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The Synchrotron Integrals with Coupling

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1 Synchrotron Integrals

The synchrotron integrals used to compute emittances, the energy spread, etc., have been analyzed assuming no coupling between the horizontal and vertical planes[1, 2]. With Mobius, these assumptions are not valid and so this paper presents the appropriate generalizations. To simplify matters it will be still be assumed that the bends are in the horizonal plane. In this case I_1, I_2, I_3, and I_5 are unchanged (but are given below for completeness). Without proof, the generalized synchrotron integrals are:

$$\begin{aligned}
I_1 &= \oint ds\, G\, \eta_x \\
I_2 &= \oint ds\, G^2 \\
I_3 &= \oint ds\, |G^3| \\
I_{4a} &= \oint ds\, (G^2 + 2K_1)\, G\, \eta_{ax} \\
I_{4b} &= \oint ds\, (G^2 + 2K_1)\, G\, \eta_{bx} \\
I_{4z} &= \oint ds\, (G^2 + 2K_1)\, G\, \eta_x \\
I_5 &= \oint ds\, |G^3|\, \mathcal{H}
\end{aligned} \tag{1}$$

where η_{ax} and η_{bx} are the horizontal components of η_a and η_b respectively. With Eq. (1), the damping partition numbers are

$$\begin{aligned}
J_a &= 1 + \frac{I_{4a}}{I_2} \\
J_b &= 1 + \frac{I_{4b}}{I_2} \\
J_z &= 1 + \frac{I_{4z}}{I_2}
\end{aligned} \tag{2}$$

$$\tag{3}$$

Since

$$\eta_{ax} + \eta_{bx} = \eta_x \ , \tag{4}$$

Robinson's theorem, $J_a + J_b + J_z = 4$, is satisfied.

2 Evaluation of the Integrals

The evaluation of I_1, I_2, I_3, and I_{4z} does not depend upon whether there is coupling or not and is given by Helm et. al[1]. Using the notation of Helm, the evaluation of the other integrals is given below.

2.1 Evaluation of I_{4a} and I_{4b}

The relation between the dispersion in eigenmode coordinates and in x—y coordinates is (cf. Sagan and Rubin[3])

$$\eta_a^{(4)} = \mathbf{V}^{-1}\, \eta_x^{(4)} \ , \tag{5}$$

where the superscript (4) is used to distinguish a 4 element vector from a two element vector (for compactness, the superscripts on the 2 element vectors will be dropped).

Through a bend were the transfer matrix between two points is of the form

$$\mathbf{T}_{12} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix} \ , \tag{6}$$

with

$$\mathbf{M} = \begin{pmatrix} \cos kl & \frac{1}{k} \sin kl \\ -k \sin kl & \cos kl \end{pmatrix} \ , \tag{7}$$

and

$$k^2 = \frac{1}{\rho^2} + k_1 \ , \tag{8}$$

k_1 being the strength of the quadrupole component of the bend. The propagation of $\eta_x^{(4)}$ is

$$\eta_{x2}^{(4)} = \mathbf{T}_{12}\, \eta_{x1}^{(4)} + \eta_{12}^{(4)} \ , \tag{9}$$

where $\eta_{12}^{(4)}$ is the contribution due to the 'generation' of dispersion within a dipole

$$\eta_{21}^{(4)} = \begin{pmatrix} \eta_{x12} \\ \mathbf{0} \end{pmatrix} \ , \tag{10}$$

with

$$\eta_{x12} = \begin{pmatrix} \rho(1 - \cos(s/\rho)) \\ \sin(s/\rho) \end{pmatrix} \ . \tag{11}$$

$\mathbf{V}^{-1}$ propagates as[3]

$$
\begin{aligned}
\mathbf{V}_2^{-1} &= \mathbf{T}_{12}\,\mathbf{V}_1^{-1}\,\mathbf{T}_{12}^{-1} \\
&= \begin{pmatrix} \gamma_1 & -\mathbf{M}\,\mathbf{C}_1\,\mathbf{N}^{-1} \\ \mathbf{N}\,\mathbf{C}_1^{+}\,\mathbf{M}^{-1} & \gamma_1 \end{pmatrix} .
\end{aligned} \tag{12}
$$

Using the above equations gives

$$
\begin{aligned}
\boldsymbol{\eta}_{a2} &= \mathbf{M}\,\boldsymbol{\eta}_{a1} + \gamma_1\,\boldsymbol{\eta}_{x12} \,, \\
\boldsymbol{\eta}_{b2} &= \mathbf{N}\,\boldsymbol{\eta}_{b1} + \mathbf{N}\,\mathbf{C}_1^{+}\,\mathbf{M}^{-1}\,\boldsymbol{\eta}_{x12} \,.
\end{aligned} \tag{13}
$$

The x components of $\boldsymbol{\eta}_a$ and $\boldsymbol{\eta}_b$ are obtained by inverting Eq. (5). For the a mode

$$
\boldsymbol{\eta}_{ax} = \gamma\,\boldsymbol{\eta}_a \,. \tag{14}
$$

Also, from Eq. (12)

$$
\gamma_2 = \gamma_1 \,. \tag{15}
$$

Using Eqs. (13), (14), and (15) then gives

$$
\boldsymbol{\eta}_{ax2} = \gamma_1\,\mathbf{M}\,\boldsymbol{\eta}_{a1} + \gamma_1^2\,\boldsymbol{\eta}_{x12} \,. \tag{16}
$$

From Eqs. (6), (9), and (10) $\boldsymbol{\eta}_x$ propagates as

$$
\boldsymbol{\eta}_{x2} = \mathbf{M}\,\boldsymbol{\eta}_{x1} + \boldsymbol{\eta}_{x12} \,. \tag{17}
$$

Comparing Eq. (16) with (17) shows that $\boldsymbol{\eta}_{ax}$ propagates like $\boldsymbol{\eta}_x$ except for extra factors of γ_1. Thus, the integrated η_{ax} can be obtained from a modification of Helm Eq. 14:

$$
\int ds\,\eta_{ax} = \gamma_0\,\eta_{a0}\,\frac{\sin kl}{k} + \gamma_0\,\eta_{a0}'\,\frac{1-\cos kl}{k^2} + \frac{\gamma_0^2}{\rho}\frac{kl-\sin kl}{k^3} \,. \tag{18}
$$

Eq. (18) can be used to evaluate I_{5a} (cf. Helm[1]). For I_{5b} the integral of $\boldsymbol{\eta}_{bx}$ can then be obtained using Eq. (4)

$$
\int ds\,\eta_{bx} = \int ds\,\eta_x - \int ds\,\eta_{ax} \,. \tag{19}
$$

2.2 Evaluation of I_{5a} and I_{5b}

To compute I_5 we go back to the equation for $\mathcal{H}$

$$
\begin{aligned}
\mathcal{H} &= \gamma\,\eta^2 + 2\,\alpha\,\eta\,\eta' + \beta\eta'^2 \\
&= \left[\eta,\,\mathbf{S}\,\mathbf{J}\,\eta\right] ,
\end{aligned} \tag{20}
$$

where $[\mathbf{A},\, \mathbf{B}] \equiv \mathbf{A}^t\mathbf{B}$, $\mathbf{S}$ is given by

$$\mathbf{S} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{21}$$

and $\mathbf{J}$ is the Twiss matrix (cf. Courant and Snyder[4])

$$\mathbf{J} \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} . \tag{22}$$

The propagation of $\mathbf{J}$ through the bend is given by

$$\begin{aligned} \mathbf{J}_{a2} &= \mathbf{M}\,\mathbf{J}_{a1}^{-1}\,\mathbf{M}^{-1} , \\ \mathbf{J}_{b2} &= \mathbf{N}\,\mathbf{J}_{b1}^{-1}\,\mathbf{N}^{-1} . \end{aligned} \tag{23}$$

In the case were there is no coupling $\mathcal{H}_x$ would propagate like

$$\begin{aligned} \mathcal{H}_{x2} &= \Big[\eta_{x2},\, \mathbf{S}\,\mathbf{J}_{x2}\,\eta_{x2}\Big] \\ &= \Big[\mathbf{M}\,\eta_{x1} + \eta_{x12},\, \mathbf{S}\,\mathbf{M}\,\mathbf{J}_{x1}\mathbf{M}^{-1}\,(\mathbf{M}\,\eta_{x1} + \eta_{x12})\Big] \\ &= \Big[\eta_{x1},\, \mathbf{S}\,\mathbf{J}_{x1}\,\eta_{x1}\Big] + 2\,\Big[\eta_{x1},\, \mathbf{S}\,\mathbf{J}_{x1}\,\mathbf{M}^{-1}\,\eta_{x12}\Big] + \\ &\qquad\qquad \Big[\eta_{x12},\, \mathbf{S}\,\mathbf{M}\,\mathbf{J}_{x1}\,\mathbf{M}^{-1}\,\eta_{x12}\Big] \end{aligned} \tag{24}$$

Where we have used the identity that for arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$

$$\Big[\mathbf{A}\,\mathbf{B},\, \mathbf{S}\Big] = \Big[\mathbf{A},\, \mathbf{S}\,\mathbf{A}^{+}\Big] . \tag{25}$$

The integration of Eq. (24) is straight–forward, if tedious, and is given by Helm Eq. 20 (reproduced here for convenience)

$$\begin{aligned} \int ds\,\mathcal{H}_x = &\Big[l\,(\gamma_{x0}\,\eta_{x0}^2 + 2\,\alpha_{x0}\,\eta_{x0}\,\eta'_{x0} + \beta_{x0}\eta'^2_{x0})\Big] + \\ &\frac{2}{\rho}\left[(\gamma_{x0}\eta_{x0} + \alpha_{x0}\eta'_{x0})\frac{\sin kl - kl}{k^3} + (\alpha_{x0}\eta_{x0} + \beta_{x0}\eta'_{x0})\frac{1-\cos kl}{k^2}\right] + \\ &\frac{1}{\rho^2}\left[\gamma_{x0}\frac{3kl - 4\sin kl + \sin kl\,\cos kl}{2k^5} - \alpha_{x0}\frac{(1-\cos kl)^2}{k^4} + \beta_{x0}\frac{kl - \cos kl\,\sin kl}{2k^3}\right] \end{aligned} \tag{26}$$

The 3 terms in Eq. (26) correspond to the integration of the 3 terms in Eq. (24). With coupling, the propagation of $\mathcal{H}_a$ is obtained with the help of Eqs. (13), (20), and (23) to be

$$\begin{aligned} \mathcal{H}_{a2} = &\Big[\eta_{a1},\, \mathbf{S}\,\mathbf{J}_{a1}\,\eta_{a1}\Big] + 2\,\gamma_1\,\Big[\eta_{a1},\, \mathbf{S}\,\mathbf{J}_{a1}\,\mathbf{M}^{-1}\,\eta_{x12}\Big] + \\ &\qquad \gamma_1^2\Big[\eta_{x12},\, \mathbf{S}\,\mathbf{M}\,\mathbf{J}_{a1}\,\mathbf{M}^{-1}\,\eta_{x12}\Big] \end{aligned} \tag{27}$$

This is similar to Eq. (24) with the addition of some factors of γ_1. The integration of $\mathcal{H}_a$ is thus obtained from Eq. (26) by inspection

$$
\begin{aligned}
\int ds\, \mathcal{H}_a = & \left[l\,(\gamma_{a0}\,\eta_{a0}^2 + 2\,\alpha_{a0}\,\eta_{a0}\,\eta'_{a0} + \beta_{a0}\eta'^2_{a0})\right] + \\
& \frac{2\gamma_0}{\rho}\left[(\gamma_{a0}\eta_{a0} + \alpha_{a0}\eta'_{a0})\frac{\sin kl - kl}{k^3} + (\alpha_{a0}\eta_{a0} + \beta_{a0}\eta'_{a0})\frac{1-\cos kl}{k^2}\right] + \\
& \frac{\gamma_0^2}{\rho^2}\left[\gamma_{a0}\frac{3kl - 4\sin kl + \sin kl\,\cos kl}{2k^5} - \alpha_{a0}\frac{(1-\cos kl)^2}{k^4} + \beta_{a0}\frac{kl - \cos kl\,\sin kl}{2k^3}\right]
\end{aligned}
\tag{28}
$$

For $\mathcal{H}_b$ Eqs. (13), (20), and (23) give

$$
\begin{aligned}
\mathcal{H}_{b2} = & \left[\eta_{b1},\, \mathbf{S}\,\mathbf{J}_{b1}\,\eta_{b1}\right] + 2\left[\eta_{b1},\, \mathbf{S}\,\mathbf{J}_{b1}\,\mathbf{C}_1^+\mathbf{M}^{-1}\,\eta_{x12}\right] + \\
& \left[\eta_{x12},\, \mathbf{S}\,\mathbf{M}(\mathbf{C}_1\,\mathbf{J}_{b1}\,\mathbf{C}_1^+)\mathbf{M}^{-1}\,\eta_{x12}\right]
\end{aligned}
\tag{29}
$$

Again the integration is straight–forward and gives

$$
\begin{aligned}
\int ds\, \mathcal{H}_b = & \left[l\,(\gamma_{b0}\,\eta_{b0}^2 + 2\,\alpha_{b0}\,\eta_{b0}\,\eta'_{b0} + \beta_{b0}\eta'^2_{b0})\right] + \\
& \frac{2}{\rho}\left[(m_1 c_{22} - m_2\,c_{21})\,\frac{\sin kl - kl}{k^3} + (m_2 c_{11} - m_1\,c_{12})\,\frac{1-\cos kl}{k^2}\right] + \\
& \frac{1}{\rho^2}\left[\gamma_{\text{eff}}\frac{3kl - 4\sin kl + \sin kl\,\cos kl}{2k^5} - \alpha_{\text{eff}}\frac{(1-\cos kl)^2}{k^4} + \beta_{\text{eff}}\frac{kl - \cos kl\,\sin kl}{2k^3}\right]
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
m_1 &\equiv \gamma_{a0}\eta_{a0} + \alpha_{a0}\eta'_{a0}\,, \\
m_2 &\equiv \alpha_{a0}\eta_{a0} + \beta_{a0}\eta'_{a0}\,,
\end{aligned}
\tag{31}
$$

and the effective Twiss parameters are defined by

$$
\mathbf{J}_{\text{eff}} \equiv \mathbf{C}_0\,\mathbf{J}_{b0}\,\mathbf{C}_0^+ \tag{32}
$$

which when multiplied out give

$$
\beta_{\text{eff}} = c_{11}^2\,\beta_{b0} - 2\,c_{11}\,c_{12}\,\alpha_{b0} + c_{12}^2\,\gamma_{b0}
$$

$$
\alpha_{\text{eff}} = -c_{21}\,c_{11}\,\beta_{b0} + (c_{11}\,c_{22} + c_{12}\,c_{21})\,\alpha_{b0} - c_{12}\,c_{22}\,\gamma_{b0} \tag{33}
$$

$$
\gamma_{\text{eff}} = c_{21}^2\,\beta_{b0} - 2\,c_{21}\,c_{22}\,\alpha_{b0} + c_{22}^2\,\gamma_{b0} \tag{34}
$$

References

[1] R. H. Helm, M. J. Lee, P. L. Morton, and M. Sands, "Evaluation of Synchrotron Radiation Integrals," IEEE Trans. Nucl. Sci. NS-20, 900 (1973).

[2] J. Jowett, "Introductory Statistical Mechanics for Electron Storage Rings," AIP Conf. Proc. **153**, p. 864, (1987).

[3] D. Sagan and D. Rubin, "Propagation of Twiss and Coupling Parameters," Cornell CBN 96-20, 1996.

[4] E. D. Courant and H. S. Snyder "Theory of Alternating–Gradient Synchrotron," Ann. Physics, **3**, p. 1-48. (1958).