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BETATRON TUNE SHIFT GENERATED SPECIFICALLY IN THE PRETZEL MACHINE

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The well-known edge focusing effect of dipole fringe field has its natural continuation in the multipole phenomenology. Namely, entrance with an angle into quadrupole field gives some *effective sextupole* at the edge, entrance with an angle into sextupole field gives some *effective octupole* and so on. In the pretzel machine the nonzero entrance angle is naturally implemented into the optical structure. Here we represent some nonlinear effect, connected with this systematic variation of the entrance angle associated with pretzel operational mode in CESR. This *nonlinear* part of betatron tune shift associated with this effect is about $\Delta Q_V \cong 10^{-3}$ for the pretzel amplitude about 2 *cm*.

1. Introduction

Specific of CESR is a pretzel operational mode, when the equilibrium betatron orbit looks like a wave around the reference orbit. The last one runs though the centers of quadrupoles and sextupoles. This wave has opposite deflection for electrons and positrons, what gave the name to this mode. This wave is generated by electrostatic electrodes, and the amplitude of the wave is a function of the voltage applied[1]. Systematic wave in the orbit changes the focusing properties of the beam optics. For example, the bigger amplitudes in the sextupoles distributed around the ring, result the bigger additional quadrupole value. This *linearly* changes the betatron frequency. However, some *nonlinear* change in betatron frequency as a function of the pretzel amplitude is measured experimentally. This moved the search for the effects that can cause such nonlinear betatron tune shift versus pretzel amplitude.

In this publication we investigate one of the effects, what yield a *quadratic* behavior of the betatron tune shift versus pretzel amplitude. This is an effect, arising from a specific magnetic field behavior at the *edge* of a multipole (including quadrupole), so called *fringe field effect*. This effect is an *additional* part to the linear shift, generated by off axis orbit location in the sextupoles. This tune shift basically arises from the lenses what are incorporated into the lattices and, additionally, from the *final focus* lenses.

2. Edge field in action

First we will prove a theorem.

Theorem:

A passage with an angle through the edge field of a multipole, acting to the particle as the ***next*** *order multipole with the power, proportional to the tangent of the entrance angle multiplied by original multipole value at the center and reduced by the order of original multipole.*

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[1] We are talking about *real* pretzel amplitude. Some *electrostatic* nonlinarities may yield a nonlinear dependence of the pretzel amplitude as a function of the voltage applied. So we are talking about tune shift as a function of the pretzel amplitude not the voltage applied.

***Demonstration**:*

Magnetic field generated by a multipole in 3D can be obtained from the complex function $W(\bar{z},z,t)$, as the following [1]

$$\overline{B} = B_x - iB_y = \frac{\partial W(\bar{z},z,t)}{\partial z} + \frac{\partial W(\bar{z},z,t)}{\partial \bar{z}}, \qquad B_s = \mathrm{Re}\int\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)W(\bar{z},z,t)ds \tag{1}$$

where $z = x + iy$, $\bar{z} = x - iy$, $x,\ y,\ s$ — are the physical coordinates and $W(\bar{z},z,t)$ satisfy the equation

$$4\frac{\partial^2 W(\bar{z},z)}{\partial z\,\partial \bar{z}} + \left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)W(\bar{z},z,t) = 0\,. \tag{2}$$

Solution of (2) for function $W(\bar{z},z,t)$ can be expressed as [1]

$$W = (-i)\sum_{m=1}^{\infty}\frac{z^m}{m}\left\{G_{m-1}(s,t) - \frac{|z|^2}{4(m+1)}\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_{m-1}(s,t) + \frac{|z|^4}{32(m+1)(m+2)}\left(\frac{\partial^2}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^2 G_{m-1}(s,t) - \ldots\right\}, \tag{3}$$

where $G_{m-1}(s,t)$ is the multipole strength at the axis; m=1 corresponds to a dipole, m=2 corresponds to a quadrupole, m=3 corresponds to a sextupole *etc*. One can see, that in 2D case, fare from the edge, the only term remains is the first one[2]. The magnetic field can be represented here as

$$\overline{B} = B_x - iB_y = \frac{\partial W(\bar{z},z)}{\partial z} + \frac{\partial W(\bar{z},z)}{\partial \bar{z}} = \frac{\partial}{\partial z}W(\bar{z},z) = \frac{\partial}{\partial z}(-i)\sum_m G_{m-1}\frac{z^m}{m} = (-i)\sum_m G_{m-1}z^{m-1}\,. \tag{4}$$

Vertical force acting on the instant particle position is proportional to the $v \times B_s(x,y,s)\cdot\sin\alpha(x,y,s)/c$, where α --is the angle between the trajectory and the normal to the edge, see Fig.1, $v \cong c$ — is the particle's speed.

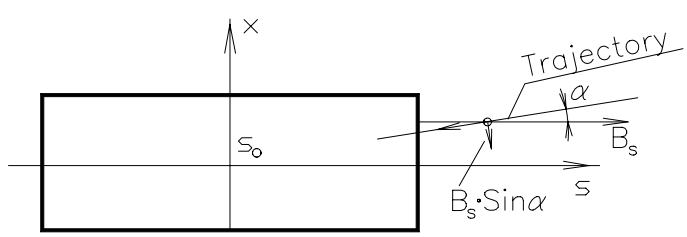

Fig.1. The top view onto the magnet. The y axis is looking up from the plane of the drawing.

Vertical kick experienced by a particle, passed though the multipole magnet off it's the center with angle α, can be expressed as the following

[2] We will be interesting in the *stati*c field, so the time derivative *will be omitted*.

$$y' \cong \frac{1}{pc}\int_{outside}^{inside}\frac{Fds}{\cos\alpha} = \frac{\int eB_s \cdot \tan\alpha \cdot ds}{pc} = \frac{\int e\,\mathrm{Re}\{\int (\partial^2 W/\partial s^2)ds\}\cdot\tan\alpha\cdot ds}{pc} \cong ,$$

$$\cong \frac{e\cdot\tan\alpha}{pc}\mathrm{Re}\{W(s=s_0)\} \cong \frac{e\cdot\tan\alpha}{pc}\mathrm{Im}\sum_m G_{m-1}(s_0)\frac{z^m}{m}, \tag{5}$$

where the point $s = s_0 = 0$ is located inside the magnet. Cosine in denominator arisen from the fact, that integration along the trajectory is transformed into integration along the longitudinal axis. So, the transverse behavior demonstrates the presence of effective multipole as $\mathrm{Re}(-i)\sum_m G_{m-1}(s_0)\tan\alpha\cdot\frac{z^m}{m}$. The principal field of any multipole defined far from the edge, see (4). An effective power of this last multipole could obtained as a result of integration the transverse field over the half of the length as $\int \overline{B}ds \cong \int \frac{\partial W}{\partial z}ds = (-i)\sum\frac{z^{m-1}}{m-1}\cdot\int G_{m-1}(s)ds$. So, compare the power of z one can conclude, that for every multipole

$$\int_{outside}^{inside} G_m(s)ds = \frac{\tan\alpha}{m}G_{m-1}(in), \tag{6}$$

where $G_{m-1}(in) \equiv G_{m-1}(s_0)$ is the value of original multipole at the center, and $G_m(s)$ is treated as *effective next order multipole*.

Q.E.D.

Well known and broadly used application of this phenomena is a *dipole* fringe field, what yields so called *edge* focusing, when the pole face has an angle to the particle's trajectory. This corresponds to $m = 1$ in our formula (6), when G_0 --is a magnet field strength inside the dipole.

We will be interesting in small angles, so in all formulas below we suggested $\tan\alpha \cong \sin\alpha \cong \alpha$.

As example, let us consider a static magnetic field, generated by sextupole *S(s)*. The field can be described according to (3), (4) as the following

$$B_x = 2S(s)xy - \frac{3x^3y + xy^3}{12}S''(s) + \ldots, \quad B_y = S(s)(x^2 - y^2) - \frac{3x^4 + 6x^2y^2 - 5y^4}{12\cdot 4}S''(s) + \ldots ,$$

$$B_s = \frac{3x^2y - y^3}{3}S'(s) - \ldots . \tag{7}$$

The kick

$$y' \cong \frac{\int F_{\parallel}\,dl}{pc} \cong \frac{\alpha}{(HR)}\frac{(3x^2y - y^3)}{3}\int S'(s)ds - \ldots = \frac{1}{(HR)}\frac{(3x^2y - y^3)\cdot\alpha\cdot S(in)}{3} - \ldots, \tag{8}$$

where $S(in)$ is the sextupole value inside the lens, *(HR)*- is the magnet rigidity of the particle. This must be compared with the octupole field behavior [1]

$$B_x = O(s)(3x^2y - y^3) - \frac{5x^4y - y^5}{20}O''(s) + \ldots, \quad B_y = O(s)(x^3 - 3xy^2) - \frac{x^5 - 5xy^4}{20}O''(s) + \ldots,$$

$$B_s = O'(s)\cdot(x^3y - xy^3) - \ldots \tag{9}$$

So, again at the edge of sextupole the integrated *octupole* appears with the value

$$\int_{outside}^{in} O(s)ds = \frac{1}{3}\alpha\cdot S(in)\,. \tag{10}$$

In the same manner the effective *sextupole* appears by the passage through one edge of a quadrupole

$$\int_{outside}^{in} S(s)ds = \frac{1}{2}\alpha\cdot G(in)\,, \tag{11}$$

where *G(in)*—is the quadrupole value *inside* the lens. In formula (11) we used the relation for the vertical kick in the quadrupole as

$$\Delta y' \cong \frac{e}{pc}\frac{2xy\cdot\alpha\cdot G(in)}{2} = \frac{1}{(HR)}\cdot\frac{2xy\cdot G(in)\cdot\alpha}{2} \tag{12}$$

In our case the angle $\alpha = \alpha(s)$ and radial displacement $x(s)$ generated by *pretzel* mode of operation. For derivation the previous formula, we supposed, that the angle does not change much during the passage through the *single edge*[3].

3. Lenses around the ring

So we know how one side of the multipole works. Formulas like described above may be interesting for the lenses what have significant variation of the trajectory in the lens, what is final focusing lenses for example, se lower. In this part we consider the action of the fringe fields inside the ring.

At the second side of the lens, the particle gets an opposite kick from the edge field, so the difference for the sextupole will be

$$\Delta y' \cong \frac{(3x^2y - y^3)\cdot\alpha\cdot S(in)}{3\cdot(HR)}\Big|_{s1} - \frac{(3x^2y - y^3)\cdot\alpha\cdot S(in)}{3\cdot(HR)}\Big|_{s2} \cong \frac{3y[x_1^2\cdot\alpha(s_1) - x_2^2\cdot\alpha(s_2)]}{3\cdot(HR)}\cdot S(in) \tag{13}$$

Supposing, that $\alpha l << x_{1,2}$[4], where l -- is the length of the sextupole, one can obtain

$$\Delta y' \cong \frac{2x_1y\cdot\alpha^2\cdot eS(in)\cdot l}{pc} = 2x_1y\cdot\alpha^2\frac{Sl}{(HR)}\,, \tag{14}$$

where x_1 is the particle's coordinate at the entrance. One can see, that this *resulting* kick acts as a gradient. This gradient, however, is a third power of the pretzel amplitude.

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[3] But this angle may change significantly for the whole lens, however, see lower

[4] This is not valid for the particles running close to the center of the lens, but here the influence of the field is also negligible.

The value $l\cdot(\frac{1}{l}\int Fds) = 2x_1 y\cdot\alpha^2\cdot eS(in)\cdot l$ also can be treated as an *integrated* force due to the passage over *whole* sextupole lens, and the average force becomes

$$\Delta F = \frac{1}{l}\int Fds = 2x_1 y\cdot\alpha^2\cdot eS(in) \tag{15}$$

For the quadrupole lens under the same assumptions

$$\Delta y' \cong \frac{e}{pc}\frac{2xy\cdot\alpha\cdot G(in)}{2}\Big|_{s1} - \frac{e}{pc}\frac{2xy\cdot\alpha\cdot G(in)}{2}\Big|_{s2} \cong y\cdot\alpha^2\frac{G\cdot l}{(HR)}. \tag{16}$$

Now one can obtain the change for the focusing parameter integrated over the passage through *whole* lens like

$$\Delta k_y \cong -\frac{1}{pc}\frac{\partial\Delta F}{\partial y} \cong \frac{e}{pc}S\alpha^2 x_1 = \frac{S}{(HR)}\alpha^2(s)x_1(s) \quad \text{-- for a sextupole,} \tag{17a}$$

$$\Delta k_y \cong \frac{G}{(HR)}\alpha^2(s) \quad \text{-- for a quadrupole.} \tag{17b}$$

Supposing, that the displacement is going according to the law

$$x \cong a\cdot\sin(2\pi s/\lambda), \qquad \alpha \cong x' \cong \frac{2\pi a}{\lambda}\cos(2\pi s/\lambda), \tag{18}$$

where a -- is the pretzel amplitude, λ -- is the wavelength of the pretzel. For square of the angle one can use $\alpha^2(s) \cong \left(\frac{2\pi a}{\lambda}\right)^2\frac{1+\cos(2\cdot 2\pi s/\lambda)}{2}$. The betatron tune shift can be evaluated as

$$\Delta Q \cong \frac{1}{4\pi}\oint\beta(s)\Delta k_y ds, \tag{19}$$

so one can see that the only shift remains arisen from the quadrupole lenses

$$\Delta Q_V \cong \frac{1}{4\pi}\frac{G(in)}{(HR)}\left(\frac{2\pi a}{\lambda}\right)^2\oint\beta(s)\cdot I(s)\frac{1+\cos(4\pi s/\lambda)}{2}ds \cong \frac{1}{8\pi}\frac{G(in)}{(HR)}\left(\frac{2\pi a}{\lambda}\right)^2\overline{\beta}\cdot C, \tag{20}$$

where $I(s) = \pm 1$ is a step function describing changing polarities of quadrupoles, C—is a circumference of the machine (C =758 m for CESR), $\overline{\beta}$-- is an average envelope function variation along the ring. Supposing $\lambda \cong C/Q_V$ we can represent (20) as

$$\Delta Q_V \cong \frac{\pi a^2}{2}\frac{G(in)}{(HR)}\frac{Q_V^2}{C}\overline{\beta}, \tag{21}$$

where, like in previous formulas, $G(in)$ is the quadrupole gradient at the center of the lens, *a*-- is a pretzel amplitude in *cm*. Dependence of the betatron tune shift is a *quadratic* function of the pretzel amplitude. Let us make some estimations. Substitute here numerical values, we obtain for $G = 1\ kG/cm$

$$\Delta Q_V \cong \frac{\pi a^2[cm^2]}{2} \frac{1[kG/cm]}{1.7\cdot 10^4[kG\cdot cm]} \frac{100}{75800[cm]} 2000[cm] \cong 2.4\cdot 10^{-4} a^2[cm^2]. \qquad (22)$$

So, for the pretzel amplitude $a = 2cm$, absolute tune shift will be $\Delta Q_V \cong 10^{-3}$. For revolution frequency $f_0 \cong 390\,kHz$, the frequency shift will be $\Delta f_V = \Delta Q_V \cdot f_0 \cong 0.39 kHz$. This shift looks as not big.

One can see, that dependence of the tune shift (and betatron frequency) proportionally as a square of displacement, allows exclude this effect, in principle, by adding an *octupole* field on the orbit. In this case the octupole must be chosen so that the tune shift from averaged octupole $\overline{O(s)}$

$$\Delta Q_V \cong \frac{3}{32\pi^2} \frac{\overline{O}\, a^2 C^2}{Q_V (HR)} \qquad (23)$$

must cancel the tune, arisen from the pretzel. Unfortunately, cancellation the nonlinearity by this way will implement the real octupole into the orbit. The formal similar behavior as a function of displacement has a different physics behind.

4. Final focus lens

For estimation the final focus lens influence we return to the formula (16) for the vertical kick and leave now exact relations

$$\Delta y' \cong \frac{\overline{F}l}{pc} \cong \frac{1}{(HR)} \frac{2xy\cdot\alpha\cdot G(in)}{2}\Big|_{s1} - \frac{1}{(HR)} \frac{2xy\cdot\alpha\cdot G(in)}{2}\Big|_{s2} \qquad (24)$$

For the last lens before the IP there is a vertically focusing lens. Interaction is going with radial crossing angle in *x* direction $\cong \pm 2.56\,mrad$. Variation of the focusing parameter is the following

$$\Delta k_y \cong -\frac{1}{pc}\cdot\frac{\partial \overline{F}}{\partial y} = \frac{G(in)}{(HR)\cdot l}\left[x\alpha\Big|_{s1} - x\alpha\Big|_{s2}\right] \qquad (25)$$

In the case of final focusing lens, the angle and coordinate variation is significant. For our purposes we can approximate the coordinate and angle change as

$$\alpha_2 \cong \alpha_1 + \frac{Glx_1}{(HR)}, \qquad x_2 \cong x_1 + \alpha_2 \cdot l. \qquad (26)$$

So one needs to substitute these formulas into (24). We, however, will make some further simplifications. As we suggested, that the angle α_1 is the crossing angle at IP, we can also suggest, that the variation of the angle in the final lens is about the same value, so $\alpha_2 \cong 0$, and $x_1 \cong L \cdot \alpha_1$, where $L \cong \frac{(HR)}{Gl}$ -- is the distance between final lens and IP, what is of the order of focal distance of the lens. So (24) yields under these conditions rather simple formula

$$\Delta k_y \cong -\frac{1}{pc} \cdot \frac{\partial \overline{F}}{\partial y} \cong \frac{G(in)\alpha_1}{(HR) \cdot l} \left[\frac{(HR)}{Gl} \alpha_1 \right] \cong -\frac{\alpha_1^2}{l^2}. \tag{27}$$

For the betatron tune variation at *one side* of the final lens one can obtain according to (19)

$$\Delta Q_{lens} \cong \frac{1}{4\pi} \frac{\alpha_1^2}{l^2} \oint \beta(s) ds \cong \frac{1}{4\pi} \frac{\alpha_1^2 \beta}{l} \cong \frac{1}{4\pi} \frac{\alpha_1^2 (\beta_0 + L^2 / \beta_0)}{l} \cong \frac{1}{4\pi} \frac{\alpha_1^2 L^2}{l \beta_0}. \tag{28}$$

Let us made some estimations. For $\alpha_1 \cong 2.56 \cdot 10^{-3}$, $L \cong 2m$, $l \cong 0.5m$, $\beta_0 \cong 0.018m$ one can evaluate the tune shift at *one end* of the final lens as $\Delta Q_{IP} \cong 2.3 \cdot 10^{-4}$. So this looks as additional *focusing* force, if the last lens is a vertically focusing one, or as *defocusing* one if the last lens is a horizontally focusing one.

So at the IP we have the reactions from the edge parts of the final lenses as a cancellation of the angle arisen from both kicks at different quads around IP. This becomes clear, if one considers, that after interaction the particle goes to the symmetrical final lens with *opposite* vertical and transverse coordinates, see Fig.2. But as the IP remains the same, what means that the tune shift arisen from changed effective quadrupole strength need to be compensated in the ring. So it looks as if the final lenses *together* change its strengths. So one can conclude, that the tune shift estimated for single edge as (27) need to be doubled.

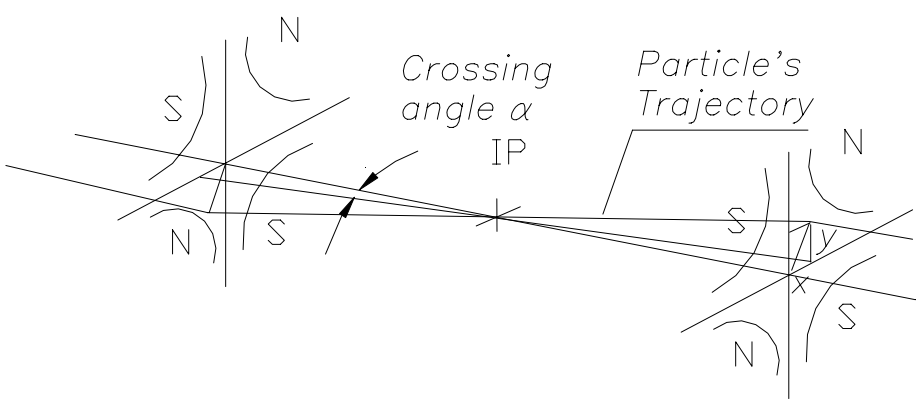

Fig.2. Geometry of the interaction region. Reference particle has opposite coordinates at the faces of final lenses looking to IP.

Total tune shift becomes somewhere within $\Delta Q_{IP} \cong 1.5 \cdot 10^{-3}$ for the pretzel amplitude about 2 *cm*. Of cause the lenses next to the IP also could give a noticeable input, but this requires more careful analysis.

5. Discussion

Formula for the effective action of the edge (Theorem) is valid under realistic conditions for any multipole. The formula for action of both ends is valid under assumption that the envelope function does not change much. This effect, however can be taken into account for the next approximation. One can see from the formula (21), that the betatron tune shift is proportional to the square of pretzel amplitude and the actual betatron tune itself, $\Delta Q_V \approx a^2 \cdot Q_V^2$. The actual betatron tune shift, however, is in it's turn is a linear function of the pretzel amplitude. So full picture may include cubic terms arisen directly from there as

$$\Delta Q_V \approx a^2 \cdot [Q_{V0} + (\partial Q_V / \partial a) \cdot a]^2 \cong a^2 Q_{V0} + 2a^3 \cdot (\partial Q_V / \partial a) + o(a^2) .$$

The picture at the central region around IP is much more complicated that what represented here and requires more detailed analyses, but we hope that the scale is correct.

6. Conclusion

This nonlinear tune shift does not look as a big one, but it remains within the ability to diagnose it. In absolute units it is within a fraction of a *kHz* for CESR.

Much more powerful nonlinear influence is associated with nonlinarities of main quadrupole, what will be investigated in other publication.

It is not clear how existing popular numerical codes take into account the fringe effects arisen from the multipoles higher than a dipole one.

In conclusion Author thanks M. Billing for useful discussion.

7. Reference

[1] A.A. Mikhailichenko, “3-D electromagnetic Field. Representation and measurements”, CBN 95-16, Cornell, 1995, 42 pp.