CBN 98-1
Developable constant perimeter surfaces: Application to the end design of a tape-wound quadrupole saddle coil

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1. Introduction

Constant perimeter surfaces are generally thought to be desirable for the winding surfaces for saddle coils for superconducting dipoles or quadrupoles. Winding multiple layers on such surfaces is facilitated, since the winding of each layer can proceed in either direction without any tendency of the turns to slip to a "shorter" path due to the winding tension.

A surface is said to be "ruled" if it is generated by moving a straight line continuously in space. A "developable" ruled surface is a surface that can be rolled on a plane, touching along the entire surface as it rolls. Such a surface has a constant tangent plane for the whole length of each ruling. Parallel geodesic loops (in a direction perpendicular to the rulings) on closed developable ruled surfaces all have the same length; such surfaces are thus "constant perimeter" surfaces.

Winding of tape conductors on such developable constant perimeter surfaces should provide for minimum strain to the tape. The tangent plane of the developable surface is fixed, so the normal to this, the direction of curvature of the surface, is constant over the whole width of the tape. Consequently, all the bending involved is in the "easy" direction (i.e., normal to the surface of the tape). The developable surface (the tape) can be unrolled to a flat surface without twisting or distortion.

2. Construction of a developable constant perimeter surface

A developable constant perimeter surface may be constructed for any given space curve $\vec{r}$ by following the prescription outlined in reference 1. The basic idea is the following. We define a space curve $\vec{r}(\beta)$ from which we will develop the constant perimeter surface. In the case of a tape conductor winding, this curve would correspond to the path followed by one edge of the tape. At each point along the space curve, we define a triplet of orthogonal unit vectors (tangent, normal, and binormal), using the following relations:

$$\vec{t} = \vec{r}', \quad \vec{n} = \frac{\vec{r}''}{k}, \quad \vec{b} = \vec{t} \times \vec{n} \tag{1}$$

in which

$$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\beta} \Big/ \frac{ds}{d\beta}, \quad \vec{r}'' = \frac{d^2\vec{r}}{ds^2} \tag{2}$$

and

$$\frac{ds}{d\beta} = \sqrt{\left[\frac{dx}{d\beta}\right]^2 + \left[\frac{dy}{d\beta}\right]^2 + \left[\frac{dz}{d\beta}\right]^2} \tag{3}$$

The curvature is

$$k = \sqrt{\left[\frac{dx}{ds}\right]^2 + \left[\frac{dy}{ds}\right]^2 + \left[\frac{dz}{ds}\right]^2} \tag{4}$$

As shown in reference 1, the equation of a ruled, developable constant perimeter surface, generated from the space curve $\vec{r}(\beta)$, is

$$\vec{\Omega}(\beta,v) = \vec{r}(\beta) + v\vec{p}(\beta) \tag{5}$$

in which

$$\vec{p} = \vec{b} + \kappa\vec{t} \tag{6}$$

and κ is the ratio of the space curve torsion τ to its curvature k:

$$\kappa = \frac{\tau}{k}; \quad \vec{b}' = -\tau\vec{n} \tag{7}$$

The torsion τ is obtained from last equation.

For the application to a quadrupole saddle coil, we define the reference space curve $\vec{r}(\beta)$ for one edge of the tape, which establishes the geometry of the coil end.

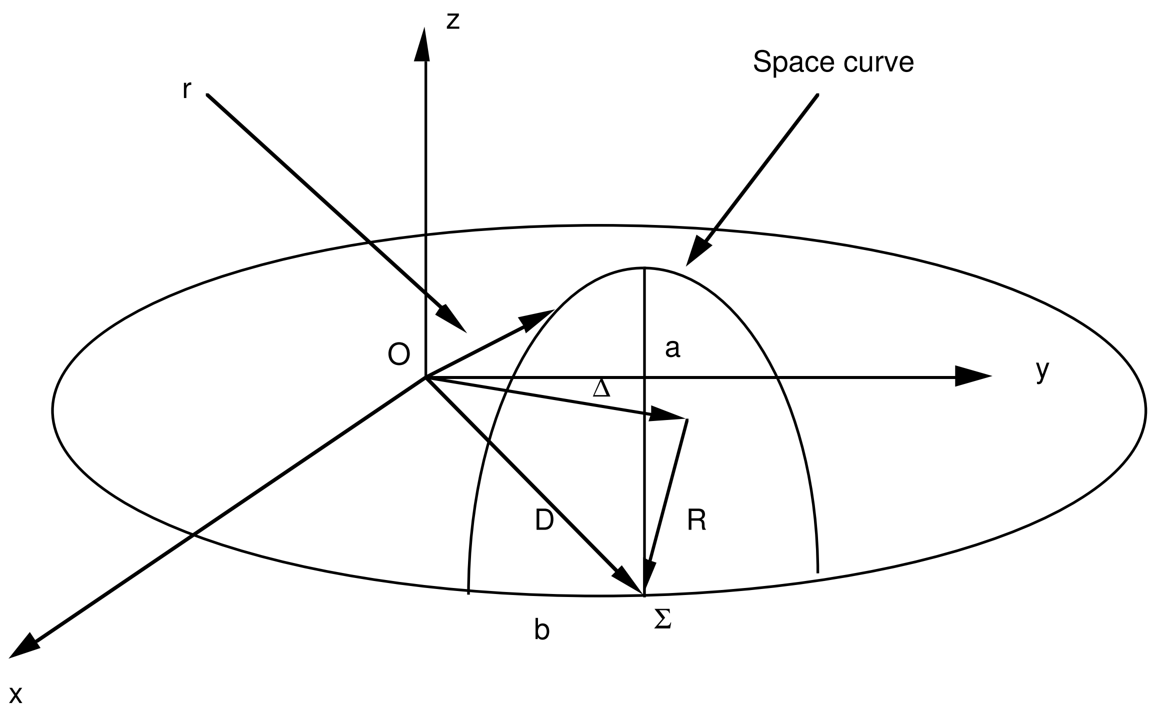

Fig. 1
Illustrating the space curve corresponding to one edge of the tape

The space curve is taken to be an ellipse, with major and minor axes a and b, on the surface of a cylinder of radius R. The cylinder axis is parallel to the z-axis, and its center is located at the head of the vector $\vec{\Delta}$ in the x-y plane. This is illustrated in Fig. 1. Fig. 2 shows the space curve on the cylinder's surface.

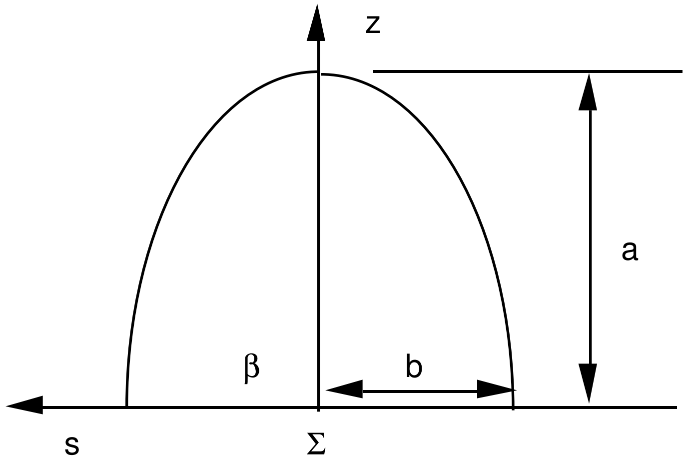

Fig. 2
The space curve on the cylindrical surface

The coordinates of the space curve on the cylinder are given in terms of the parametric angle β by

$$s = b\cos\beta$$
$$z = a\sin\beta \tag{8}$$

in which s is the arc length on the cylinder.

Fig. 3 shows the projection of the space curve onto the x-y plane. The strategy in specifying the space curve is the following. For a space curve defined as an ellipse on a fixed radius (R) cylinder, centered on the origin, we would have $\vec{\Delta}$ =0 and D=R in Fig. 3. In order to accommodate coil blocks in which the winding surface in the straight part of the coil is parallel to the x-axis, rather than radial (as in a "keystoned" coil), we allow the center of the cylinder to move as the space curve transits around the ellipse; that is, we allow the vector $\vec{\Delta}$ to be a function of β. The vector $\vec{\Delta}(\beta)$ is parameterized as

$$\vec{\Delta}(\beta) = \Delta\left(\hat{i}\,sin^2\frac{\beta}{2} + \hat{j}\,cos^2\frac{\beta}{2}\right) \tag{9}$$

To insure quadrupole symmetry, the radius vector of the reference cylinder must always point from the head of $\vec{\Delta}$ to the point Σ, which lies on the intersection of the cylinder with the 45° line. Hence we have the constraint

$$\vec{D} = \vec{R} + \vec{\Delta}(\beta) \tag{10}$$

where

$$\vec{D} = d(\beta)\left(\hat{i} + \hat{j}\right) \tag{11}$$

is a vector whose direction lies along the 45° line. The quantity d(β) is given by solving Eqs. (9), (10), and (11) simultaneously; the result is

$$d(\beta) = \frac{\Delta \cos\beta}{2} + \frac{\sqrt{2}}{4}\sqrt{4R^2 - \Delta^2\left(1 + \cos[2\beta]\right)} \tag{12}$$

In Fig.3, the vector $\vec{\rho}$ is the projection of $\vec{r}$ onto the x-y plane. From the geometry of Fig. 3, we have

$$\vec{\rho} = \vec{v} + \vec{\Delta} \tag{13}$$

in which $\vec{v}$ is a vector of length R.

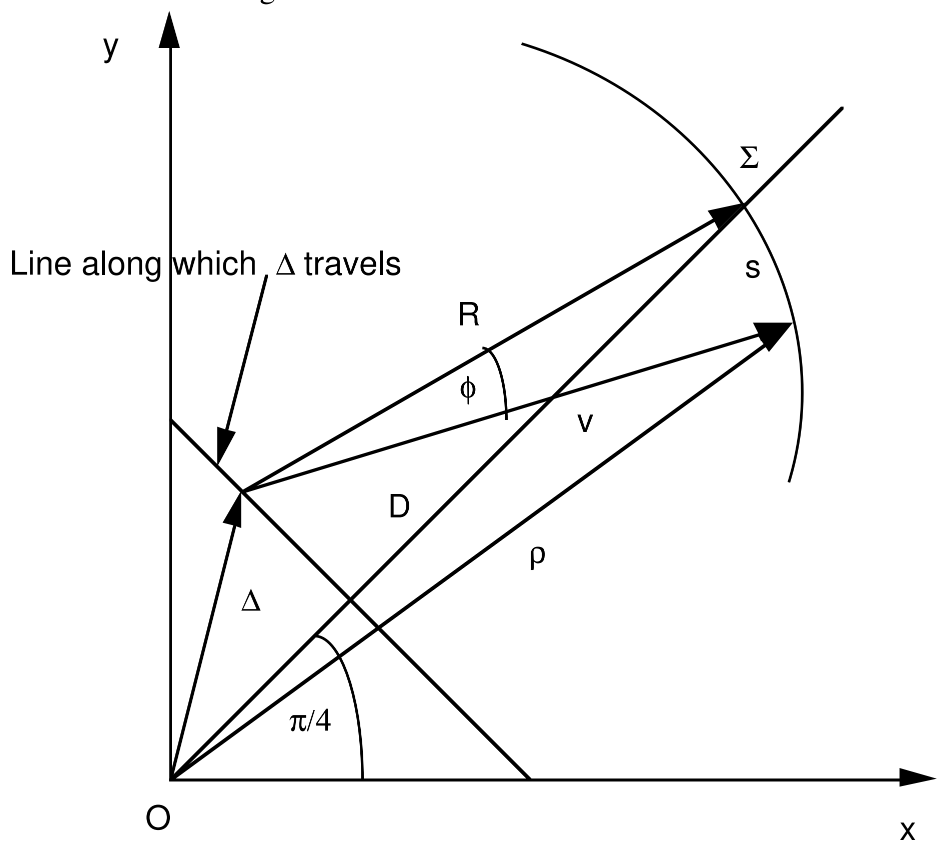

Fig. 3
Projection of the space curve onto the x-y plane, and associated defining geometry

The relation between $\vec{R}$, $\vec{v}$, and the angle $\phi = \frac{s}{R}$ is illustrated in Fig. 4

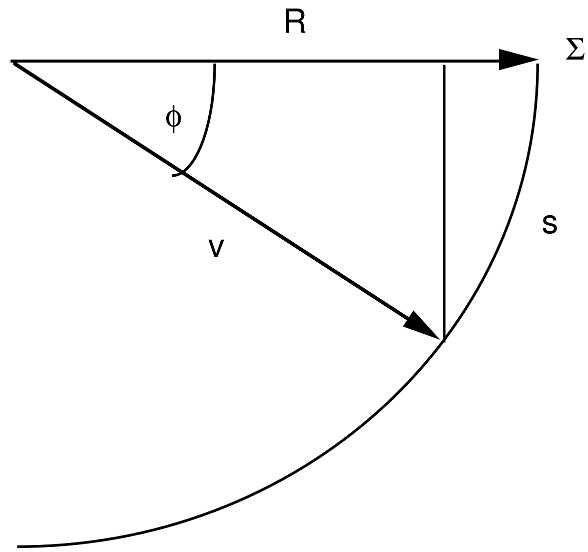

Fig. 4

Relation between $\vec{v}$ and $\vec{R}$

From Fig. 4, we have

$$\vec{v} = \vec{R}cos\,\phi - (\hat{k} \times \vec{R})\,sin\,\phi \tag{14}$$

So, Eq. (13) becomes

$$\vec{\rho} = \vec{R}cos\frac{s}{R} - (\hat{k} \times \vec{R})\,sin\frac{s}{R} + \vec{\Delta} \tag{15}$$

The vector $\vec{R}$ is given in Eq. (10). Substituting into Eq. (15) gives

$$\vec{\rho}(\beta) = \left(\vec{D} - \vec{\Delta}(\beta)\right)cos\left[\frac{b\,cos\,\beta}{R}\right] - \left(\hat{k} \times \left(\vec{D} - \vec{\Delta}(\beta)\right)\right)sin\left[\frac{b\,cos\,\beta}{R}\right] + \vec{\Delta}(\beta) \tag{16}$$

Eqs. (9), (11), (12) and (16) give the complete specification of the space curve in terms of the ellipse parameters a and b, the cylinder radius R, and the offset Δ.

The values for a, b, R and Δ are determined by the geometry of the coil's straight section. Fig. 5 shows the x-y plane projection at β=0, the point where the end meets the straight section. The rectangular block, a distance x_0 from the origin, and of height y_0, represents the cross section of the tape conductor coil block in the coil's straight section. The tape width is T.

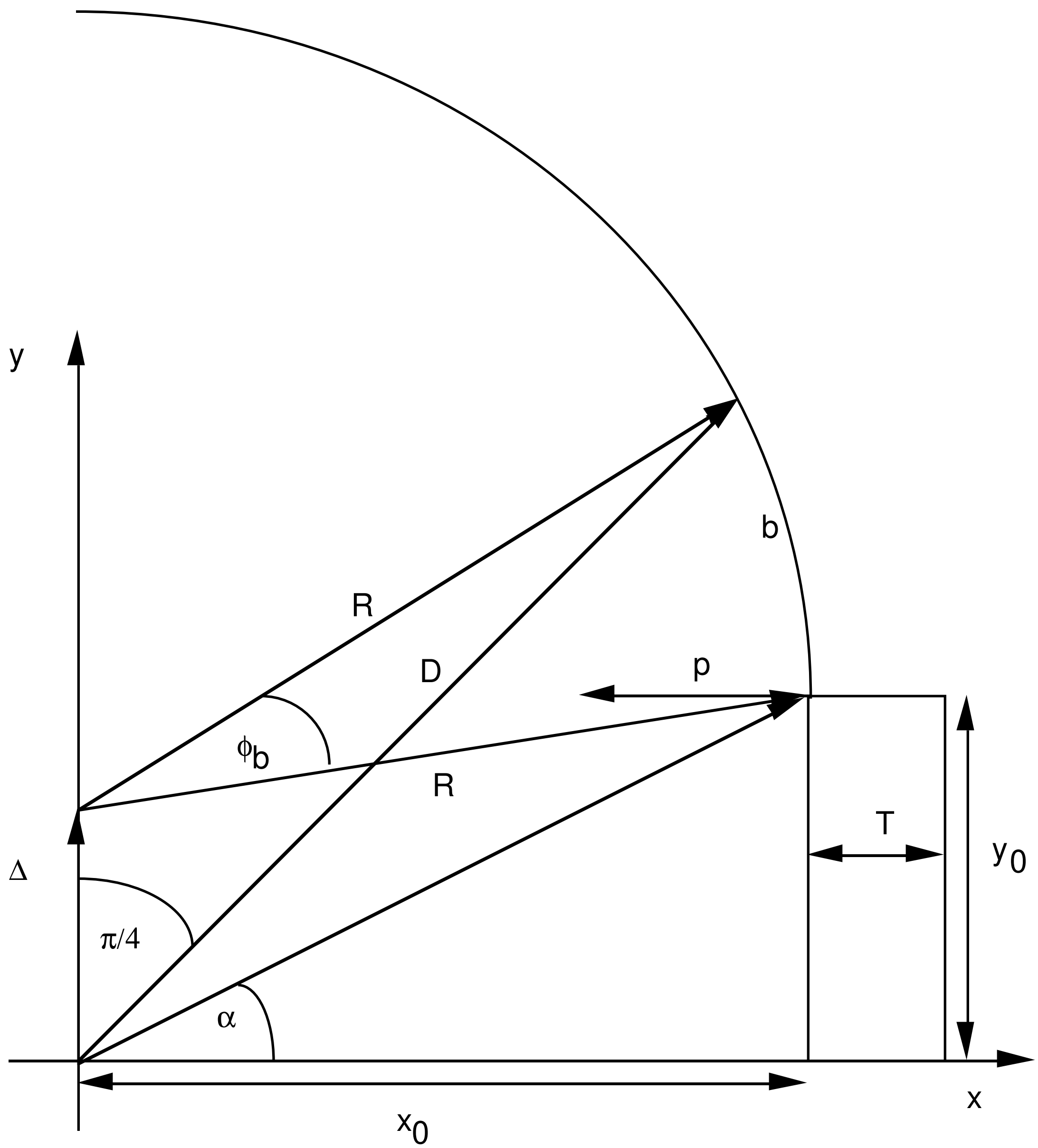

Fig. 5
Projection on the x-y plane and β=0;
geometry to define R, Δ, and b.

As discussed above, the instantaneous direction of the developable constant perimeter surface is given by the vector $\vec{p}$ defined in Eq. (6). We require that, at the points at which the space curve describing the end of the coil meets the straight section (at $\beta=0$, and at $\beta=\pi$), the direction of the winding surface must be the same as that in the straight section (i.e., it must only have a component in the x-direction, as shown in Fig. 5). Since the tangent vector $\vec{t}$ points along the z-axis at this point, we require that $\kappa(0)=0$, and that $\vec{b}$ must only have an x-component at $\beta=0$.

When $z(\beta)$ has the form given in Eq. (8), the requirement that $\kappa(0)=0$ is satisfied if

$$\left.\frac{d\vec{\rho}(\beta)}{d\beta}\right|_{\beta=0} = 0 \;\; and \;\; \left.\frac{d^3\vec{\rho}(\beta)}{d\beta^3}\right|_{\beta=0} = 0 \tag{17}$$

This condition is satisfied for all even functions of β with continuous derivatives. Since $\vec{\rho}(\beta)$ as given in Eq. (16) is such a function, this requirement is satisfied by construction.

The requirement that $\vec{b}$ have only an x-component provides the following constraint:

$$\left(\Delta^2 R + b\left[\Delta^2 - 2R^2\right]\right)cos\,\phi_b + \left(\Delta^2 R - \Delta^2 b + 2bR^2\right)sin\,\phi_b +$$
$$\Delta\sqrt{2R^2 - \Delta^2}\left(R + [b-R]cos\,\phi_b + [b+R]sin\,\phi_b\right) = 0 \tag{18}$$

in which

$$\phi_b = \frac{b}{R} \tag{19}$$

From the geometry of Fig. 5, we have the additional relations:

$$D^2 = 2d(0)^2 = \Delta^2 + R^2 + 2\Delta R\,cos\left(\frac{\pi}{2} - \phi_b - cos^{-1}\frac{x_0}{R}\right)$$
$$= \Delta^2 + R^2 + 2\Delta x_0\,sin\,\phi_b + 2\Delta\left(y_0 - \Delta\right)cos\,\phi_b \tag{20}$$

in which, from Eq. (5),

$$d(0) = \frac{\Delta}{2} + \frac{\sqrt{2R^2 - \Delta^2}}{2} \tag{21}$$

and

$$R^2 = x_0^2 + (y_0 - \Delta)^2 \tag{22}$$

Solving (18), (20) and (22) simultaneously gives b, R and Δ. We are still free to choose ε, the ellipticity of the ellipse describing the end curve, to fix a=εb.

3. Application to the end design of a tape-wound quadrupole saddle coil

The parameters for the developable constant perimeter surface for such a coil are given in Table 1.

Coil parameter	**Value**
x_0	35 mm
y_0	15 mm
T	3 mm
ε	1.5
Δ	8.14 mm
R	35.67 mm
b	15.33 mm
ϕ_b	0.4298 rad

Table 1: Parameters for the constant perimeter surface of a tape-wound quadrupole saddle coil

Fig. 6 shows the projection of the space curve, from which the winding surface is developed, in the x-y plane (the upper curve). The space curve projection is indicated. The rectangular blocks represent the tape coil blocks in the straight section of the coil, the curve above the space curve is the projection of the upper edge of the tape as it goes around the coil end. The quarter-circle is a 35 mm radius arc.

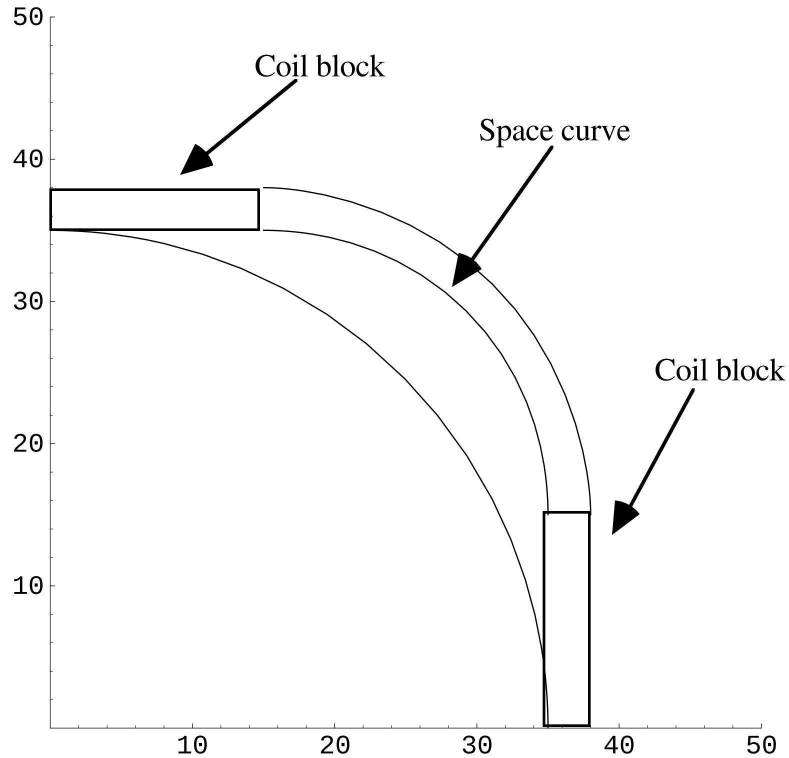

Fig. 6

Projection of the space curve and elements of the tape onto the x-y plane.

Fig. 7 shows the space curve in three dimensions, with the triplet of orthogonal unit vectors shown at $\beta=0$, $\pi/2$, and π.

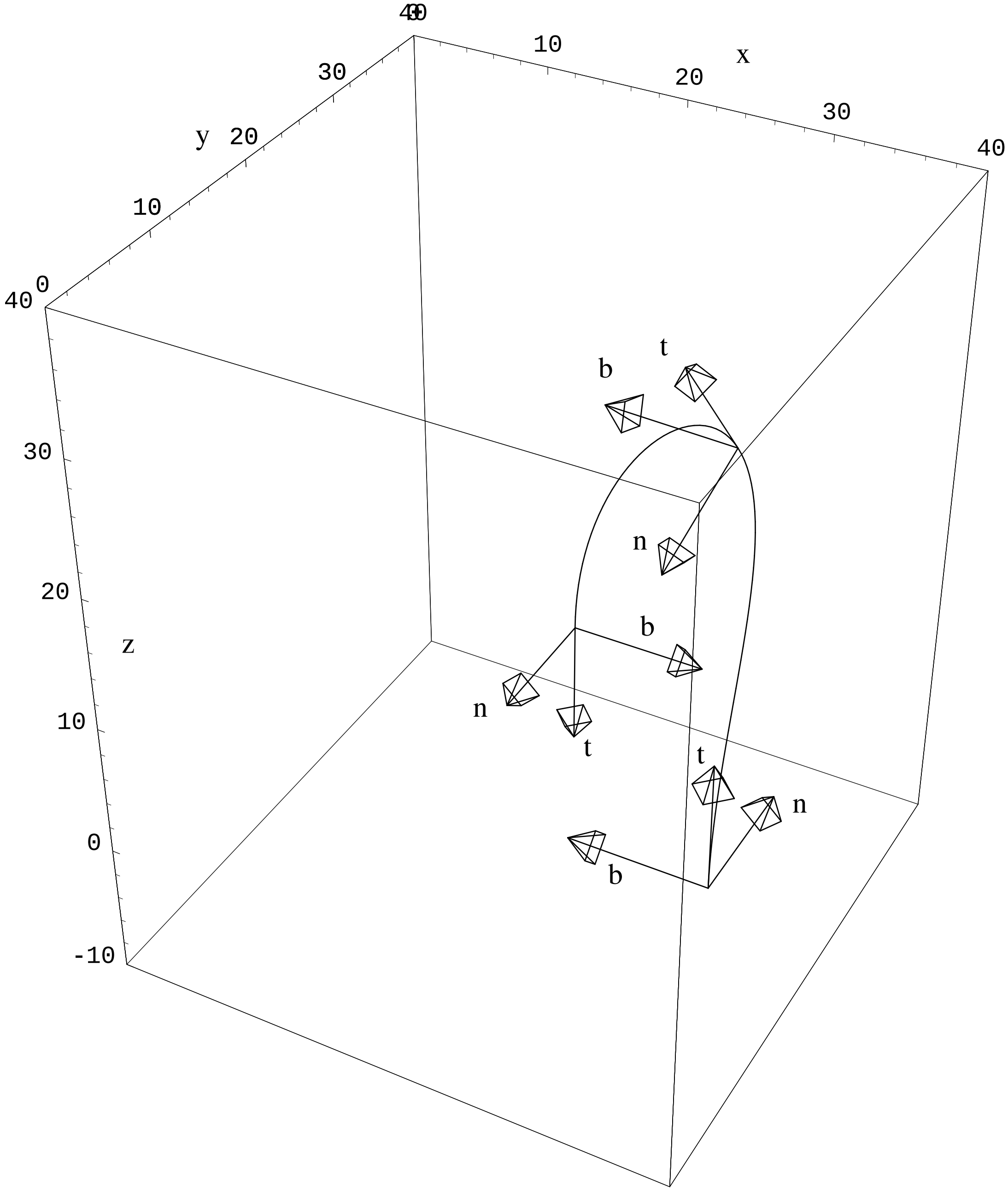

Fig. 7
The space curve in three dimensions, with the triplet of orthogonal unit vectors shown at β=0, π/2, and π.

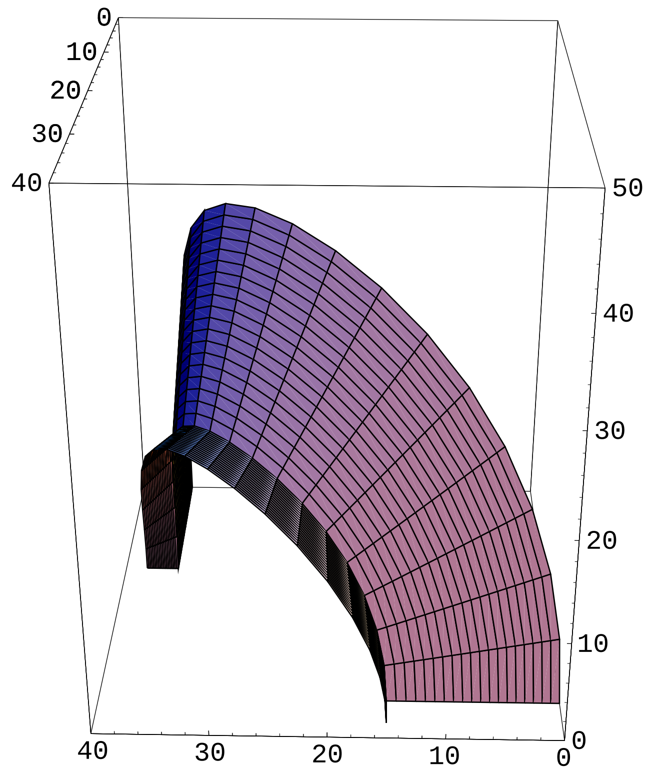

Fig. 8
Constant perimeter tape winding surface (thin strip) and the tape edge guiding surface

Fig. 8 shows the constant perimeter surface (the thin strip), as well as a surface generated from the space curves by allowing the angle α in Fig. 5 to vary from 0 to $tan^{-1}\frac{y_0}{x_0}$. This surface represents the winding surface which guides the edges of the tape elements comprising the coil block.

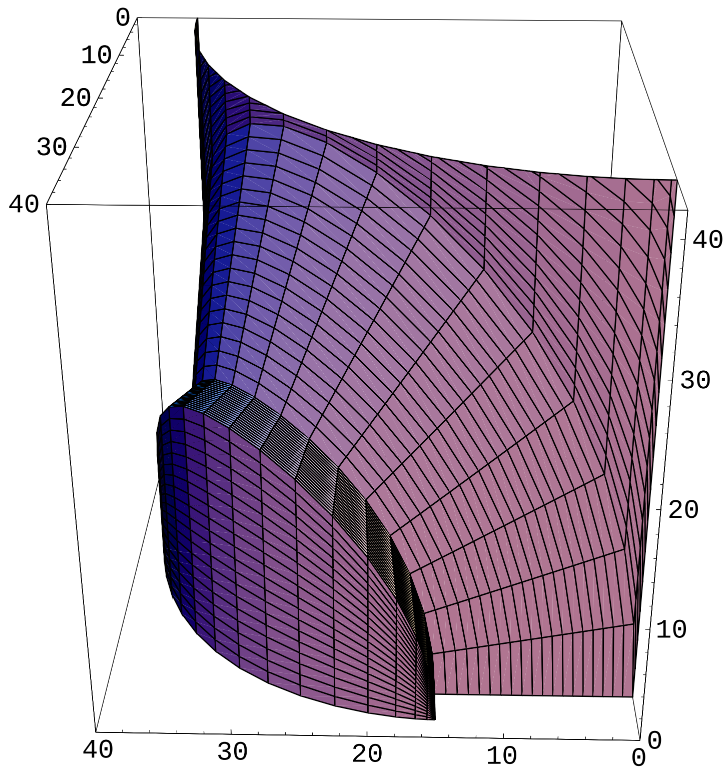

Fig. 9
End surface of the coil winding form

Fig. 9 shows the complete end surface for the coil winding form. The bottom surface is extended in z using a cylinder of 35 mm radius, centered on the origin. The top surface is formed by extruding the upper edge of the constant perimeter surface in z. This surface is joined to a straight section. The surface shown in Fig. 9 is duplicated at the other end of the straight section, and the whole forms the complete coil winding form for one saddle coil of the quadrupole.

4. Acknowledgments

I'd like to thank Jim Welch for helpful suggestions, for uncovering reference 1, which forms the basis of this work, and for translating the surfaces generated by Mathematica into CAD files.

5. Reference

1. T. C. Randle, "A Few Mathematical Notes on Constant Perimeter Layers in Coil Windings", Rutherford Lab internal note RL-74-165 (Dec., 1974)