# Dependence of Luminosity in CESR on Bunch Length for Flat and Round Beams

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## I. INTRODUCTION

In this paper, general comparisons will be made between the luminosity potential of CESR with round beams, as compared to flat beams. The comparisons will highlight the "geometric" effects of a finite bunch length and crossing angle on the luminosity and on the head-on beam-beam tune shift. These geometric effects have also been extensively discussed in ref [3].

## II. LUMINOSITY RELATIONS

I will let $\varepsilon_{x,y}$= the (unnormalized) rms transverse emittance in the x,y plane; $I$ = the current per bunch; $B$=the number of bunches; $f_0$= the revolution frequency; $\beta^*_{x,y}$ = the beta function at the interaction point in the x,y planes; and $\sigma_s$ =the rms bunch length. For round beams, I assume that

$$\varepsilon_x = \varepsilon_y = \frac{\varepsilon}{2}; \quad \beta^*_x = \beta^*_y = \beta^* \tag{1}$$

in which ε is the rms equilibrium emittance, and I take the crossing angle to be zero (see Section III below for a discussion of this). Starting from results given in ref. [2], it is shown in Appendix I that the total luminosity of the collider can be written as

$$L_{round} = \frac{2I^2 B}{4\pi e^2 f_0 \varepsilon \beta^*} H_{round}(r) \tag{2}$$

in which the effects of a finite bunch length are described by the "hourglass" function

$$H_{round}(r) = r\sqrt{\pi}\, Exp\left[r^2\right] Erfc[r] \tag{3}$$

where

$$r = \frac{\beta^*}{\sigma_s}. \tag{4}$$

For "flat beams", (mostly uncoupled), I take

$$\varepsilon_x = \varepsilon; \quad \varepsilon_y = k\varepsilon; \quad \beta^*_y = \beta^*; \quad \beta^*_x = \frac{\beta^*}{k'}; \quad \nu_0 = \frac{\alpha\sigma_s}{\sqrt{\beta^*_x \varepsilon}} = \alpha\sigma_s\sqrt{\frac{k'}{\beta^* \varepsilon}} \tag{5}$$

where $\varepsilon$ is the equilibrium rms emittance, $k<<1$ is the coupling parameter, $k' = \frac{\beta^*_y}{\beta^*_x} << 1$, and α is the crossing (half) angle. In Appendix I, it is shown that, to lowest order in $k$ and $k'$, the total luminosity can be written as

$$L_{flat} = \frac{I^2 B}{4\pi e^2 f_0 \varepsilon \beta^*}\sqrt{\frac{k'}{k}} H_{flat}(r, \nu_0, \alpha) \tag{6}$$

in which the effects of a finite bunch length and a crossing angle are described by

$$H_{flat}(r, \nu_0, \alpha) = Cos^2(\alpha)\frac{r}{\sqrt{\pi}} Exp\left[\frac{r^2}{2}(1+\nu_0^2)\right] K_0\left[\frac{r^2}{2}(1+\nu_0^2)\right] \tag{7}$$

Fig. 1 plots $H_{flat}(r, \nu_0, \alpha)$ and $H_{round}(r)$ vs. r. Note that the "hourglass" reduction is larger for round than for flat beams, for the same value of r.

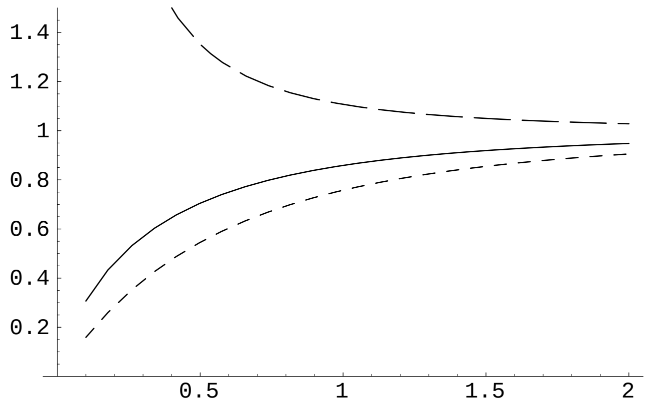


Fig. 1

Plot of $H_{flat}(r, \nu_0, \alpha)$ (solid line), $H_{round}(r)$(short dashed lines), and $G_{flat}(r, \nu_0)$(long dashed lines), vs. r, for $\alpha$=0.0025 and $\nu_0$ =0.056.

## III. CROSSING ANGLES

I have assumed a zero crossing angle for the round beam, but not the flat beam, for the following reason. To inhibit synchrobetatron coupling through the beam-beam interaction, the crossing angle $\alpha$ is required to satisfy

$$\alpha << \frac{\sigma_x}{\sigma_s} = \frac{\sqrt{\beta_x^* \varepsilon_x}}{\sigma_s} \tag{8}$$

for crossing in the horizontal plane. This implies that the parameter $\nu_0 << 1$. For flat beams, using the values $\varepsilon_x$ = 0.2 µm-rad, $\beta_x^*$ = 1 m, $\sigma_s$ = 1 cm, the requirement is

$$\alpha_{flat} << \frac{\sqrt{\beta_x^* \varepsilon_x}}{\sigma_s} = \frac{1}{.01}\sqrt{2x10^{-7}} = 45 \;\; mrad \tag{9}$$

For round beams, with $\varepsilon_x$ = 0.1 µm-rad, $\beta_x^*$ = 3 cm, $\sigma_s$ = 1 cm, the crossing angle requirement is

$$\alpha_{round} << \frac{\sqrt{\beta_x^* \varepsilon_x}}{\sigma_s} = \frac{1}{.01}\sqrt{.03x1x10^{-7}} = 5.5 \;\; mrad \tag{10}$$

Because of this more restrictive requirement in the round beam case, I will assume here exactly zero crossing angle for the round beam. For the flat beam, I assume a 2.5 mrad crossing half-angle for the flat beams, which gives $\nu_0 = \frac{2.5}{45} = .056$.

## IV. HEAD-ON TUNE SHIFTS

The relation for the head-on crossing tune shift for zero amplitude particles, including the bunch length and crossing angle dependence, is derived, starting from ref. [1], in Appendix II; the result for the round beam, with no crossing angle, is

$$\xi_{round} = \frac{Ir_e}{2\pi e f_0 \gamma \varepsilon} \tag{11}$$

in which $r_e$ is the classical radius of the electron, and $\gamma$ is $E/m_0c^2$. For a flat beam, in the limit *k->0* and *k'->0*, but with a finite crossing angle, to lowest order in $\nu_0$, the horizontal tune shift is

$$\xi_{flat,x} = \frac{Ir_e\left(1-\nu_0^2\right)}{2\pi e f_0 \gamma \varepsilon} . \tag{12}$$

In the vertical plane, to lowest order in *k* and *k'*,

$$\xi_{flat,y} = \frac{Ir_e}{2\pi e f_0 \gamma \varepsilon}\sqrt{\frac{k'}{k}}G_{flat}(r,v_0) \tag{13}$$

in which the bunch length dependence results in an enhancement factor for the vertical tune shift, given (to lowest order in $v_0$) by

$$G_{flat}(r,v_0) = \frac{U[-\frac{1}{2},0,2r^2]}{r\sqrt{2}} - \frac{v_0^2 r Exp[r^2]K_1[r^2]}{\sqrt{2\pi}} \tag{14}$$

where $U(a,b,c)$ is a confluent hypergeometric function. $G_{flat}(r,v_0)$ is plotted in Fig. 1

# V. TOTAL-CURRENT-LIMITED LUMINOSITY

## A. Flat Beam Luminosity

Let me first consider the flat beam case. I will suppose that the total current per beam $\hat{I} = BI$ in the machine is limited to a value $\hat{I}_{max}$, and that the equilibrium emittance ε and the vertical beta function $\beta^*$ are fixed. I will consider *k'* and *B* to be free parameters. In this case, the head-on tune shifts in the x and y plane will be

$$\xi_{flat,x} = \frac{\hat{I}_{max} r_e (1-v_0^2)}{2B\pi e f_0 \gamma \varepsilon}; \quad \xi_{flat,y} = \frac{\hat{I}_{max} r_e}{2B\pi e f_0 \gamma \varepsilon}\sqrt{\frac{k'}{k}}G_{flat}(r_{flat},v_0) \tag{15}$$

and the luminosity will be

$$L_{flat} = \frac{\hat{I}^2{}_{max}}{4B\pi e^2 f_0 \varepsilon \beta^*_{flat}}\sqrt{\frac{k'}{k}}H_{flat}(r_{flat},v_0,\alpha) \tag{16}$$

in which $r_{flat} = \frac{\beta^*{}_{flat}}{\sigma_s}$ and $v_0 = \frac{\sqrt{k'}\alpha\sigma_s}{\sqrt{\beta^*{}_{flat}\varepsilon}}$. The luminosity may be increased by increasing the parameter *k'* (decreasing $\beta_x^*$) This also increases $\xi_{flat,y}$. The head-on tune shift is beam-dynamically limited at some value $\xi_{flat,y,max}$; increasing the tune shift beyond this value results in an increase in the beam size, the beam lifetime starts to decrease significantly due to the beam-beam interaction, and machine operation becomes very difficult. The value for *k'* which results in $\xi_{flat,y} = \xi_{flat,y,max}$ is

$$\sqrt{\frac{k'}{k}} = \frac{2B\pi e f_0 \gamma \varepsilon \xi_{flat,y,max}}{G_{flat}(r_{flat},v_0)\hat{I}_{max} r_e} \tag{17}$$

(For simplicity, the dependence of $v_0$ on *k'* in $G_{flat}$ has been ignored). The luminosity per unit total current $\hat{L} = \frac{L}{\hat{I}_{max}}$ is then

$$\hat{L}_{flat} = \frac{\gamma}{2er_e}\frac{\xi_{flat,y,max}}{\beta^*{}_{flat}}\frac{H_{flat}(r_{flat},v_0,\alpha)}{G_{flat}(r_{flat},v_0)} \tag{18}$$

Setting *k'* to the value given in Eq. (17) does not affect the horizontal tune shift. Let me call the maximum dynamically-allowed value of this tune shift $\xi_{flat,x,max}$. For the horizontal tune shift to be less than this value, the number of bunches B must be greater than

$$B_{flat,min} = \frac{\hat{I}_{max} r_e (1-v_0^2)}{2\xi_{flat,x,max}\pi e f_0 \gamma \varepsilon} \tag{19}$$

If $B = B_{flat,min}$, then combining Eq. (17) and (19) gives

$$\sqrt{\frac{k'}{k}}\Bigg|_{min} = \frac{\left(1-\nu_0^2\right)}{G_{flat}(r_{flat},\nu_0)}\frac{\xi_{flat,y,max}}{\xi_{flat,x,max}} \tag{20}$$

which implies, using Eq. (5),

$$\beta^*_{x,max} = \frac{\beta^*_{flat}}{k}\left[\frac{\xi_{flat,x,max}}{\xi_{flat,y,max}}\right]^2 \frac{G^2_{flat}(r_{flat},\nu_0)}{\left(1-\nu_0^2\right)^2} \tag{21}$$

For $B > B_{flat,min}$, as long as we can decrease $\beta^*_x$ from the value given in Eq. (2) to make $k'$ scale up with $B$ according to Eq.(17), the vertical tune shift and the luminosity will both be independent of B; the horizontal tune shift in this case will be less than $\xi_{flat,x,max}$.

## B. Round Beam Luminosity

For the round beam, I again suppose that the total current is limited to $I_{max}$ , and that the equilibrium emittance ε and the beta function $\beta^*$ are fixed. The number of bunches $B$ is a free parameter. The head-on tune shift in this case is

$$\xi_{round} = \frac{\hat{I}_{max} r_e}{2B\pi e f_0 \gamma\varepsilon} \tag{22}$$

and the luminosity will be

$$L_{round} = \frac{2\hat{I}^2_{max}}{4B\pi e^2 f_0 \varepsilon\beta^*_{round}} H_{round}(r_{round}) \tag{23}$$

in which $r_{round} = \frac{\beta^*{}_{round}}{\sigma_s}$. In this case, the luminosity is maximized when the number of bunches $B$ is such that the tune shift, Eq. (22), reaches its dynamically allowed maximum, $\xi_{round,max}$:

$$B_{round} = \frac{\hat{I}_{max} r_e}{2\pi e f_0 \gamma\varepsilon\xi_{round,max}} \tag{24}$$

Then the luminosity per unit total current $\hat{L} = \frac{L}{\hat{I}_{max}}$ is

$$\hat{L}_{round} = \frac{\gamma}{e r_e}\frac{\xi_{round,max}}{\beta^*_{round}} H_{round}(r_{round}) \tag{25}$$

## C. Relative Luminosity and Current per Bunch

The ratio of the two luminosities, Eqs. (18) and (25), is

$$\frac{\hat{L}_{round}}{\hat{L}_{flat}} = 2\frac{\dfrac{\xi_{round,max}}{\beta^*{}_{round}}}{\dfrac{\xi_{flat,y,max}}{\beta^*{}_{flat}}} \frac{G_{flat}(r_{flat},\nu_0) H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)} \tag{26}$$

In order to proceed any further, I must choose values for the tune shifts and $\beta^*$ for the flat and round beams. Somewhat arbitrarily, I speculate that the round beam can achieve $\xi_{round,max}$=0.1 but is limited optically to $\beta^*{}_{round}$=3 cm. For the flat beam, I use $\xi_{flat,y,max}$=0.05, and $\beta^*{}_{flat}$=1 cm, as optimistic values. I take the half-crossing angle α to be 2.5 mrad for the flat beam, and zero for the round beam. With these choices, I plot in Fig. 2 the ratio $\frac{H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)}$ vs. $\sigma_s$.

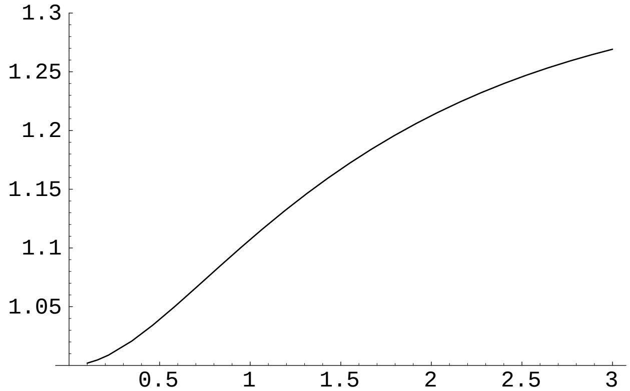


Fig. 2

$\dfrac{H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)}$ (ordinate)vs. $\sigma_S$ (cm)(abscissa)

The ratio is greater than 1, because although the hourglass luminosity reduction is larger for the round beam case than for the flat beam for the same $r$, the round beam, with a larger $\beta^*$, has a larger value of $r$ for the same $\sigma_S$, and this effect more than compensates the other. In Fig. 3, I plot $\dfrac{G_{flat}(r_{flat},\nu_0)H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)}$ vs. $\sigma_S$:

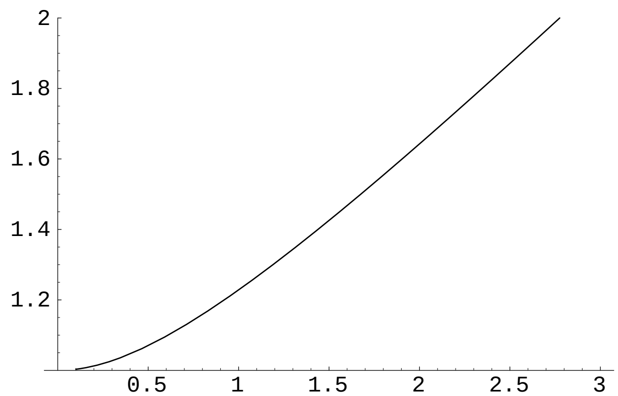


Fig. 3

$\dfrac{G_{flat}(r_{flat},\nu_0)H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)}$ (ordinate)vs. $\sigma_S$(cm) (abscissa)

Comparison with Fig. 2 shows the effect of the vertical tune shift enhancement factor present in the flat beam case, which further reduces the flat beam luminosity (at fixed tune shift) for small r.

To get a relative comparison of the luminosities, I must include the ratios $\dfrac{\xi_{round,max}}{\beta^*_{round}}$ and $\dfrac{\xi_{flat,max}}{\beta^*_{flat}}$. With the parameter choices noted above, the ratio is

$$\frac{\hat{L}_{round}}{\hat{L}_{flat}} = 2\frac{\frac{.1}{3}}{\frac{.05}{1}}\frac{G_{flat}(r_{flat},\nu_0)H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)} = 1.33\frac{G_{flat}(r_{flat},\nu_0)H_{round}(r_{round})}{H_{flat}(r_{flat},\nu_0,\alpha)} \quad (27)$$

This is plotted in fig. 4

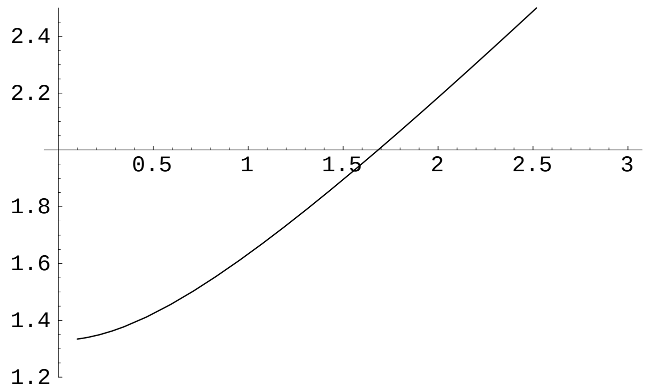


Fig. 4

$\frac{\hat{L}_{round}}{\hat{L}_{flat}}$(ordinate) vs. $\sigma_S$(cm) (abscissa)

With these assumptions, the round beam has a 60-90% advantage in luminosity per unit current over the flat beam, for $\sigma_S$ in the range of 1.-1.5 cm.

The maximum current per bunch $I_{max} = \frac{\hat{I}_{max}}{B}$ for the flat beam, from Eq. (19), is

$$I_{flat,max} = \frac{\hat{I}_{max}}{B_{flat,min}} = \frac{2\xi_{flat,x,max}\pi e f_0 \gamma \varepsilon}{r_e\left(1 - \nu_0^2\right)} \tag{28}$$

and for the round beam, from Eq.(24), the current per bunch should be

$$I_{round} = \frac{\hat{I}_{max}}{B_{round}} = \frac{2\xi_{round,max}\pi e f_0 \gamma \varepsilon}{r_e} \tag{29}$$

Their ratio is

$$\frac{I_{round}}{I_{flat,max}} = \frac{\xi_{round,max}}{\xi_{flat,x,max}}\left(1 - \nu_0^2\right) = 2\left(1 - \nu_0^2\right) \tag{30}$$

if we take $\xi_{flat,x,max}$=0.05.

## D. Absolute Luminosity

Using

$$\frac{\gamma}{e r_e} = \frac{E}{m_e c^2 e r_e} = \frac{5.29x10^9}{0.511x10^6 x1.6x10^{-19} x2.82x10^{-13}} C^{-1}cm^{-1} = 2.29x10^{35} C^{-1}cm^{-1} \tag{31}$$

the luminosity is

$$\hat{L}_{flat}(10^{33}cm^{-2}\sec^{-1}A^{-1}) = 114.5\frac{\xi_{flat,max}}{\beta^*{}_{flat}(cm)}\frac{H_{flat}(r_{flat},\nu_0,\alpha)}{G_{flat}(r_{flat},\nu_0)}$$

$$\hat{L}_{round}(10^{33}cm^{-2}\sec^{-1}A^{-1}) = 229\frac{\xi_{round,max}}{\beta^*{}_{round}(cm)}H_{round}(r_{round}) \tag{32}$$

Using $\xi_{round,max}$=0.1, $\beta^*{}_{round}$=3 cm,

$$\hat{L}_{round}(10^{33}cm^{-2}\sec^{-1}A^{-1}) = 7.63 H_{round}(r_{round}) \tag{33}$$

Similarly, using $\xi_{flat,max}$=0.05, $\beta^*{}_{flat}$ = 1 cm,

$$\hat{L}_{flat}(10^{33}cm^{-2}\sec^{-1}A^{-1}) = 5.73\frac{H_{flat}(r_{flat},\nu_0,\alpha)}{G_{flat}(r_{flat},\nu_0)} \tag{34}$$

In Fig. 5, $\hat{L}_{round}(10^{33} cm^{-2} sec^{-1} A^{-1})$ (dashed line) and $\hat{L}_{flat}(10^{33} cm^{-2} sec^{-1} A^{-1})$ (solid line) are plotted vs. $\sigma_S$.

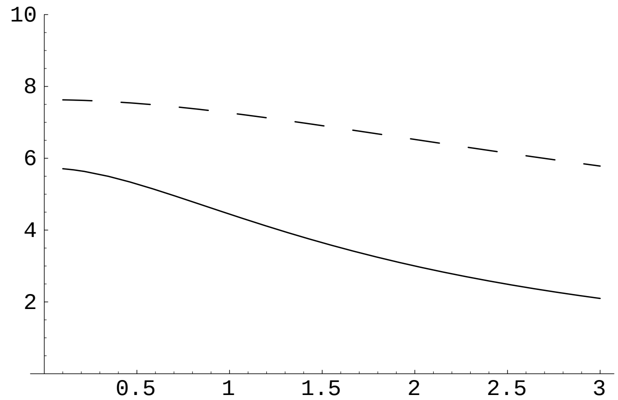


Fig.5
$\hat{L}_{round}(10^{33} cm^{-2} sec^{-1} A^{-1})$ (dashed line) and $\hat{L}_{flat}(10^{33} cm^{-2} sec^{-1} A^{-1})$ (solid line)
vs. $\sigma_S$(cm) (abscissa)

In Fig. 6, I plot the value of $\beta^*_{x,max}$ vs. $\sigma_S$, for the flat beam, from Eq. (21), with the tune shift parameters as given above, and with $\beta^*$=1 cm and $k$=0.01.

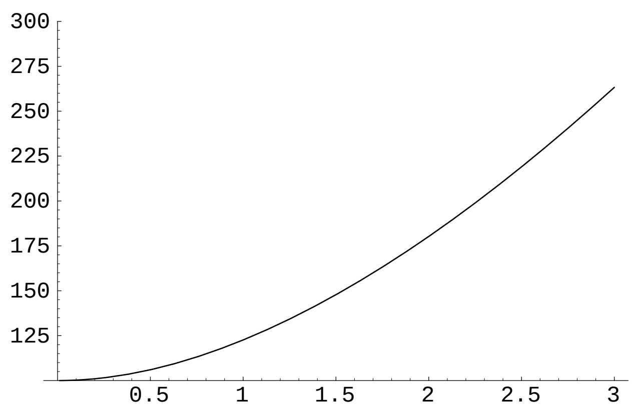


Fig. 6
$\beta^*_{x,max}$(cm) (ordinate) vs. $\sigma_S$ (cm) (abscissa) for the flat beam
with $\beta^*$=1 cm and $k$=0.01

## E. Current per Bunch and Number of Bunches

In order to saturate the beam-beam limit, the current per bunch in the round beam case, from Eq.(29), must satisfy

$$I_{round}(mA) = 1438\xi_{round,max}\varepsilon(\mu rad - m) \tag{35}$$

For an equilibrium emittance of $\varepsilon$ = 0.2 μm-rad, and the above assumed maximum tune shifts, the saturation current is $I_{round} = 28.8 \;\; mA$.

In the flat beam case, the maximum current per bunch, from Eq. (25), is given by

$$I_{flat,max}(mA) = \frac{1438}{\left(1-\nu_0^2\right)}\xi_{flat,x,max}\varepsilon(\mu rad - m) \tag{36}$$

For the above assumed maximum tune shift and $\varepsilon$ = 0.2 μm-rad, the maximum current per bunch is $I_{flat,max} \approx 14.4 \;\; mA$. Fig. 7 plots the number of bunches, required to maximize the luminosity, vs. the total current, for the flat and round beam cases (for the flat beam case, this is the *minimum* number of bunches). For this plot, I use the values quoted above for tune shifts and the equilibrium emittance.

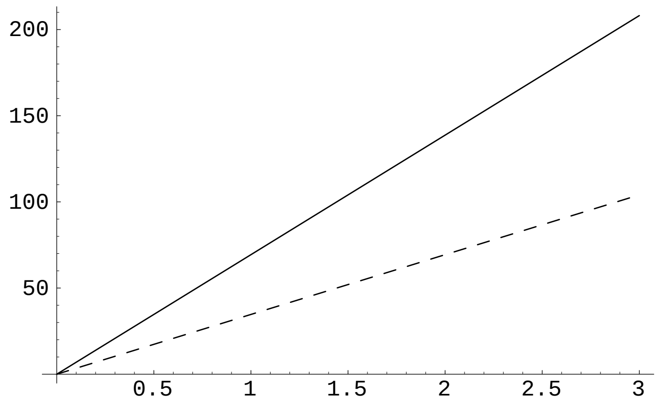


Fig. 7

Number of bunches to saturate the beam-beam limit (ordinate), vs. total current per beam (Amp) (abscissa), (dashed: round beam; solid: flat beam)

for $\varepsilon$ = 0.2 $\mu$m-rad

With no crossing angle in the round beam case, the bunch separation would have to be at least twice the distance from the IP to the closest electrostatic separator. Taking this distance to be 5 m in an optimized IR design, the number of round beam bunches would be limited to about 80. For operation at the tune shift limit (29 mA per bunch) the total current would then be about 2.3 A per beam. From Fig. 5, the luminosity in this case is about $1.7 \times 10^{34}$ $cm^{-2}sec^{-1}$ for $\sigma_S$=1 cm.

In the flat beam case, where a crossing angle is used, the maximum number of bunches depends on how large a crossing angle is possible, which depends in turn on the details of the optics in the interaction region. For a crossing angle of 2.5 mrad, a bunch spacing of 14 nsec appears to be tolerable, allowing 180 bunches per beam. Operation at the tune shift limit of about 14 ma per bunch gives a total current of 2.5 A per beam, and from Fig. 5, a luminosity of $1.1 \times 10^{34}$ $cm^{-2}sec^{-1}$.

As Eq. (35) and (36) show, the required current per bunch scales with the equilibrium emittance, and may be reduced or increased by changing $\varepsilon$ (modifying the lattice). Thus, in both the round and flat beam cases, if more total current is available, these luminosities could be increased by increasing the equilibrium emittance.

## VI. CONCLUSION

Estimates have been made of the bunch length dependence of the luminosity per unit current for round and flat beams in CESR. I have taken the flat beam vertical tune shift limit to be 0.05, and used $\beta_y^*$ =1 cm for the flat beam. For a round beam tune shift limit of 0.1, and assuming that $\beta^*$ for the round beam optics is limited to 3 cm, the round beam luminosity per unit current is expected to be greater than that of the flat beam by a factor of 1.6 to 1.9, for $\sigma_S$ in the range of 1-1.5 cm. The dependence of luminosity on $\sigma_S$ is weaker for the round beam than for the flat beam.

The number of bunches is limited in the round beam case because the crossing angle must be very small. Round beam operation at the tune shift limit, with $\varepsilon$=0.2 $\mu$m-rad, with 80 bunches, corresponds to 2.3 A/beam and gives a luminosity of about $1.7 \times 10^{34}$ $cm^{-2}sec^{-1}$ for $\sigma_S$ =1 cm. With a 2.5 mrad crossing angle and 180 bunches, flat beam operation at the tune-shift limit, with $\varepsilon$=0.2 $\mu$m-rad, corresponds to 2.5 A/beam and gives a luminosity of about $1.1 \times 10^{34}$ $cm^{-2}sec^{-1}$ for $\sigma_S$ =1 cm. The round beam thus has a 50% luminosity advantage, for this particular set of parameters. Neglecting the bunch length dependence, this advantage scales roughly as $\frac{\xi_{round,max}}{\xi_{flat,y,max}} \frac{\beta^*_{flat}}{\beta^*_{round}}$.

## VIII. REFERENCES

[1] K. Takayama, " A New Method for the Potential of a 3-Dimensional Nonuniform Charge Distribution", *Nuovo Cimento Letters*, 34,7 p 190 (1982)
[2] G. Guignard, "Selection of Formulae Concerning Proton Storage Rings", CERN 77-10
[3] S. Milton, "Calculation of How the Ratio $\beta^*/\sigma_{\text{bunch length}}$ Affects the Maximum Luminosity Obtainable: The "Hourglass Effect", CBN 89-1

Appendices I and II are available from the author on request.