10 GeV Synchrotron Longitudinal Dynamics

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In this note, I provide some estimates of the parameters of longitudinal dynamics in the 10 GeV synchrotron. The equations used to make these estimates are given in section I. In section II, the equations are applied to the case of electron dynamics; in section III, positron dynamics is considered.

I. Longitudinal dynamics equations

The overall time variation of the energy of a synchronous particle in the synchrotron, is taken as

$$E(t) = E_{ps} - E_{off} + \frac{\left(E_{poc} - E_{ps} + E_{off}\right)}{2}\left(1 - \cos\tilde{\omega}t\right) \tag{1}$$

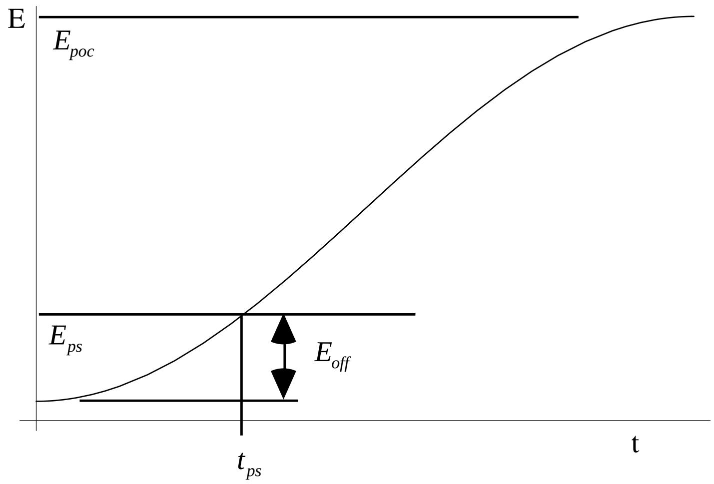

Fig. 1
Energy variation with time

in which the peaking strip energy, E_{ps}, the maximum energy, E_{poc}, and the offset energy, E_{off}, are defined as shown in Fig. 1. The frequency $\tilde{\omega}=2\pi\tilde{f}$, where $\tilde{f}$ is the cycle frequency of the synchrotron. The beam is injected at energy E_{inj} and time t_{inj}, which is a fixed (short) time after t_{ps}:

$$t_{ps} = \frac{\cos^{-1}\left[\dfrac{E_{poc} - E_{off} - E_{ps}}{E_{poc} + E_{off} - E_{ps}}\right]}{\tilde{\omega}} \tag{2}$$

The acceleration rate a is

$$a(t) = \frac{dE(t)}{dt} = \frac{E_{poc} - E_{ps} + E_{off}}{2}\tilde{\omega}\sin\tilde{\omega}t \tag{3}$$

and the energy gain per turn associated with this acceleration is

$$\Delta E_a(t) = T_0 a(t) \tag{4}$$

in which T_0 is the revolution period. Varying E_{ps} adjusts the injection energy, which must match the energy of the Linac. Varying E_{poc} adjusts the extraction energy, which must match the energy of CESR. Varying E_{off} adjusts the slope of the E vs. t curve at t_{ps}, which changes the acceleration rate at injection, according to the following equation:

$$a(t_{ps}) = \tilde{\omega}\sqrt{E_{off}\left(E_{poc} - E_{ps}\right)} \tag{5}$$

The energy vs. time relation can also be expressed in term of E_{AC}, the energy amplitude of the AC sine wave of the synchrotron, and E_{DC}, the DC offset energy. In these terms, the energy vs. time relation is

$$E(t) = E_{DC} - E_{AC}\cos\tilde{\omega}t \tag{6}$$

Comparing this equation with Eq. (1) leads to the relations

$$\begin{aligned} E_{DC} &= \frac{E_{poc} + \left(E_{ps} - E_{off}\right)}{2} \\ E_{AC} &= \frac{E_{poc} - \left(E_{ps} - E_{off}\right)}{2} \end{aligned} \tag{7}$$

The synchronous rf phase is

$$\phi_s(t) = \pi - \sin^{-1}\left[\frac{\Delta E(t)}{eV_{rf}(t)}\right] \tag{8}$$

in which $V_{rf}(t)$ is the peak rf voltage at time t. In this equation ΔE is given by

$$\Delta E(t) = \Delta E_a(t) + \Delta E_\gamma(t)$$

in which ΔE_a is given by Eq. (4) above, and

$$\Delta E_\gamma(t) = C_\gamma \frac{E(t)^4}{c\rho} \tag{9}$$

is the energy loss due to radiation. C_γ=8.85x10^{-5} m/GeV^3, and ρ is the magnet bending radius. The synchrotron frequency is

$$f_s(t) = f_0\sqrt{\frac{eV_{rf}(t)\left|\cos\phi_s(t)\right|h\alpha}{2\pi E(t)}} \tag{10}$$

in which f_0 is the revolution frequency, h is the harmonic number, and α is the momentum compaction factor. The rf bucket area is

$$A_b(t) = \frac{8}{\pi f_0}\sqrt{\frac{eV_{rf}(t)E(t)}{2\pi h^3\alpha}}\,g(\pi - \phi_s(t)) \tag{11}$$

in which the "moving bucket" factor $g(x)$ is shown in Fig. 2.

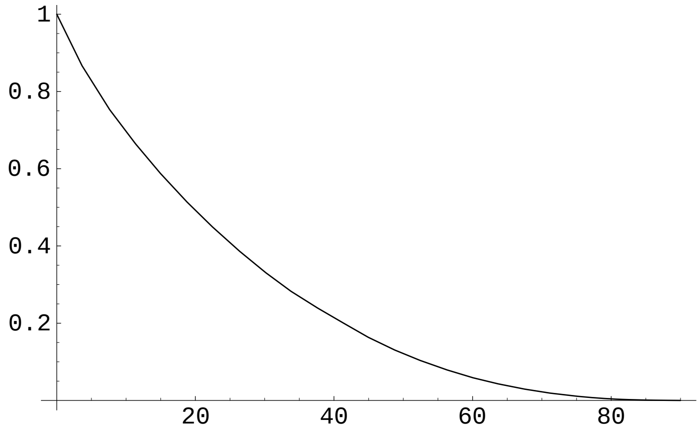

Fig. 2
Moving bucket factor *g(x)* vs. *x* (degrees)

The relative energy acceptance of the bucket is

$$\Delta_b(t) = \frac{\Delta E_b(t)}{E(t)} = \sqrt{\frac{eV_{rf}(t)}{E(t)h\alpha}}\sqrt{-\frac{2}{\pi}\cos\phi_s(t) + \left(1 - \frac{2}{\pi}\phi_s(t)\right)\sin\phi_s(t)} \quad (12)$$

The rms bunch length is

$$\sigma_s(t) = c\sqrt{\frac{\varepsilon}{x(t)f_0}} \quad (13)$$

in which ε is the rms longitudinal emittance (in eV-sec), and

$$x(t) = \sqrt{\frac{2\pi eV_{rf}(t)E(t)h}{\alpha\left|\cos\phi_s(t)\right|}} \quad (14)$$

The rms relative energy spread in the bunch is

$$\sigma_\delta(t) = \frac{c\varepsilon}{\sigma_s(t)E(t)} \quad (15)$$

The longitudinal beta function is

$$\beta_s(t) = \frac{\sigma_s(t)}{\sigma_\delta(t)} = \frac{cE(t)}{x(t)f_0} \quad (16)$$

The average current in the bunch is

$$I = Nef_0 \quad (17)$$

in which N is the number of particles in the bunch. The peak current (assuming a Gaussian bunch) is

$$I_p(t) = \frac{Ic}{\sqrt{2\pi}\sigma_s(t)f_0} = \frac{Nec}{\sqrt{2\pi}\sigma_s(t)} \quad (18)$$

The longitudinal emittance at injection is given by the initial rms bunch length σ_{s0} and rms relative energy spread $\sigma_{\delta 0}$:

$$\varepsilon_0 = E_{inj}\sigma_{\delta 0}\frac{\sigma_{s0}}{c} \quad (19)$$

For a correct longitudinal match, the longitudinal beta function of the machine at injection,

$$\beta_s(t_{inj}) = \frac{\sigma_s(t_{inj})}{\sigma_\delta(t_{inj})} = \frac{cE_{inj}}{x(t_{inj})f_0} \tag{20}$$

should match that of the injected beam,

$$\beta_{s0} = \frac{\sigma_{s0}}{\sigma_{\delta 0}} \tag{21}$$

This leads to the following equation for the voltage at injection:

$$\beta_{s0}^2 = \frac{c^2 E_{inj}\alpha}{2\pi h f_0^2} \frac{\left|\cos\phi_s(t_{inj})\right|}{eV_{rf}(t_{inj})} \tag{22}$$

II. Calculations for electrons

The parameters used for electrons in the synchrotron are presented in Table 1. The parameters for the injected electron bunch have been estimated as follows. Measurements of the energy spread of the electron beam [1] indicate a FWHM of 0.93 MeV. For a 330 MeV beam , with a Gaussian energy distribution, this would correspond to σ_δ = 0.93/330/2.2=0.0013. The energy distribution has a long low energy tail, however, (see Fig. 3), so I have (arbitrarily) used a somewhat larger value of σ_δ =0.002 for the rms relative energy spread. The rms bunch length of 3 mm corresponds to an rms phase width of about 10^o at S-band.

Dipole bending radius ρ	98	m
Injection energy E_{inj}	0.33	GeV
Final energy E_{poc}	5.2	GeV
Magnet cycle frequency $\tilde{f}$	1/60	sec
Magnet cycle half-period	8.33	msec
Momentum compaction α	9.6x10^{-3}	
Harmonic number	1800	
Mean radius R	120.162	m
Injected beam rms relative energy spread $\sigma_{\delta 0}$	0.002	
Injected beam rms bunch length σ_{s0}	3	mm
Injected longitudinal emittance	6.6	μeV-sec
Nominal intensity	5	10^8

Table 1
Electron longitudinal parameters in the synchrotron

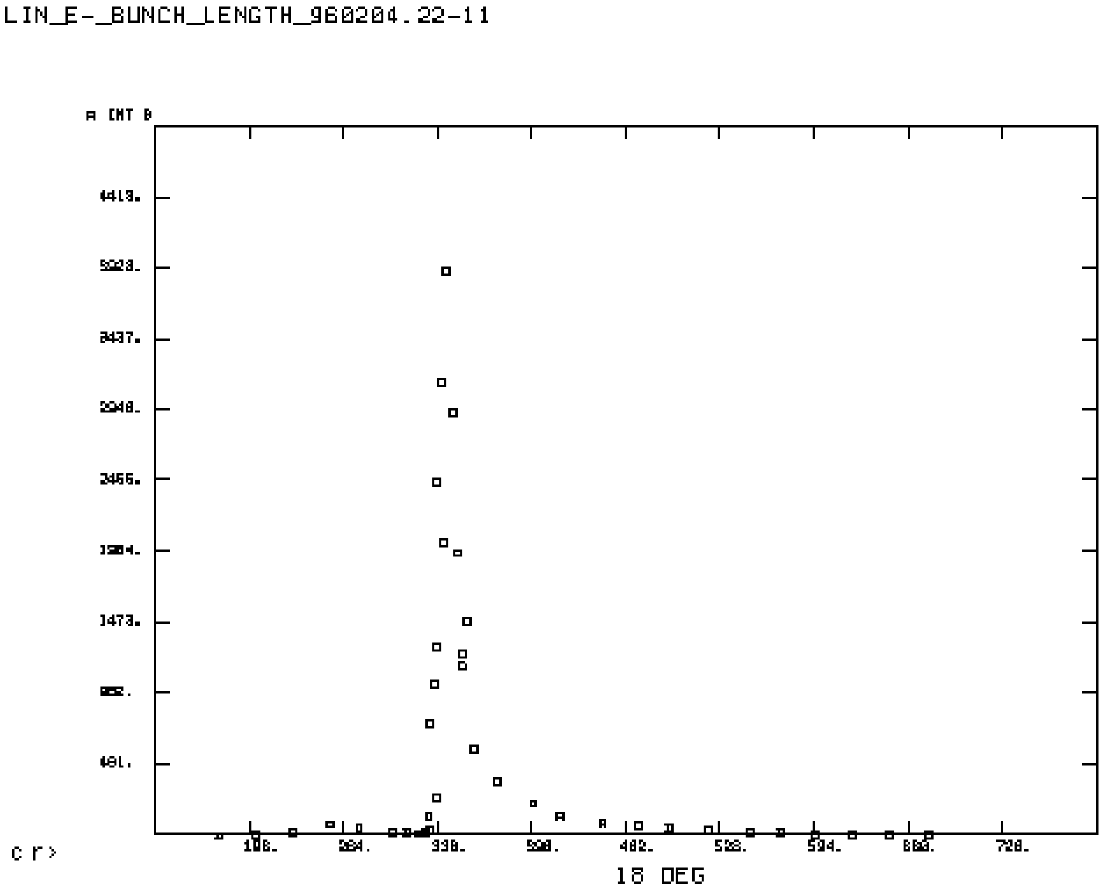

Fig. 3
Linac beam energy distribution from [1]
Scale calibration: 1 unit = -62 keV

From Eq. (22), the voltage required in the machine for longitudinal matching at injection is $V_{rf}(t_{inj}) = 70$ MV. This voltage is clearly not achievable, and a longitudinal mismatch is inevitable. The emittance growth which occurs as a result of the mismatch will have no consequence, provided this emittance can be efficiently transmitted through the synchrotron and injected into CESR. A minimum requirement for efficient transport through the synchrotron is that the bucket half-height should be substantially larger than the rms energy spread. Let the ratio of bucket height to rms bunch length be

$$R = \frac{\Delta_b(t_{inj})}{\sigma_{\delta 0}} \tag{23}$$

Requiring R=3 determines a voltage at injection. For electrons in the synchrotron, this is 0.72 MV. The voltage throughout the cycle can then be specified by requiring that the same ratio of bucket height to rms energy spread be maintained. The resulting voltage profile is shown in Fig. 4.

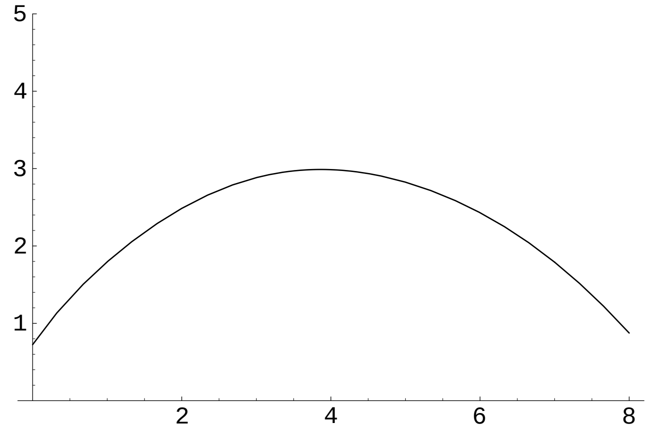

Fig. 4
RF voltage required maintain a bucket height/rms energy spread ratio of 3 throughout the acceleration cycle. The ordinate is in MV, the abcissa in msec.

The actual voltage, which is used in typical operation, is shown in Fig. 5

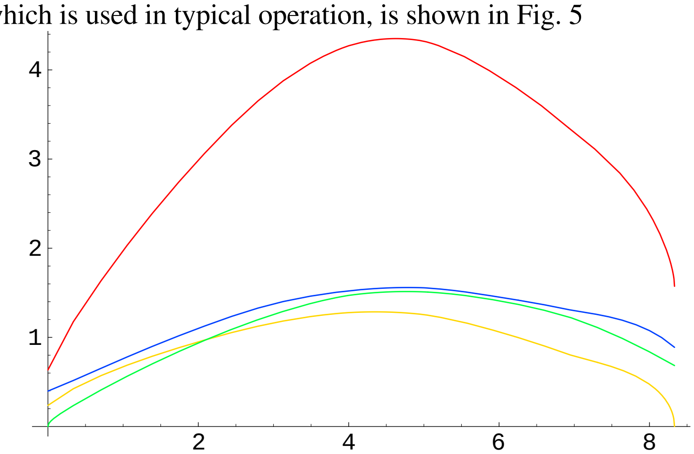

Fig. 5
Actual RF used during operation. The ordinate is in MV, the abcissa in msec.
The red curve is the total voltage; the voltages from L1, L2 and L5 are yellow, green and blue, respectively.

The voltage for each RF cavity was calculated in the following way. A hard copy of the forward power curve, as displayed on the control room scope, was obtained, and the shape of the power curve vs. time, *P(t)* , was measured. Simultaneously, the average forward power $\langle P \rangle$ was recorded. To determine the scale factor *S* to calibrate the power curve in kW, the following equation was used:

$$S = \frac{2T\langle P \rangle}{\int_0^T P(t)dt} \tag{24}$$

in which T is the magnet cycle half-period. Then, the voltage vs. time was obtained from the power using

$$V(t)[MV] = 0.308\sqrt{SP(t)[kW]} \tag{25}$$

(This relation is computed from $V = \sqrt{2\tau\, Pr\, L}\left[\frac{1-e^{-\tau}}{\tau}\right]$, with r= 29 MΩ/m, L= 4.48 m, and τ=0.717 [2]). The total voltage is the sum of that from L1, L2, and L5.

Since this voltage does not provide a longitudinal match, the injected bunch will execute quadrupole longitudinal oscillations in the bucket until it decoheres. The longitudinal emittance will increase. The final longitudinal emittance will be that which corresponds to a match with the bucket, with the same energy spread as that of the injected bunch. In the small-oscillation limit, this emittance is

$$\varepsilon = \frac{\left(\sigma_{\delta 0} E_{inj}\right)^2}{x(t_{inj})\, f_0} \tag{26}$$

assuming that the decoherence occurs soon after injection. For the parameters given above, this corresponds to an emittance of 65 μeV-sec, about a factor of 10 greater than the injected emittance. However, this is still much smaller than the initial bucket area, which is 1803 μeV-sec. The emittance growth leads to an increase in the rms bunch length to about 30 mm.

Figs. 6 through 14 show the total energy per turn supplied to the beam, the relative energy per turn, the synchronous phase, the synchrotron tune, the synchrotron period (in turns), the bunch length, the bucket half-width and the bunch rms energy spread, the longitudinal beta function, and the peak current, respectively, throughout the acceleration cycle.

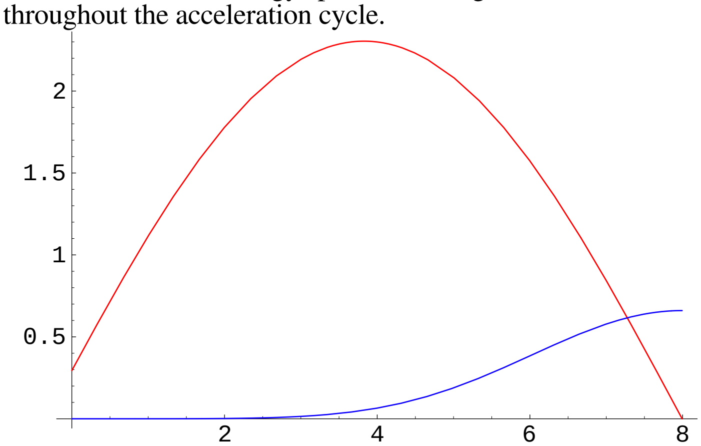

Fig. 6
Energy gain per turn required for acceleration (red) and energy loss per turn due to radiation (blue), both in MeV, vs. cycle time (msec)

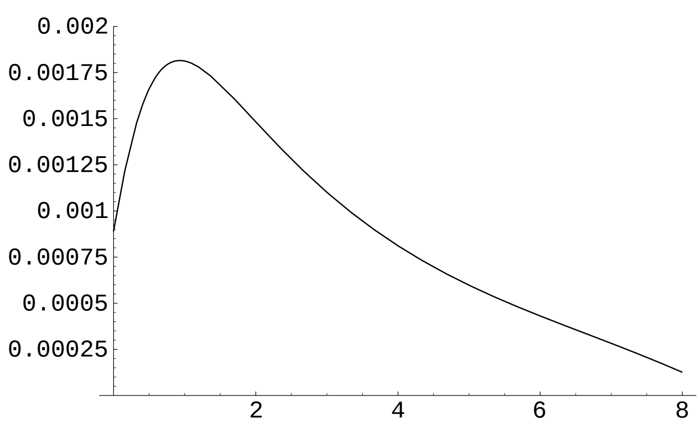

Fig. 7
Relative energy gain per turn vs. cycle time (msec)

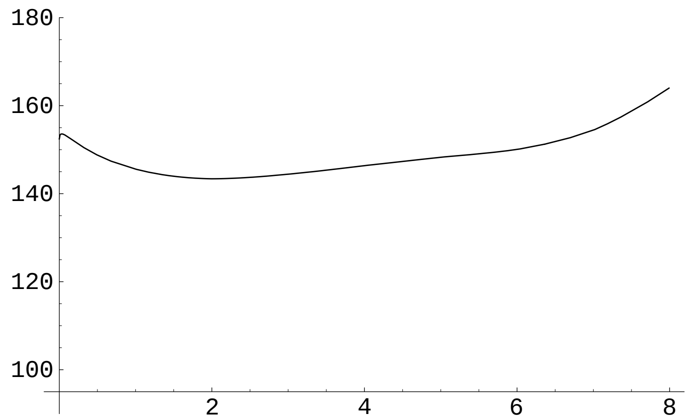

Fig. 8
Synchronous phase (degrees) vs. cycle time (msec)

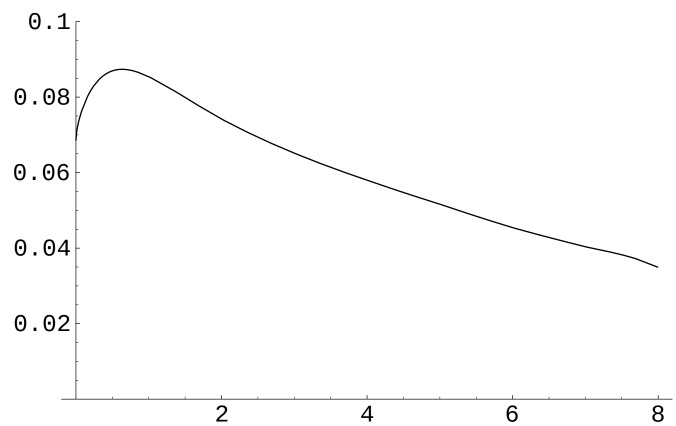

Fig. 9
Synchrotron tune vs. cycle time (msec)

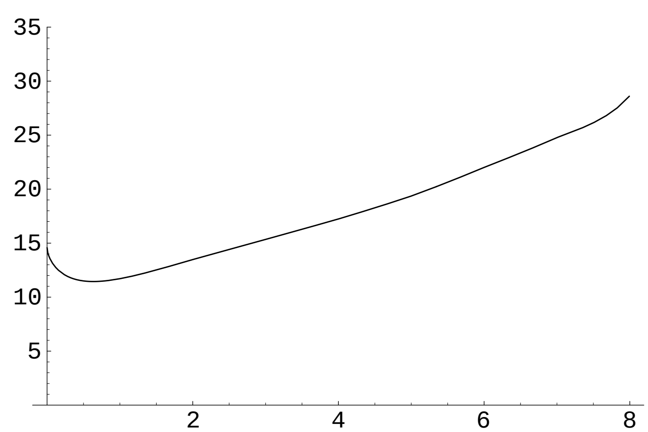

Fig. 10
Synchrotron period (turns) vs. cycle time (msec)

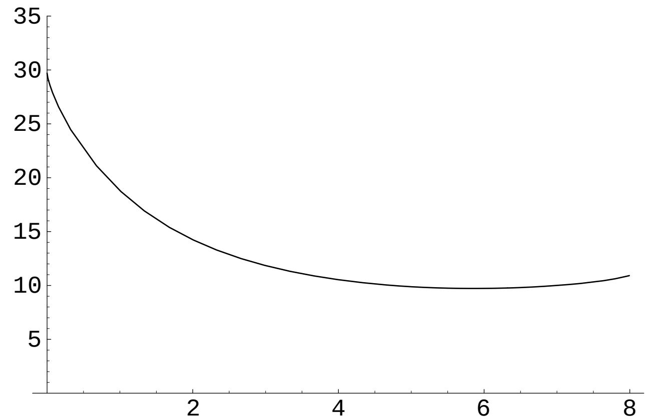

Fig. 11
Bunch length (mm) vs. cycle time (msec)

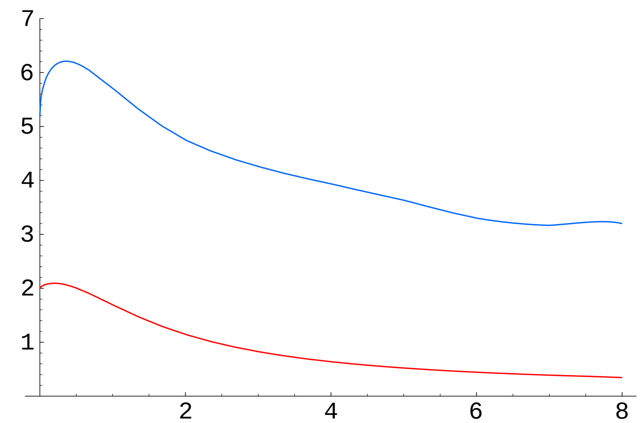

Fig. 12
Bucket relative half-width (upper) and bunch rms relative energy spread (lower), $x10^3$, vs. cycle time (msec)

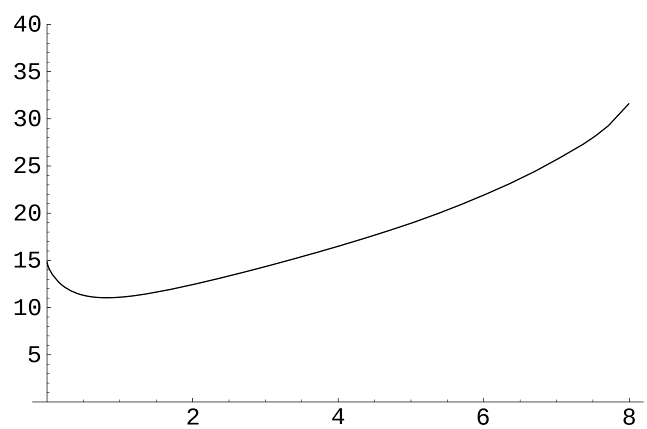

Fig. 13
Longitudinal beta function β_s(m) vs. cycle time (msec)

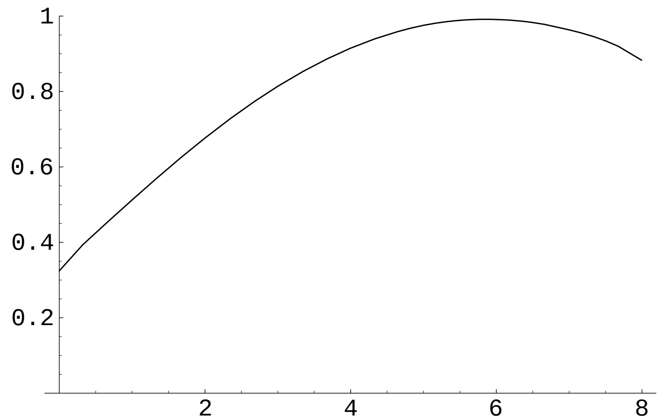

Fig. 14
Peak current (A), vs. cycle time (msec)

III. Calculations for positrons

The parameters used for positrons in the synchrotron are presented in Table 2. The injected beam energy spread is taken from an estimate in [3]. The rms bunch length is taken to be 30 mm . (The bunch length will be much larger than that of the electrons because of the energy compressor cavity in the positron snout. Since the energy compressor reduces the positron energy spread by roughly a factor of 10, I have taken the bunch length increase to be the same factor).

Dipole bending radius ρ	98	m
Injection energy E_{inj}	0.20	GeV
Final energy E_{poc}	5.2	GeV
Magnet cycle frequency $\tilde{f}$	1/60	sec
Magnet cycle period	8.33	msec
Momentum compaction α	$9.6\text{x}10^{-3}$	
Harmonic number	1800	
Mean radius R	120.162	m
Injected beam rms momentum spread $\delta(t_{inj})$	0.006	
Injected beam rms bunch length	30	mm
Injected longitudinal emittance	120	μeV-sec
Nominal intensity	1	10^8

Table 2
Positron longitudinal parameters in the synchrotron

From Eq. (22), the voltage required in the machine for longitudinal matching at injection is $V_{rf}(t_{inj})$ = 3.8 MV. This voltage gives a bucket area at injection of about 7,875 μeV-sec, and a bucket half-height of about 2.5%. Fig. 15 shows the voltage required to maintain this same bucket area throughout the acceleration cycle.

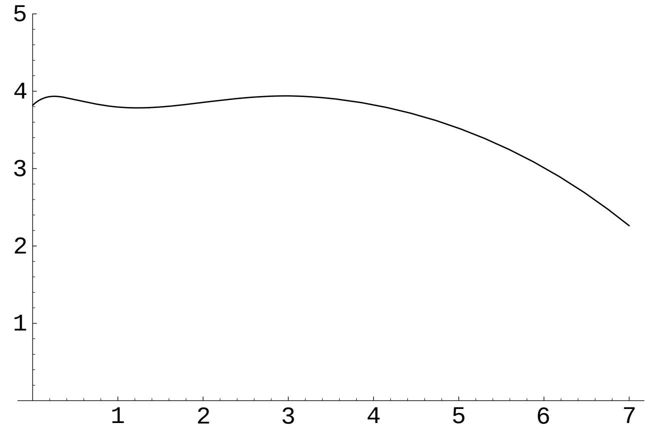

Fig. 15
RF voltage required to match at injection and to maintain the same bucket area throughout acceleration. The ordinate is in MV, the abscissa in msec.

As with electrons, the emittance growth which occurs as a result of the mismatch may not have much consequence, provided this emittance can be efficiently transmitted through the synchrotron and injected into CESR. A minimum requirement for efficient transport through the synchrotron is that the bucket half-height should be substantially larger than the rms energy spread. If we require that the ratio of bucket height to bunch length be R=3 at injection, then Eq. (23) determines a voltage at injection. For positrons in the synchrotron, this is 2.2 MV. The voltage throughout the cycle can then be specified by requiring that the same ratio of bucket height to rms energy spread be maintained. The resulting voltage profile is shown in Fig. 16.

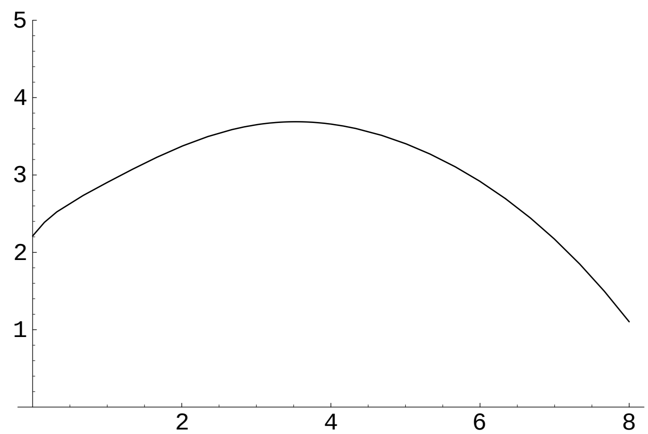

Fig. 16
RF voltage required maintain a bucket height/rms energy spread ratio of 3 throughout the acceleration cycle. The ordinate is in MV, the abscissa in msec.

The actual voltage, which is used in typical operation, is shown in Fig. 17. The method of calculation is the same as that described above for the electrons.

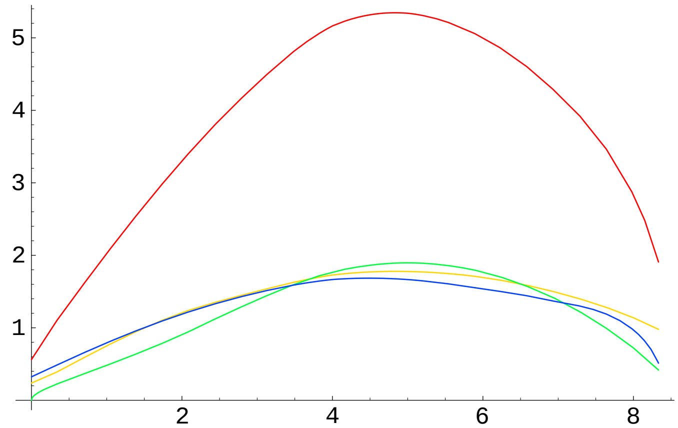

Fig. 17
Actual RF used during operation. The ordinate is in MV, the abscissa in msec.
The red curve is the total voltage; the voltages from L1, L2 and L5 are yellow, green and blue, respectively.

As is the case for electrons, since this voltage does not provide a longitudinal match, the injected bunch will execute quadrupole longitudinal oscillations in the bucket until it decoheres. The longitudinal emittance will increase, and the final longitudinal emittance will be that which corresponds to a match with the bucket, with the same energy spread as that of the injected bunch. In the small-oscillation limit, this emittance is given above by Eq. (26), assuming that the decoherence occurs soon after injection. For the positron parameters, this corresponds to an emittance of 289 μeV-sec, about a factor of 2.4 greater than the injected emittance. However, this is still smaller than the initial bucket area, which is 1093 μeV-sec. The emittance growth leads to an increase in the rms bunch length to about 72 mm.

Figs. 18 through 26 show the total energy per turn supplied to the beam, the relative energy per turn, the synchronous phase, the synchrotron tune, the synchrotron period (in turns), the bunch length, the bucket half-width and the bunch rms energy spread, the longitudinal beta function, and the peak current, respectively, throughout the acceleration cycle.

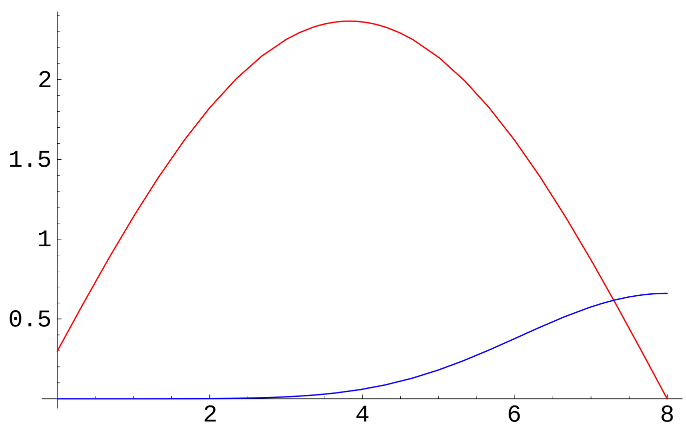

Fig. 18
Total energy gain per turn (MeV) vs. cycle time (msec)

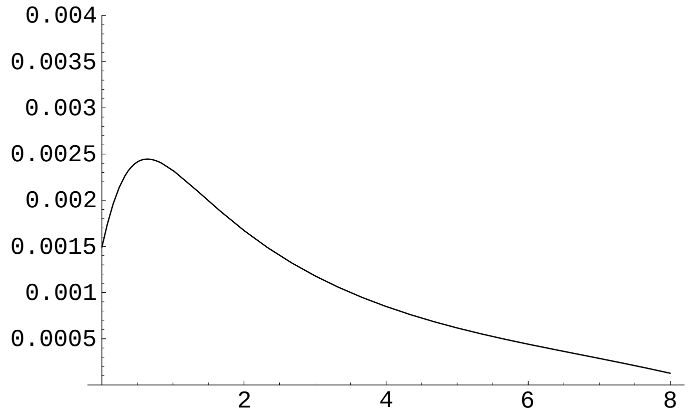

Fig. 19
Relative energy gain per turn vs. cycle time (msec)

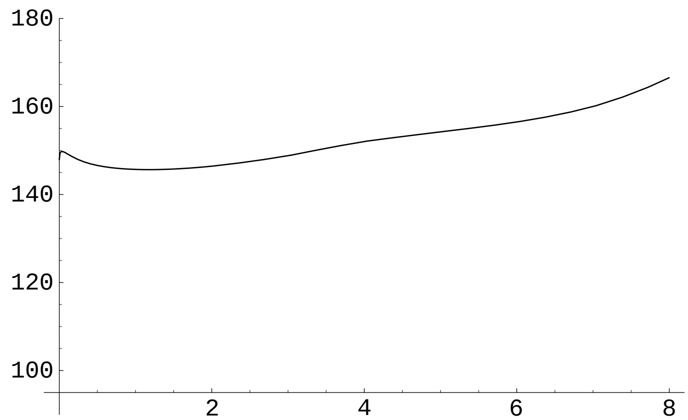

Fig. 20
Synchronous phase (degrees) vs. cycle time (msec)

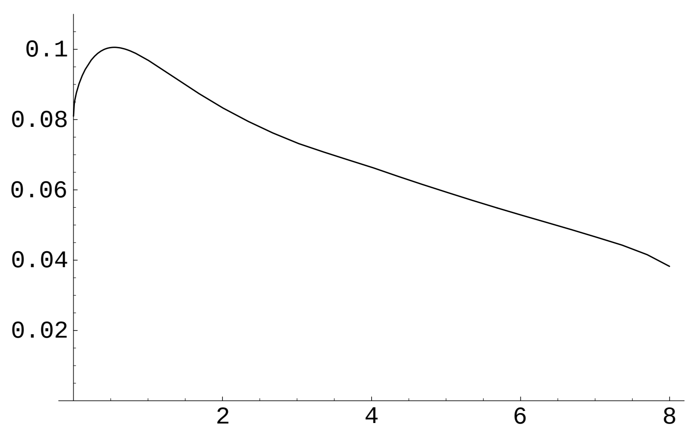

Fig. 21
Synchrotron tune vs. cycle time (msec)

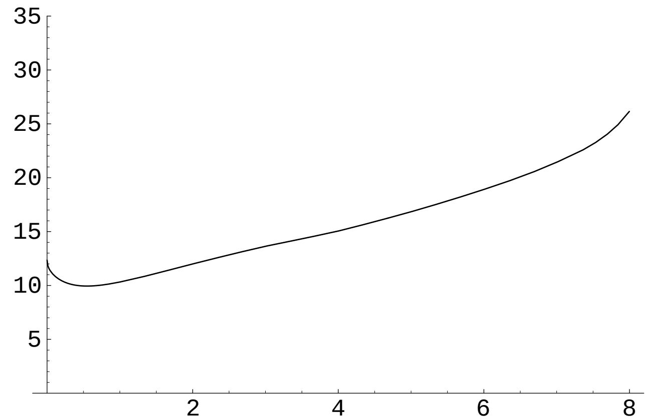

Fig. 22
Synchrotron period (turns) vs. cycle time (msec)

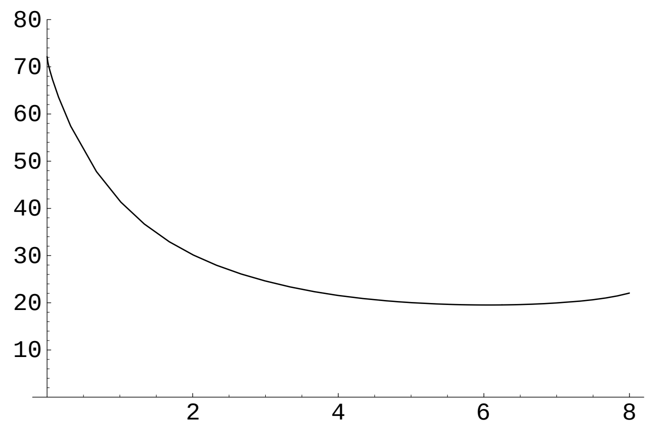

Fig. 23
Bunch length (mm) vs. cycle time (msec)

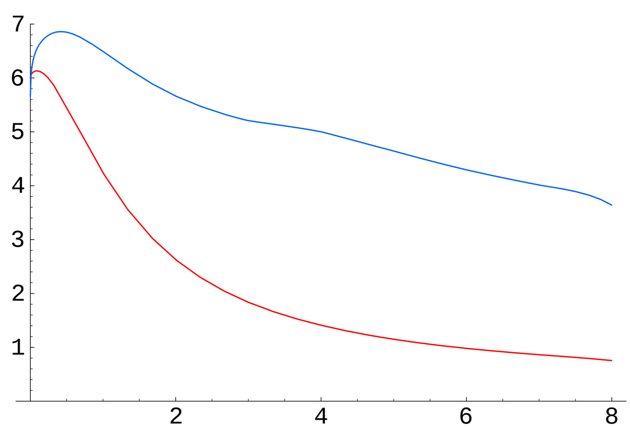

Fig. 24
Bucket relative half-width (upper) and bunch rms relative energy spread (lower), x10^3, vs. cycle time (msec)

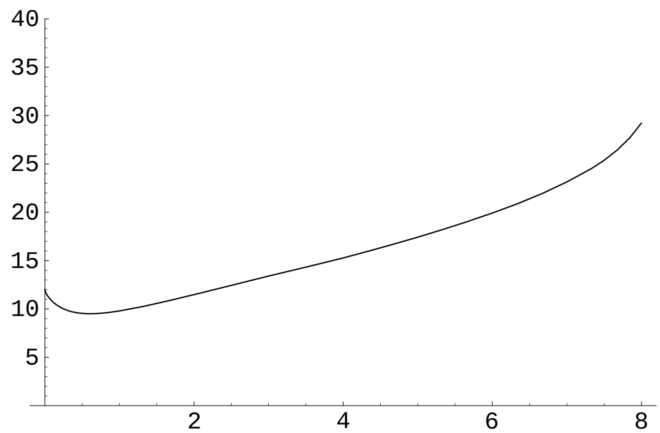

Fig. 25
Longitudinal beta function β_s(m) vs. cycle time (msec)

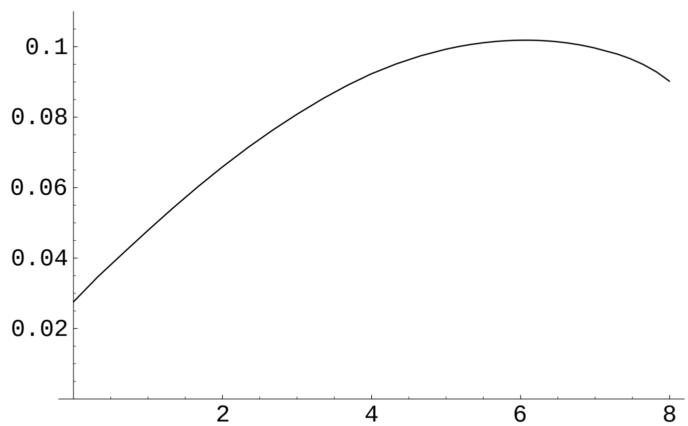

Fig. 26
Peak current (A), vs. cycle time (msec)

IV. Conclusions

Estimates have been made of parameters relevant to longitudinal dynamics in the 10 GeV synchrotron. This has been done for both electrons and positrons, based on measured forward power curves used during operation, and estimates of the injected beams' longitudinal parameters.

The nominal voltage curves used in operation do not provide sufficient voltage at injection to achieve a longitudinal match. However, since the bucket area is adequate, this is most likely not detrimental to performance.

A more relevant criterion for performance is the bucket height, in comparison with the rms beam energy spread. For electrons, the operational voltages are large enough to provide adequate (>3) ratio of bucket height to rms energy spread, if the energy distribution were Gaussian (compare Figs. 4 and 5). However, the electron energy distribution has a long, low energy tail extending to >1% energy spread; if this tail is to be fully captured (see Fig. 12), the voltage should be increased from that shown in Fig. 5 during the first two msec of the cycle.

For positrons, the voltage is insufficient during the first two msec to provide adequate bucket height (defined as 3 times the rms energy spread; compare Figs. 16 and 17, and see Fig. 24). To provide sufficient bucket height, the voltages at injection should be increased from that shown in Fig. 17 by roughly a factor of 3 at injection.

Acknowledgements

Thanks to Mike Billing for pointing out to me the data on Linac electron beam energy measurements

References

[1] http://cesrelog.lns.cornell.edu/MSLOG/linac/bunching/linac_bunch_length_960209.html
[2] M. Tigner, “Accelerating System for the Cornell 10 GeV Electron Synchrotron”, CS-32 (1967)
[3] G. R. Hemmie, “Some Remarks on Status and Improvements of the CESR Synchrotron”, pg. 34, unpublished note (1995)