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Propagation Of Twiss and Coupling Parameters

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1 Introduction

The linear properties of the transverse phase space are described by the normal mode Twiss parameters β_i and α_i, the betatron phase ϕ_i, and the 2×2 coupling matrix $\mathbf{C}$. An efficient method for computing these parameters is presented along with prescriptions for determining derivatives with respect to quadrupole rotations and quadrupole strength changes.

2 Propagation of Twiss and Coupling Parameters

the normal mode Twiss parameters are obtained from the one-turn matrix $\mathbf{T}$ by writing $\mathbf{T}$ in the form[1]

$$\mathbf{T} = \mathbf{V}\,\mathbf{U}\,\mathbf{V}^{-1} \,. \tag{1}$$

The normal mode matrix $\mathbf{U}$ is block diagonal

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix} , \tag{2}$$

and the coupling matrix $\mathbf{V}$ is of the form

$$\mathbf{V} = \begin{pmatrix} \gamma \mathbf{I} & \mathbf{C} \\ -\mathbf{C}^+ & \gamma \mathbf{I} \end{pmatrix} , \tag{3}$$

where

$$\gamma^2 = 1 - \|\mathbf{C}\| \,. \tag{4}$$

The normal mode twiss parameters are obtained from $\mathbf{U}$ using the standard equations (cf. Bovet et al.[2] pg. 16). Given that the normal mode analysis has been done at point 1, and given the transfer matrix $\mathbf{T}_{12}$ between points 1 and 2, how can the normal mode analysis be propagated from 1 to 2? One straightforward way is simply to form the one–turn matrix at point 2:

$$\mathbf{T}_2 = \mathbf{T}_{12}\,\mathbf{T}_1\,\mathbf{T}_{12}^{-1}\,, \tag{5}$$

and the Twiss and coupling analysis can be computed as indicated above. The problem here is that when a design/analysis program does an optimization it can easily have to reevaluate the entire lattice 10^3 times or so. Since there are of order 10^3 elements in the CESR lattice that means that there are of order 10^6 propagations. It is thus worthwhile to find a more efficient way to propagate the normal modes and this can be achieved by working directly with the 2×2 submatrices.

Using Eqs. (1) and (5) the propagation of the normal mode matrix $\mathbf{U}$ is given by

$$\begin{aligned}\mathbf{U}_2 &= \mathbf{V}_2^{-1}\,\mathbf{T}_2\,\mathbf{V}_2 \\ &= (\mathbf{V}_2^{-1}\,\mathbf{T}_{12}\,\mathbf{V}_1)\,\mathbf{U}_1\,(\mathbf{V}_1^{-1}\,\mathbf{T}_{12}^{-1}\,\mathbf{V}_2) \\ &\equiv \mathbf{W}_{12}\,\mathbf{U}_1\,\mathbf{W}_{12}^{-1}\end{aligned} \tag{6}$$

where

$$\mathbf{W}_{12} \equiv \mathbf{V}_2^{-1}\,\mathbf{T}_{12}\,\mathbf{V}_1\,. \tag{7}$$

$\mathbf{W}_{12}$ is the similarity transformation connecting the eigenmode matrices $\mathbf{U}_1$ and $\mathbf{U}_2$. Since the eigenmodes are independent there cannot be any terms in $\mathbf{W}_{12}$ that connect the two modes. Since the $\mathbf{U}_i$ are block diagonal this means that $\mathbf{W}_{12}$ is either block diagonal or is "off–block diagonal" (has zeros on the 2×2 block diagonals). A formal proof of this is given in Appendix A. The exceptions to the above statement come when the modes can "mix" at the coupling resonance or at the stop–band resonance. We will not consider these exceptional cases further. $\mathbf{W}_{12}$ will be off–block diagonal when the mode associated with the upper-left hand block in $\mathbf{U}_1$ moves to the lower-right hand block in $\mathbf{U}_2$ and vice versa for the other mode. This case can occur, for example, in a Mobius lattice but for ordinary CESR lattices this is not the case. Thus we consider first the case where $\mathbf{W}_{12}$ is block diagonal:

$$\mathbf{W}_{12} = \begin{pmatrix}\mathbf{E}_{12} & 0 \\ 0 & \mathbf{F}_{12}\end{pmatrix}\,. \tag{8}$$

Write

$$\mathbf{T}_{12} = \begin{pmatrix}\mathbf{M}_{12} & \mathbf{m}_{12} \\ \mathbf{n}_{12} & \mathbf{N}_{12}\end{pmatrix}\,. \tag{9}$$

From Eq. (7) we have $\mathbf{V}_2\mathbf{W}_{12} = \mathbf{T}_{12}\mathbf{V}_1$ which gives with Eqs. (8) and (9)

$$\begin{pmatrix}\gamma_2\mathbf{E}_{12} & \mathbf{C}_2\mathbf{F}_{12} \\ -\mathbf{C}_2^{+}\mathbf{E}_{12} & \gamma_2\mathbf{F}_{12}\end{pmatrix} = \begin{pmatrix}\gamma_1\mathbf{M}_{12} - \mathbf{m}_{12}\mathbf{C}_1^{+} & \mathbf{M}_{12}\mathbf{C}_1 + \gamma_1\mathbf{m}_{12} \\ \gamma_1\mathbf{n}_{12} - \mathbf{N}_{12}\mathbf{C}_1^{+} & \mathbf{n}_{12}\mathbf{C}_1 + \gamma_1\mathbf{N}_{12}\end{pmatrix}\,. \tag{10}$$

Consider first the (2,2) component in Eq. (10). Since $\mathbf{W}_{12}$ is made up of symplectic matrices $\mathbf{W}_{12}$ is symplectic and thus $\|\mathbf{F}_{12}\| = 1$. Thus

$$\gamma_2^2 = \|\mathbf{n}_{12}\mathbf{C}_1 + \gamma_1\mathbf{N}_{12}\|\,. \tag{11}$$

Equating the rest of the terms in Eq. (10) gives

$$\begin{aligned}\mathbf{E}_{12} &= (\gamma_1 \mathbf{M}_{12} - \mathbf{m}_{12}\mathbf{C}_1^{+})/\gamma_2 \ ,\\ \mathbf{F}_{12} &= (\mathbf{n}_{12}\mathbf{C}_1 + \gamma_1 \mathbf{N}_{12})/\gamma_2 \ ,\\ \mathbf{C}_2 &= (\mathbf{M}_{12}\mathbf{C}_1 + \gamma_1 \mathbf{m}_{12})\,\mathbf{F}_{12}^{-1} \ .\end{aligned} \tag{12}$$

From Eq. (8) the normal mode vectors propagate as

$$\begin{aligned}\mathbf{a}_2 &= \mathbf{E}_{12}\,\mathbf{a}_1 \ ,\\ \mathbf{b}_2 &= \mathbf{F}_{12}\,\mathbf{b}_1 \ .\end{aligned} \tag{13}$$

If $\mathbf{W}_{12}$ is off–block diagonal so that

$$\mathbf{W}_{12} = \begin{pmatrix} 0 & \mathbf{F}_{12} \\ \mathbf{E}_{12} & 0 \end{pmatrix} \ , \tag{14}$$

The appropriate equation for γ_2 is now

$$\gamma_2^2 = \|\mathbf{M}_{12}\mathbf{C}_1 + \gamma_1 \mathbf{m}_{12}\| \ , \tag{15}$$

and the formulas for $\mathbf{E}_{12}$, $\mathbf{F}_{12}$, and $\mathbf{C}_2$ are

$$\begin{aligned}\mathbf{E}_{12} &= (\gamma_1 \mathbf{n}_{12} - \mathbf{N}_{12}\mathbf{C}_1^{+})/\gamma_2 \ ,\\ \mathbf{F}_{12} &= (\mathbf{M}_{12}\mathbf{C}_1 + \gamma_1 \mathbf{m}_{12})/\gamma_2 \ ,\\ \mathbf{C}_2 &= (\gamma_1 \mathbf{M}_{12} - \mathbf{m}_{12}\mathbf{C}_1^{+})\,\mathbf{E}_{12}^{-1}\end{aligned} \tag{16}$$

and the normal mode vectors propagate as

$$\begin{aligned}\mathbf{b}_2 &= \mathbf{E}_{12}\,\mathbf{a}_1 \ ,\\ \mathbf{a}_2 &= \mathbf{F}_{12}\,\mathbf{b}_1 \ .\end{aligned} \tag{17}$$

Faced with Eqs. (11) and (12) on the one hand and Eqs. (15) and (16) on the other, which set of equation should be used? The problem here is that, in general, both solutions are possible. Faced with two possibilities one possible answer is to choose the solution with the largest γ_2 thus minimizing the coupling matrix $\mathbf{C}_2$. This prescription is equivalent to Billing's[1] choice of signs for γ in his Eq. (14). That is, with this choice one gets the same answer as would have been obtained by forming $\mathbf{T}_2$ from Eq. (5) and then using the recipe given in [1]. It should be noted that if the RHS of Eq. (11) or Eq. (15) is zero or negative then there is only one solution and the choice is unambiguous.

In the special case where $\mathbf{T}_{12}$ is block diagonal, i.e. $\mathbf{n}_{12} = \mathbf{m}_{12} = \mathbf{0}$, then $\mathbf{W}_{12}$ is block diagonal and Eqs. (11) and (12) reduce to

$$\begin{aligned}\gamma_2 &= \gamma_1 \ ,\\ \mathbf{W}_{12} &= \mathbf{T}_{12} \ ,\\ \mathbf{C}_2 &= \mathbf{M}_{12}\,\mathbf{C}_1\,\mathbf{N}_{12}^{-1} \ .\end{aligned} \tag{18}$$

3 Propagation of the Dispersion

The dispersion in normal mode coordinates is related to the despersion in x–y coordinates by

$$\tilde{\eta}_a = \widetilde{\mathbf{V}}^{-1}\tilde{\eta}_x \tag{19}$$

where the tilde denotes the use of the full 6-dimensional space

$$\tilde{\eta}_a = (\eta_a, \eta'_a, \eta_b, \eta'_b, 0, 1)^t \, ,$$

$$\widetilde{\mathbf{V}} = \left(\begin{array}{c:cc} \mathbf{V} & \multicolumn{2}{c}{0} \\ \hdashline \multirow{2}{*}{0} & 1 & \\ & & 1 \end{array} \right) . \tag{20}$$

The propagation for $\tilde{\eta}_a$ is then

$$\tilde{\eta}_{a2} = \widetilde{\mathbf{W}}_{12}\tilde{\eta}_{a1} \tag{21}$$

where $\widetilde{\mathbf{W}}_{12}$ is obtained from the appropriate analog of Eq. (7)

4 Real Space Motion With Coupling

At a specific point in the ring how does the position of the beam change in the x–y plane on a turn-by-turn basis? This question has been analyzed by Bagley and Rubin[3]. The intention here is to clarify the sign of the motions.

For a mode shaking the x–y position on the n$^{\text{th}}$ turn is given by Bagley and Rubin[3]:

$$x = \widehat{A}\gamma \sqrt{\beta_a} \cos \phi_a(n) \tag{22}$$

$$y = \widehat{A} \left(-\sqrt{\beta_b}\, \overline{C}_{22} \cos \phi_a(n) - \sqrt{\beta_b}\, \overline{C}_{12} \sin \phi_a(n) \right) , \tag{23}$$

where $\phi_a(n)$ is the total phase advance from turn 0. For $\overline{C}_{22} > 0$ and $\overline{C}_{12} = 0$ the motion is along a line that is rotated clockwise with respect to the x–axis as shown in figure 1a. For $\overline{C}_{12} > 0$ and $\overline{C}_{22} = 0$ the motion is elliptical with the beam rotating in a clockwise manner as shown in figure 1b.

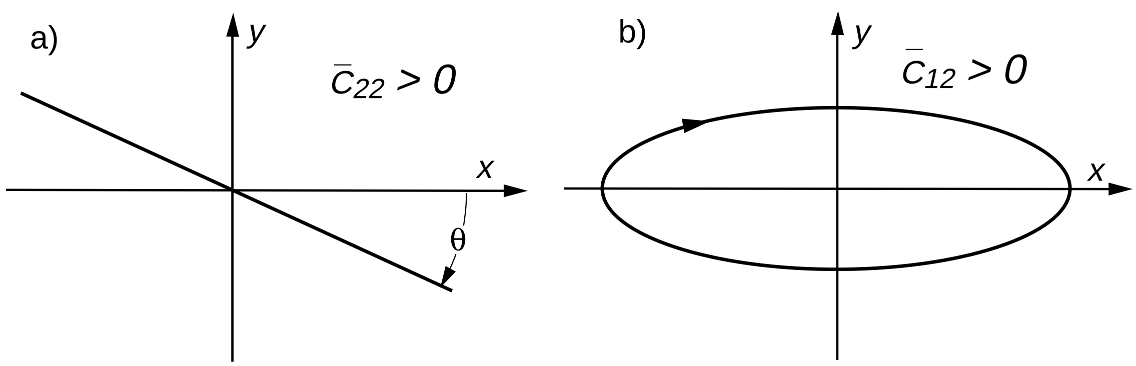

Figure 1: Coupled motion in the x—y plane at one point in the ring. (a) $\overline{C}_{22} > 0$, $\overline{C}_{12} = 0$. (b) $\overline{C}_{12} > 0$, $\overline{C}_{22} = 0$.

5 $\overline{\mathbf{C}}$ Propagation Between Couplers

As shown in Appendix A any 2×2 matrix can be decomposed into rotational and anti-rotational matrices. We can thus write the $\overline{\mathbf{C}}$ matrix at point 1 as

$$\overline{\mathbf{C}}_1 = \lambda_1 \, \mathbf{S}(\phi_1) + \kappa_1 \, \mathbf{R}(\theta_1) \tag{24}$$

Between couplers, $\overline{\mathbf{C}}$ propagation is derived from Eq. (18):

$$\overline{\mathbf{C}}_2 = \overline{\mathbf{M}}_{12} \, \overline{\mathbf{C}}_1 \, \overline{\mathbf{N}}_{12}^{-1} \,, \tag{25}$$

where $\overline{\mathbf{M}}_{12}$ and $\overline{\mathbf{N}}_{12}$ are rotation matrices with rotations angles ϕ_{a12} and ϕ_{b12} respectively. Thus

$$\overline{\mathbf{C}}_2 = \lambda_1 \, \mathbf{S}(\phi_1 - \phi_{a12} - \phi_{b12}) + \kappa_1 \, \mathbf{R}(\theta_1 + \phi_{a12} - \phi_{b12}) \,. \tag{26}$$

Thus, from Assertion (1) of Appendix A, $\overline{C}_{11} + \overline{C}_{22}$ and $\overline{C}_{12} - \overline{C}_{21}$ propagate as the difference between the normal mode phases and $\overline{C}_{11} - \overline{C}_{22}$ and $\overline{C}_{12} + \overline{C}_{21}$ propagate as the sum of the normal mode phases.

One important point here: The way we have defined rotation matrices in Eq. (53) means that a positive phase advance implies a *clockwise* rotation. Not a counter–clockwise one.

6 Perturbation From a Rotated Quadrupole

How does changing the focusing strength or body rotation angle of a rotated quadrupole affect things? For simplicity we will assume the quad has zero length. For a quad rotated by an angle θ_q in the x–y plane the transfer matrix $\mathbf{T}_q$ through the quad is obtained from the real space rotation matrix $\mathbf{R}_s$ and the kick matrix $\mathbf{K}$ by

$$\begin{aligned} \mathbf{T}_q &= \mathbf{R}_s(\theta_q) \, \mathbf{K} \, \mathbf{R}_s^{-1}(\theta_q) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k\cos(2\theta_q) & 1 & -k\sin(2\theta_q) & 0 \\ 0 & 0 & 1 & 0 \\ -k\sin(2\theta_q) & 0 & k\cos(2\theta_q) & 1 \end{pmatrix} , \end{aligned} \tag{27}$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{pmatrix} \tag{28}$$

and

$$\mathbf{R}_s(\theta_q) = \begin{pmatrix} \cos\theta_q & 0 & -\sin\theta_q & 0 \\ 0 & \cos\theta_q & 0 & -\sin\theta_q \\ \sin\theta_q & 0 & \cos\theta_q & 0 \\ 0 & \sin\theta_q & 0 & \cos\theta_q \end{pmatrix} \tag{29}$$

[Note that here positive real space rotations are defined to be counterclockwise.] We want to know what happens when k or θ_q is varied. For k variation Eq. (27) gives

$$\delta\mathbf{T}_q = \delta k \begin{pmatrix} -\mathbf{q}_c & -\mathbf{q}_s \\ -\mathbf{q}_s & \mathbf{q}_c \end{pmatrix} , \tag{30}$$

and for θ_q variation

$$\delta\mathbf{T}_q = 2\,\delta\theta_q\,k \begin{pmatrix} \mathbf{q}_s & -\mathbf{q}_c \\ -\mathbf{q}_c & -\mathbf{q}_s \end{pmatrix} , \tag{31}$$

where

$$\mathbf{q}_c \equiv \begin{pmatrix} 0 & 0 \\ \cos(2\theta_q) & 0 \end{pmatrix}$$

$$\mathbf{q}_s \equiv \begin{pmatrix} 0 & 0 \\ \sin(2\theta_q) & 0 \end{pmatrix} . \tag{32}$$

Consider the variation of the one–turn matrix at a point just after the quad. Let $\mathbf{T}_{\mathrm{arc}}$ be the transfer matrix from just after the quad to just before the quad so the one-turn matrix $\mathbf{T}_1$ at a point just after the quad is

$$\mathbf{T}_1 = \mathbf{T}_q\,\mathbf{T}_{\mathrm{arc}} \tag{33}$$

The variation of $\mathbf{T}_1$ is

$$\begin{aligned} \delta\mathbf{T}_1 &= \delta\mathbf{T}_q\,\mathbf{T}_{\mathrm{arc}} \\ &= \delta\mathbf{T}_q\,\mathbf{T}_q^{-1}\,\mathbf{T}_1 \end{aligned} \tag{34}$$

For both k and θ_q variation it is easily shown that

$$\delta\mathbf{T}_q\,\mathbf{T}_q^{-1} = \delta\mathbf{T}_q \tag{35}$$

thus

$$\delta\mathbf{T}_1 = \delta\mathbf{T}_q\,\mathbf{T}_1 \tag{36}$$

To simplify matters it will be assumed that the coupling is weak and we will only keep terms up to first order in the coupling. Thus, from Eq. (4), to first order $\gamma = 1$. From Eqs. (1), (2), and (3) to first order

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{CB} - \mathbf{AC} \\ \mathbf{BC}^+ - \mathbf{C}^+\mathbf{A} & \mathbf{B} \end{pmatrix} \tag{37}$$

Using Eq. (37) with Eq. (36) and either (30) or (31) gives $\delta \mathbf{T}_1$ in terms of δk or $\delta \theta_q$. For k variation this gives

$$\begin{pmatrix} \delta \mathbf{A} & \delta(\mathbf{CB} - \mathbf{AC}) \\ \delta(\mathbf{BC}^+ - \mathbf{C}^+\mathbf{A}) & \delta \mathbf{B} \end{pmatrix} = \\ \delta k \begin{pmatrix} -\mathbf{q}_c\mathbf{A} - \mathbf{q}_s(\mathbf{BC}^+ - \mathbf{C}^+\mathbf{A}) & -\mathbf{q}_s\mathbf{B} - \mathbf{q}_c(\mathbf{CB} - \mathbf{AC}) \\ -\mathbf{q}_s\mathbf{A} + \mathbf{q}_c(\mathbf{BC}^+ - \mathbf{C}^+\mathbf{A}) & \mathbf{q}_c\mathbf{B} - \mathbf{q}_s(\mathbf{CB} - \mathbf{AC}) \end{pmatrix} \tag{38}$$

Using the (1,1) component of Eq. (38) gives an equation for $\delta \mathbf{A}$. $\mathbf{A}$ can be cast in the standard form:

$$\mathbf{A} = \begin{pmatrix} \cos\theta_a + \alpha_a \sin\theta_a & \beta_a \sin\theta_a \\ -\gamma_a \sin\theta_a & \cos\theta_a - \alpha_a \sin\theta_a \end{pmatrix} , \tag{39}$$

and taking the trace of $\delta \mathbf{A}$ gives the variation of the a eigenmode tune:

$$\begin{aligned} \delta\theta_a &= \frac{\delta k}{2} \left[\beta_a \cos(2\theta_q) + \frac{\sin(2\theta_q)}{\sin\theta_a} (B_{12}C_{11} - B_{11}C_{12} + A_{22}C_{12} - A_{12}C_{22}) \right] \\ &\equiv \frac{\delta k}{2} \beta_a(\text{eff}) . \end{aligned} \tag{40}$$

For the b eigenmode the corresponding equation is

$$\begin{aligned} \delta\theta_b &= \frac{-\delta k}{2} \left[\beta_b \cos(2\theta_q) + \frac{\sin(2\theta_q)}{\sin\theta_b} (A_{11}C_{12} + A_{12}C_{22} - B_{12}C_{11} - B_{22}C_{12}) \right] \\ &\equiv \frac{-\delta k}{2} \beta_b(\text{eff}) . \end{aligned} \tag{41}$$

For an upright quad with $\theta_q = 0$ Eqs. (40) and (41) reduce to the standard uncoupled formulas for the tune shift. As shown in Appendix B, between any two points the normal mode transfer matrix $\mathbf{W}_{12}$ is, to first order, independent of the coupling. This means that Eq. (40) and (41) can be used to define effective betas which can be used in perturbation formulas that are derived without coupling. For example, without coupling, for a given mode, the change in phase $\delta\phi_2$ at point 2 given a quadrupole perturbation δk at point 1 is

$$\delta\phi_2 = \frac{\beta_1 \, \delta k}{4 \sin\theta} \left[\sin\theta - \sin(\theta - 2|\phi_2 - \phi_1|) \right] \begin{cases} -1 & \phi_2 < \phi_1 \\ 1 & \phi_2 > \phi_1 \end{cases} , \tag{42}$$

with θ being the tune. With coupling the appropriate formula is obtained by substituting the effective beta as calculated from Eq. (40) or (41) for β_1 in Eq. (42). A similar situation holds for the formula for a beta wave.

For the variation of $\mathbf{C}$ the (1,2) component of Eq. (38) is added to the symplectic complement of the (2,1) component. Using the fact that $\mathbf{A} + \mathbf{A}^+ = 2\mathbf{I} \cos\theta_a$ after some algebra one finds

$$\delta \mathbf{C} = \frac{\delta k}{2(\cos\theta_a - \cos\theta_b)} \Big[(\mathbf{q}_s \mathbf{B} - \mathbf{A}^+ \mathbf{q}_s) + \tag{43}$$
$$\mathbf{q}_c(\mathbf{CB} - \mathbf{AC}) + (\mathbf{CB}^+ - \mathbf{A}^+\mathbf{C})\mathbf{q}_c + \cos(2\theta_q)\,(A_{12} + B_{12})\,\mathbf{C} \Big]$$

The normalized $\overline{\mathbf{C}}$ is[3]

$$\overline{\mathbf{C}} = \mathbf{G}_a\, \mathbf{C} \mathbf{G}_b^{-1} \tag{44}$$

Using this in Eq. (43) gives

$$\delta \overline{\mathbf{C}} = \frac{\delta k}{2(\cos\theta_a - \cos\theta_b)} \Big[\sqrt{\beta_a \beta_b}\, \sin(2\theta_q) \begin{pmatrix} \sin\theta_a & 0 \\ \cos\theta_b - \cos\theta_a & \sin\theta_b \end{pmatrix} + \tag{45}$$
$$\beta_a \mathbf{q}_c(\overline{\mathbf{C}}\,\overline{\mathbf{B}} - \overline{\mathbf{A}}\,\overline{\mathbf{C}}) + \beta_b\,(\overline{\mathbf{C}}\,\overline{\mathbf{B}}^+ - \overline{\mathbf{A}}^+\,\overline{\mathbf{C}})\mathbf{q}_c + \cos(2\theta_q)\,(A_{12} + B_{12})\,\overline{\mathbf{C}} \Big]$$

This is the variation of $\overline{\mathbf{C}}$ just after the quad and this variation can be propagated by the formulas developed earlier. For the zeroth order contribution to $\delta\overline{\mathbf{C}}$ (the first term on the RHS of Eq. (45)) the decomposition into rotational and anti–rotational matrices is:

$$\begin{pmatrix} \sin\theta_a & 0 \\ \cos\theta_b - \cos\theta_a & \sin\theta_b \end{pmatrix} = \sin(\theta_+) \begin{pmatrix} \cos(\theta_-) & \sin(\theta_-) \\ -\sin(\theta_-) & \cos(\theta_-) \end{pmatrix} - \tag{46}$$
$$\sin(\theta_-) \begin{pmatrix} \cos(\theta_+) & \sin(\theta_+) \\ \sin(\theta_+) & -\cos(\theta_+) \end{pmatrix}$$

where

$$\theta_+ = \frac{1}{2}(\theta_b + \theta_a)$$
$$\theta_- = \frac{1}{2}(\theta_b - \theta_a) \tag{47}$$

The above analysis has been done for k variation. For variation of θ_a the above analysis can be used with the substitution:

$$\delta k\ \cos(2\theta_q) \longrightarrow -2\,k\,\delta\theta_q\ \sin(2\theta_q)$$
$$\delta k\ \sin(2\theta_q) \longrightarrow \quad 2\,k\,\delta\theta_q\ \cos(2\theta_q) \tag{48}$$

Appendix A: Block Diagonal Proof

To show that $\mathbf{W}_{12}$ is block diagonal or off–block diagonal we normalize $\mathbf{U}_1$ and $\mathbf{U}_2$ with the standard similarity transformation (Billing[1] Eq. (3)):

$$\overline{\mathbf{U}} = \mathbf{G}\,\mathbf{U}\,\mathbf{G}^{-1}\,, \tag{49}$$

where the $\mathbf{G}$ are of the form

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_a & 0 \\ 0 & \mathbf{G}_b \end{pmatrix} , \tag{50}$$

and $\mathbf{U}_1$ and $\mathbf{U}_2$ are of the form

$$\overline{\mathbf{U}} = \begin{pmatrix} \mathbf{R}(\theta_a) & 0 \\ 0 & \mathbf{R}(\theta_b) \end{pmatrix} , \tag{51}$$

or are of the form

$$\overline{\mathbf{U}} = \begin{pmatrix} \mathbf{R}(\theta_b) & 0 \\ 0 & \mathbf{R}(\theta_a) \end{pmatrix} , \tag{52}$$

where θ_a and θ_b are the tunes of the a and b modes respectively, and $\mathbf{R}$ is a rotation matrix

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{53}$$

Notice that since the rotation angle of the normal modes is dependent only on the tune, and not on any particular starting point, $\mathbf{U}_1$ and $\mathbf{U}_2$ can only be as given by Eq. (51) or (52). Using Eqs. (6) and (49) we can write

$$\overline{\mathbf{U}}_2 = \overline{\mathbf{W}}_{12}\,\overline{\mathbf{U}}_1\,\overline{\mathbf{W}}_{12}^{-1} \tag{54}$$

where

$$\overline{\mathbf{W}}_{12} = \mathbf{G}_2^{-1}\,\mathbf{W}_{12}\,\mathbf{G}_1 \tag{55}$$

Obviously $\mathbf{W}_{12}$ is block diagonal (or off-block diagonal) if and only if $\overline{\mathbf{W}}_{12}$ is.

We now assume that the a eigenmode stays in the top-left corner in going from point 1 to point 2 so that $\overline{\mathbf{U}}_1$ and $\overline{\mathbf{U}}_2$ are of the form Eq. (51):

$$\overline{\mathbf{U}}_1 = \overline{\mathbf{U}}_2 = \overline{\mathbf{U}} \equiv \begin{pmatrix} \mathbf{R}(\theta_a) & 0 \\ 0 & \mathbf{R}(\theta_b) \end{pmatrix} \tag{56}$$

Thus from Eq. (54)

$$\overline{\mathbf{W}}_{12} = \overline{\mathbf{U}}\,\overline{\mathbf{W}}_{12}\,\overline{\mathbf{U}}^{-1} \tag{57}$$

Writing

$$\overline{\mathbf{W}}_{12} = \begin{pmatrix} \mathbf{p} & \mathbf{q} \\ \mathbf{r} & \mathbf{s} \end{pmatrix} \tag{58}$$

gives for $\mathbf{q}$

$$\mathbf{q} = \mathbf{R}(\theta_a)\,\mathbf{q}\,\mathbf{R}^{-1}(\theta_b) . \tag{59}$$

We want to show that $\mathbf{q} = 0$. To see this we note the following without proof:

1) Any 2×2 matrix $\mathbf{z}$ can be decomposed into "rotational" and "anti–rotational" matrices:

$$\begin{aligned}\mathbf{z} &= \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} z_{11}-z_{22} & z_{12}+z_{21} \\ z_{12}+z_{21} & -(z_{11}-z_{22}) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} z_{11}+z_{22} & z_{12}-z_{21} \\ -(z_{12}-z_{21}) & z_{11}+z_{22} \end{pmatrix} \\ &= \lambda\,\mathbf{S}(\phi) + \kappa\,\mathbf{R}(\theta)\end{aligned} \tag{60}$$

where λ and κ are constants, $\mathbf{R}$ is a rotational matrix given by Eq. (53), and

$$\mathbf{S}(\phi) \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \tag{61}$$

The inverse is

$$\mathbf{z}^{-1} = \frac{1}{\lambda^2-\kappa^2}\left[\lambda\,\mathbf{S}(\phi) - \kappa\,\mathbf{R}(-\theta)\right] . \tag{62}$$

2) Given $\mathbf{z}$, the product $\lambda\mathbf{S}(\phi)$ is unique and the product $\kappa\mathbf{R}(\theta)$ is also unique so that if

$$\begin{aligned}\mathbf{z}_1 &= \lambda_1\,\mathbf{S}(\phi_1) + \kappa_1\,\mathbf{R}(\theta_1)\,, \\ \mathbf{z}_2 &= \lambda_2\,\mathbf{S}(\phi_2) + \kappa_2\,\mathbf{R}(\theta_2)\,,\end{aligned} \tag{63}$$

then $\mathbf{z}_1 = \mathbf{z}_2$ if and only if

$$\begin{aligned}&\lambda_1\mathbf{S}(\phi_1) = \lambda_2\mathbf{S}(\phi_2) \text{ and} \\ &\kappa_1\mathbf{R}(\theta_1) = \kappa_2\mathbf{R}(\theta_2)\,.\end{aligned} \tag{64}$$

3) Rotation matrices and anti–rotation matrices multiply as:

$$\begin{aligned}\mathbf{R}(\theta_1)\,\mathbf{R}(\theta_2) &= \mathbf{R}(\theta_1+\theta_2) \\ \mathbf{S}(\phi_1)\,\mathbf{R}(\theta_2) &= \mathbf{S}(\phi_1+\theta_2) \\ \mathbf{R}(\theta_2)\,\mathbf{S}(\phi_1) &= \mathbf{S}(\phi_1-\theta_2) \\ \mathbf{S}(\phi_1)\,\mathbf{S}(\phi_2) &= \mathbf{R}(\phi_2-\phi_1)\end{aligned} \tag{65}$$

Writing $\mathbf{q} = \lambda\mathbf{S}(\phi) + \kappa\mathbf{R}(\theta)$ and using this with Assertions (2) and (3) in Eq. (59) gives

$$\lambda\mathbf{S}(\phi) = \lambda\mathbf{S}(\phi-\theta_a-\theta_b)\,, \tag{66}$$

$$\kappa\mathbf{R}(\theta) = \kappa\mathbf{R}(\theta+\theta_a-\theta_b)\,, \tag{67}$$

Since we are assuming that we are not at the stop–band resonance where $\theta_a+\theta_b = 2\pi m$ for some integer m, Eq. (66) can only be true if $\lambda = 0$. Additionally, since we are assuming that we are not at the coupling resonance where $\theta_a - \theta_b = 2\pi n$ for some integer n, Eq. (67) can only be true if $\kappa = 0$. Thus $\mathbf{q} = \mathbf{0}$. Similarly, it can be shown that $\mathbf{r} = 0$ and thus $\overline{\mathbf{W}}_{12}$ and $\mathbf{W}_{12}$ are block diagonal. For the case where there is a switch so that $\mathbf{U}_1$ is of the form (51) and $\mathbf{U}_2$ is of the form (52) then is can similarly be shown that $\mathbf{p} = \mathbf{s} = 0$ and hence $\mathbf{W}_{12}$ is off–block diagonal.

Appendix B: Independence of $\mathbf{W}_{12}$

We want to show that $\mathbf{W}_{12}$ between two given points 1 and 2 is, to first order, independent of the coupling. To see this consider first the situation where there is only one thin coupler at point 1. The transfer matrix $\mathbf{T}_{12}$ is then (Cf. Billing[1] Eq. (7))

$$\begin{aligned}
\mathbf{T}_{12} \equiv& \begin{pmatrix} \mathbf{M}_{12} & \mathbf{m}_{12} \\ \mathbf{n}_{12} & \mathbf{N}_{12} \end{pmatrix} \\
=& \begin{pmatrix} \mathbf{M}_0 & 0 \\ 0 & \mathbf{N}_0 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{f} \\ -\mathbf{f}^+ & \mathbf{I} \end{pmatrix} \\
=& \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_0\,\mathbf{f} \\ -\mathbf{N}_0\,\mathbf{f}^+ & \mathbf{N}_0 \end{pmatrix} ,
\end{aligned} \tag{68}$$

where $\mathbf{f}$ determines the strength of the coupler at point 1 and $\mathbf{M}_0$ and $\mathbf{N}_0$ are the transfer matrices without any coupling. The diagonal matrices ($\mathbf{M}_{12}$ and $\mathbf{N}_{12}$) of $\mathbf{T}_{12}$ are seen to be independent of the coupling while the off–diagonal matrices $\mathbf{m}_{12}$ and $\mathbf{n}_{12}$ are first order in the coupling. This can easily be generalized to the case where there is an arbitrary number of coupling elements at arbitrary positions. Using this in Eqs. (12) one sees that $\mathbf{E}_{12}$ and $\mathbf{F}_{12}$ are also, to first order, independent of the coupling. QED.

References

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