideband frequency with a duration of 2.5 msec.

frequency) decaying with different time constants. Waiting until the other eigen modes have decayed, we observe the component with the longest time constant, i.e. the least stable mode. An example of this may be seen in Figure 2. Here we are exciting a synchrotron sideband at 32.7 MHz for about 2.5 ms and we observe the response of the longitudinal position detector. After the excitation, the beam's amplitude decays with at least two different time constants.

Another important effect is also clear when we recall that the wakefields experienced by different bunches will cause different tune shifts (as well as the different damping rates) for each of the eigen modes of the beam. When we observe the amplitude of the beam's response to excitation at a range of frequencies including a particular betatron sideband, we will generally see multiple spectral peaks having amplitudes and tune shifts which correspond to the eigen modes which have projections at this sideband. These spectral peaks will not be able to be resolved if their half widths are larger than or comparable to the tune shifts of the different eigen modes which contribute to the given sideband. However, in the case when one or more eigen modes are only marginally stable, the spectral peaks from these modes will appear as narrow, large amplitude peaks superposed on the broad spectral peak coming from the superposition of the other eigen modes. Thus observation of the mode spectrum itself (without the necessity of driving the beam) will give useful information about the onset of an instability.

4. Basis Modes for Nearly Equally Spaced Trains of Bunches

In CESR the spacing between lead bunches in the trains is not exactly equal. Consequently the results from the analysis performed above does not accurately describe this situation. However, since the train spacing in CESR is not too far from being equal, this analysis does suggest trying the same basis set of vectors as is used for equally spaced trains. To motivate this line of thought we will consider the case of 9 trains of 3 bunches circulating in CESR as an example.

Since we fill every seventh RF bucket in CESR, there are 183 of these 71.4 MHz super-buckets around the circumference of the ring. If we first consider nine trains of one bunch each, these bunches occur repeatedly with a spacing of 20, 20 and 21 of the super-buckets. As was seen above, with nine equally spaced trains, the frequency spectrum of a position signal would have every ninth of the upper sidebands of the rotation harmonics corresponding to the same basis mode of oscillation of the beam and, in particular, the 0-th and 180-th harmonics would correspond to the same mode. In the actual case the upper sidebands of the 0 and 183 harmonics correspond to the same mode of oscillation. Therefore, at intermediate sidebands the frequencies correspond not to a single mode, but to an admixture of three of the basis modes. This mixing is necessary for the "identification" of a sideband with one basis mode to "slide" forward three rotation harmonics after every 183. As we will see this underlying structure persists when we add more bunches to the trains.

For a known set of basis vectors we can compute the frequency spectrum of each vector by superposing the effect of each bunch. For the current distribution shown in Figure 3, we assign a delay of $t(n_t,n_b)$ to the n_b-th bunch in the n_t-th train with respect to the first bunch in the fill. Then the signal v[t] from a bunch on one turn may be summed to give the signal from all bunches in the fill pattern,

$$v_{Tr\,.\,B}^{(m_t\,.\,m_b)}(t) = \sum_{n_t=0}^{N_t-1} \sum_{n_b=0}^{N_b-1} v[\,t + t(n_t\,,\,n_b)\,] \;*$$

$$\exp\left[j\omega_\beta t(n_t\,,\,n_b) + 2\pi\, j\,\frac{n_t\;m_t}{N_t} + 2\pi\, j\,\frac{n_b\;m_b}{N_b} \right]$$

where N_t is the number of trains, N_b is the number of bunches,
$t(n_t\,,\,n_b)$ is the time delay for train, n_t, and bunch n_b
$m_t = 0, .., N_t - 1$; $m_b = 0, .., N_b-1$
$(m_t\,.\,m_b)$ is the coupled bunch mode number

This bunch signal may then be Fourier analyzed to give the spectrum for this particular basis set of modes,

$$v_{Tr\,,\,B}^{(m_t\,,\,m_b)}(\omega) = \sum_{n_t=0}^{N_t-1} \sum_{n_b=0}^{N_b-1} \hat{v}(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\}) \;*$$

$$\exp\left[j\omega_\beta t(n_t\,,\,n_b) - j\omega\;t(n_t\,,\,n_b) + 2\pi\, j\,\frac{n_t\;m_t}{N_t} + 2\pi\, j\,\frac{n_b\;m_b}{N_b} \right]$$

$$= \hat{v}(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\}) \quad *$$

$$\sum_{n_t=0}^{N_t-1} \sum_{n_b=0}^{N_b-1} \exp\left[- j\, n\, \omega_r\, t(n_t, n_b) + 2\pi j \frac{n_t\, m_t}{N_t} + 2\pi j \frac{n_b\, m_b}{N_b} \right]$$

where $\hat{v}(\omega)$ is the envelope of $v(\omega)$.

The last expression gives the spectral components at each of the upper sidebands of the rotation harmonics. Notice that the complex phase is evaluated at $\{n\ \omega_r\}$ and not at $\{n\ \omega_r + \omega_\beta\}$ as one might be inclined to expect. (The sum of these complex phases may also be treated as a sum over phasors.)

This last, more general expression has been used to calculate the spectral components for the different basis modes of oscillation for 9 trains of 3 bunches having a spacing of 28 ns between bunches within the train. Results of the calculations are plotted in Figures 4 through 6 for all of the train modes which correspond to the bunch modes number 0 through 2, respectively. These figures may be compared with Figure 1 which has the

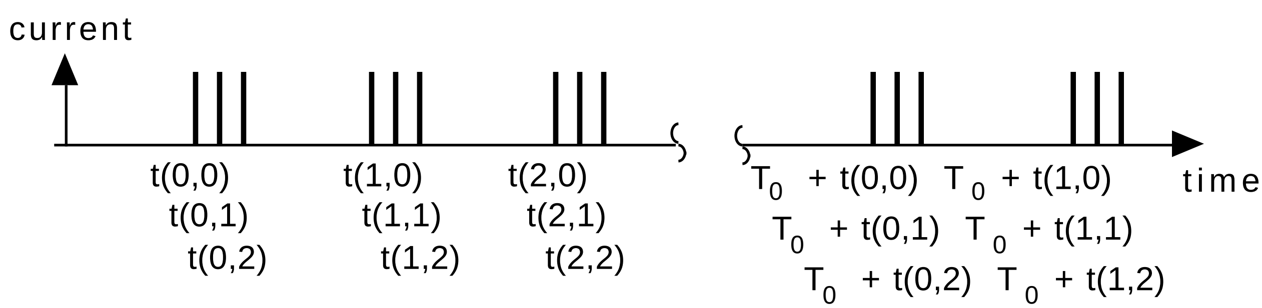


Figure 3. Temporal distribution of current in a fill.

envelope function for 3 equal spaced bunches within a train. Notice in Figures 4 through 6 that there are generally non-zero mode amplitudes for three different basis modes at each frequency, as is expected. Also, with a 28 ns spacing of bunches, we would expect the spectrum to repeat in 36 MHz intervals. All three figures show that, with the unequally spaced trains in CESR, the spectral amplitudes of the three different basis modes become comparable to each other at frequencies near 36 MHz producing a relatively smaller envelope for the spectrum than for the single (or equally

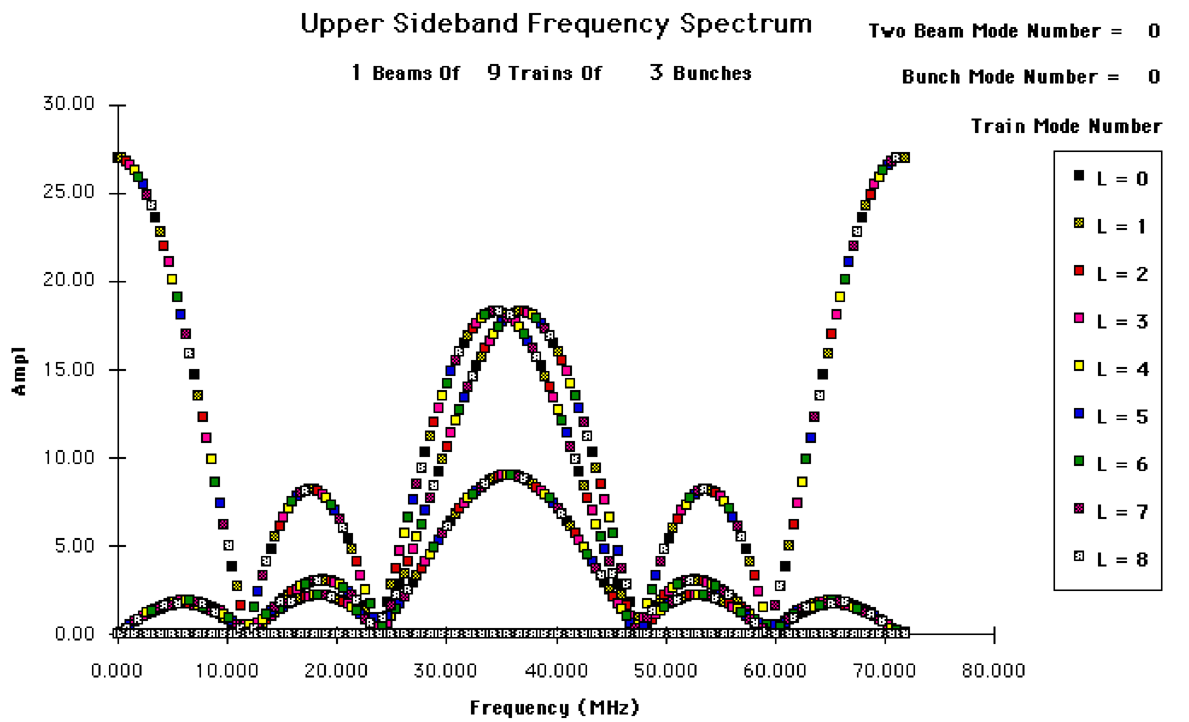


Figure 4. Spectral components 9 trains of 3 bunches (28 ns spaced) for all train modes having a bunch mode number of 0.

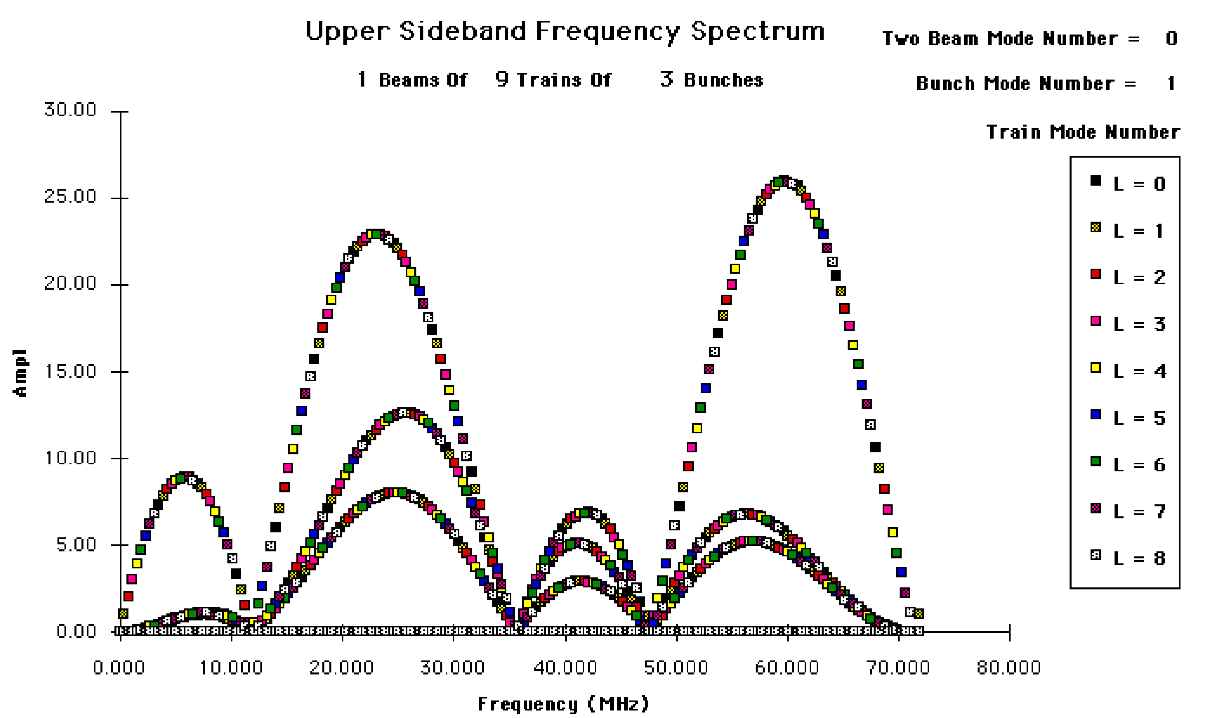


Figure 5. Spectral components 9 trains of 3 bunches (28 ns spaced) for all train modes having a bunch mode number of 1.

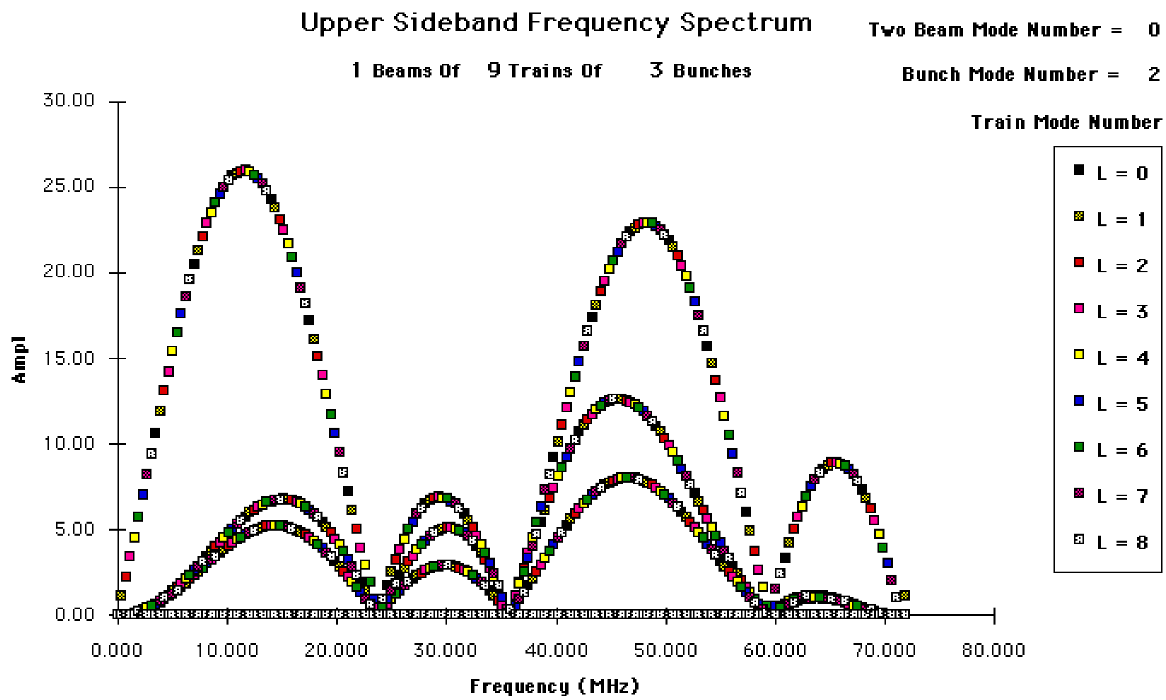


Figure 6. Spectral components 9 trains of 3 bunches (28 ns spaced) for all train modes having a bunch mode number of 2.

spaced) train case. Note also that the square root of the sum of the squares of all the spectral amplitudes at each frequency will give a spectral envelope of the same shape as the single (or equally spaced) train case. Finally, it is clear that we can select a best upper sideband frequency for each bunch train mode (basis mode) which has a significant fraction of the spectral amplitude originating from the mode of interest. This implies that the bunch train basis modes of equally spaced bunches are still a good choice for basis modes for nearly equally spaced bunches.

5. Basis Modes for Two Counter-Rotating Beams

In CESR it is also necessary to study instabilities which occur with stored current in the two counter-rotating beams (electrons and positrons.) The analysis in section 2 may be easily extended so that we may identify half of the components of λ_i as those from one beam and the other half from the other beam. The eigen equation may be solved and the set of eigen vectors determined. It is natural with two beams to consider basis vectors which have displacements for the two beams "in phase", the zero mode, and "out of phase", the π mode. However, unlike the single beam case where the betatron oscillations "travel" with the beam, with two counter-rotating beams the traveling waves for each beam produce a combination of traveling and standing waves as observed in a position monitor at any point in the ring. For a given mode of oscillation the standing waves will cause the amplitude of the oscillation to vary around the ring, going through maxima and nulls. If the beams have equal currents and equal excitations, then there will be only standing waves. Since the deflections from the wakefields (present in Z) and from a kicker (if it is exciting the beams during the measurements) ultimately determine the relative phase of oscillation between the two beams at every point in the ring, this implies that there will be places in the ring where the signal from a zero or π mode will be at a null and at a maximum $\pi/2$ in betatron phase from these points. Therefore, it is possible for a position monitor to be placed at a location where either the zero or π mode may have no signal when observing the total position monitor signal for both beams.

There are different ways to resolve the difficulty cased by the standing waves from the two beams. The simpler conceptually is to have a second position monitor at approximately $\pi/2$ away in betatron phase and to observe the signal from both monitors at the frequencies normally used for observing bunch train basis oscillations. This method requires two complete sets of monitoring systems, but will work. Another method would use only a single monitor and by gating would look at signal from only one of the beams. Since there are no standing modes when only one beam is observed, the motion from all modes of oscillation will be observed albeit with some reduction in sensitivity (since the two beams are not contributing coherently to the signal in the best case.) If is possible to choose either or both beams for observation at the single beam bunch train basis vector frequencies and if it is possible to drive either or both beams with an excitation, then the problem with possible standing waves in the motion for two beams obscuring an instability may be overcome. In CESR since we are most interested in the observation of the least stable modes, we have chosen the latter method for initially finding theses modes which we can then observe more closely during operations or machine studies.

So it is possible to construct a basis set of oscillation vectors for two beams, these being the zero and π modes of oscillation for one beam with respect to the other. Since the exact phasing of the standing waves caused by the counter-rotating beams is not entirely determined by the excitation of the beam by a kicker, the resulting standing waves from these two beam basis modes cause difficulties for guaranteeing that an unstable mode will be observed. it is probably simpler to just observe the motion of one of the beams and, if the instability is not observed, then measure the other beam. With an adequate signal to noise ratio this should generally allow the observation of marginally stable modes of oscillation of the beams.

6. Conclusions

This paper has presented a basic analysis of the modes of oscillation of a beam of trains of bunches. There is enough symmetry even in the case of nearly equally spaced trains of bunches that a basis set of modes of oscillation and their frequency spectra may be constructed. These basis modes determine a specific set of sideband frequencies which provide a set of references from which marginally stable modes of oscillation of the beams may be observed. As was mentioned at the beginning of this paper, the results here are derived for dipole betatron motion of the bunches, but this is only an example since the results may equally well be applied to longitudinal and transverse motion in dipole, quadrupole etc. modes of oscillation.

7. Acknowledgments

I wish to thank R. Meller, J. Sikora and G. Codner for useful discussions and for some of the measurement results presented in this paper. A part of the formalism used in the paper is motivated by a paper by R. Siemann[1].

8. References

[1] Siemann, R.H., "Bunched Beam Dynamics", Physics of Particle Accelerators, AIP Conference Proceedings 184, p. 430 ff.

Appendix 1. Description of Signals from Single Bunches

A single bunch of particles of current I_b, which executes betatron motion of amplitude x_0 at an angular frequency ω_β, has a transverse dipole moment d(t) as it passes a position monitor at some point in the ring. The signal from the beam position monitor will be proportional to the dipole moment of the charge distribution. Modeling the bunch as a delta function in time and realizing that the bunch passes the measurement point with a period T_r, the dipole moment may be written as

$$d(t) = x_0 \; I_b \; e^{j\omega_\beta t} \sum_{n=-\infty}^{\infty} \delta(t + nT_r)$$

Making use of the results of Appendix 2., this periodic dipole moment may be Fourier transformed to give the frequency spectrum, $d(\omega)$,

$$d(\omega) = x_0 \; I_b \sum_{n=-\infty}^{\infty} e^{j\, nT_r(\omega_\beta - \omega)}$$

$$= x_0 \; I_b \; \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\})$$

which is a line spectrum with lines occurring at the upper betatron sidebands of revolution harmonics. The Fourier transform yields a set of lines which appear at negative frequencies. In a spectrum analyzer these lines will appear to be reflected through the origin (zero frequency) to become positive lower sidebands of the rotation harmonics.

The peak signal produced by the beam in the position monitor will equal $d(t)$ times some characteristic resistance R_x. This signal is often processed by sampling the peak signal and holding it for a time $2\;\Delta T$ to a voltage $v(t)$,

$$v(t) = x_0 \; I_b \; R_x \sum_{n=-\infty}^{\infty} e^{j\omega_\beta n T_r} \; \{ \, U(t - nT_r) - U(t - nT_r - 2\Delta T) \, \}$$

where $U(t)$ is the unitary step function. (U equals 0 if t is less than 0, and U equals 1 otherwise.) The frequency spectrum of $v(t)$ is thus

$$v(\omega) = \int_{-\infty}^{\infty} dt \;\; e^{-j\omega t} \;\; v(t)$$

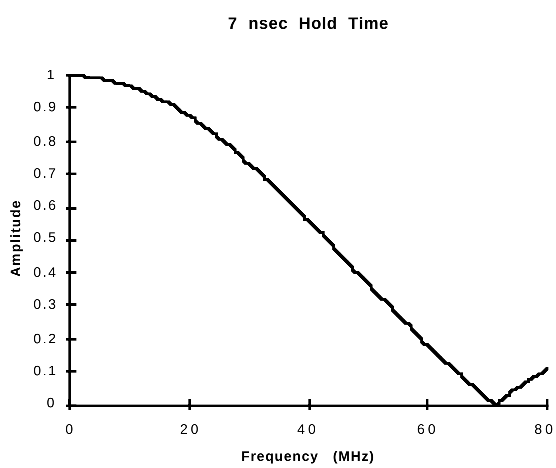
$$= \; x_0 \; I_b \; R_x \sum_{n=-\infty}^{\infty} e^{j\omega_\beta n T_r} \int_{nT_r}^{nT_r \,+\, 2\Delta T} dt \;\; e^{-j\omega t}$$

$$= \ x_0 \ I_b \ R_x \sum_{n=-\infty}^{\infty} e^{j\omega_\beta n T_r} \ [\ e^{-j\omega(nT_r+2\Delta T)} - e^{-j\omega n T_r}\ ]$$

$$= \ 2 \ x_0 \ I_b \ R_x \ e^{-j\omega\Delta T} \sum_{n=-\infty}^{\infty} e^{j(\omega_\beta-\omega)nT_r} \ \frac{\sin(\omega \ \Delta T)}{\omega}$$

$$= \ 2 \ x_0 \ I_b \ (\omega_r \Delta T) \ R_x \ e^{-j\omega\Delta T} \ \frac{\sin(\omega \ \Delta T)}{\omega \ \Delta T} \sum_{n=-\infty}^{\infty} \delta(\omega - \{\omega_\beta + n\omega_r\})$$

This is a line spectrum with an envelope which peaks at ω equal to zero frequency and has nulls occurring for angular frequencies at multiples of $1/\Delta T$. Most position monitors produce a bipolar signal which has an envelope with very little amplitude at low frequencies and a spectral maximum at fairly high frequencies. Thus the advantage of sampling and holding the peak signal is that the envelop of the spectrum will be altered to give the maximum of the spectrum at low frequency. Examples of the envelope function for the frequency response of circuits with 7 nsec and 28 nsec sample and hold times may be found in Figure A1.1. We see that the envelope remains fairly constant up to angular frequencies of 1/3 to 1/2 of $1/\Delta T$.



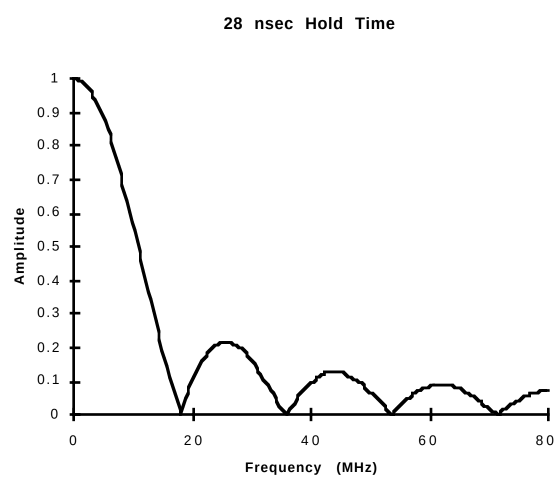


Figure A1.1 Spectral envelope for circuits with 7 nsec and 28 nsec sample and hold times.

Appendix 2. Fourier Analysis Conventions

Some useful results from Fourier analysis which are used in this paper are presented here. The Fourier transform of f(t) is defined as

$$F(\omega) = \int_{-\infty}^{\infty} dt\ e^{-j\omega t}\ f(t)$$

The inverse Fourier transform is therefore

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\ e^{j\omega t}\ F(\omega)$$

A function of time equal to another function delayed by a time T has this transform

$$g(t) = f(t+T) \quad \text{-->}$$

$$G(\omega) = \int dt\ e^{-j\omega t}\ f(t+T) = \int dt'\ e^{-j\omega(t' - T)}\ f(t')$$
$$= e^{-j\omega T}\ F(\omega)$$

The sum of an infinite number of periodic complex exponentials can be written as an infinite sum of delta-functions,

$$\sum_{n=-\infty}^{\infty} e^{-j\omega n T_r} = \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega + n\omega_r)$$

This result may be proven using

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi\ n}{\alpha}\right)$$

$$\text{where } F(\omega) = 2\pi\ \delta(\omega + \omega_r) \ \ \& \quad \alpha = \frac{2\ \pi}{\omega_r} = T_r$$