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The Effect of Coupling on Luminosity Performance

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1 Introduction

In a storage ring the existence of skew quadrupoles, solenoids, and other coupling elements breaks the independence of the horizontal and vertical motions. With the flat beams used in electron/positron colliding beam storage rings this coupling results in an increase in the vertical beam size with an attendant loss in luminosity. It is useful in dealing with coupling to be able to relate how severe the luminosity degradation is for a given amount of coupling. To this end it is useful to define a 'badness' parameter B_c given by

$$B_c \equiv \frac{\mathcal{L}(\text{BBI}) - \mathcal{L}(\text{BBI+Coup})}{\mathcal{L}(\text{BBI})} \, , \tag{1}$$

where $\mathcal{L}$(BBI+Coup) is the luminosity obtained with coupling present, and $\mathcal{L}$(BBI) is the luminosity without coupling and only the beam–beam interaction to determine the beam size (and hence the luminosity). With this definition for B_c the condition needed so that the coupling is negligible is simply

$$B_c \ll 1 \, . \tag{2}$$

The usefulness of B_c comes when we can relate it directly to the coupling so that by measuring the coupling it is possible to determine whether the coupling is strong enough to cause harm. This is the problem to be addressed in the rest of the paper. To keep the calculations simple, it will be assumed, as is generally true in practice, that the coupling is weak.

As will be seen, there are two components to B_c. One component is due to vertical blowup of the beams mentioned above. The other component is due to the beams being tilted with respect to one another in the transverse (x–y) plane. This second component is not present if the opposing beams follow the same trajectory since, in

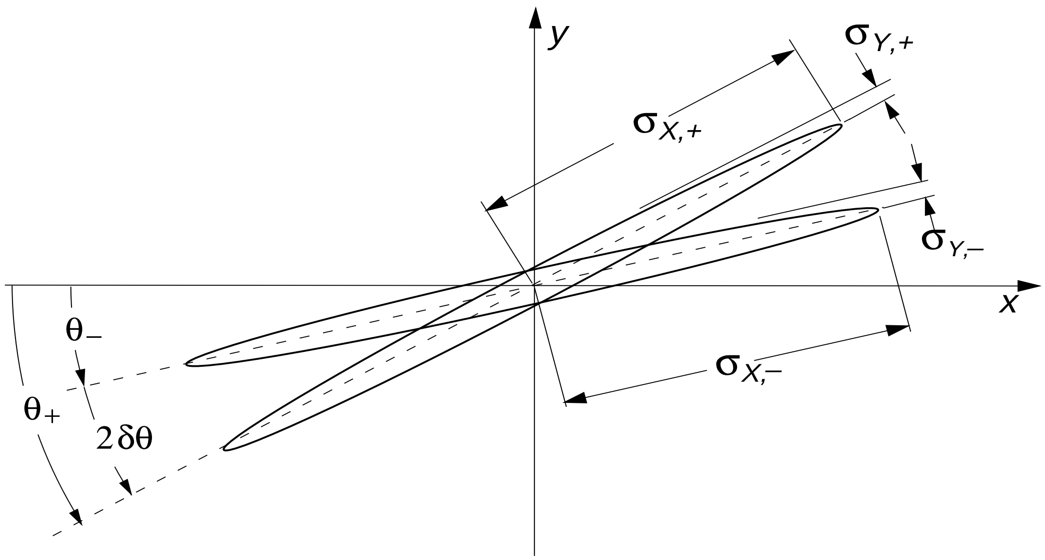

Figure 1: 1σ beam envelopes

this case, there is time reversal symmetry. However, with a pretzeled orbit, or with a two ring machine, the coupling each beam sees is different and the symmetry is lost. The calculation of B_c starts with a calculation of the luminosity which given by

$$\mathcal{L} = f \cdot \mathcal{O} \,, \tag{3}$$

where f is the frequency of collisions and $\mathcal{O}$ is the overlap integral. Given beam densities ρ_+ and ρ_-, and taking $\sigma_X \gg \sigma_Y$, the overlap integral is given by (cf. Figure 1)

$$\begin{aligned}\mathcal{O} &\equiv \int dx \int dy \, \rho_+(x,y)\, \rho_-(x,y) \\ &\approx \int dx \int dy \, \frac{N_+}{2\pi\,\sigma_X\,\sigma_Y}\, e^{-x^2/2\sigma_X^2}\, e^{-(y-x\theta_+)^2/2\sigma_Y^2} \cdot \\ &\qquad\qquad \frac{N_-}{2\pi\,\sigma_X\,\sigma_Y}\, e^{-x^2/2\sigma_X^2}\, e^{-(y-x\theta_-)^2/2\sigma_Y^2} \\ &\approx \frac{N_+ N_-}{4\pi\sigma_X\sigma_Y} \left(1 + \left(\frac{\sigma_X \cdot \delta\theta}{\sigma_Y}\right)^2\right)^{-1/2} \,,\end{aligned} \tag{4}$$

where N_+ and N_- are the number of particles in each beam, σ_X and σ_Y are the beam sigmas along the principal axes, and $2\delta\theta \equiv (\theta_+ - \theta_-)$ is the angle between the beams due to the coupling. In Eq. (4) it has been assumed that the beam sizes are the same so that $\sigma_{X,+} = \sigma_{X,-} \equiv \sigma_X$ and $\sigma_{Y,+} = \sigma_{Y,-} \equiv \sigma_Y$. If this is not true then, for example,

σ_Y in the last line must be replaced by the average beam height $\bar{\sigma}_Y$ given by

$$\bar{\sigma}_Y \equiv \sqrt{\frac{\sigma_{Y,+}^2 + \sigma_{Y,-}^2}{2}}\,. \tag{5}$$

Using Eq. (4) in Eq. (1) and using that fact that for weak coupling σ_X is independent of the coupling gives

$$B_c \approx \frac{\sigma_Y^*(\mathrm{BBI{+}Coup}) - \sigma_Y^*(\mathrm{BBI})}{\sigma_Y^*(\mathrm{BBI{+}Coup})} + \frac{1}{2}\left(\frac{\sigma_X^* \cdot \delta\theta^*}{\sigma_Y^*(\mathrm{BBI{+}Coup})}\right)^2 , \tag{6}$$

where '$*$' indicates the quantity must be evaluated at the IP. The first term on the RHS of Eq. (6) is due to the vertical blow-up of the beams and the second term is due to the decrease in overlap when the beams are rotated with respect to one another. The two terms will be respectively examined in the next two sections.

2 Vertical Beam Blowup

Consider first the vertical blow-up term in Eq. (6). The problem with this term is that it is not an easy matter to compute $\sigma_Y(\mathrm{BBI{+}Coup}) - \sigma_Y(\mathrm{BBI})$. The reason for this is that the beam blowup due to coupling is essentially a linear phenomena while the beam–beam induced blowup is highly nonlinear in nature. It is therefore a nontrivial matter to say how the beam–beam interaction couples with the coupling to affect the beam height. One way around this is to simply assume that the beam–beam interaction and the coupling can be taken to be independent processes so that the beam height scales in quadrature:

$$\sigma_Y^2(\mathrm{BBI{+}Coup}) = \sigma_Y^2(\mathrm{BBI}) + \sigma_Y^2(\mathrm{Coup})\,, \tag{7}$$

where $\sigma_Y(\mathrm{Coup})$ is the vertical beam height with coupling but without the beam–beam interaction. The problem of computing $\sigma_Y(\mathrm{BBI{+}Coup}) - \sigma_Y(\mathrm{BBI})$ is now simpler since $\sigma_Y(\mathrm{BBI{+}Coup})$ can be approximated using the design or observed beam-beam tune shift parameter and, as will be shown, $\sigma_Y(\mathrm{Coup})$ can be obtained from coupling data. In order to test Eq. (7) computer simulations were performed using the weak–strong model developed by Krishnagopal and Siemann[1] modified to include coupling. The results of the simulations show more of a linear rather than a quadratic dependence. This is reasonable since the coupling changes the strength of some of the resonances driven by the beam–beam interaction. A more conservative formula would then be to take

$$\sigma_Y(\mathrm{BBI{+}Coup}) = \sigma_Y(\mathrm{BBI}) + \sigma_Y(\mathrm{Coup})\,. \tag{8}$$

In the spirit that B_c is to be used as a first check on whether the coupling is significantly degrading the luminosity, as an order of magnitude estimate Eq. (8) will be used. Putting Eq. (8) in Eq. (6) gives

$$B_c \approx \frac{\sigma_Y^*(\mathrm{Coup})}{\sigma_Y^*(\mathrm{BBI{+}Coup})} + \frac{1}{2}\left(\frac{\sigma_X^*\cdot\delta\theta^*}{\sigma_Y^*(\mathrm{BBI{+}Coup})}\right)^2 . \tag{9}$$

The computation of $\sigma_Y(\mathrm{Coup})$ is relatively straightforward. The normal mode coordinate transformation for the 4x4 coupled one–turn transfer matrix $\mathbf{T}$ is written as[2, 5]

$$\begin{aligned}\mathbf{T} &= \mathbf{V}\cdot\mathbf{U}\cdot\mathbf{V}^{-1}\\ &= \begin{pmatrix}\mathbf{I}\gamma & \mathbf{C}\\ -\mathbf{C}^\dagger & \mathbf{I}\gamma\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{pmatrix}\begin{pmatrix}\mathbf{I}\gamma & -\mathbf{C}\\ \mathbf{C}^\dagger & \mathbf{I}\gamma\end{pmatrix},\end{aligned} \tag{10}$$

where $\mathbf{I}$ is the identity matrix, '$\dagger$' denotes the symplectic conjugate, and γ is given by

$$\gamma^2 + ||\mathbf{C}|| = 1\,. \tag{11}$$

Eigenmode a is the nearly horizontal mode and b is the nearly vertical mode. Matrix $\mathbf{A}$ can be written in the standard Twiss form

$$\mathbf{A} = \begin{pmatrix}\cos 2\pi\nu_a + \alpha_a \sin 2\pi\nu_a & \beta_a \sin 2\pi\nu_a\\ -\gamma_a \sin 2\pi\nu_a & \cos 2\pi\nu_a - \alpha_a \sin 2\pi\nu_a\end{pmatrix}, \tag{12}$$

and similarly for $\mathbf{B}$. The normal mode vector $\mathbf{a} = (a, p_a, b, p_b)^t$ is related to the laboratory coordinates $\mathbf{x}$ by

$$\mathbf{a} = \mathbf{V}^{-1}\mathbf{x}\,. \tag{13}$$

To remove the beta dependence $\mathbf{a}$ can be transformed to $\overline{\mathbf{a}}$ via

$$\overline{\mathbf{a}} = \mathbf{G}\,\mathbf{a}\,, \tag{14}$$

where

$$\mathbf{G} = \begin{pmatrix}\mathbf{G}_a & \mathbf{0}\\ \mathbf{0} & \mathbf{G}_b\end{pmatrix}, \tag{15}$$

and

$$\mathbf{G}_a \equiv \begin{pmatrix}\frac{1}{\sqrt{\beta_a}} & 0\\ \frac{\alpha_a}{\sqrt{\beta_a}} & \sqrt{\beta_a}\end{pmatrix}, \tag{16}$$

and similarly for $\mathbf{G}_b$. $\mathbf{T}$ is now written in terms of the normalized normal modes as

$$\mathbf{T} = \mathbf{G}^{-1}\,\overline{\mathbf{V}}\,\overline{\mathbf{U}}\,\overline{\mathbf{V}}^{-1}\,\mathbf{G}\,, \tag{17}$$

where

$$\begin{aligned}\overline{\mathbf{V}} &= \mathbf{G}\,\mathbf{V}\,\mathbf{G}^{-1}\\ &= \begin{pmatrix} \mathbf{I}\,\gamma & \mathbf{G}_a\,\mathbf{C}\,\mathbf{G}_b^{-1} \\ -\mathbf{G}_b\,\mathbf{C}^\dagger\,\mathbf{G}_a^{-1} & \mathbf{I}\,\gamma \end{pmatrix}\\ &\equiv \begin{pmatrix} \mathbf{I}\,\gamma & \overline{\mathbf{C}} \\ -\overline{\mathbf{C}}^\dagger & \mathbf{I}\,\gamma \end{pmatrix}\,.\end{aligned} \tag{18}$$

This defines the normalized coupling matrix $\overline{\mathbf{C}}$ Since the coupling is weak the following approximations can be made:

$$\begin{aligned}\beta_a &= \beta_{X,a} = \beta_x\\ \beta_b &= \beta_{Y,b} = \beta_y\\ \epsilon_a &= \epsilon_x\ ,\end{aligned} \tag{19}$$

where β_x and β_y are the horizontal and vertical betas without coupling, and $\beta_{X,a}$ and $\beta_{Y,b}$ are the betas for the a and b modes projected onto the X and Y axes respectively with X and Y lying along the principal axes of a beam (cf. figure 2).

Without the beam–beam interaction consider the motion along the Y–axis for a given particle of a given beam:

$$Y = \sqrt{A_a\,\beta_{Y,a}}\,\cos(2\pi\,\nu_a n + \phi_a) + \sqrt{A_b\,\beta_{Y,b}}\,\cos(2\pi\,\nu_b n + \phi_b)\,, \tag{20}$$

where ν_a and ν_b are the tunes of the modes, and $\beta_{Y,a}$ and $\beta_{Y,b}$ are the betas projected along the Y-axis[3, 4]. In the above equation n is the turn number, A_a and A_b are the amplitudes, and ϕ_a and ϕ_b are the phases of the oscillations. Averaging Y^2 over all particles gives $\sigma_Y(\text{Coup})$ is

$$\begin{aligned}\sigma_Y^2(\text{Coup}) &\equiv \left\langle Y^2 \right\rangle\\ &= \epsilon_a\,\beta_{Y,a} + \epsilon_b\,\beta_{Y,b}\\ &\equiv \sigma_{Y,a}^2 + \sigma_{Y,b}^2\,,\end{aligned} \tag{21}$$

where $\langle \cdots \rangle$ indicates an average over all particles, $\epsilon_a \equiv \langle A_a \rangle$ and $\epsilon_b \equiv \langle A_b \rangle$ are the emittances for the modes, and it has been assumed that $\nu_a \pm \nu_b \neq$ integer so that the

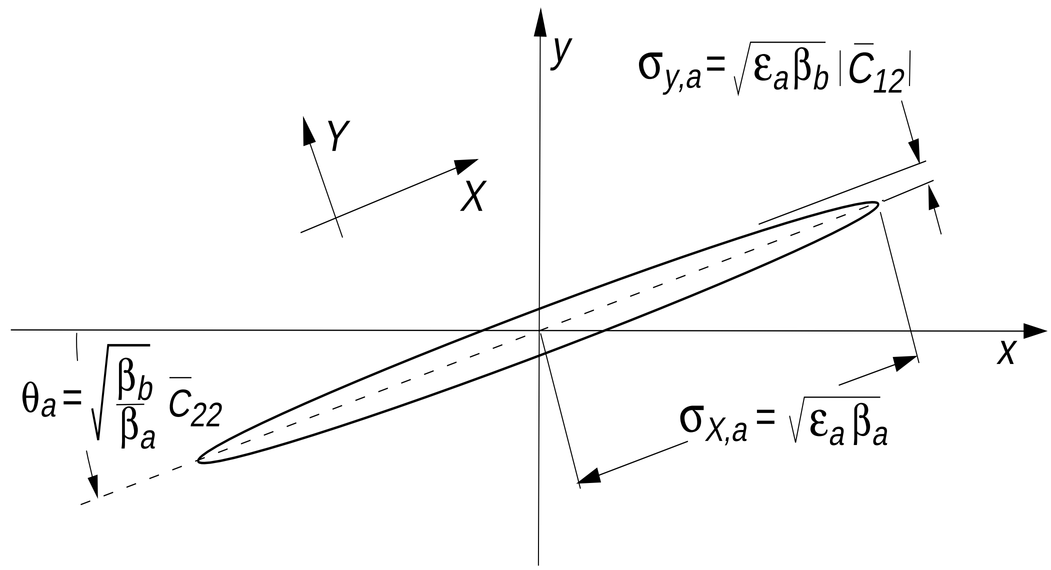

Figure 2: 1σ envelope for eigenmode a. Adapted from Bagley and Rubin figure 1.

cross term between modes averages to zero. σ_Y is thus made up of two components, $\sigma_{Y,a}$ and $\sigma_{Y,b}$ corresponding to the a and b eigenmotions respectively.

Consider first the a eigenmode. Since $\epsilon_a \gg \epsilon_b$ the motion due to the a mode dominates so, to a good approximation, the Y–axis coincides with the minor axis of the a mode. It is shown in reference [2] that

$$\sigma_{Y,a} = \sqrt{\epsilon_a \beta_b}\, |\overline{C}_{12}| \, , \tag{22}$$

For the b motion $\sigma_{Y,b}$ is calculated from Eq. (21):

$$\sigma_{Y,b} = \sqrt{\epsilon_b \beta_b} \, . \tag{23}$$

Combining Eqs. (21), (22), and (23), and using Eq. (19) gives

$$\sigma_Y^*(\mathrm{Coup}) = \sqrt{\epsilon_a \beta_y^*} \left(\overline{C}_{12}^{*2} + \frac{\epsilon_b}{\epsilon_a} \right)^{1/2} \, . \tag{24}$$

This is almost the desired result. What remains is to relate ϵ_b/ϵ_a to the coupling. This is shown in Appendix A for the special case of a single coupler and uniform Twiss parameters. For the general case the reader is referred to Billing[6] or Orlov and Sagan[7].

How does the contribution to σ_Y^* from $\sigma_{Y,a}^*$ and $\sigma_{Y,b}^*$ compare? From Eqs. (22), (23), (43), (44), and (45) it is seen that both $\sigma_{Y,a}$ and $\sigma_{Y,b}$ scale linearly with $\overline{C}$ in

Parameter	Value
β_x^*	$1.0\,\mathrm{m}$
β_y^*	$0.015\,\mathrm{m}$
ϵ_x	$0.16\,\mu\mathrm{m}$
σ_X^*	$400\,\mu\mathrm{m}$
$\sigma_Y^*(\mathrm{BBI+Coup})$	$8\,\mu\mathrm{m}$
$\epsilon_y^*(\mathrm{BBI+Coup})$	$4.3\,\mathrm{nm}$

Table 1: 'Typical' CESR Values Under Normal Colliding Beam Conditions.

Conditions	Bagley & Rubin Figure	$\left\langle \overline{C}_{12}^2 \right\rangle_s^{1/2}$	B_c
Two skew quads powered	2	0.04	0.3
After globally decoupling	4	0.03	0.2
After local decoupling	5	0.01	0.07

Table 2: B_c as estimated from coupling data from Bagley and Rubin[2].

the sense that if the $\overline{C}$ around the ring are scaled by some factor then both $\sigma_{Y,a}$ and $\sigma_{Y,b}$ will be scaled by the that factor. However, it is important to remember that $\sigma_{Y,b}^*$ is dependent upon the coupling matrix around the ring as opposed to $\sigma_{Y,a}^*$ which is determined solely by the coupling matrix at the IP. Thus, it is always possible to make the a mode contribution to σ_Y^* equal to zero by using a single skew but the b mode contribution will always be present unless the ring is totally ('locally') decoupled.

Ignoring the tilt term for the moment, the calculation of B_c from Eqs. (9) and (24) and from knowledge of the coupling is straight forward if somewhat cumbersome. If one only wants a rough number one can assume that the $\sigma_{Y,a}$ contribution has been zeroed out using a skew quad and then use Eq. (47) to give

$$B_c \approx \sqrt{\frac{2\epsilon_x}{\epsilon_y(\mathrm{BBI+Coup})}} \left\langle \overline{C}_{12}^2 \right\rangle_s^{1/2} . \tag{25}$$

where $\epsilon_y(\mathrm{BBI+Coup}) \equiv \sigma_Y^2(\mathrm{BBI+Coup})/\beta_y$, and the $\overline{C}_{12}$ matrix element has been chosen to be averaged over since it is the easiest matrix element to obtain from measurement. To show the usefulness of Eq. (25) we consider the coupling data presented by Bagley and Rubin[2]. For example figure 2 from Bagley and Rubin shows

$\overline{C}_{12}$ around the ring due to two symmetrically placed, anti-symmetrically powered skew quads. From their figure the RMS for $\overline{C}_{12}$ is 0.04 which, using the 'typical' CESR parameters given in table 1, gives a B_c of 0.4. Similarly Bagley and Rubin show data for $\overline{C}_{12}$ after globally decoupling and after local decoupling. Table 2 shows the results. It is only after the local decoupling that B_c drops to a value that is small enough so that the coupling will only have a small effect on the luminosity. How does this compare with observations of the luminosity? No quantitative measurements have been made but qualitatively observation indicates that local decoupling is necessary for obtaining the greatest luminosity in line with the above calculations[8].

3 $\delta\theta$ Calculation

For a given beam since $\epsilon_a \gg \epsilon_b$ the a eigenmotion dominates so that with negligible error we can take the angle of a beam, θ to correspond to θ_a — the angle for the a mode ellipse. θ_a is related to $\overline{C}_{22}$ as shown in figure 2. $\delta\theta^*$ is then

$$\delta\theta^* \equiv \frac{1}{2}(\theta^*_+ - \theta^*_-) = \sqrt{\frac{\beta^*_y}{\beta^*_x}}\,\delta\overline{C}^*_{22}\,, \tag{26}$$

where

$$\delta\overline{C}_{22} \equiv \frac{1}{2}(\overline{C}_{22,+} - \overline{C}_{22,-})\,. \tag{27}$$

Since $\delta\theta^*$ depends upon the difference in the $\overline{C}^*_{22}$, with pretzeled orbits $\delta\theta^*$ may be zeroed using a single skew sextupole.

The critical $\delta\theta^*$ is defined as the angle needed to give a badness of 0.1. From Eq. (9) this is found to be

$$\delta\theta^*_{crit} = 0.46\,\frac{\sigma^*_Y(\text{BBI+Coup})}{\sigma^*_X}\,. \tag{28}$$

Combining Eq. (28) with Eq. (26) gives

$$\begin{aligned}\delta\overline{C}^*_{22,crit} &= 0.46\sqrt{\frac{\beta^*_x}{\beta^*_y}\,\frac{\sigma^*_Y(\text{BBI+Coup})}{\sigma^*_X}} \\ &= 0.46\sqrt{\frac{\epsilon_y(\text{BBI+Coup})}{\epsilon_x}}\,. \end{aligned} \tag{29}$$

Using the values in table 1 gives

$$\delta\overline{C}^*_{22,crit} = 0.15 \tag{30}$$

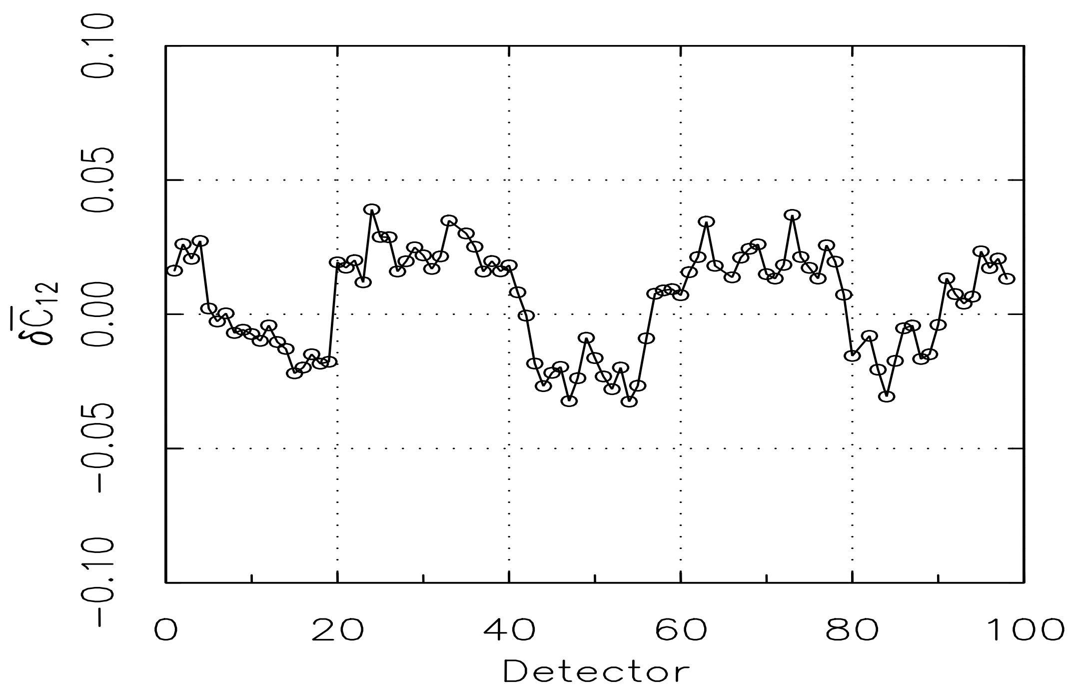

Figure 3: Differential coupling around the ring.

Eq. (29) shows that $\delta\overline{C}^*_{22,crit}$ is independent of β^*_x or β^*_y. This is just a reflection of the fact that the $\overline{C}$'s are properly normalized. This is an important point: From measurement of the $\delta\overline{C}_{12}$ 'wave' outside of the IP one can get a sense of whether $\delta\overline{C}^*_{22}$ is too large. Furthermore, for a given $\delta\overline{C}_{22}$ at any point in the ring, it is easily shown that the percentage change in the overlap integral due to a finite $\delta\theta$ is independent of the local β_x and β_y. The conclusion is that a quick visual inspection as to whether the beams overlap at the synch light ports will give a good indication of how the beams are overlapping at the IP. One must always remember, however, that it is possible for the phases to be such that there is no tilt at the synch light ports but unacceptable tilt at the IP (or vice versa).

$\overline{C}_{22}$ is hard to measure accurately[2]. However, $\overline{C}_{12}$ is relatively easy to measure and since the $\overline{\mathbf{C}}$ matrix can be represented as the superposition of two rotating phasors[5] the magnitude of the $\delta\overline{C}_{12}$ wave should be very close to the magnitude of the $\delta\overline{C}_{22}$ wave. Figure 3 shows a measurement of $\delta\overline{C}_{12}$ around the ring. Since $|\delta\overline{C}_{12}|$ is everywhere much less than $\delta\overline{C}^*_{22,crit}$ it gives a good indication that $\delta\overline{C}_{22}$ is small in this case. On the other hand, it has been observed that at times that the tilt of the beams at the synch light ports has been unacceptably large.

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Appendix A: Calculation of $\sigma_{Y,b}$

From Billing[6] or Orlov and Sagan[7]:

$$\frac{\epsilon_b}{\epsilon_a} = \frac{J_a \left\langle G^3 \mathcal{H}_b \right\rangle_s}{J_b \left\langle G^3 \mathcal{H}_b \right\rangle_s} \,, \tag{31}$$

where G is the inverse of the bending radius, J_a and J_b are the damping partition numbers, $\mathcal{H}_a$ and $\mathcal{H}_b$ are the emittance generating functions, and $\langle \cdots \rangle_s$ is an average over the entire ring. Concentrating on the b mode for the moment $\mathcal{H}_b$ is obtained from the action I_b via

$$\mathcal{H}_b = \frac{\partial^2}{\partial p_z^2} I_b \,, \tag{32}$$

with $p_z \equiv \Delta E/E$ being the conical longitudinal momentum and I_b is given by[9]

$$I_b(s, b, p_b) = \frac{1}{2}(\beta_b \, p_b^2 + 2\alpha_b \, b \, p_b + \gamma_b \, b^2) \,. \tag{33}$$

Putting Eq. (33) in Eq. (32) and using the fact (cf. Eq. (35)) that b and p_b are linear in p_z gives

$$\mathcal{H}_b = \beta_b \left(\frac{\partial p_b}{\partial p_z} \right)^2 + 2\alpha_b \left(\frac{\partial b}{\partial p_z} \right) \left(\frac{\partial p_b}{\partial p_z} \right) + \gamma_a \left(\frac{\partial b}{\partial p_z} \right)^2 \,, \tag{34}$$

In terms of the laboratory x–y coordinate system the normal modes are given by Eqs. (13) and (18)

$$\begin{pmatrix} a \\ p_a \\ b \\ p_b \end{pmatrix} = \mathbf{G}^{-1} \cdot \begin{pmatrix} \mathbf{I}\gamma & -\overline{\mathbf{C}} \\ \overline{\mathbf{C}}^\dagger & \mathbf{I}\gamma \end{pmatrix} \cdot \mathbf{G} \cdot \begin{pmatrix} x - \eta_x \, p_z \\ p_x - \eta'_x \, p_z \\ y - \eta_y \, p_z \\ p_y - \eta'_y \, p_z \end{pmatrix} \,. \tag{35}$$

For the a mode the appropriate equations are obtained via a change of variables: $b \to a$, $p_b \to p_a$ in Eqs. (32), (34), and (35). Using Eq. (35) in the a mode equivalent of Eq. (34), and assuming the coupling is small (cf. (19)) gives

$$\mathcal{H}_a = \beta_x \eta_x'^2 + 2\alpha_x \eta_x \eta_x' + \gamma_x \eta_x^2 + o(\overline{C}^2) \,. \tag{36}$$

The first 3 terms on the right hand side of Eq. (36) are equal to the uncoupled $\mathcal{H}_x$ (cf. Sands[10] Eq. 5.71). Thus, to first order in the coupling, $\mathcal{H}_a = \mathcal{H}_x$ which is essentially the last equation in (19).

One can now grind through the arithmetic for the calculation for $\mathcal{H}_b$. In order to be able to see the forest through the trees we will simplify the calculation by assuming that $\beta_x(s)$, $\beta_y(s)$, $\eta_x(s)$ and $G(s)$ are constants independent of s. From Eqs. (19) and (35) b and p_b are given to first order in the $\overline{C}$ by

$$\begin{aligned} b &= (y - \eta_y p_z) + \sqrt{\frac{\beta_y}{\beta_x}}\overline{C}_{22}(x - \eta_x p_z) - \sqrt{\beta_x \beta_y}\overline{C}_{12}\, p_x \\ p_b &= (p_y - \eta_y' p_z) - \frac{\overline{C}_{21}}{\sqrt{\beta_x \beta_y}}(x - \eta_x p_z) + \sqrt{\frac{\beta_x}{\beta_y}}\overline{C}_{11}\, p_x \, . \end{aligned} \tag{37}$$

Using this in Eq. (34) gives

$$\mathcal{H}_b = \beta_y \left(\frac{\overline{C}_{21}\eta_x}{\sqrt{\beta_x \beta_y}} - \eta_y' \right)^2 + \frac{1}{\beta_y} \left(\eta_y + \sqrt{\frac{\beta_y}{\beta_x}}\overline{C}_{22}\eta_x \right)^2 . \tag{38}$$

To simplify matters further it will be assumed that the only coupling element is a single skew quad of strength q_{sk} at $s = 0$. For a single skew quad the $\overline{\mathbf{C}}$ matrix is given by Billing Eqs. (32) and (36) to be

$$\begin{aligned} \overline{\mathbf{C}}(\mathrm{s}) = \frac{\sqrt{\beta_x \beta_y}\, q_{sk}}{2(\cos(2\pi\nu_x) - \cos(2\pi\nu_y))} \cdot & \\ \Big[\sin\pi(\nu_x + \nu_y)\, \mathbf{R}^\dagger[\pi(\nu_x - \nu_y)] \cdot \mathbf{R}[\phi_x(s) - \phi_y(s)] + & \\ \sin\pi(\nu_x - \nu_y)\, \mathbf{R}^\dagger[\pi(\nu_x + \nu_y)] \cdot \mathbf{R}[\phi_x(s) + \phi_y(s)] \cdot \mathbf{J}\Big] \, , & \end{aligned} \tag{39}$$

where $\mathbf{R}$ are rotation matrices, $\phi(s)$ is the phase advance from $s = 0$, and $\mathbf{J}$ is the inverting matrix:

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{40}$$

To find η_y consider a off–momentum particle. The particle will have a closed horizontal orbit of $x_\eta = \eta_x p_z$ which gives a vertical kick at the skew quad of

$$\Delta y'_{sk} = q_{sk}\eta_x p_z \, . \tag{41}$$

This vertical kick generates a closed vertical orbit. Using the standard formula (Sands[10] Eq. (2.92)) the vertical η_y is

$$\eta_y(s) = \frac{q_{sk}\eta_x}{2\sin(\pi\nu_y)}\beta_y \cos(\phi_y(s) - \pi\nu_y) \, . \tag{42}$$

The two terms for $\overline{C}$ in Eq. (39) oscillate as the sum and difference frequencies $\nu_x \pm \nu_y$. Since η_y oscillates as ν_y (cf. Eq. (42)) the cross terms in Eq. (38) tend to average to zero when $\mathcal{H}_b$ is averaged over the ring. Using this in Eq. (31) and assuming $J_a = J_b$ gives

$$\frac{\epsilon_b}{\epsilon_a} \equiv r_x + r_y \, , \tag{43}$$

where

$$\begin{aligned} r_x &\equiv \left\langle \overline{C}_{21}^2 + \overline{C}_{22}^2 \right\rangle_s \, , \\ r_y &\equiv \frac{\left\langle \eta_y^2/\beta_y + \beta_y \eta_y'^2 \right\rangle_s}{\eta_x^2/\beta_x} \, . \end{aligned} \tag{44}$$

r_y represents the contribution to ϵ_b due to the finite η_y and r_x represents the contribution to ϵ_b due to the finite η_x and the finite $\overline{C}$ that couples the b mode motion into the horizontal. Using Eqs. (39) and (42) gives

$$\begin{aligned} r_x &= \frac{q_{sk}^2 \beta_x \beta_y}{4(\cos(2\pi\nu_x) - \cos(2\pi\nu_y))^2} \left[\sin^2 \pi(\nu_x + \nu_y) + \sin^2 \pi(\nu_x - \nu_y)\right] \\ r_y &= \frac{q_{sk}^2 \beta_x \beta_y}{4 \sin(\pi\nu_y)} \, . \end{aligned} \tag{45}$$

r_x has a resonant denominator at the sum and difference frequencies $\nu_x \pm \nu_y =$ integer. Owing to the operation of CESR with tunes close to the half–integer r_x will dominate over r_y. For example, with fractional tunes of $\nu_x = 0.526$ (205 kHz) and $\nu_x = 0.602$ (235 kHz) one finds

$$\begin{aligned} \frac{r_y}{r_x} &= \frac{\sin^2 \pi\nu_y (\cos 2\pi\nu_x - \cos 2\pi\nu_y)^2}{\sin^2 \pi(\nu_x + \nu_y) + \sin^2 \pi(\nu_x - \nu_y)} \\ &= 0.15 \, . \end{aligned} \tag{46}$$

With tunes near the half–integer one can ignore r_y and write

$$\frac{\epsilon_b}{\epsilon_a} \approx 2 \left\langle \overline{C}_{ij}^2 \right\rangle_s \, , \tag{47}$$

where $\left\langle \overline{C}_{ij}^2 \right\rangle_s$ is an average over any of the $\overline{C}$ matrix elements (they all give the same average in this example). Of course in real life there will be more than one coupler and the Twiss parameters will not be constant. However, as long as we neglect any local coupling bump in the IR (which, because of no bends, will not contribute to ϵ_b), and as long as the number of significant couplers in the arcs is small, then Eq. (47) should be a valid approximation.

References

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