Approximations for large angle synchrotron radiation

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Introduction

A large-angle beamstrahlung detector at CESR appears feasible[1] except for the unknown synchrotron radiation (SR) emitted at large ($\gamma\theta >> 1$) angle by the beam line magnets, most notably the quadrupoles adjacent to the CLEO IR. The purpose of this note is to describe a simple algorithm to extend the validity of the classical SR approach to angles far exceeding the typical angle $\theta = 1/\gamma$.

The beamstrahlung detector could monitor instantaneously the beam-beam interaction. By extracting information on the beam-beam overlap, it could drive a feedback system that maintains optimal beam-beam overlap. Gains in integrated luminosity of 10-20% at CESR could conceivably be had, with larger gains obtainable if the device can allow detailed studies of the beam-beam limit and further machine optimization[1]. Larger gains can be had at two-ring machines, such as PEP II, where machine optimization is far more complex.

The problems tackled by this note are the following two. First, large angle radiation involves a coherent Fourier transform of the magnetic field of the entire magnet. Our current algorithms work by propagating a beam in small steps through magnetic fields, and adding the radiation from each step incoherently. Thus a method must be developed to simulate coherent radiation as the sum of many incoherent steps. Second, a cutoff in phase space must be established, below which the classical approximation is valid, and above which new formulae will be used.

The method presented here is valid for one electron only. The complications of predicting the exact beam shape and trajectory through various magnets are not tackled. The formulae here need to be convoluted with beam shape etc. to arrive at meaningful conclusions. This can be done either via Monte Carlo or numerically.

In CGS units, the classical[2] spectrum is as follows:

$$\frac{d^2U}{d\omega d\Omega} = \frac{3e^2\gamma^{10}}{\pi^2 c} t^2 \left(\frac{K^2_{2/3}(t)}{(1+u^2)} + \frac{u^2 K^2_{1/3}(t)}{(1+u^2)^2}\right) \tag{1}$$

where ρ is the radius of curvature, equal to

$$\rho = \frac{\gamma mc^2}{eB},$$

and

$$t = \frac{\omega\rho}{3c}(1/\gamma^2 + \theta^2)^{3/2}, \quad u = \gamma\theta.$$

Other quantities of interest are the radiated power and the energy loss per unit length. They are defined as follows

$$P = \frac{2e^2\gamma^4 c}{3\rho^2}, \quad \frac{dU}{dx} = \frac{P}{c}.$$

When the observation angle is much larger than the deflection angle, the short magnet approximation[3] is valid. In the short magnet approximation, the length scale ρ is no longer the only relevant scale. The length of the magnet L enters into the expressions, and is typically very different from the radius of curvature. This will make matters more complicated. On the other hand, the angle being so large it is fixed as a constant and taken out of the Fourier transform, as described below.

Classical radiation approximation.

A simpler derivation of the method is afforded if one already decides that the cutoff will be at angles much larger than $1/\gamma$, which vastly simplifies the equations, and then proves post facto that it is so.

Unfortunately, the choice of angles in Ref.[2] is non-standard, θ being the latitude instead of the zenith. θ can vary from $+\pi/2$ to $-\pi/2$, so that a factor of two is to be added if θ is to be a positive-defined quantity (the convention chosen in this note). Integrating over ϕ yields 2π, and the solid angle factor reduces to $4\pi d\theta$.

Using known approximations for the modified Bessel functions[4], Eq.(1) becomes at large angle ($u >> 1$)

$$d^2U \sim \frac{4e^2}{c}(\frac{\omega\rho\theta}{c}) \exp{(-2\omega\rho\theta^3/3c)} d\omega d\theta. \qquad (2)$$

Substituting for the total radiated power, the final double-differential distribution in (t, u) is obtained,

$$d^2P = \frac{27}{\pi^2} Pt^2 (\frac{K^2_{2/3}(t)}{(1+u^2)} + \frac{u^2 K^2_{1/3}(t)}{(1+u^2)^2}) du dt. \qquad (3)$$

Short magnet approximation.

Although the original paper is Ref.[3], I use Eq.(16) of Ref.[5], where several misprints have been fixed (there is still a factor of two missing in that equation).

The general form for short magnet radiation is

$$\frac{d^2U}{d\omega d\Omega} = \frac{2e^2\gamma^2 f}{\pi m^2 c^5} F^2(k),$$

where

$$k = \frac{\omega(1+\gamma^2\theta^2)}{2c\gamma^2}.$$

F^2 is the squared Fourier transform of the bending force, integrated along the longitudinal coordinate z. f is the angular part, factorizing out of the Fourier transform. Its complete form, for the unpolarized case, is

$$f = \frac{1 + \gamma^4\theta^4 - 2\gamma^2\theta^2 \cos 2\phi}{(1+\gamma^2\theta^2)^4}. \qquad (4)$$

The magnetic field shape along the beam line is assumed to be of the form

$$B(z) = B_0 g(z),$$

where $g(z)$ is a dimensionless function, describing the magnetic field profile, and ρ_0 the curvature radius corresponding to B_0. The Fourier transform of g is defined as G. Substituting for eB_0, the following expression is obtained

$$\frac{d^2U}{d\omega d\Omega} = \frac{2e^2 f \gamma^4}{\pi c \rho_0^2} G^2. \qquad (5)$$

In the short magnet approximation, angles are defined as usual, so that

$$d\Omega \sim \theta d\theta d\phi.$$

Dipole magnet.

In the case of a dipole, $g(z) = 1$ inside the dipole length L, and zero elsewhere. The modified Fourier transform, squared, has the form

$$G^2 = \frac{8 \sin^2 kL/2}{\pi k^2},$$

a familiar result.

The oscillatory term G^2 oscillates at least 10^2 times for visible frequencies ($\omega \sim 10^{16} sec^{-1}$), a curvature radius of $10^2 - 10^3$ meters, angles in excess of 1 mrad, $\gamma \sim 10^4$ and $L \sim 1$ meter. Thus it can be averaged in the large angle approximation to give

$$G^2 \sim \frac{4}{\pi k^2}.$$

This is important because the length scale disappears from the formulae. Integrating over the azimuth, the large-angle approximation for short magnet radiation now becomes

$$d^2U \sim \frac{32e^2c}{\pi\rho^2}\frac{1}{\omega^2\theta^7}d\omega d\theta. \tag{6}$$

The steps of the previos section are repeated, but this time $t' = kL$. Upon substituting for P, integrating over ϕ, and using the (t', u) variables of the previous section, the result is (large angle approximation)

$$d^2P(dipole) = \frac{24P}{\pi}\frac{\sin^2 t'/2}{t'^2}\frac{1+u^4}{(1+u^2)^5}dudt' \tag{7}$$

Algorithm.

The fact that the length scale L disappeared from the problem means that smaller sections of the magnet add up incoherently. This affords vast simplifications. First, Eqs.(2) and (6) can now be compared directly to give a cutoff which is independent of L, and second, the current algorithm that propagates the beam step-by-step does not need to be encumbered with a Fourier transform taken over the whole magnetic field (I correct below for fringe effects). Indeed, a magnet, such as a quadrupole, that presents a variable magnetic field to the beam, can be broken into many small dipoles, which is the method used now.

Thus the final algorithm is simply to replace the current step-by-step calculation of the field with one that computes the angle between the source and the detector, and uses the classical or short-magnet approximation depending on cutoff.

Setting Equations (2) and (6) to be equal the cutoff condition is obtained:

$$K = \frac{27\pi}{8\theta} t^3 e^{-2t} = 1,$$

which is the main result of this note. When K is greater than one, the classical approximation is used, otherwise the short magnet approximation is used. For visible frequencies, a curvature radius of 100 meters, and $\gamma = 10^4$, the condition is satisfied at $\theta \sim 2.5$mrad, which is 25 times larger that $1/\gamma$, thus proving that the large angle approximation was valid throughout.

Fringe effects and quadrupole approximation.

At large angle, the magnet is seen as a whole by the observer, therefore all length scales associated with the magnet have to be studied. The exact fall-off of the magnetic field around the magnet ends affects dramatically the high-part of the Fourier transform. If the fall-off is very sudden, there will be no exponential cutoff in frequency. If the fall-off is gradual, there will be an exponential cutoff. The following simple modeling of edge effects provides an answer.

The step function $g(z)$ of the previous sections is now replaced by two error functions

$$g(z) = \frac{1}{2}(erf(\frac{L+2z}{\sigma}) + erf(\frac{L-2z}{\sigma})),$$

where L is again the length of the magnet and σ the length over which the magnetic field falls off (meters and centimeters are their typical sizes). Integrating by parts gives (large angle approximation)

$$G^2 = G^2_{dipole} \exp{(-k^2\sigma^2)}. \tag{8}$$

The exponential correction is small, and does not affect appreciably the total radiated power, but becomes eventually large at angles of order a few mrad (indeed it should. Both the beam length and the edges of the magnet are a few centimeters. Thus they should be experiencing a fall-off at similar angles). This correction is potentially important, if the radiation at the chosen angle is seen to come predominantly from the center of the beam. It is not important if radiation is seen coming predominantly from the tails.

In a quadrupole, the magnetic field seen by a particle is (assuming the for simplicity that the quadrupole extends from 0 to L)

$$g(z) = (1 - \sin(\beta z/c)),$$

with β being the oscillation frequency about the axis of the quadrupole. The Fourier transform of this object is very simple, if $\beta << \omega$:

$$G^2 = G^2_{dipole} \cos^2(\frac{\beta L}{2c}).$$

Notice the independence of the correction on ω. If the $\sin(\beta z/c)$ term is not too large, which is true in most practical cases, then the step-propagation through the magnet will average the radiated power in such a way that to first and second order in $\beta L/2c$ the result will be the same to the one obtained above. Therefore, a program with step propagation of particles computes this correction to high accuracy. In conclusion the final algorithm to be used in the short magnet region is

$$d^2P = \frac{24P}{\pi} \frac{\sin^2 t'/2}{t'^2} \frac{1+u^4}{(1+u^2)^5} \exp(-t'^2\sigma^2/L^2) du dt'. \tag{9}$$

Assuming that the magnetic field fringe is of order 1 centimeter, the exponential factor is close to one at around 2 mrad and for optical frequencies, and therefore the previous derivation of the cutoff is still valid.

Polarization.

At these angles, the classical approximation clearly fails badly to reproduce the correct polarization, in particular, its azimuthal dependence[3]. I believe we can postpone this problem until we learn more about quadrupole radiation. The problem is also somewhat secondary, our primary interest is the total level of backgrounds. The obvious solution is to model all of the polarization, inside and outside of the cutoff, according to the short magnet approximation, using the f_π and f_σ polarization factors described in Ref.[5].

Conclusions.

I was surprised that the algorythmic modifications to the code, to implement short magnet radiation, could be so simple. Choosing a spectrum (Eq. 3) or another (Eq. 9) based on a simple cutoff decision will suffice. While the magnet length disappears from the solution, the magnetic field fall-off length at the magnet's edges cannot be made to disappear. The model extends its accuracy to very large angle by implementing fall-off corrections, which become important only far above the cutoff. Corrections for the exact type of magnet being studied are not necessary.

An algorythm to sum up radiation in the detector was not developed. There are no particular problems in writing one. The choice of fringe fall-off, σ, also needs to be made to complete the algorythm, but is generally available from magnetic field measurements.

I notice in passing that, if the backgrounds at the chosen location for the beamstrahlung detector (6 mrad) are seen to be coming mainly from the beam core, they can be suppressed by several orders of magnitude by simply tapering off the magnetic field at the ends of the quadrupoles.

References

[1] G. Bonvicini and J. Welch, CLNS-97-1523, to be published in Nuclear Instrument and Methods A.

[2] J. D. Jackson, "Classical Electrodynamics", Chapter 14.

[3] R. Coisson, Phys. Rev. A 20, 524, 1979. There are some misprints in the formulae, and the Fourier transform as defined in the paper differs from the common one by a factor of $(2\pi)^{-1/2}$.

[4] J. D. Jackson, "Classical Electrodynamics", Chapter 3.

[5] M. Bassetti *et al.*, IEEE Trans. Nucl. Science 30: 2182, 1983.