A Time Domain method for the Calculation of Loss Factors and Wake Potentials

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ABSTRACT: We present a method for calculating the loss factors of a cavity using a time domain method. We compare the results thus obtained with that of the more traditional frequency domain method and show that they yield almost identical results. Results obtained from this method are used to predict the wake potentials produced for different length bunches. The predictions agree in the long range with calculations obtained from from the frequency domain. This method becomes the basis for an extremely fast algorithm for calculating the long range wake potentials of different bunch lengths for long time scales (but short compared to the damping time of the particular mode) that would otherwise incur large computational overheads. In any particular application, only a single MAFIA T2 run will be necessary.

INTRODUCTION

We present in this paper a method for calculating the loss factors of a cavity[1] using a time domain method. We will assume that the bunch is travelling at $\beta = 1$ throughout this paper. The method consists of the following steps

(*i*) Calculation of the wake potential excited in a cavity due to a ring of charge with azimuthal moment m. The ring beam is assumed to have a δ-function charge distribution in the longitudinal directio

(*ii*) Identification of the strongest resonances of the cavity by examining the power spectrum of the wake potential in frequency space. This spectrum is deconvolved to approximate the spectrum of a corresponding δ-function longitudinal charge distribution.

(*iii*) Using a linear least squares fit on the wake potential using only the strong resonances. The coeffi- cients from the least squares fit will then give us the loss factors λ_k (where k is the mode index).

We will apply this to the prebuncher cavity of Wilson Laboratory. The results are compared to the more traditional frequency domain method. We will show that these two methods yield almost identical results. B using only the λ_k's calculated from the method above, we then made predictions about the wake potentials for beams with different longitudinal charge distributions. We compared results from predictions to tho calculated by MAFIA. These showed excellent agreement between the two methods. The method used for the predictions incurs negligible computation time as compared to calculations of the wake potenti using MAFIA. In practice, we need only run T2 once, apply the algorithm outlined above, and we can predict the long range, long time (but short compared to the damping time of the mode) wake potentials for any longitudinal charge distribution. Thus, this method forms a basis for an extremely fast algorit for calculating long range wake potentials for long time scales.

The work horses for calculating the wake potentials and eigenmodes of the prebuncher cavity are MAFIA T2 and MAFIA E modules. The wake field formalism is thoroughly discussed in an upcoming paper and will not be repeated here. Our notation follows that[1] of Chao.

The geometry of the prebuncher cavity used in all the MAFIA calculations is shown in Figure 1.

MAFIA T2 Calculations

MAFIA calculates the wake potentials Φ using the T2 module. Unfortunately, we can only use ring beams that are gaussian in ζ, the longitudinal direction. In order to use the results from MAFIA for th

[1] Physics of Collective Beam Instabilities In High Energy Accelerators, A. W. Chao, Wiley Series in Bea Physics and Accelerator Technology.

Figure 1 The geometry of prebuncher used in all the MAFIA calculations.

time domain calculation, we will have to deconvolve out Φ from the the wake potential calculated by MAFIA. The flowchart for doing this shown in Figure 2. We show the theory below.

The gaussian beam in MAFIA can be written as

$$\mathcal{D}(\zeta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\zeta-\bar{\zeta})^2}{2\sigma^2}\right] \tag{1}$$

so that $\int_{-\infty}^{\infty} d\zeta\, \mathcal{D}(\zeta) = 1$. The Fourier transform† of (1) is

$$\tilde{\mathcal{D}}(k) = e^{-ik\bar{\zeta}} e^{-\sigma^2 k^2/2} \tag{3}$$

Let ϕ be the wake potential generated by MAFIA, therefore by superposition

$$\phi(\zeta) = \int_{-\infty}^{\infty} d\zeta'\, \mathcal{D}(\zeta-\zeta')\Phi(\zeta') \tag{4}$$

† Throughout this paper, we will define the Fourier transform as

$$f(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\zeta} \tilde{f}(k)\, dk \tag{2}$$

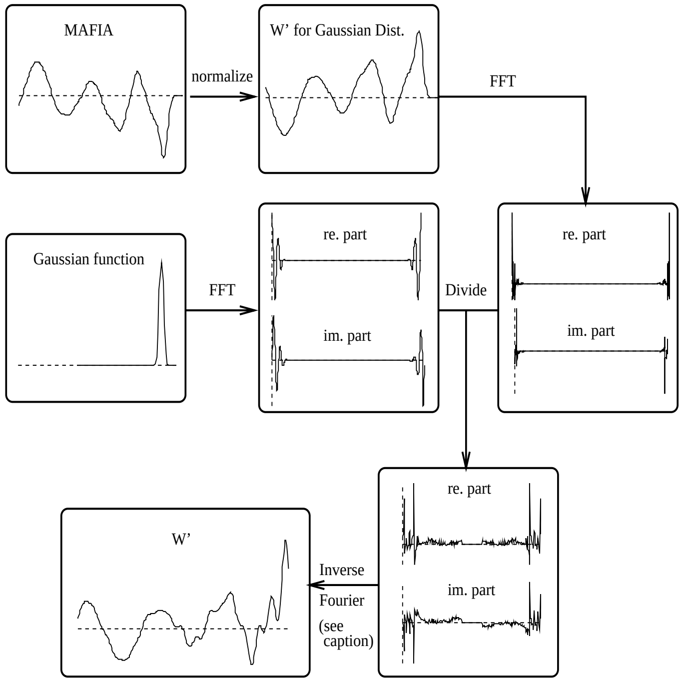

Figure 2 The above flowchart shows the method for deconvolving out the δ-function response of the cavity. It is important that the final Fourier transform be done with the definition shown in this paper in order to get the correct normalization.

Fourier transforming the above by using our definition shown in (2), we have

$$\begin{aligned}\tilde{\Phi}(k) &= \tilde{\phi}(k)/\tilde{\mathcal{D}}(k) \\ &= \tilde{\phi}(k)e^{\sigma^2k^2/2}e^{ik\bar{\zeta}}\end{aligned} \tag{5}$$

Thus by inverse Fourier transforming (5), we can obtain Φ. We then use

$$\left.\begin{aligned}\Phi_r &= -eI_mW_m(\zeta)mr^{m-1}\cos m\theta \\ \Phi_\theta &= eI_mW_m(\zeta)mr^{m-1}\sin m\theta \\ \Phi_s &= -eI_mW'_m(\zeta)r^m\cos m\theta\end{aligned}\right\} \tag{6}$$

to extract W'_m and W_m.†

MAFIA T2 results for $m=0$ and $m=1$ excitations

The result of the wake potential $\phi_{\|}$ due to an $m=0$ excitation calculated by MAFIA T2 is shown in Figure 3. After deconvolving the result from MAFIA, and doing the required renormalization, $W'_0(\zeta)$ is

† The notation throughout this paper is from Chao.

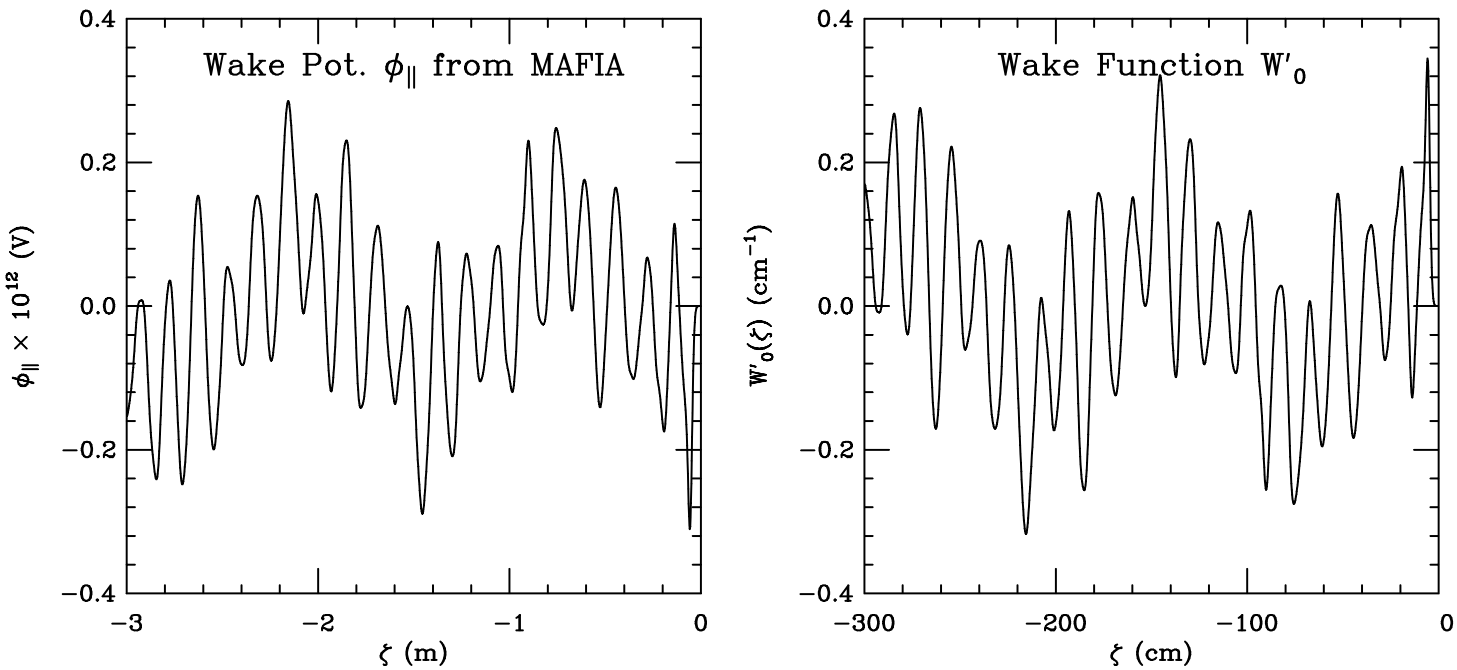

Figure 3 The wake potential $\phi_{\parallel}$ of the prebuncher calculated using MAFIA T2 for m=0 is shown on the left. The gaussian bunch used to create the wake field has σ_{ζ} =1 cm and is centred at ζ =−5 cm. It has a total charge of 1 C (Note: MAFIA uses MKS units). The figure on the right shows $W'_0(\zeta)$ obtained from the graph on the left by normalizing out the charge using (6) and converting it to gaussian units.

shown in Figure 7 for a point charge (i.e. a δ-function) moving along the axis of symmetry. As noted in the caption of the figure, $W'_0(\zeta)$ satisfies the properties outlined on page 62 of Chao. A similar deconvolution is also applied to $\phi_{\parallel}$ and $\phi_{\perp}$ due to an m=1 excitation. Again it is clear from Figure 6 that W'_1 is cosine-like and W_1 is sine-like.

MAFIA E Calculations

MAFIA calculates the modes of the cavity using the E module. We calculated the first 40 monopole and dipole modes. Their values are shown graphically in Figure 8. The numarical values are shown in the Appendix.

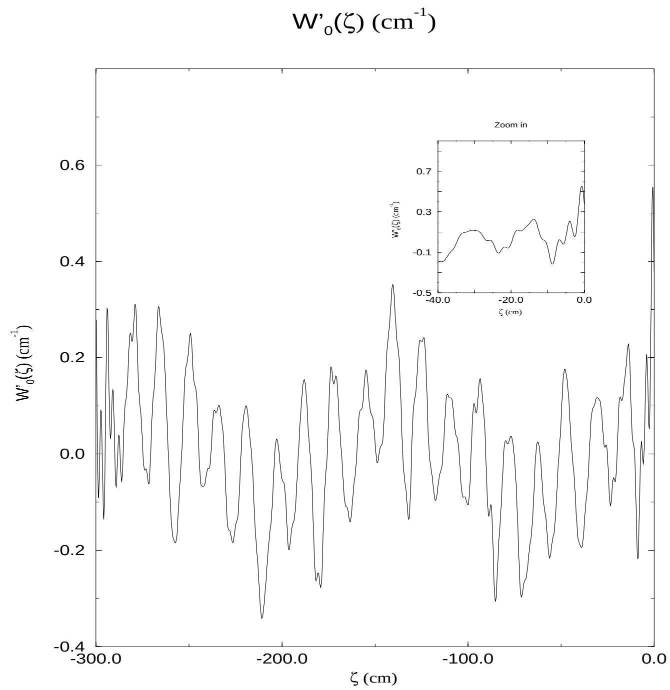

Figure 4 W_0' was calculated by the algorithm outlined in the text. Notice that at $\zeta = 0$, $W' > 0$ as required by the properties of the wake function shown on page 61 of Chao. The inserted graph is a zoom in for $-40 < \zeta < 0$ cm. It is clear from the zoom in that W_0' has a cosine-like behaviour. The slight roll-off at $\zeta = 0$ is because of the discontinuity at $\zeta = 0$ which is difficult to model numerically. The jagged edges that are seen at around -300 cm result from noise introduced into the data from the deconvolution.

CALCULATION OF LOSS FACTORS

We shall describe how the loss factors are calculated using the time domain method as well as the frequency domain method. The results from the time domain method are used to compare with the frequency domain method. We will see that both methods yield results that agree very well.

Calculation of Loss Factors using the Time Domain method

To calculate the loss factors, we will first identify all the significant resonances by Fourier transforming the wake functions that were calculated in the previous section. For comparison purposes, we will identify these strong resonances with the corresponding resonances calculated by MAFIA E because MAFIA E has a better resolution compared to the Fourier transform method. Using linear least squares, we can fit the

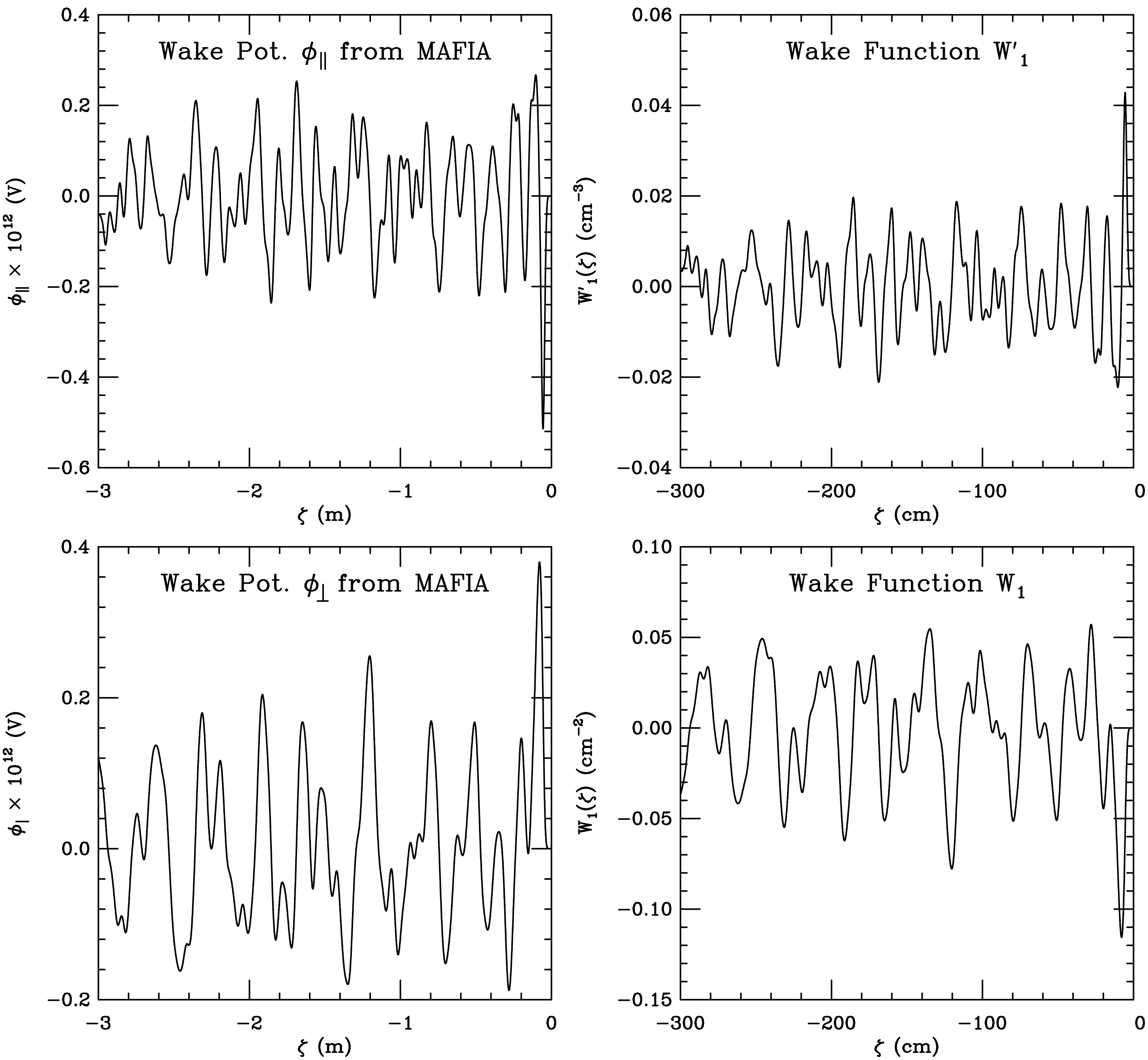

Figure 5 The wake potential $\phi_{\|}$ and $\phi_{\perp}$ of the prebuncher calculated using MAFIA T2 for $m = 1$ are shown in the first column. The gaussian bunch used to create the wake field has $\sigma = 1$ cm and is centred at $\zeta = -5$ cm. It has a total charge of 1 C (Note: MAFIA uses MKS units). The column on the right shows $W'_1(\zeta)$ and $W_1(\zeta)$ obtained from the MAFIA data in gaussian units. Notice the dimensions on the ordinates.

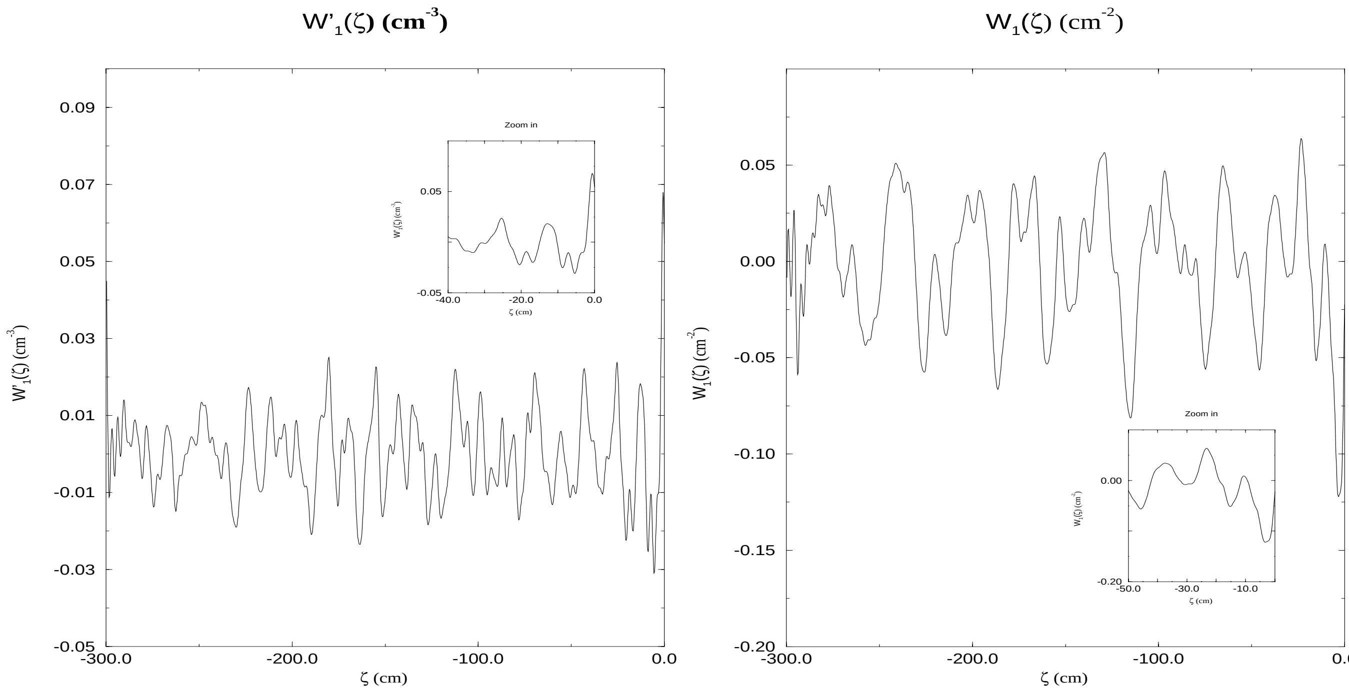

Figure 6 W_1' and W_1 as calculated by the algorithm outlined in the text. Notice that at $\zeta = 0$, $W_1' > 0$ as required by the properties of the wake function shown on page 61. The inserted graphs are zoom ins for $-40 < \zeta < 0$ cm. Again W_1' has a cosine-like behaviour. W_1 also behaves like a sine function as shown in Chao.

wake potential with the significant resonances, thus identifying all the loss factors $k_{m\lambda}$. We will only be interested in resonances below 3 GHz because 3 GHz is the lowest cutoff frequency of a 3 inch diameter pipe. The flowchart for this calculation is shown in Figure 9.

Figures 10 and 11 show the Fourier transforms of the wake functions. The strongest resonances are then used via the linear least squares method to fit the following functions w_m' and w_m defined to be

$$w_m'(\zeta) = \sum_{\substack{\text{significant} \\ \text{resonances: } \lambda}} 2k_{m\lambda} \cos \frac{2\pi f_\lambda}{c} \zeta \tag{7}$$

and

$$w_m(\zeta) = \sum_{\substack{\text{significant} \\ \text{resonances: } \lambda}} \frac{2k_{m\lambda} c}{2\pi f_\lambda} \sin \frac{2\pi f_\lambda}{c} \zeta \tag{8}$$

to W_0', W_1' and W_1 shown in Figures 3 and 5. The factor of 2 in the sums of (7) and (8) comes from Chao's definition of power which is $P = V_{\text{peak}}^2/2R$. The more usual definition is $P = V_{\text{peak}}^2/R$ which is used in most American contexts. Tables 1, 2 show the resonances chosen and the $k_{m\lambda}$ that were used to fit the data.

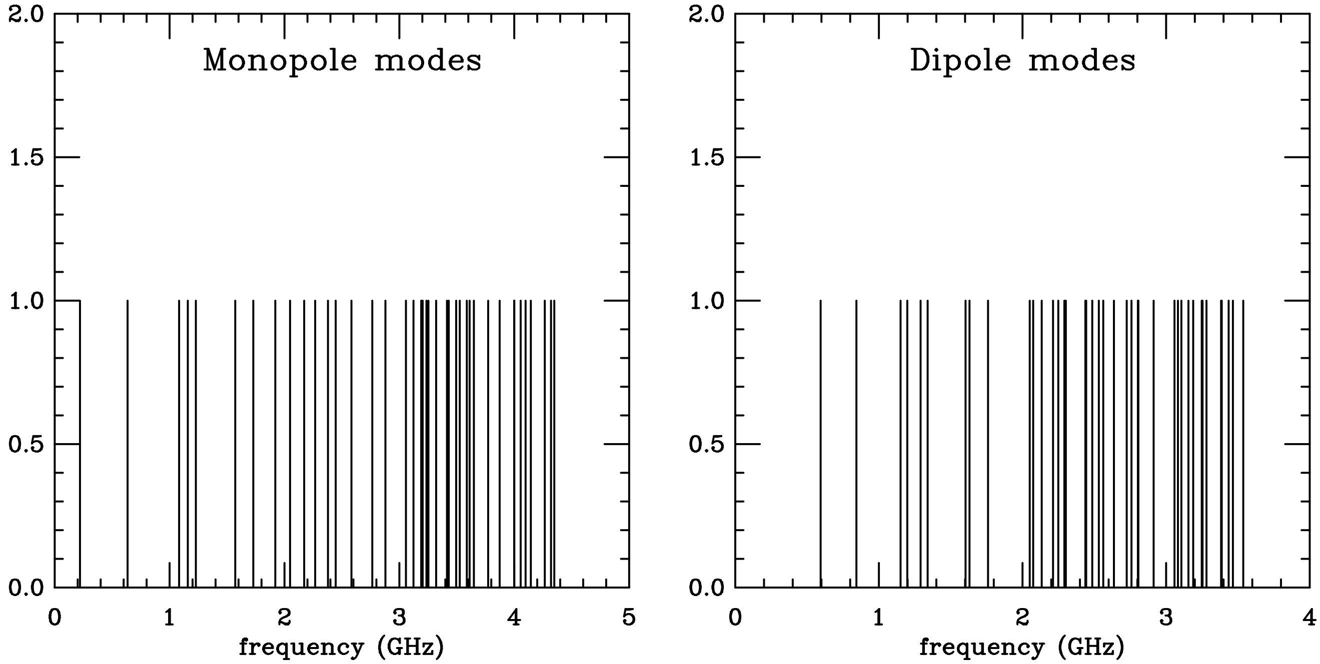

Figure 8 This figure shows the distribution of the first forty monopole and dipole modes calculated by MAFIA E. The actual numerical values of the modes are shown in the Appendix A.

Figures 12, 13, and 14 show the fits. Note: *only* the significant resonances identified from the power spectrum are used in the summation.

For the case of $m = 1$, we can obtain $k_{1\lambda}$ by fitting $W_1(\zeta)$ and $W_1'(\zeta)$. $k_{1\lambda}^{long}$ is the loss factor obtained by fitting $W_1(\zeta)$ while $k_{1\lambda}^{trans}$ is the loss factor obtained by fitting $W_1'(\zeta)$. Finally $k_{1\lambda}^{ave}$ is the average of the previously mentioned loss factors.

Table 1. $m = 0$ **Loss Factors** $k_{0\lambda}$

λ	**Frequency** (GHz)	$2 \times k_{0\lambda}$ (cm^{-1})	$(R/Q)_\lambda \omega_\lambda$ (cm^{-1})
1	0.220	0.103	0.106
2	0.634	0.036	0.036
3	1.229	0.040	0.043
4	1.920	0.151	0.157

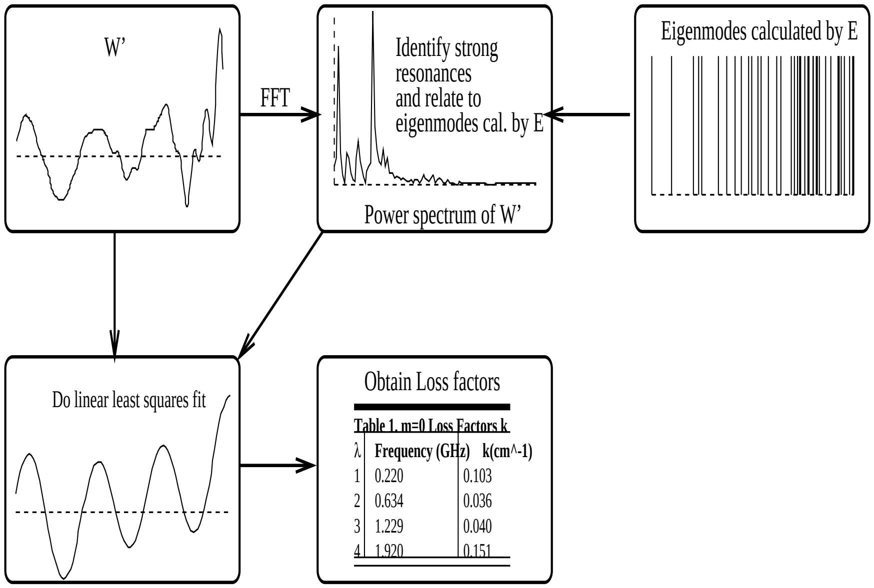

Figure 9 The flowchart for calculating the loss factors $k_{m\lambda}$ is shown.

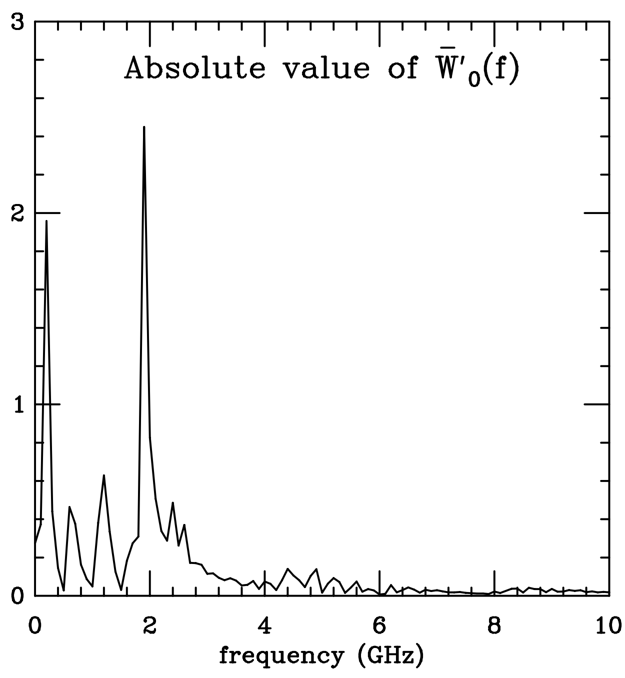

Figure 10 The Fourier transforms of the wake function enables us to identify the significant resonances that are present in the problem. All significant resonances of W_0' occur below 3 GHz.

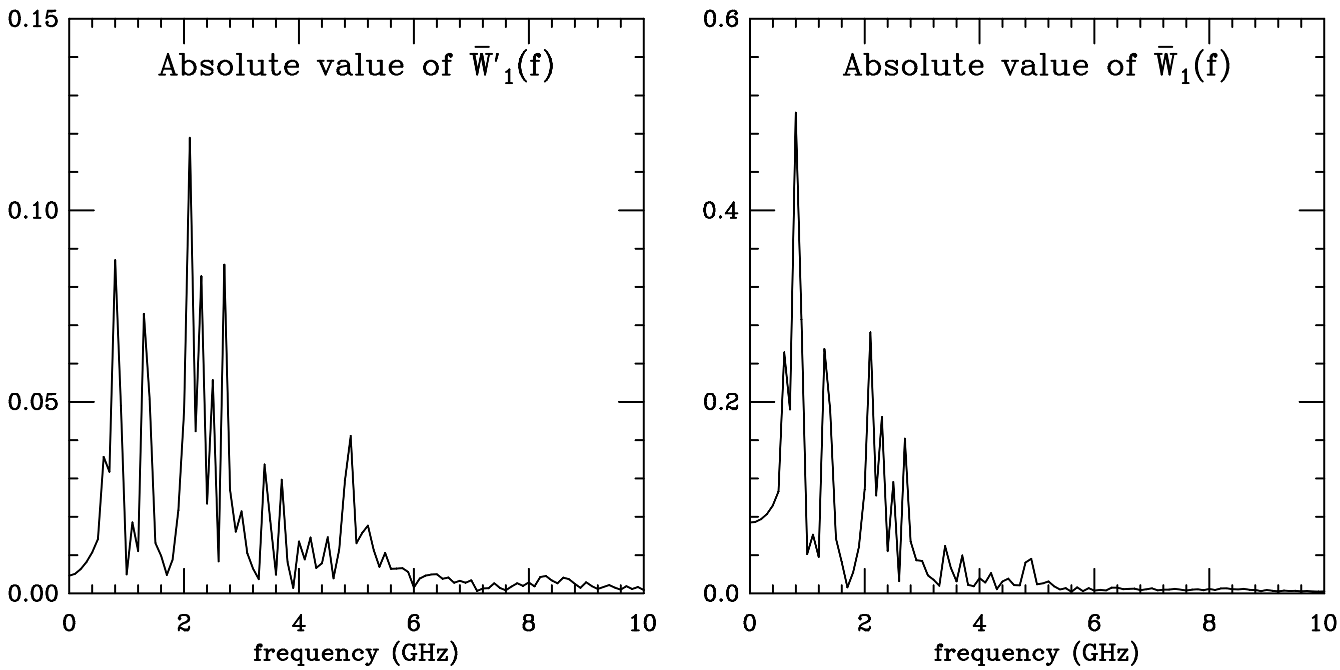

Figure 11 These figures show the Fourier transform of the wakefunction of W'_1 and W_1. The significant resonances occur below 6 GHz. However, we are only interested in trapped resonances. Resonances higher than 3 GHz propagate out of the prebuncher because 3 GHz is the lowest cutoff frequency for a 3 inch diameter pipe.

Table 2. $m = 1$ **Loss Factors** $k_{1\lambda}$

λ	**Frequency** (GHz)	$(2 \times k_{1\lambda}^{long}) \times 10^{-2}$ (cm^{-3})	$(2 \times k_{1\lambda}^{trans}) \times 10^{-2}$ (cm^{-3})	$(2 \times k_{1\lambda}^{ave}) \times 10^{-2}$ (cm^{-3})	$(R/Q)_\lambda \omega_\lambda \times 10^{-2}$ (cm^{-3})
1	0.594	0.135	0.145	0.140	0.135
2	0.844	0.551	0.569	0.560	0.574
3	1.340	0.512	0.512	0.512	0.544
4	2.134	0.849	0.850	0.850	0.899
5	2.213	0.316	0.350	0.333	0.316
5	2.292	0.554	0.618	0.586	0.423
6	2.486	0.331	0.386	0.356	0.303
7	2.726	0.090	0.090	0.090	0.147

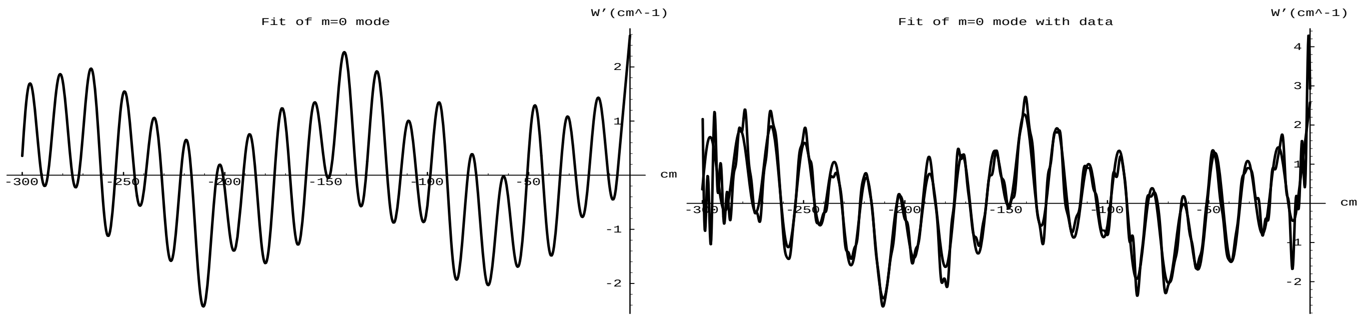

Figure 12 The figure on the left shows the linear least squares fit using the coefficients shown in Table 1. The figure on the right shows the superposition of W'_0 and the least squares fit.

Calculation of Loss Factors using the Frequency Domain method

In the frequency domain, the loss factor is simply given by

$$2 \times k_{m\lambda} = (R/Q)_{m\lambda}\,\omega_\lambda \tag{9}$$

However, we must note that the dimension of $[R] = \Omega/2^m$. So to distinguish this from ordinary resistance, we introduce the notation $\mathcal{R}$ with dimension Ω. For the prebuncher cavity which is made of copper, we will assume that it has a surface resistance of $1.725 \times 10^{-8}\ \Omega m$. With this number, we can then calculate the Q of the cavity as well as the power loss per cycle U due to heating of the cavity walls. We demonstrate this below.

Q is relatively easy to calculate from its definition

$$Q = \omega\,\frac{\text{energy stored in cavity}}{\text{power loss per cycle}} \tag{10}$$

To calculate R, we have to calculate $\mathcal{R}$ first from U by using the relationship

$$U = \frac{V^2}{2\mathcal{R}} \tag{11}$$

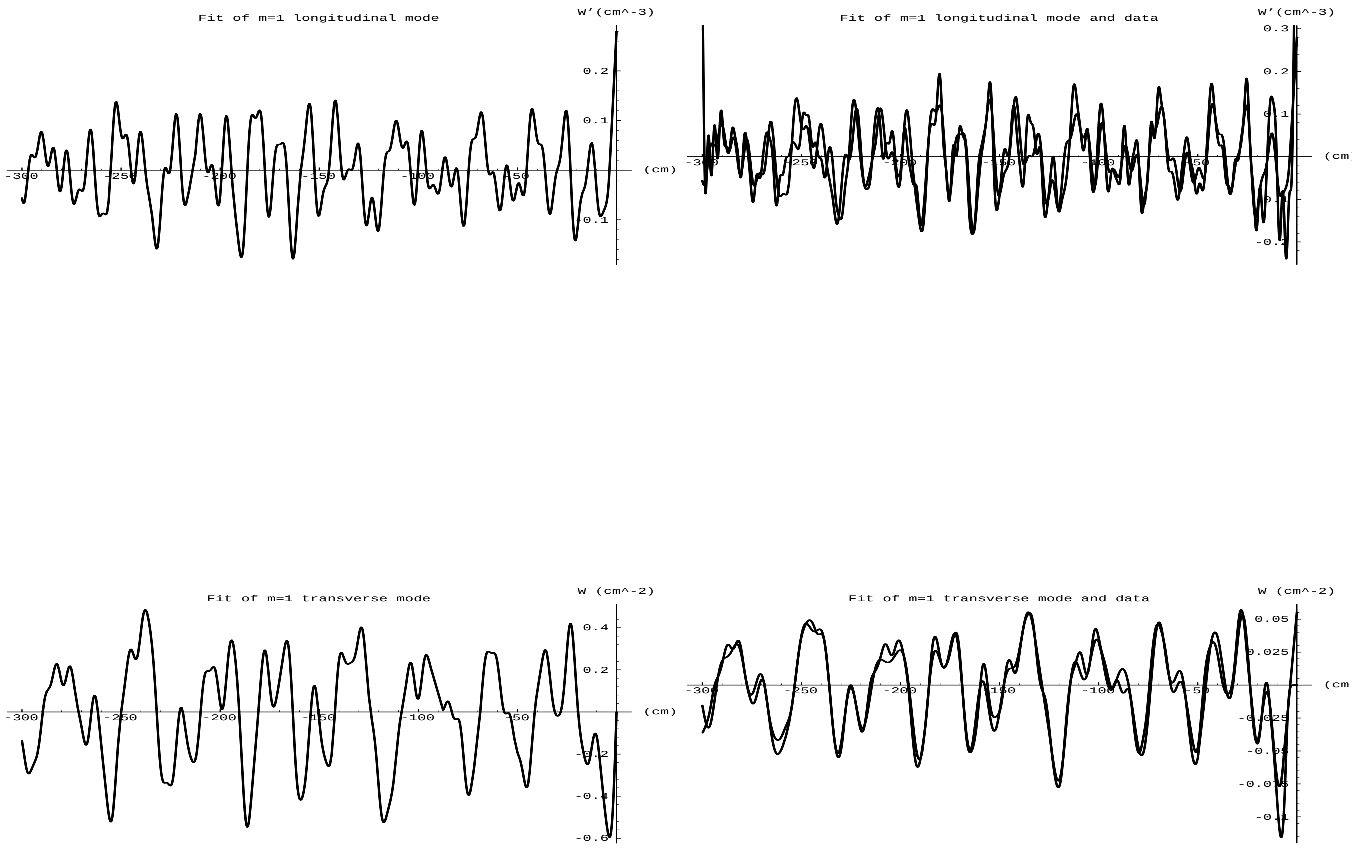

Figure 13 The figures in the left column shows the linear least squares fit using the coefficients shown in Table 2. The figures in the right column show the superposition of W_1 and W_1 and the least squares fit.

where

$$V = \int_{-L/2}^{L/2} ds\, E(r = a, s) \tag{12}$$

is the peak voltage for a *standing* wave $E = E(r, s)e^{i\omega t}$, a is some convenient off-axis r where we do the integration. (Example: for dipole mode $m = 1$, if we integrate on axis (12) will be zero. Thus we have to

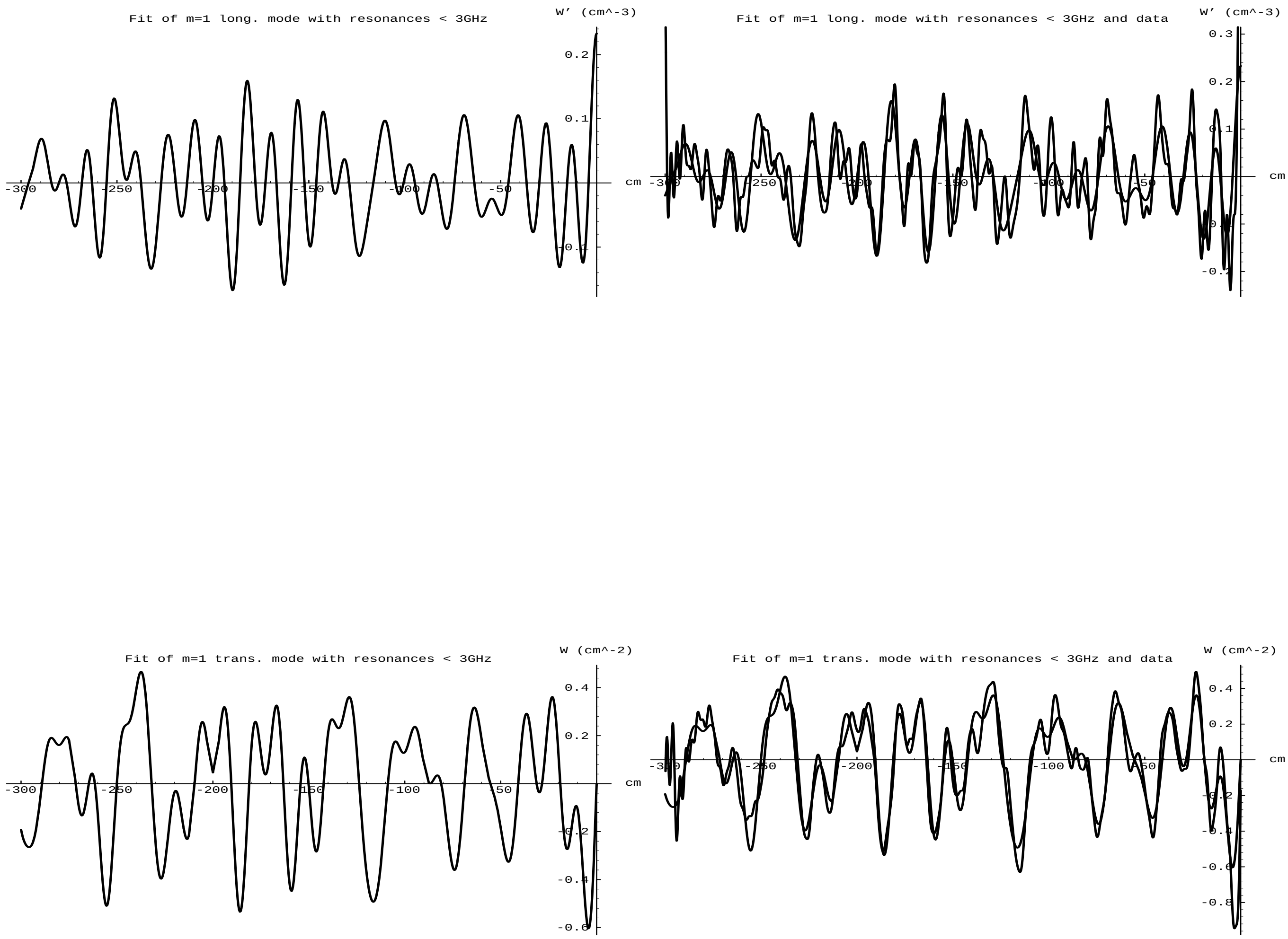

Figure 14 The figures in the left column show the linear least squares fit using *only* the coefficients below 3 GHz shown in Tables 2. The figures in the right column show the superposition of W_1' and W_1 and the least squares fits. Although resonances above 3 GHz were removed from the fit, the superposition of the data and fit shows that long range effects are still retained.

integrate off-axis.). Therefore,

$$\mathcal{R} = \frac{V^2}{2U} \tag{13}$$

at $r = a$. Now, $\mathcal{R}$ is calculated for a unit ring of charge at $r = a$ with *no* azimuthal dependence. Now R has an azimuthal dependence of $\cos m\theta$. Thus the relationship between $\mathcal{R}$ and R is

$$R = \frac{\mathcal{R}}{a^{2m}\langle \cos^{2m} m\theta \rangle} \tag{14}$$

Thus for $m = 0$

$$R = \mathcal{R} \tag{15}$$

and for $m = 1$

$$R = 2\mathcal{R} \tag{16}$$

PREDICTIONS

To demonstrate the use of the longitudinal loss factors that we have calculated, we used Table 1 to generate the $m = 0$ wake function for a $\sigma = 2$ cm gaussian beam centred at $\zeta = 10$ cm. We then compared this prediction with the wake function calculated by MAFIA T2. As can be seen from Figure 15, the agreement between the two methods is extremely good far from the head of the beam. Short range agreement is, however, poor. This is not surprising because we have truncated higher order resonances (i.e. we have neglected short range effects) in the generation of Table 1.

A similar calculation was done for $m = 1$. Again the predicted result and the MAFIA T2 calculation differed near the head of the bunch, but agreed well in the long range.

CONCLUSION

We have shown that the loss factors calculated using our time domain method agrees well with that obtained from the frequency domain. By using only the significant loss factors calculated from the linear least squares fit, we can make predictions of wake potentials for different bunch lengths and for long time scales that are less amenable from more traditional methods.

ACKNOWLEDGEMENTS

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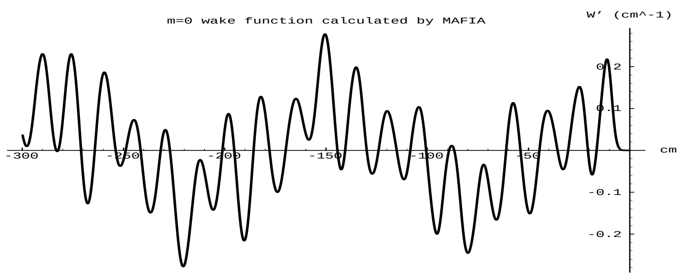
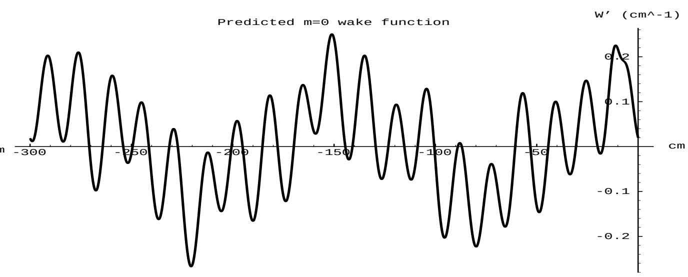
APPENDIX A

The tables in this Appendix were calculated using the MAFIA E module for the prebuncher. We have assumed that the prebuncher is made of copper which has a surface conductivity of 5.8×10^7 mho m^{-1}. For resonances higher than 3 GHz, the Q values calculated are unreliable because these modes are no longer trapped. To convert resistance in MKS units to cgs units, $\Omega \rightarrow 1/(9 \times 10^{11})$ cm^{-1} s.

Table A.1 $m = 0$ Resonances identified by MAFIA E module

Frequency (GHz)	Q	R/Q (Ω)	**Frequency** (GHz)	Q	R/Q (Ω)
0.220	19107	69.177	3.235	32388	14.259
0.634	25684	8.203	3.253	38677	4.612
1.083	27271	2.522	3.320	30648	4.385
1.158	32324	0.103	3.414	42615	0.007
1.229	25195	4.988	3.430	51380	0.023
1.571	27139	0.685	3.494	39782	0.916
1.729	38917	0.780	3.526	31617	12.997
1.920	32219	11.724	3.587	39525	1.011
2.049	32432	0.611	3.612	36585	7.392
2.171	36115	0.390	3.649	40005	0.843
2.267	35916	0.414	3.772	41440	7.552
2.378	43669	1.631	3.873	37138	2.747
2.446	31562	0.284	4.000	56320	0.208
2.583	37659	0.893	4.000	46969	1.857
2.764	37120	0.097	4.056	39605	1.093
2.878	56376	0.133	4.098	38947	0.546
3.057	40047	0.075	4.144	42204	0.040
3.122	40471	0.118	4.266	42781	6.739
3.190	41151	1.722	4.320	44978	0.072
3.203	30681	8.442	4.348	49383	1.506





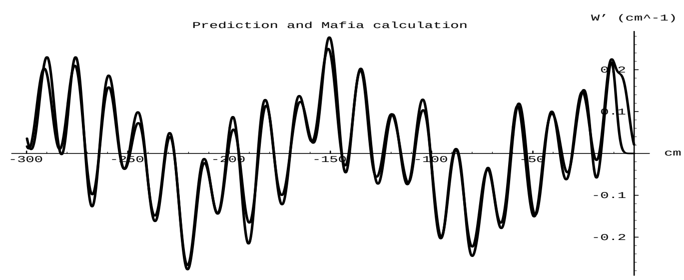


Figure 15 This figure shows the longitudinal wake function that is generated for a gaussian beam with $\sigma = 2$ cm centred at $\zeta = 10$. We calculated the wake function using Table 1 and MAFIA T2. As can be seen the wake function generated by the two methods agree very well. The agreement near the head of the bunch is poor. This is not surprising because we have truncated higher order resonances (i.e. we have neglected short range effects) in the generation of Table 1.

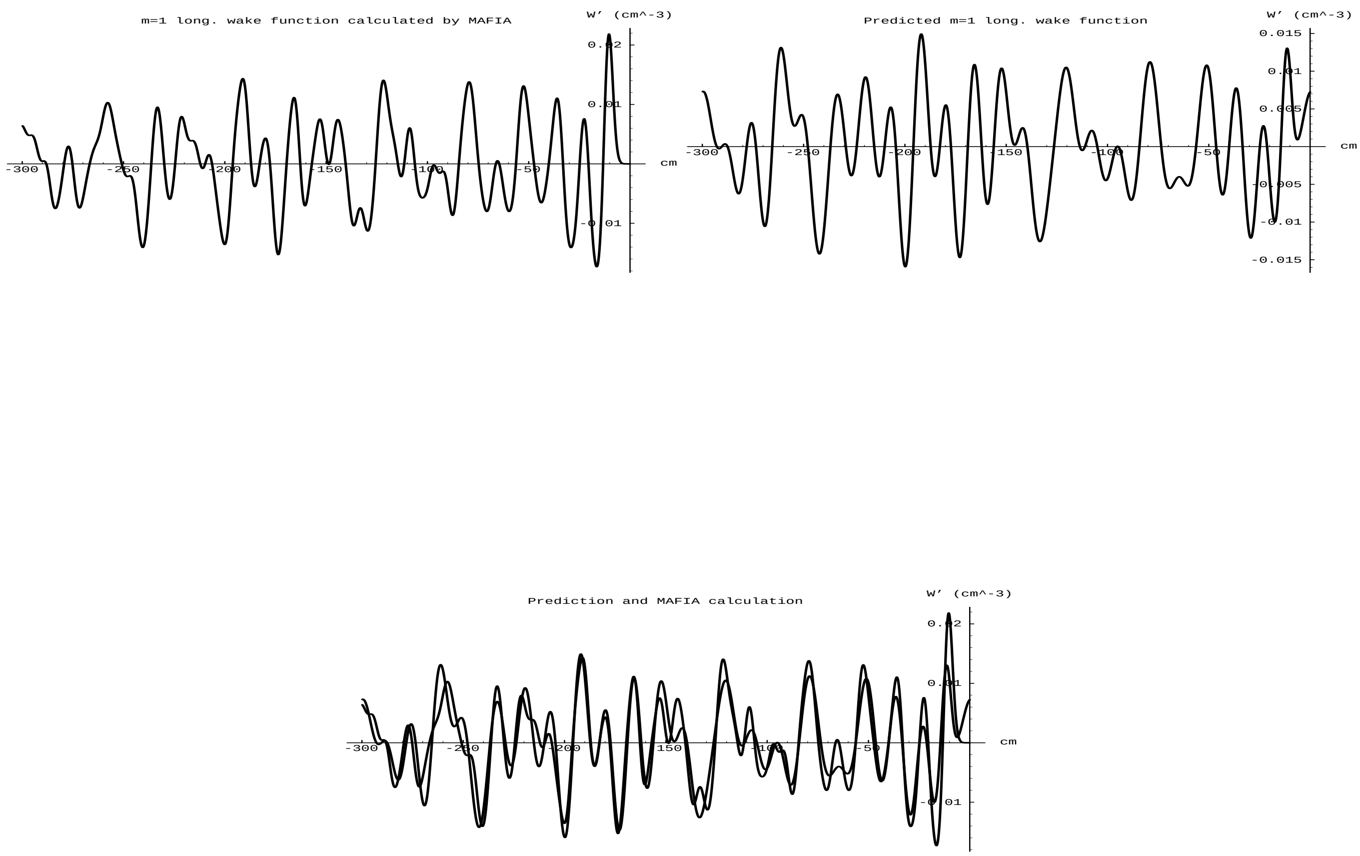


Figure 16 This figure shows the longitudinal wake function W' that is generated for a gaussian beam with $\sigma = 2$ cm centred at $\zeta = 10$. We calculated the wake function using Table 2 and MAFIA T2.

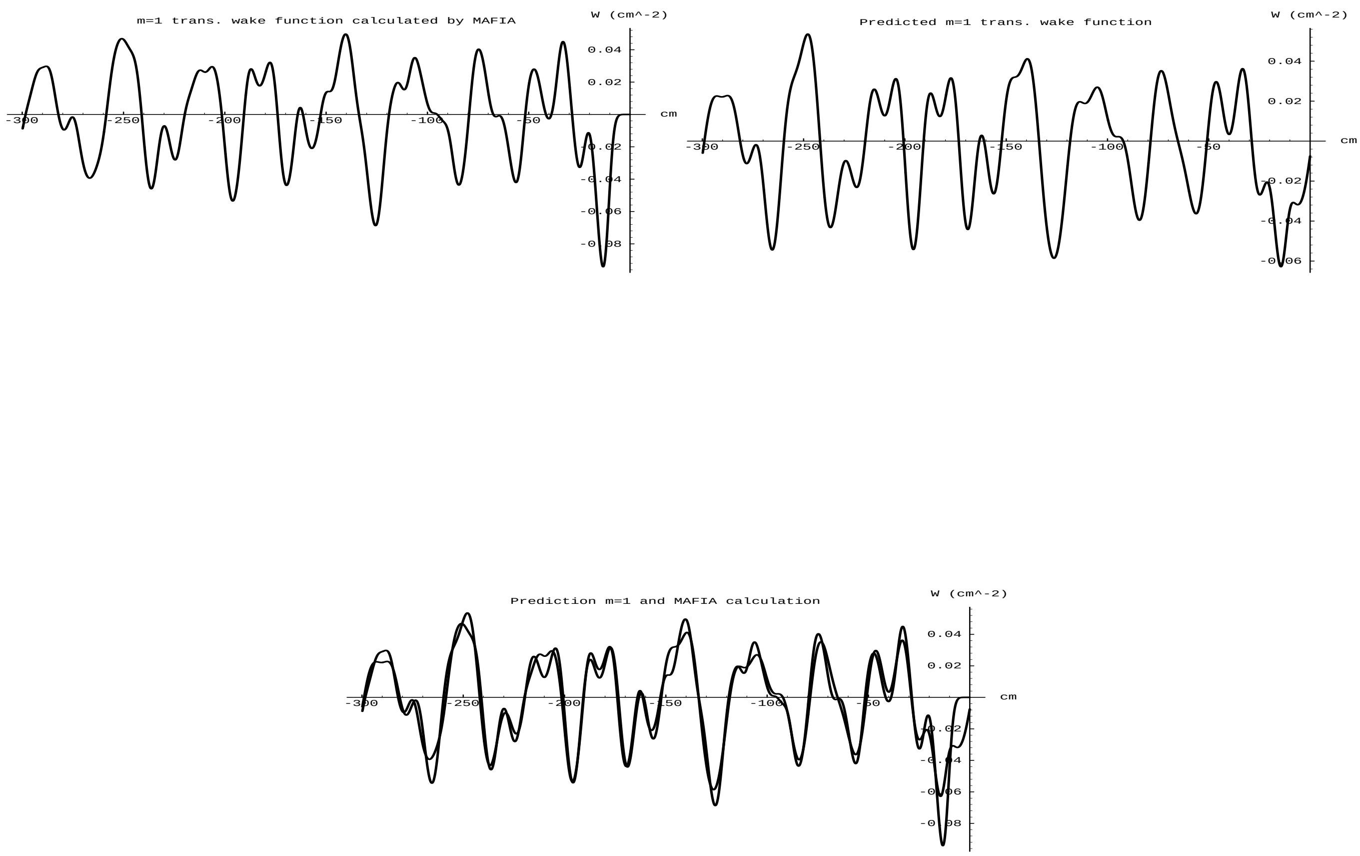


Figure 17 This figure shows the transverse wake function W that is generated for a gaussian beam with $\sigma = 2$ cm centred at $\zeta = 10$. We calculated the wake function using Table 2 and MAFIA T2.

Table A.2 $m=1$ Resonances identified by MAFIA E module

Frequency (GHz)	Q	R/Q (Ω/m^2)	Frequency (GHz)	Q	R/Q (Ω/m^2)
0.594	25531	3261.4	2.563	82616	90.9
0.844	25477	9747.4	2.636	27553	1142.6
1.151	29209	1487.1	2.726	29423	2199.2
1.198	34924	70.7	2.758	69214	32.2
1.291	44050	292.2	2.803	40154	1069.5
1.340	27985	5815.1	2.806	24715	407.4
1.604	34033	314.6	2.913	55328	759.6
1.630	43070	228.1	3.058	30732	114.5
1.760	44268	269.0	3.082	80884	45.9
2.051	41038	683.8	3.107	41449	230.2
2.074	47627	853.0	3.155	79692	5.6
2.134	39456	6032.9	3.188	42619	250.1
2.213	35508	2047.1	3.248	56993	18.9
2.249	52838	16.2	3.253	45416	133.5
2.292	22127	2646.5	3.281	34729	67.3
2.301	42682	788.3	3.383	63601	47.3
2.438	49857	583.6	3.387	44631	1429.1
2.443	42289	1254.3	3.436	41597	56.8
2.486	28728	1743.3	3.466	54838	370.4
2.531	22098	351.2	3.537	34425	131.4