CBN 98–14
July 10, 1998

Twiss Analysis With a Mobius Lattice

David Sagan

1 Introduction

The standard analysis for a coupled lattice is based upon the formalism of Edwards and Teng[1]. This is fine for weak coupling but with the strong coupling that arises in a Mobius lattice there are complications that must be addressed. To see this lets back step a moment and consider how the Twiss parameters are affected by a coordinate transformation. To simplify matters consider the 1–dimensional case where the laboratory coordinates (x, x') are transformed to coordinates (a, a') using the matrix $\mathbf{G}$:

$$\begin{pmatrix} a \\ a' \end{pmatrix} = \mathbf{G} \begin{pmatrix} x \\ x' \end{pmatrix} . \tag{1}$$

For example, if $\mathbf{G}$ is taken to be

$$\mathbf{G} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} , \tag{2}$$

then, in terms of the new coordinates, the Twiss parameters have been normalized with $\beta_a = 1$ and $\alpha_a = 0$. This example shows that the Twiss parameters are dependent upon the particular choice of the coordinate axes. Since many standard formulas are based upon the assumption that the axes are oriented along position and velocity coordinates these formulas need to be modified when using different coordinates. For example, the standard formulas

$$\begin{aligned} \sigma &\propto \sqrt{\beta} \, , \\ \frac{d\phi}{ds} &= \frac{1}{\beta} \, , \end{aligned} \tag{3}$$

are no longer valid. Additionally, formulas for the emittance, damping partition numbers, etc. need to be modified.

With a highly coupled lattice we have a similar problem: The normal mode axes do not correspond (even approximately) to the usual position and velocity axes. This is not just due to the way Edwards and Teng compute the normal modes. It is inherent in the normal modes themselves. In the case of a Mobius lattice[4], a possibility is to compute the Twiss parameters using the 2-turn transfer matrix instead of the 1-turn transfer matrix. However, with the 2-turn transfer matrix, the two modes have the same tune (that is, you are at the coupling resonance) so an arbitrarily small perturbation can cause the normal mode axes to become just as convoluted as in the 1-turn analysis and you are back to the same problem.

The only remedy, as far as I can see, is to just live with the complications. Once correctly analyzed, and after the pain of rewriting software, they should not be too hard to live with.

2 Mode Flips

Following Billing[2], the Edwards and Teng decomposition of the 1–turn matrix is written as

$$\begin{aligned}\mathbf{T} &= \begin{pmatrix} \mathbf{M} & \mathbf{m} \\ \mathbf{n} & \mathbf{N} \end{pmatrix} \\ &= \mathbf{V}\,\mathbf{U}\,\mathbf{V}^{-1} \\ &= \begin{pmatrix} \gamma\mathbf{I} & \mathbf{C} \\ -\mathbf{C}^{+} & \gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \begin{pmatrix} \gamma\mathbf{I} & -\mathbf{C} \\ \mathbf{C}^{+} & \gamma\mathbf{I} \end{pmatrix} .\end{aligned} \tag{4}$$

where

$$\gamma = \sqrt{1/2 \pm 1/2\sqrt{\frac{(\mathrm{Tr}[\mathbf{M}-\mathbf{N}])^2}{(\mathrm{Tr}[\mathbf{M}-\mathbf{N}])^2 + 4\det\mathbf{H}}}} \tag{5}$$

$$\mathbf{C} = \mp\frac{\mathbf{H}\cdot\mathrm{Sgn}(\mathrm{Tr}(\mathbf{M}-\mathbf{N}))}{\gamma\sqrt{(\mathrm{Tr}(\mathbf{M}-\mathbf{N}))^2 + 4\det\mathbf{H}}} \tag{6}$$

$$\mathbf{H} \equiv \mathbf{m} + \mathbf{n}^{+} \tag{7}$$

$$\mathbf{C}^{+} = \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \tag{8}$$

$$\gamma^2 + \det\mathbf{C} = 1 \tag{9}$$

and

$$\mathrm{Sgn}\,x = \begin{cases} -1 & \text{For } x \le 0 \\ +1 & \text{For } x \ge 0 \end{cases} . \tag{10}$$

In general, it is possible to construct two distinct solutions for Eq. (4). One solution uses the $+$ sign in Eq. (5) and the $-$ sign in Eq. (6), and vice versa for the other solution. Physically, the two solutions can be explained as follows: Call the two eigenmodes of the system "h" and "v" (for a physical picture think, for example, of the h eigenmode as the mode with the higher tune and the v mode as the mode with the lower tune). The difference between the two solutions is whether the h mode is associated with the a mode (*i.e.* gets placed in the upper left hand corner of the $\mathbf{U}$ matrix) or whether the h mode is associated with the b mode. From a physics standpoint it does not matter whether the h mode is associated with the a mode or the b mode but the eigenaxes for the h mode (and the v mode) depend upon it and therefore the Twiss parameters will be different depending upon the solution used. Furthermore, if $\det\mathbf{H} < 0$ then Eq. (5) allows only 1 solution. It can thus happen that at one place in the ring with $\det\mathbf{H} < 0$ the h mode is forced to be associated with the a mode and at another place in the ring the h mode is forced to be associated with the b mode. This is not simply a theoretical matter. CESR Mobius lattices typically have this property. The switching of the h and v modes between the a and b modes as one propagates the twiss parameters through the ring is something I call "mode flipping." Since γ is constant when propagated through an uncoupled element[6] mode flipping will only be forced when propagating through an element that couples the x–y motions.

From the above discussion it is seen that the bookkeeping for a highly coupled lattice is more complicated than if the lattice is only slightly coupled. For example, with mode flipping you can have different programs compute the Twiss parameters at a point and come up with different numbers depending upon how the programs handle flips. In fact, if you try to compare two lattices to see how similar they are you cannot just compare the Twiss parameters but have to look in detail what the eigenaxes are.

Some additional points: First, at a point where the modes flip the Twiss parameters will be discontinuous. Furthermore, the integer part of the phase is not well defined where there is a mode flip. The reason for this is the same reason why the integer part of the tune is not well defined when you are only given the full 1–turn matrix but are not given the full lattice description. Lastly, a lattice should always have an even number of mode flips so that the eigenaxes you end up with when you get back to the starting point are the same as the axes that you started out with. If this is not done then even the fractional part of the tune will not be correctly computed.

3 The Projected Twiss Parameters

If the a mode is excited the position of the beam can be derived from Eq. (4) (cf. Bagley and Rubin[3])

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_n = A_a \begin{pmatrix} \gamma\sqrt{\beta_a}\cos\theta_a \\ \frac{-\gamma}{\sqrt{\beta_a}}(\sin\theta_a + \alpha_a\cos\theta_a) \\ -C_{22}\sqrt{\beta_a}\cos\theta_a - \frac{C_{12}}{\sqrt{\beta_a}}(\sin\theta_a + \alpha_a\cos\theta_a) \\ C_{21}\sqrt{\beta_a}\cos\theta_a + \frac{C_{11}}{\sqrt{\beta_a}}(\sin\theta_a + \alpha_a\cos\theta_a) \end{pmatrix}, \tag{11}$$

and for the b mode

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_n = A_b \begin{pmatrix} C_{11}\sqrt{\beta_b}\cos\theta_b - \frac{C_{12}}{\sqrt{\beta_b}}(\sin\theta_b + \alpha_b\cos\theta_b) \\ C_{21}\sqrt{\beta_b}\cos\theta_b - \frac{C_{22}}{\sqrt{\beta_b}}(\sin\theta_b + \alpha_b\cos\theta_b) \\ \gamma\sqrt{\beta_b}\cos\theta_b \\ \frac{-\gamma}{\sqrt{\beta_b}}(\sin\theta_b + \alpha_b\cos\theta_b) \end{pmatrix}, \tag{12}$$

where the phase on the n$^{\mathrm{th}}$ turn and at position s is

$$\begin{aligned} \theta_a(n,s) &= 2\pi Q_a n + \phi_a(s) + \phi_{a0}\,, \\ \theta_b(n,s) &= 2\pi Q_b n + \phi_b(s) + \phi_{b0}\,, \end{aligned} \tag{13}$$

where Q_a and Q_b are the tunes, ϕ_a, and ϕ_b are the betatron phases, and ϕ_{a0} and ϕ_{b0} are calculated from the initial conditions.

From the above equations the beam sigmas of the a mode are

$$\begin{aligned} \sigma_{x,a} &= \gamma\sqrt{\epsilon_a\beta_a} \\ \sigma_{y,a} &= \sqrt{\epsilon_a\beta_a}\left[\left(C_{22} + \frac{C_{12}\alpha_a}{\beta_a}\right)^2 + \left(\frac{C_{12}}{\beta_a}\right)^2\right]^{1/2} \end{aligned} \tag{14}$$

And for the b mode

$$\begin{aligned} \sigma_{x,b} &= \sqrt{\epsilon_b\beta_b}\left[\left(C_{11} - \frac{C_{12}\alpha_b}{\beta_b}\right)^2 + \left(\frac{C_{12}}{\beta_b}\right)^2\right]^{1/2} \\ \sigma_{y,b} &= \gamma\sqrt{\epsilon_b\beta_b} \end{aligned} \tag{15}$$

The total sigma in a plane is then just the the quadrature sum of the mode sigmas

$$\sigma^2 = \sigma_a^2 + \sigma_b^2 \,. \tag{16}$$

The normal mode betas are not simply related to the beam sizes. To remedy this one can define "projected" betas. For simplicity it will be assumed that the emittances are equal

$$\epsilon_a = \epsilon_b \equiv \epsilon \,. \tag{17}$$

The projected betas β_p are then defied by

$$\sigma^2 \equiv \epsilon \beta \,. \tag{18}$$

From Eqs. (14) and (15)

$$\begin{aligned} \beta_{px} &= \gamma^2 \beta_a + \beta_b C_{11}^2 - 2\alpha_b C_{11} C_{12} + \gamma_b C_{12}^2 \,, \\ \beta_{py} &= \gamma^2 \beta_b + \beta_a C_{22}^2 + 2\alpha_a C_{22} C_{12} + \gamma_a C_{12}^2 \,. \end{aligned} \tag{19}$$

By extension, one can also define projected alphas and gammas

$$\begin{aligned} \sigma_{xx'} &\equiv -\epsilon \, \alpha_x \,, \\ \sigma_{x'}^2 &\equiv \epsilon \, \gamma_x \,, \end{aligned} \tag{20}$$

with similar equations for the vertical plane. The equations for the projected Twiss parameters can be put in a more transparent form by writing the Twiss in matrix form (Cf. Courant and Snyder[5] Eq. (2.23))

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \,. \tag{21}$$

The projected Twiss parameters are then given by

$$\begin{aligned} \mathbf{J}_{px} &\equiv \gamma^2 \, \mathbf{J}_a + \mathbf{C} \, \mathbf{J}_b \, \mathbf{C}^+ \,, \\ \mathbf{J}_{py} &\equiv \gamma^2 \, \mathbf{J}_b + \mathbf{C}^+ \, \mathbf{J}_a \, \mathbf{C} \,. \end{aligned} \tag{22}$$

The problem with defining the projected twiss parameters in this way is that the normalization condition

$$\det \mathbf{J} = 1 \tag{23}$$

(which is equivalent to $\beta\gamma = 1 + \alpha^2$) is not necessarily satisfied. The proper normalization can be obtained by dividing all the Twiss parameters by a normalization factor obtained by taking the determinate of Eqs. (22). In this case, the emittance used in Eqs. (18) and (20) would have to be multiplied by the same factor to keep these equations satisfied. The problem here is that, in general, the emittance factor would no longer be a constant around the ring. Since a constant emittance factor is to be preferred over proper normalization it is the unnormalized definition for the Twiss parameters that is presented here.

At the IP it is desired that the beams be round. Equal $\beta_{px} = \beta_{py}$ does not guarantee a round beam. For example, the beams could be flat and tilted at a 45° angle. For this one can define a correlation function C_{xy}. Using Eqs. (11) and (12) one finds

$$\begin{aligned} C_{xy} &\equiv \frac{\langle xy \rangle}{\langle x^2 \rangle + \langle y^2 \rangle} \\ &= \frac{\gamma\left[C_{11}\beta_b - C_{22}\beta_a - C_{12}(\alpha_a - \alpha_b)\right]}{\beta_{px} + \beta_{py}} . \end{aligned} \tag{24}$$

where $\langle \cdots \rangle$ means an average over all particles in the beam and it has been assumed that $\epsilon_a = \epsilon_b$. The condition for the beams to be round is for $\beta_{px} = \beta_{py}$ and

$$|C_{xy}| \ll 1 . \tag{25}$$

Given a lattice with a matched Mobius insertion[4], and given no couplers outside the insertion, then outside the insertion region the x and y projected Twiss parameters are the equal to the normal mode Twiss parameters which in turn are equal to the Twiss parameters that would be had if the mobius section where not there. At the match points the Twiss parameters without the insertion are

$$\begin{aligned} \beta_x &= \beta_y = \beta_R \\ \alpha_x &= \alpha_y = 0 , \end{aligned} \tag{26}$$

and transfer matrix $\mathbf{T}_0$ outside the insertion is given by

$$\mathbf{T}_0 = \begin{pmatrix} \mathbf{R}(\theta_x) & 0 \\ 0 & \mathbf{R}(\theta_y) \end{pmatrix} , \tag{27}$$

where $\mathbf{R}$ is a the appropriate rotation matrix

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \beta_R \sin\theta \\ \dfrac{-\sin\theta}{\beta_R} & \cos\theta \end{pmatrix} . \tag{28}$$

The matched Mobius insertion is

$$\mathbf{T}_M = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} , \tag{29}$$

where $s = \pm 1$. With the insertion the 1–turn matrix $\mathbf{T}_1$ at the match point after after the insertion is

$$\mathbf{T}_1 = \mathbf{T}_M \, \mathbf{T}_0 = \begin{pmatrix} 0 & s\,\mathbf{R}(\theta_y) \\ \mathbf{R}(\theta_x) & 0 \end{pmatrix} . \tag{30}$$

Using Eqs. (4), (5), (6), (7), and (9) gives at this match point

$$\gamma = \frac{1}{\sqrt{2}} \tag{31}$$

$$\mathbf{C} = \gamma \, \mathbf{R} \left(\frac{\theta_y - \theta_x}{2} \right) \tag{32}$$

$$-\mathbf{A} = \mathbf{B} = \mathbf{R} \left(\frac{\theta_x + \theta_y}{2} \right) \tag{33}$$

where the following identity has been used:

$$\mathbf{R}(\theta) + \mathbf{R}(\phi) = 2 \, \cos \left(\frac{\theta - \phi}{2} \right) \, \mathbf{R} \left(\frac{\theta + \phi}{2} \right) . \tag{34}$$

At the match point the normal mode Twiss parameters with the insert are given by Eq. (33)

$$\begin{aligned} \beta_a &= \beta_b = \beta_R \\ \alpha_a &= \alpha_b = 0 \, . \end{aligned} \tag{35}$$

Using Eqs. (22) then gives the projected twiss parameters

$$\begin{aligned} \beta_{px} &= \beta_{py} = \beta_R \\ \alpha_{px} &= \alpha_{py} = 0 \, . \end{aligned} \tag{36}$$

[Notice that the projected Twiss parameters turn out to be properly normalized in this case.] Eqs. (26), (35), (36) are identical to each other. it remains to be shown that the equality remains when the Twiss parameters are propagated through the rest of the ring. The propagation here is not through the Mobius insert so the transfer matrix $\mathbf{T}_{12}$ from the match point to any point 2 is uncoupled

$$\mathbf{T}_{12} = \begin{pmatrix} \mathbf{M}_{12} & 0 \\ 0 & \mathbf{N}_{12} \end{pmatrix} . \tag{37}$$

The propagation of the normal mode betas (with or without the insert) can be derived from Sagan and Rubin[6] Eq. (17) to give

$$\begin{aligned} \gamma_2 &= \gamma_1 \\ \mathbf{J}_{a2} &= \mathbf{M}_{12} \, \mathbf{J}_{a1} \mathbf{M}_{12}^{-1} \\ \mathbf{J}_{b2} &= \mathbf{N}_{12} \, \mathbf{J}_{b1} \mathbf{N}_{12}^{-1} \\ \mathbf{C}_2 &= \mathbf{M}_{12} \, \mathbf{C}_1 \mathbf{N}_{12}^{-1} \end{aligned} \tag{38}$$

and using this with Eqs. (22) gives

$$\begin{aligned}
\mathbf{J}_{px2} &= \gamma_2^2\,\mathbf{J}_{a2} + \mathbf{C}_2\,\mathbf{J}_{b2}\,\mathbf{C}_2^+ \\
&= \gamma_1^2\,\mathbf{M}_{12}\,\mathbf{J}_{a1}\mathbf{M}_{12}^{-1} + \mathbf{M}_{12}\,\mathbf{C}_1\,\mathbf{J}_{b1}\,\mathbf{C}_1^+\mathbf{M}_{12}^{-1} \\
&= \mathbf{M}_{12}\,\mathbf{J}_{px1}\,\mathbf{M}_{12}^{-1}
\end{aligned} \tag{39}$$

With a similar equation for $\mathbf{J}_{py}$. Eqs. (38) and (39) show that the projected and normal mode Twiss parameters, along with the twiss parameters without the mobius insert, propagate exactly in the same manner and so they will remain equal everywhere outside of the insert.

4 Twiss Measurement and Analysis

In the standard phase measurement a normal mode is excited and the phase and amplitude of the signal at the beam buttons is recorded. From the button data the phase and amplitude of oscillation of the horizontal and vertical components of the motion is computed[3]

$$\begin{aligned}
x &= A_x e^{i\theta_x} \, , \\
y &= A_y e^{i\theta_y} \, ,
\end{aligned} \tag{40}$$

where, for the h mode oscillations

$$\begin{aligned}
\theta_x(n,s) &= 2\pi Q_h n + \phi_{xh}(s) + \phi_{x0} \, , \\
\theta_y(n,s) &= 2\pi Q_h n + \phi_{yh}(s) + \phi_{y0} \, ,
\end{aligned} \tag{41}$$

with similar equations for the v mode. The question is now asked as to what the quadrupole strengths and rotations must be in order to get what is observed. The simplest way to proceed would be to vary a set of "fit" quadrupole strengths and rotations within the lattice model of the ring until the calculated oscillation amplitudes and phases, as calculated from Eqs. (11) and (12), match the measured ones. On the other hand, the standard analysis (using the program CESR_FIT) matches measured and fit betatron phases and $\overline{C}$'s. The final result of the fitting essentially does not matter upon what is matched but it is easier on the user if what is matched does not depend upon the type of lattice. In order to make the coupled lattice analysis look like the standard analysis one must be able to convert from oscillation amplitudes and phases to betatron phases and $\overline{C}$'s and vice versa. This is not difficult but the presence of strong coupling and mode flips complicates things somewhat so the procedure used by CESR_FIT is outlined below.

It turns out to be simpler to use normalized $\overline{C}$'s so the x and y components of Eq. (11) are rewritten as[3]

$$\begin{aligned} x &= A_a \gamma \sqrt{\beta_a} \cos\theta_a \,, \\ y &= -A_a \sqrt{\beta_b} (\overline{C}_{22} \cos\theta_a + \overline{C}_{12} \sin\theta_a) \,. \end{aligned} \tag{42}$$

Comparing Eqs. (41) and (42) gives the phases of the horizontal and vertical motions

$$\begin{aligned} \phi_{xa} &= \phi_a \,, \\ \phi_{ya} &= \phi_a + \phi_{ya0} \,, \\ \tan\phi_{ya0} &= \frac{\overline{C}_{12}(\mathrm{th})}{-\overline{C}_{22}(\mathrm{th})} \,, \end{aligned} \tag{43}$$

where the notation is used here that $\tan\theta = c/d$ implies that $\sin\theta$ has the same sign as c and $\cos\theta$ has the same sign as d. For the b mode

$$\begin{aligned} x &= A_b \sqrt{\beta_a} (\overline{C}_{11} \cos\theta_b - \overline{C}_{12} \sin\theta_b) \,, \\ y &= A_b \gamma \sqrt{\beta_b} \cos\theta_b \,. \end{aligned} \tag{44}$$

Thus

$$\begin{aligned} \phi_{xb} &= \phi_b + \phi_{xb0} \\ \phi_{yb} &= \phi_b \\ \tan\phi_{xb0} &= \frac{\overline{C}_{12}(\mathrm{th})}{\overline{C}_{11}(\mathrm{th})} \end{aligned} \tag{45}$$

where (th) indicates the theoretical design lattice. To convert the measured A_x, A_y, ϕ_x, and ϕ_y into betatron phases and $\overline{C}$'s one compares Eq. (40) with Eq. (42) or Eq. (44). For the a mode

$$\begin{aligned} \phi_a(\mathrm{meas}) &= \frac{w_{xa}\phi_{xa} + w_{ya}(\phi_{ya} - \phi_{ya0})}{w_{xa} + w_{ya}} \\ \overline{C}_{12}(\mathrm{meas}) &= \frac{\gamma\sqrt{\beta_a(\mathrm{th})}}{\sqrt{\beta_b(\mathrm{th})}} \, \frac{A_{ya}\sin(\phi_{ya} - \phi_{xa})}{A_{xa}} \\ \overline{C}_{22}(\mathrm{meas}) &= \frac{-\gamma\sqrt{\beta_a(\mathrm{th})}}{\sqrt{\beta_b(\mathrm{th})}} \, \frac{A_{ya}\cos(\phi_{ya} - \phi_{xa})}{A_{xa}} \end{aligned} \tag{46}$$

where the values for the weights w_{xa} and w_{ya}, and the value of the phase offset ϕ_{ya} are based on the theoretical lattice

$$\begin{aligned} w_{xa} &= \gamma\sqrt{\beta_a(\mathrm{th})} \\ w_{ya} &= \sqrt{\beta_b(\mathrm{th})}\sqrt{\overline{C}_{12}^2(\mathrm{th}) + \overline{C}_{22}^2(\mathrm{th})} \end{aligned} \tag{47}$$

where the notation is used here that $\tan\theta = c/d$ implies that $\sin\theta$ has the same sign as c and $\cos\theta$ has the same sign as d. For the b mode the appropriate equations are

$$\begin{aligned}
\phi_b(\text{meas}) &= \frac{w_{yb}\,\phi_{yb} + w_{xb}(\phi_{xb} - \phi_{xb0})}{w_{xb} + w_{yb}} \\
\overline{C}_{12}(\text{meas}) &= \frac{\gamma\sqrt{\beta_b(\text{th})}}{\sqrt{\beta_a(\text{th})}} \frac{A_{xb}\sin(\phi_{xb} - \phi_{yb})}{A_{yb}} \\
\overline{C}_{11}(\text{meas}) &= \frac{\gamma\sqrt{\beta_b(\text{th})}}{\sqrt{\beta_a(\text{th})}} \frac{A_{xb}\cos(\phi_{xb} - \phi_{yb})}{A_{yb}}
\end{aligned} \tag{48}$$

with

$$\begin{aligned}
w_{xb} &= \sqrt{\beta_a(\text{th})}\sqrt{\overline{C}_{12}^2(\text{th}) + \overline{C}_{11}^2(\text{th})} \\
w_{yb} &= \gamma\sqrt{\beta_b(\text{th})}
\end{aligned} \tag{49}$$

Notice that Eqs. (46) and (48) are only approximations. Only in the limit where the actual lattice corresponds to the theoretical one are the equations exact. Since, in practice, the actual lattice is "close" to the theoretical this is not a problem.

The above equations are used by the PHASE_ANAL program as follows:

1. PHASE_ANAL reads in the two raw button data sets for the two modes.

2. By definition, if the modes are not flipped, the h mode corresponds to the a mode and the v mode corresponds to the b mode.

3. PHASE_ANAL chooses whether a data set is associated with the h or v modes of the theoretical lattice by seeing which mode has its tune closest to the measured tune of the data set.

4. PHASE_ANAL converts the raw button data to horizontal and vertical amplitude and phase values.

5. Depending upon which mode the data is associated with and whether the theoretical lattice is mode flipped or not PHASE_ANAL uses the appropriate equations to convert phase and amplitude to phase and $\overline{C}$s.

6. PHASE_ANAL writes a data file of the results. This file is used by CESR_FIT.

In CESR_FIT, the first problem to be able to associate the h and v modes of the theoretical lattice with the h and v modes of some fit lattice. This could possibly be done by matching the tunes as closely as possible but what is done in practice is to match the eigen planes. This is done because I believe it to be more robust. [I don't have a proof of this in general, but for weak coupling it is definatly true.] The procedure for eigen plane matching is as folows. The matching procedure is done at the starting point of the ring. Given $\mathbf{V}_1$ and $\mathbf{V}_2$ from two lattices we look at the determinate of the two matrices

$$\begin{aligned} \det_{aa} &= \begin{vmatrix} \gamma_1 \mathbf{I} & \mathbf{C}_1 \\ -\mathbf{C}_2^+ & \gamma_2 \mathbf{I} \end{vmatrix} \\ \det_{ab} &= \begin{vmatrix} \gamma_1 \mathbf{I} & \mathbf{C}_1 \\ \gamma_2 \mathbf{I} & \mathbf{C}_2 \end{vmatrix} \end{aligned} \tag{50}$$

If $|\det_{aa}| > |\det_{ab}|$ then the modes are flipped relative to one another.

at a given detector, the comparison of the measured phase and $\overline{C}$'s with the fit ones is straight forward except if the fit decomposition is mode flipped and the theoretical decomposition is not (or vice versa). In this case one must compute new "flip fit" phases and $\overline{C}$'s. This is done by requiring that the a mode motion with the flip fit look like the b mode motion with the original fit. Equating Eqs. (42) with (44) then gives

$$\begin{aligned} \phi_a(\text{flip fit}) &= \phi_b(\text{fit}) + \tan^{-1} \frac{\overline{C}_{12}(\text{fit})}{\overline{C}_{11}(\text{fit})} \\ \phi_b(\text{flip fit}) &= \phi_a(\text{fit}) + \tan^{-1} \frac{\overline{C}_{12}(\text{fit})}{-\overline{C}_{22}(\text{fit})} \\ \overline{C}_{11}(\text{flip fit}) &= -\gamma(\text{fit}) \sqrt{\frac{\beta_a(\text{fit})}{\beta_a(\text{th})}} \frac{\overline{C}_{12}(\text{fit})}{\sqrt{\overline{C}_{22}^2(\text{fit}) + \overline{C}_{12}^2(\text{fit})}} \\ \overline{C}_{12}(\text{flip fit}) &= -\gamma(\text{fit}) \sqrt{\frac{\beta_a(\text{fit})}{\beta_a(\text{th})}} \frac{\overline{C}_{12}(\text{fit})}{\sqrt{\overline{C}_{22}^2(\text{fit}) + \overline{C}_{12}^2(\text{fit})}} \\ &= -\gamma(\text{fit}) \sqrt{\frac{\beta_b(\text{fit})}{\beta_b(\text{th})}} \frac{\overline{C}_{11}(\text{fit})}{\sqrt{\overline{C}_{11}^2(\text{fit}) + \overline{C}_{12}^2(\text{fit})}} \\ \overline{C}_{22}(\text{flip fit}) &= -\gamma(\text{fit}) \sqrt{\frac{\beta_b(\text{fit})}{\beta_b(\text{th})}} \frac{\overline{C}_{11}(\text{fit})}{\sqrt{\overline{C}_{11}^2(\text{fit}) + \overline{C}_{12}^2(\text{fit})}} \end{aligned} \tag{51}$$

Again Eqs. (51) are only exact in the limit where the fit equals the theoretical design. Note that the two equations for $\overline{C}_{12}$(flip fit) in Eq. (51) correspond to the two equations for $\overline{C}_{12}$(meas) given in Eqs. (46) and (48). As an alternative to Eqs. (51) the measured phases and $\overline{C}$'s could have been flipped to correspond to the fit state. The advantage with what is actually done is that one can then directly compare the fit with the theory without any more flipping.

A How to mode flip

Given one decomposition solution to Eq. (4) with $\gamma < 1$ the other "flipped" solution is related to the first via

$$
\begin{aligned}
\gamma_f &= \sqrt{1-\gamma^2} \\
\mathbf{C}_f &= \frac{-\gamma}{\gamma_f}\,\mathbf{C} \\
\mathbf{U}_f &= \mathbf{W}\,\mathbf{U}\,\mathbf{W}^{-1}
\end{aligned}
\tag{52}
$$

where

$$
\mathbf{W} = \begin{pmatrix} 0 & \frac{\mathbf{C}}{\gamma_f} \\ \frac{-\mathbf{C}^+}{\gamma_f} & 0 \end{pmatrix}
\tag{53}
$$

Which gives for the $x \times 2$ eigenmatrices

$$
\begin{aligned}
\mathbf{A}_f &= \frac{1}{\gamma_f^2}\mathbf{C}\,\mathbf{B}\,\mathbf{C}^+ \\
\mathbf{B}_f &= \frac{1}{\gamma_f^2}\mathbf{C}^+\,\mathbf{A}\,\mathbf{C}
\end{aligned}
\tag{54}
$$

References

[1] D. Edwards and L. Teng, "Parameterization of Linear coupled Motion in Periodic Systems", IEEE trans. Nuc. Sci, Vol. NS 20, **3**, June 1973.

[2] M. Billing, "The Theory of Weakly, Coupled Transverse Motion in Storage Rings," Cornell CBN 85–2, (1985).

[3] P. Bagley and D. Rubin, "Correction of Transverse Coupling in a Storage Ring," Proc. 1989 Part. Acc. Conf. (San Francisco) p. 874, (1989).

[4] R. Talman, "The Mobius–Twisted Accelerator," Cornell CBN 94-1 (1994).

[5] E. D. Courant and H. S. Snyder "Theory of Alternating–Gradient Synchrotron," Ann. Physics, **3**, p. 1-48. (1958).

[6] D. Sagan and D. Rubin, "Propagation of Twiss and Coupling Parameters," Cornell CBN 96-20, 1996.