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Localized Multibunch Modes

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It has been suggested[1,2] that the phenomenon of intrinsic localized modes in anharmonic condensed matter systems[3,4,5,6,7] may also be observed in the excitations of multibunch modes in accelerators, in the presence of significant lattice nonlinearities. In this note, this possibility is explored quantitatively, for multibunch coupling produced by the resistive wall impedance, and an octupole-generated nonlinearity. The conditions under which such modes might be produced at CESR are discussed.

1. Introduction

Consider M equally spaced bunches in a ring, of equal population. Let $y_n(t)$ (n=0,1,...,M-1) be the "snapshot" transverse (vertical) displacement of the nth bunch. The displacement is given by

$$y_n(t) = \tilde{y}_n \, exp(-i\Omega t) \tag{1}$$

where $\tilde{y}_n$ (a complex number) represents the amplitude and phase of bunch n at time t=0.

The equation of motion[8] for bunch n, in the rigid-beam approximation (i.e., neglecting internal motion of the bunch and the length of the bunch):

$$\frac{d^2 y_n(t)}{dt^2} + \omega_\beta^2 y_n(t) = -\frac{N r_0 c}{\gamma T_0} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W_1\left(-kC - \frac{m-n}{M} C\right) y_m\left(t - kT_0 - \frac{m-n}{M} T_0\right) \tag{2}$$

In this equation, $W_1(z)$ is the transverse dipole wake function, $\omega_\beta = \nu\omega_0$ is the vertical betatron frequency, N is the number of particles per bunch, C is the ring circumference, $T_0 = \frac{c}{C} = \frac{2\pi}{\omega_0}$ is the revolution period, and $\gamma = \frac{E}{m_0 c^2}$. All quantities and equations in this note are given in SI units. Using the above form for $y_n(t)$, we can write this (for $\Omega \sim \omega_\beta$) as

$$\frac{d^2 y_n(t)}{dt^2} + \omega_\beta^2 y_n(t) = -\frac{N r_0 c}{\gamma T_0} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W_1\left(-kC - \frac{m-n}{M} C\right) exp\left(i\omega_\beta T_0 \left(k + \frac{m-n}{M}\right)\right) y_m(t) \tag{3}$$

Transforming to the frequency domain and introducing the transverse impedance $Z_1^\perp(\omega)$, we have

$$\frac{d^2 y_n(t)}{dt^2} + \omega_\beta^2 y_n(t) - \sum_{m=0}^{M-1} y_m(t) L(m-n) = 0 \tag{4}$$

with

$$L(m-n) = i\frac{4\pi N r_0}{\mu_0 c\gamma T_0^2}\sum_{p=-\infty}^{\infty} Z_1^{\perp}\left(\omega_\beta + p\omega_0\right) exp\left(-2\pi i p\frac{m-n}{M}\right) \quad (5)$$

2. Normal Modes

The normal modes of the M bunches are obtained by the usual technique. Let the normal modes q_n given in terms of y_n by the relation

$$\mathbf{q} = \mathbf{C} \bullet \mathbf{y} \quad (6)$$

in which **C** is a matrix. In matrix form, the equation of motion (4) is

$$\ddot{\mathbf{y}} + \mathbf{S} \bullet \mathbf{y} = 0 \quad (7)$$

in which

$$S_{mn} = \omega_\beta^2 \delta_{mn} - L(n-m) \quad (8)$$

Using (5) to introduce the normal modes, this becomes

$$\ddot{\mathbf{q}} + \mathbf{C} \bullet \mathbf{S} \bullet \mathbf{C}^{-1}\mathbf{q} = 0 \quad (9)$$

The matrix **C** diagonalizes **S.** The eigenvalues are the normal mode frequencies. The required matrix is

$$C_{\mu n} = \frac{1}{\sqrt{M}} exp\left(\frac{-2\pi i \mu n}{M}\right) \quad (10)$$

The matrix **C** obeys the following orthonormality condition:

$$\sum_{\alpha=0}^{N-1} C_{\mu\alpha}C_{\alpha n}^{*} = \frac{1}{M}\sum_{\alpha=0}^{N-1} exp\left(\frac{2\pi i\alpha(n-\mu)}{M}\right) = \delta_{n-\mu, rM} \quad (11)$$

in which r is any integer. Since $C_{\mu n}^{*} = C_{n\mu}^{*} = C_{n\mu}^{-1}$, **C** is a unitary matrix. Using relation (11), it follows that the eigenvectors are

$$\Omega_\mu^2 = \omega_\beta^2 - i\frac{4\pi M N r_0}{\mu_0 c\gamma T_0^2}\sum_{r=-\infty}^{\infty} Z_1^{\perp}\left(\omega_\beta + (rM+\mu)\omega_0\right) \quad (12)$$

in which μ is the normal mode index. This relation between the mode frequency and the mode number is the analog of the $\omega(k)$ dispersion relation encountered in condensed matter systems. The relation between the displacement of the nth bunch, and the normal modes q_μ, is given by inverting Eq. (6), using (11):

$$y_n(t) = \sum_{\mu=0}^{n} C_{n\mu}^{-1} q_\mu(t) = \frac{1}{\sqrt{M}}\sum_{\mu=0}^{n} exp\left(\frac{2\pi i\mu n}{M}\right) q_\mu(t) \quad (13)$$

We can define the frequency shift of normal mode μ using the approximation (for $\Omega_\mu \sim \omega_\beta$)

$$\Omega_{\mu}^{2}-\omega_{\beta}^{2}=(\Omega_{\mu}-\omega_{\beta})(\Omega_{\mu}+\omega_{\beta})\approx 2\Delta\Omega_{\mu}\omega_{\beta} \tag{14}$$

So we have

$$\Delta\Omega_{\mu}=-i\frac{2\pi MNr_{0}}{\omega_{\beta}\mu_{0}c\gamma T_{0}^{2}}\sum_{r=-\infty}^{\infty}Z_{1}^{\perp}\left(\omega_{\beta}+(rM+\mu)\omega_{0}\right) \tag{15}$$

The above expression is correct only for a point bunch, with zero chromaticity. To include the effects of a finite bunch length (assumed Gaussian, of width σ_z) and a finite chromaticity ξ, we make the replacement [10]

$$\sum_{r=-\infty}^{\infty}Z_{1}^{\perp}\left(\omega_{\beta}+(rM+\mu)\omega_{0}\right)\rightarrow\frac{cT_{0}}{2M\sigma_{z}\sqrt{\pi}}\left(Z_{1}^{\perp}\right)_{eff} \tag{16}$$

in which

$$\left(Z_{1}^{\perp}\right)_{eff}=\frac{\sum_{r=-\infty}^{\infty}Z_{1}^{\perp}\left(\omega_{\beta}+(rM+\mu)\omega_{0}\right)exp\left(-\frac{\left(\omega_{\beta}+(rM+\mu)\omega_{0}-\omega_{\xi}\right)^{2}\sigma_{z}^{2}}{c^{2}}\right)}{\sum_{r=-\infty}^{\infty}exp\left(-\frac{\left(\omega_{\beta}+(rM+\mu)\omega_{0}-\omega_{\xi}\right)^{2}\sigma_{z}^{2}}{c^{2}}\right)} \tag{17}$$

and

$$\omega_{\xi}=\frac{\xi\omega_{\beta}}{\eta} \tag{18}$$

So we get

$$\Delta\Omega_{\mu}=-i\frac{\sqrt{\pi}Nr_{0}}{\omega_{\beta}\mu_{0}\gamma\sigma_{z}T_{0}}\left(Z_{1}^{\perp}\right)_{eff} \tag{19}$$

3. Development of the Green's function.

In the analysis of localized modes[3,4,5], we will need to develop the Green's function for the equation of motion, Eq.(4). In that equation, if we let $y_{n}(t)=\tilde{y}_{n}\,exp(-i\Omega t)$, we have

$$-\Omega^{2}\tilde{y}_{n}+\omega_{\beta}^{2}\tilde{y}_{n}-\sum_{m=0}^{M-1}\tilde{y}_{m}L(m-n)=0 \tag{20}$$

If we define the matrix

$$R_{mn}(\Omega)=\left(-\Omega^{2}+\omega_{\beta}^{2}\right)\delta_{mn}-L(n-m) \tag{21}$$

then the equation of motion is

$$\sum_{m=0}^{M-1} R_{nm}(\Omega)\tilde{y}_m = 0 \tag{22}$$

The Green's function is $G_{mn}(\Omega) = R_{mn}^{-1}(\Omega)$ and it satisfies the equation

$$\sum_{m=0}^{M-1} R_{nm}(\Omega) G_{mn'}(\Omega) = \delta_{nn'} \tag{23}$$

To find it, we expand the Green's function using the normal mode eigenvectors as a basis set

$$G_{mn}(\Omega) = \frac{1}{M}\sum_{l=0}^{M-1}\sum_{l'=0}^{M-1} exp\left(\frac{2\pi iml}{M}\right) exp\left(\frac{2\pi inl'}{M}\right)\tilde{G}_{ll'}(\Omega)$$

$$\tilde{G}_{ll'}(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} exp\left(-\frac{2\pi iml}{M}\right) exp\left(-\frac{2\pi inl'}{M}\right) G_{mn}(\Omega) \tag{24}$$

substitute into Eq.. (23) and make use of (11) to get

$$G_{mn}(\Delta\Omega) = -\frac{1}{2M\omega_\beta}\sum_{l=0}^{M-1}\frac{exp\left(\frac{2\pi il(m-n)}{M}\right)}{(\Delta\Omega - \Delta\Omega_l)} \tag{25}$$

in which $\Delta\Omega = \Omega - \omega_\beta$, and $\Delta\Omega_l$ is given by Eq. (19).This result will be used below in section 7.

4. Coupling Impedance

We will only consider the resistive wall impedance, for the sake of simplicity. The existence and general character of the localized modes is not expected to depend sensitively on the details of the frequency dependence of the impedance.

The transverse impedance associated with the resistive wall[9] is, for low frequencies $\omega << \left(\frac{\mu_0\sigma}{4\pi}\frac{c^4}{b^2}\right)^{\frac{1}{3}}$,

$$Z_1^{\perp}(\omega) = \frac{Z_0 C}{4\pi b^3}\sqrt{\frac{8}{\mu_0\sigma}}|\omega|^{\frac{1}{2}}\frac{(1 - sign(\omega)i)}{\omega} \tag{26}$$

corresponding to the wake function

$$W_1(z) = -\frac{2}{\pi b^3}\sqrt{\frac{4\pi}{c\mu_0\sigma}}\frac{C}{|z|^{\frac{1}{2}}} \quad for \quad \sigma\mu_0 cb^2 >> |z| >> \left(\frac{b^2}{\mu_0 c\sigma}\right)^{\frac{1}{3}} \tag{27}$$

In these equations, σ is the conductivity of the vacuum chamber wall, b is the radius of the (assumed round) vacuum chamber, and Z_0 is the impedance of free space (377 Ω).

For the resistive wall, then, we have, combining Eq. (19) and (26),

$$\Delta\Omega_\mu = -i\frac{Nr_0}{4\sqrt{\pi}\omega_\beta\mu_0\gamma T_0^2}\frac{Z_0}{\sigma_z b^3}\sqrt{\frac{8}{\mu_0\sigma}}\frac{\sum_{p=-\infty}^{\infty}\left\{\begin{array}{l}\left|\omega_\beta+(pM+\mu)\omega_0\right|^{\frac{1}{2}}\frac{\left(1-sign(\omega_\beta+(pM+\mu)\omega_0)i\right)}{\omega_\beta+(pM+\mu)\omega_0}\times\\ exp\left(-\frac{\sigma_z^2\left(\omega_\beta+(pM+\mu)\omega_0-\omega_\xi\right)^2}{c^2}\right)\end{array}\right\}}{\sum_{p=-\infty}^{\infty}exp\left(-\frac{\sigma_z^2\left(\omega_\beta+(pM+\mu)\omega_0-\omega_\xi\right)^2}{c^2}\right)} \tag{28}$$

5. Spectrum of the Normal Modes

We choose the following specific case to calculate the normal mode spectrum for CESR. We take $N=1.3\times10^{11}$ (corresponding to about 8 ma per bunch). Using C=778 m, we take an aluminum ($\sigma=3.5\times10^7$ mho/m) vacuum chamber of radius b=25 mm. We take $\eta=0.01$, and a bunch length of $\sigma_z=20$ mm. The chromaticity is set to $\xi=2$, which makes all of the multibunch modes stable (i.e., they have a negative imaginary part). Figs. 1 and 2 give the real and imaginary parts of the frequency shift (Eq. (28)), as a function of the mode number, for M=21 bunches. Fig. 3 shows the frequencies at which the various modes would be observed on a spectrum analyzer. Figs. 4, 5 and 6 present the same information, but for M=9 bunches.

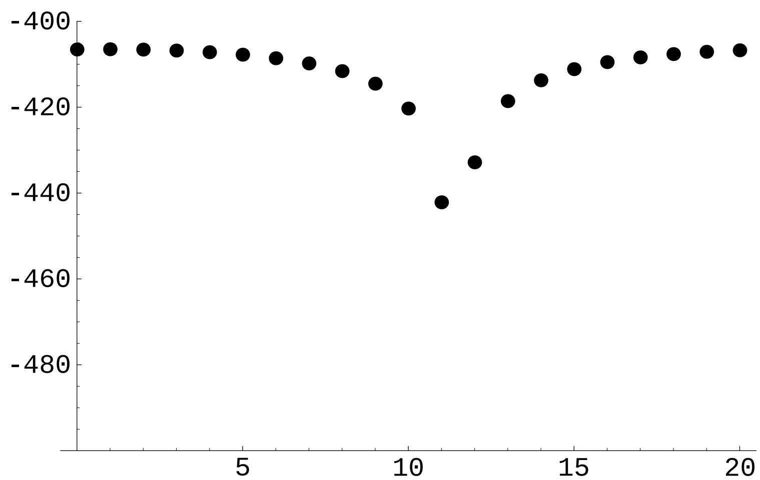

Fig. 1
Real part of the frequency shift (abscissa, rad/sec) vs. mode number (ordinate), for M=21

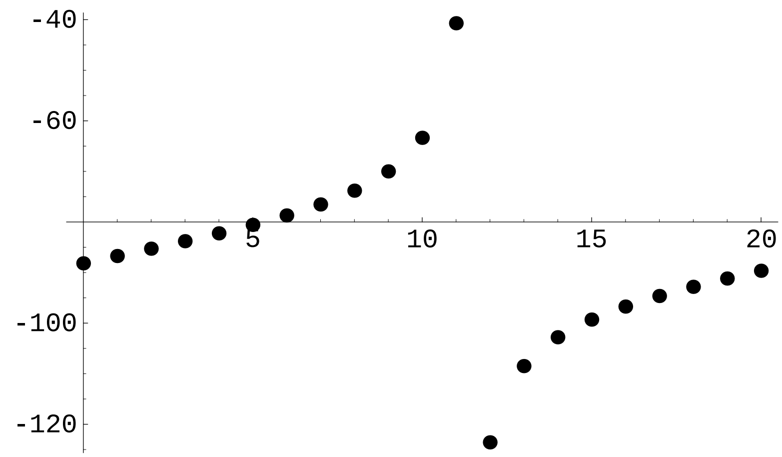

Fig. 2
Imaginary part of the frequency shift (abscissa, rad/sec) vs. mode number (ordinate), for M=21

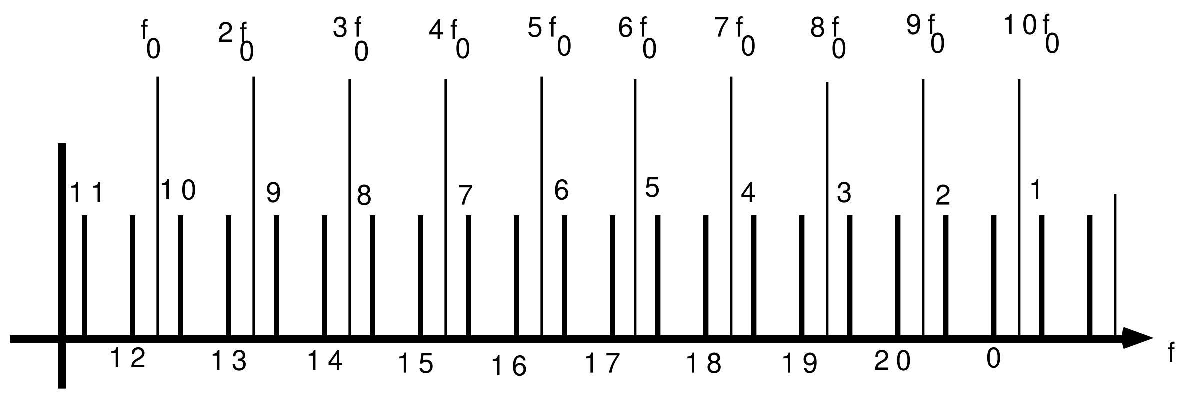

Fig. 3
Coupled-bunch mode spectrum for M=21

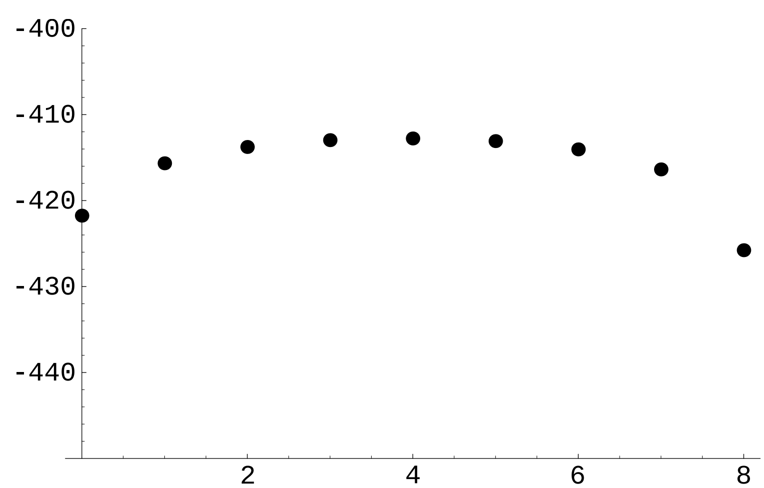

Fig. 4
Real part of the frequency shift (abscissa, rad/sec) vs. mode number (ordinate), for M=9

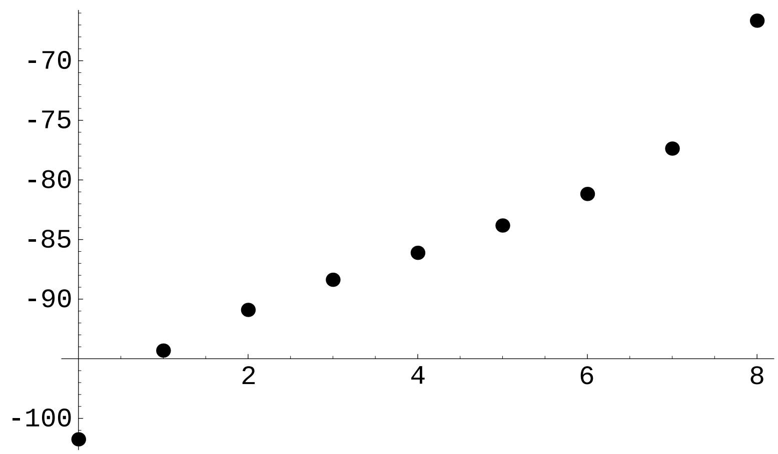

Fig. 5
Imaginary part of the frequency shift (abscissa, rad/sec) vs. mode number (ordinate), for M=9

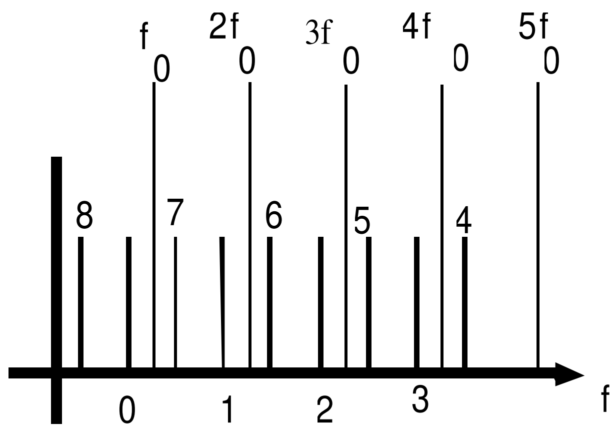

Fig. 6
Coupled-bunch mode spectrum for M=9

6. Octupole Nonlinearity

We introduce an octupole into the ring at the location s_1. Provided that we are not operating close to a second or fourth order resonance, the octupole field perturbation results primarily in a dependence of the betatron tune, $\nu = \frac{\omega_\beta}{\omega_0}$, on amplitude:

$$\nu(a) = \nu_0 + \mu a^2 \tag{29}$$

in which

$$\mu = \frac{3}{16\pi}\frac{k_3 L}{6}\frac{\beta(s_1)^2}{\beta_0} \tag{30}$$

In this equation, $k_3 = \frac{1}{B\rho}\frac{d^3B}{dy^3}$ is the octupole strength, L is the octupole length, $\beta(s_1)$ is the beta function at the location of the octupole, and β_0 is the beta function at the point at which the oscillation amplitude is a.

7. Localized Modes

Returning to the original equation of motion, we have, including the octupole nonlinearity,

$$\frac{d^2y_n(t)}{dt^2} + \omega_\beta^2(a_n)y_n(t) - \sum_{m=0}^{M-1} y_m(t)L(m-n) =$$
$$\frac{d^2y_n(t)}{dt^2} + \omega_\beta^2 y_n(t) + \mu' a_n^2 y_n(t) - \sum_{m=0}^{M-1} y_m(t)L(m-n) = 0 \tag{31}$$

where

$$\mu' = 2\mu\omega_\beta\omega_0 \tag{32}$$

and we assume that $\mu a^2 << \nu$.

We now look for solutions of the form, $y_n(t) = a_n\, exp(-i\Omega t)$, in which a_n is the oscillation amplitude. Equation (31) becomes

$$-\Omega^2 a_n + \omega_\beta^2 a_n - \sum_{m=0}^{M-1} a_m L(m-n) = -\mu' a_n^3 \tag{33}$$

The equation with μ' =0 is the same one for which we have found the Green's function. Thus, referring to Eq. (22) above, we have

$$\sum_{m=0}^{M-1} R_{nm}(\Omega)a_m = -\mu' a_n^3 \tag{34}$$

for which we have the solution, using the Green's function,

$$a_n = -\mu' \sum_{m=0}^{M-1} R_{nm}^{-1}(\Omega)a_m^3 = -\mu' \sum_{m=0}^{M-1} G_{mn}(\Delta\Omega)a_m^3 = \frac{\mu'}{2M\omega_\beta} \sum_{m=0}^{M-1}\sum_{l=0}^{M-1} \frac{exp\left(\frac{2\pi i l(m-n)}{M}\right)}{\left(\Delta\Omega - \Delta\Omega_l\right)} a_m^3 \tag{35}$$

Using Eq. (32), we have

$$a_n = \frac{\mu\omega_0}{M} \sum_{m=0}^{M-1}\sum_{l=0}^{M-1} \frac{exp\left(\frac{2\pi i l(m-n)}{M}\right)}{\left(\Delta\Omega - \Delta\Omega_l\right)} a_m^3 \tag{36}$$

This is essentially a nonlinear eigenvalue equation. There will be normal modes, corresponding to some linear combination of the a_n amplitudes, and for each mode there will be a frequency shift ΔΩ. Because of the nonlinearity, the mode pattern is complex, and the mode eigenvector and frequency shift will depend on the amplitude of the oscillation.

We are interested in specific modes, which are "localized" to one or two bunches; they are like solitons in continuum field theories. If μ is negative, the localized mode will have a frequency below the lowest normal mode frequency shown in Fig. 1 (which is mode # 11 for M=21). If μ is positive, the localized mode will have a frequency above the highest normal mode frequency shown in Fig. 1 (which is mode # 1 for M=21).

Following reference 5, let $a_n = \alpha\xi_n$, with α a measure of the oscillation amplitude, and Max(ξ_n)=1. Then we have

$$\xi_n = \frac{\mu\alpha^2\omega_0}{M}\sum_{m=0}^{M-1}\sum_{l=0}^{M-1}\frac{exp\left(\frac{2\pi il(m-n)}{M}\right)}{(\Delta\Omega-\Delta\Omega_l)}\xi_m^3 \tag{37}$$

Note that $\mu\alpha^2\omega_0$ is just the frequency shift produced by the octupole for an oscillation of amplitude α. This equation can be solved iteratively as follows. We first assume that only the n=0 bunch is excited, so ξ_0=1, and all other ξ_n are zero. Then we have

$$1 = \frac{\mu\alpha^2\omega_0}{M}\sum_{l=0}^{M-1}\frac{1}{\Delta\Omega-\Delta\Omega_l} \tag{38}$$

This equation gives the zero-order eigenfrequency shift $\Delta\Omega_0$. We then substitute this into Eq. (37) to get

$$\xi_1 = \frac{\mu\alpha^2\omega_0}{M}\sum_{l=0}^{M-1}\frac{exp\left(-\frac{2\pi il}{M}\right)}{\Delta\Omega_0-\Delta\Omega_l} \tag{39}$$

Then, with this value for ξ_1, we solve

$$1 = \frac{\mu\alpha^2\omega_0}{M}\left(\sum_{l=0}^{M-1}\frac{1}{\Delta\Omega_1-\Delta\Omega_l}+\xi_1^3\sum_{l=0}^{M-1}\frac{exp\left(-\frac{2\pi il}{M}\right)}{\Delta\Omega_1-\Delta\Omega_l}\right) \tag{40}$$

to get the 1st order eigenvalue shift $\Delta\Omega_1$. This is then used, with ξ_1, to get ξ_2, and so on.

Another approach is to observe that Eq. (37) is a set of M simultaneous nonlinear equations; the M unknowns are the ξ_n (except for ξ_0=1) and the frequency shift $\Delta\Omega$. The M equations can then be solved for the M unknowns. This procedure gives the same result as the iteration technique for the example discussed below.

8. Results

In CESR, there are octupoles at 45W, 48W, 48E, and 45E. Each has a length L=0.391 m. The vertical beta functions at 45W and 45E are 23.56 m and 22.68 m respectively; the vertical beta functions at the other two octupoles are much smaller. The strength of each is $k_3[m^{-4}]$=0.213 CU/E(GeV), where CU refers to computer units. The frequency shift which appears in Eq. (37) can be written as

$$\Delta\omega_{oct} = \mu\alpha^2\omega_0 = \omega_0 \frac{3}{16\pi}\frac{k_3L}{6}\beta(s_1)^2\frac{\alpha^2}{\beta_0} = \omega_0\frac{3\varepsilon_y}{16\pi}\frac{k_3L}{6}\beta(s_1)^2\left(\frac{\alpha}{\sigma_y}\right)^2 \tag{41}$$

in which the rms vertical emittance is $\varepsilon_y = \frac{\sigma_y^2}{\beta_0}$ and σ_y is the rms vertical beam size.

Plugging in the above numbers, and using ω_0=385,000 rad/sec, $\varepsilon_y=10^{-8}$ m-rad, we get, summing over the two octupoles at 45W and 45E,

$$\Delta\omega_{oct} = 0.0041\left(\frac{\alpha}{\sigma_y}\right)^2 CU \quad rad/sec \tag{42}$$

For example, for -1000 computer units in both octupoles and $\alpha=5\sigma_y$ (2 mm @β=20 m), we have $\Delta\omega_{oct} = -103\ rad/sec$.

With this nonlinearity, the solution of Eq. (37) for M=21 gives a localized mode with a frequency shift of -504.9-85.6i rad/sec. It is shifted down by about 63 rad/sec from the mode #11 frequency. The bunch pattern corresponding to this localized mode is shown in Fig. 7.

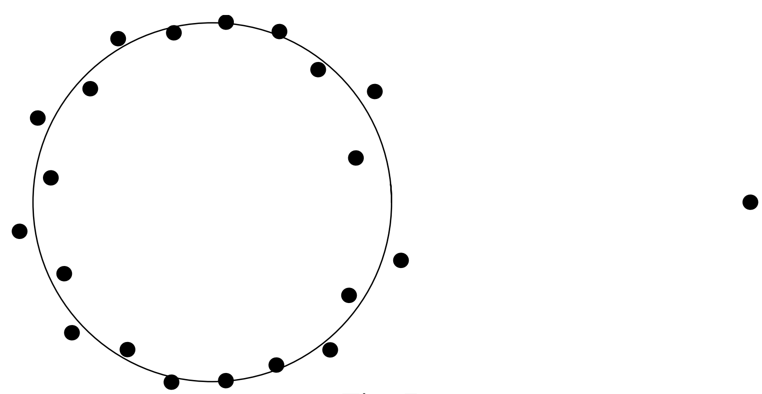

Fig. 7:
Localized mode bunch pattern for $\Delta\omega_{oct} = -103\ rad/sec$ and M=21

For +1000 computer units in both octupoles and $\alpha=5\sigma_y$, we have $\Delta\omega_{oct} = 103\ rad/sec$. With this nonlinearity, the solution of Eq (34) with M=21 gives a localized mode with a frequency shift of –317.7-86.6i rad/sec. It is shifted up by about 89 rad/sec from the mode #1 frequency. The bunch pattern corresponding to this localized mode is shown in Fig. 8.

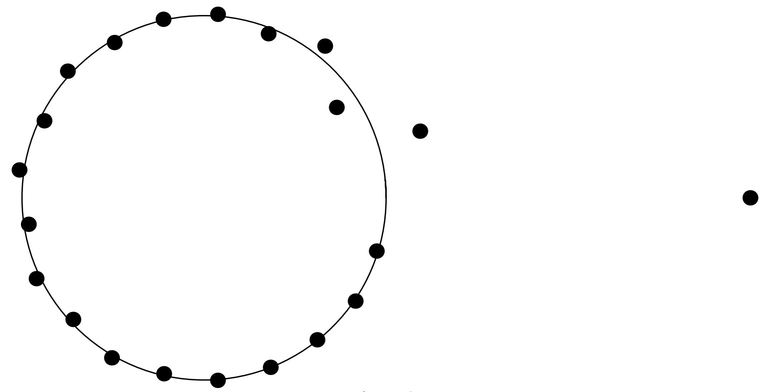

Fig. 8:
Localized mode bunch pattern for $\Delta\omega_{oct} = 103\ rad/sec$ and M=21

For M=9, $\Delta\omega_{oct} = -103\ rad/sec$ gives a frequency shift of –511.3-85.3i rad/sec (shifted down by 85.5 rad/sec from mode #8); the localized mode pattern is shown in Fig. 9. For M=9 and $\Delta\omega_{oct} = 103\ rad/sec$, we get a frequency shift of –320.9-85.9i rad/sec (shifted up by 91.9 rad/sec from mode #4). The localized mode pattern is shown in Fig. 10.

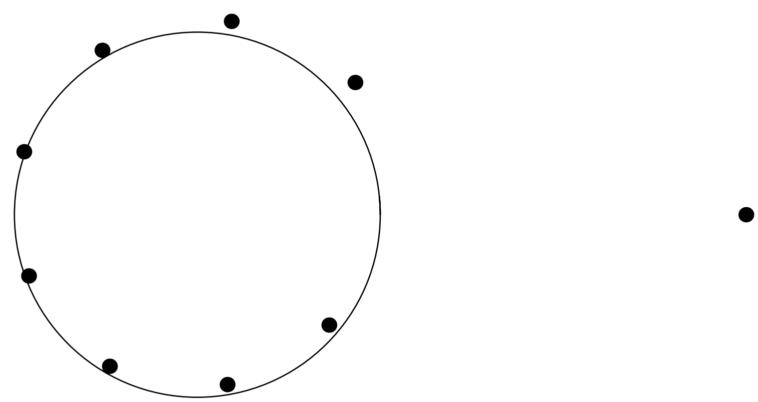

Fig. 9:
Localized mode bunch pattern for $\Delta\omega_{oct} = -103\ rad/sec$ and M=9

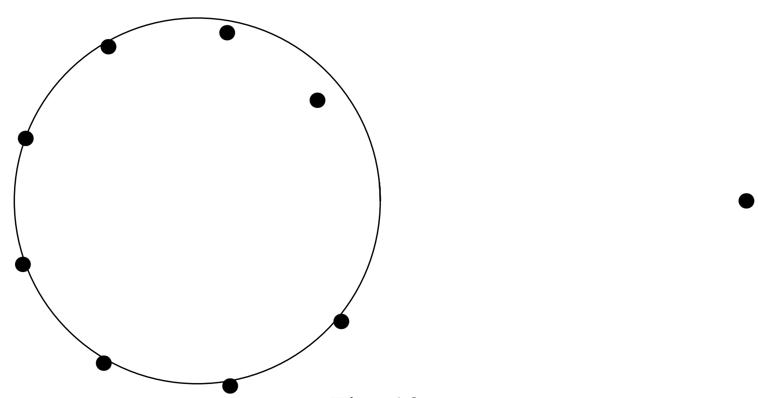

Fig. 10:
Localized mode bunch pattern for $\Delta\omega_{oct} = -103\ rad/sec$ and M=9

9. Conclusion

Localized modes may be possible in the coherent motion of arrays of bunches in CESR, in the presence of nonlinearities generated by octupoles. With a bunch-to-bunch coupling produced by the resistive wall impedance, with vertical oscillation amplitudes of about 2 mm (at a β_y of about 20 m) and for octupole strengths corresponding to about 1000 computer units in both the 45W and 45E octupoles, these modes appear for both 21 and 9 equally spaced, equally populated bunches.

10. References

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