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The MÖBIUS ACCELERATOR

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ABSTRACT

Any existing circular accelerator can be converted inexpensively for "Möbius operation" by introducing one "twist" element that interchanges horizontal and vertical betatron oscillations on each particle passage. Two, not one, traversals of the ring are required to return to a corresponding state. The ring exhibits properties different from and, in important ways, superior to the original. Beam brightness can be increased while preserving large amplitude stability, and the (necessarily round) beams are robust against beam-beam interaction in colliding beam operation.

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The term "Möbius-twisted lattice" will mean a lattice with an element present that interchanges horizontal and vertical betatron oscillations in an otherwise-ordinary, uncoupled, accelerator lattice. Before the twist is inserted, because of the inherent periodicity of a closed ring, "betatron" oscillations (transverse) can be described with the help of periodic Floquet functions $\beta_x(s)$ and $\beta_y(s)$, depending on arc length coordinate s. Thinking of β_x as plotted on one side of a strip and β_y on the other, the ends of the strip are joined, but with a single twist so that β_x connects to β_y and β_y connects to what previously was β_x (or $-\beta_x$). Since the β-functions are inherently positive quantities, one of these joins entails a phase shift of π. This topology is the basis of the name "Möbius".

After submission of this paper, a paper by Filippov *et. al.,* [1] describing a round beam scheme resembling that of the present paper, was brought to my attention. Their proposal uses solenoids rather than quadrupoles to rotate betatron amplitudes, a difference only in detail, but because a pair of $\pi/2$ rotations occurs, their once-around lattice topology does not have the Möbius topology of this paper. Rather, viewed at a single point in their lattice, betatron oscillations behave just like those in a conventional accelerator. While their scheme can reasonably be expected to have some of the round beam advantages claimed in this paper, the distribution and strengths of resonances are entirely different.

Here it is shown how Möbius revision can be accomplished with realizable elements compatible with any pre-existing accelerator. It simplifies analysis to treat this as occurring in a zero-length interval, and such a rotation, though discontinuous, is symplectic, but naturally the actual elements have non-zero length. However, without essential loss of generality, by redefining slightly what constitutes "the rest of the lattice", it is not wrong to treat the twist as occurring over zero length. Letting x and y be horizontal and vertical coordinates of a particle at that point, when connecting the lattice, there are two seemingly different possibilities. One is $x \rightarrow y$, $y \rightarrow -x$, the other is $x \rightarrow y$, $y \rightarrow x$. For issues considered important in this paper, these two possibilities are essentially equivalent, differing only by a π phase shift that can be subsumed in the rest of the ring.

The only essential dependence on particle type (electron, proton, or ion) arises from the fact that, whereas hadron beams are approximately round, electron beams in existing accelerators are ribbon-like, much wider than they are high. More precisely, width $\times$ angular-width product "emittances", ϵ_x and ϵ_y, conserved as the beam traverses the lattice, satisfy $\epsilon_y << \epsilon_x$. For a beam of N particles, the "brightnesses", $N/\epsilon_{x,y}$ are the appropriate measures of beam density in cases where minimizing spot size is important. With the twist element present, each particle toggles regularly between horizontal and vertical phases. If it is an electron beam, this has the effect of halving ϵ_x and making $\epsilon_y = \epsilon_x$; such a beam can be called "round".

What are the effects of such a twist, first supposing that no other changes are made? For a colliding beam accelerator, round beams are advantageous because they permit increased luminosity and interaction rate. For electron accelerators used as sources of synchrotron radiation beam brightness is similarly important. Some experiments need a bright horizontal line source, and for those the twist would have to be turned off, as the increase in ϵ_y due to its presence would be undesirable. But for applications requiring a point source, for example to perform a raster scan, the halving of ϵ_x would be advantageous. Next consider the effects of other lattice changes the twist makes possible. Greater benefits can be achieved. One is the freedom to increase the focusing strength of all lattice elements, to be justified below, since this can result in greatly reduced emittances. Reductions $\epsilon_x \to \epsilon_x/5, \epsilon_y \to \epsilon_y/5$ are practical.

Traditional betatron theory relies on the motion being restricted to one transverse plane in a periodic lattice. With altered symmetry significant changes are to be expected. The introduction of x, y coupling normally requires the use of 4×4 matrices, but the coupling proposed in this paper has the property that the transfer matrix is still described by two 2×2 matrices. This permits a closed-form analysis, only moderately more complicated than traditional theory. Of course, imperfections in the coupling elements will violate this representation, but such effects can be analysed perturbatively, as is customary.

Ideally, the lattice should be "matched" at the twist location; that is, the β-functions should satisfy $\beta_x = \beta_y \equiv \beta$ and $\beta'_x = \beta'_y = 0$, where primes indicate d/ds. In that case a beam suffers no phase space distortion or density dilution in passing the element. Our analysis assumes that this matching condition is approximately satisfied. The 4×4, partitioned, once-around transfer matrix is

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{-Y} \\ \mathbf{X} & 0 \end{pmatrix}, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are 2×2 matrices describing pure x and pure y motion in the uncoupled lattice. This matrix acts on the phase space column vector $(x, x', y, y')^T$ to describe once-around particle evolution in the twisted lattice.

Useful in describing coupled motion is the 2×2 off-diagonal combination $\mathbf{E} = \mathbf{X} - \mathbf{Y}^{-1}$, and its determinant $\mathcal{E} = \det|\mathbf{E}|$. Because $\mathbf{M}$ is symplectic, its eigenvalues are known to form reciprocal pairs, λ and $1/\lambda$, both lying on the unit circle for stable motion. As a result, their sum $\Lambda = \lambda + 1/\lambda = 2\cos\mu \equiv 2\cos(2\pi Q)$ is known to be real. The physically measureable (by spectrum analysis of transverse beam position signals) normal mode tunes are

$$Q_{\pm} = \frac{1}{2\pi} \cos^{-1} \frac{\pm\sqrt{\mathcal{E}}}{2}. \tag{2}$$

One implication is that the very existence of stable oscillations is keyed to the sign and magnitude of $\mathcal{E}$; for $0 \leq \mathcal{E} \leq 4$, both tunes are real and the motion is stable. Another is that, the cosine function being anti-symmetric about $\pi/2$, the two tunes are symmetrically placed, above and below 0.25. This differs from the accustomed two tune lines that can be adjusted independently, both in frequency and in amplitude. The presence of split lines indicates the motion is not simple harmonic (on a turn-by-turn basis.)

In a conventional lattice, any particle orbit is a superposition of four (two equal-tune pairs) normal mode orbits. With the twist present, any solution, because it "toggles" regularly between horizontal and vertical, yields another solution, identical except for having horizontal and vertical intervals interchanged. Clearly, when viewed over an even number of turns, these two orbits must exibit identical tunes. Since the orbit differential equation has a definite number of independent solutions, this new symmetry imposes a reduction in the number of independently controllable parameters, as observed above. This new degeneracy persists even with nonlinear elements present.

Errors will inevitably cause the lattice to be unmatched to some degree, but this will not cause qualitative difference. The eigenvalues of the once-around transfer matrix are "locked" onto the unit circle in the complex plane. Escape and "bifurcation" is possible only at two points (intersections with the real axis) of this circle. In the neighborhood of an operating point remote from these two points, there is substantial robustness of qualitative behavior. This means that minor imperfection of the elements making up the twist will not change the qualitative behavior. On the other hand, the Möbius lattice clearly lives on a presently uninhabited "branch" of the parameter space. This may account for behavior that seems anti-intuitive.

The standard "Twiss" parameterization of $\mathbf{X}$ is

$$\mathbf{X} = \begin{pmatrix} C_x + \alpha_x S_x & \beta_x S_x \\ -\gamma_x S_x & C_x - \alpha_x S_x \end{pmatrix}, \tag{3}$$

where $\alpha_x = -\beta_x'/2$, $\gamma_x = (1+\alpha_x^2)/\beta_x$, $C_x = \cos\mu_x$, $S_x = \sin\mu_x$, and similarly for y. In terms of these parameters $\mathcal{E}$ is given by $\mathcal{E} = 2 - 2\cos(\mu_x + \mu_y) + S_x S_y R$ where

$$R = \left(\sqrt{\frac{\beta_x}{\beta_y}} - \sqrt{\frac{\beta_y}{\beta_x}}\right)^2 + \left(\sqrt{\frac{\beta_x}{\beta_y}}\alpha_y - \sqrt{\frac{\beta_y}{\beta_x}}\alpha_x\right)^2. \tag{4}$$

By design, since $R = 0$, we have $0 \le \mathcal{E} \le 4$, making the twisted lattice stable, with $\mu_\pm = (\mu_x + \mu_y \pm \pi)/2$. (After aliasing into the tune range from 0 to 0.5 these appear symmetric about 0.25.) Because some mismatch is inevitable, in practice $0 < R << 1$. In Fig. 1, to exaggerate the effect, a "gross" mismatch $\beta_y/\beta_x = 2$, $\alpha_x = \alpha_y = 0$, is assumed and stability boundaries are plotted. Though ten times worse than could be achieved easily, and perhaps one hundred times what could be reliably obtained in practice, instability excludes only a modest region of tune space. On the other hand, there is an essential singularity at the origin and slivers of instability lying in the wedges between the straight lines $Q_y = Q_x(-1 - R/2 \pm \sqrt{R(1+R/4)})$. These are indicated in Fig. 1.

To predict the performance of an accelerator it is necessary to be able to calculate the effects of small imperfections. Let $X_t = (x_t, x_t')^T$ and $Y_t = (y_t, y_t')^T$ be phase space coordinates on turn t and let $\Delta X_t'(P) = (0, \Delta x_t'(x_t(P), y_t(P)))^T$ and $\Delta Y_t'(P)$ be deflections or "perturbations" occurring at location P on turn t. For the Möbius accelerator it is convenient to analyse the effects of these perturbations after two complete turns rather than one. To analyse the effect of a lattice imperfection at P the transfer matrices can be factorized $\mathbf{X} = \mathbf{X_2 X_1}$, $\mathbf{Y} = \mathbf{Y_2 Y_1}$ where $\mathbf{X_1}$ represents horizontal propagation (phase advance μ_{x1}) from the twist to the point P, and $\mathbf{X_2}$ from there on around to the twist. In terms of these matrices, neglecting terms quadratic in deflections, the following fifth-order

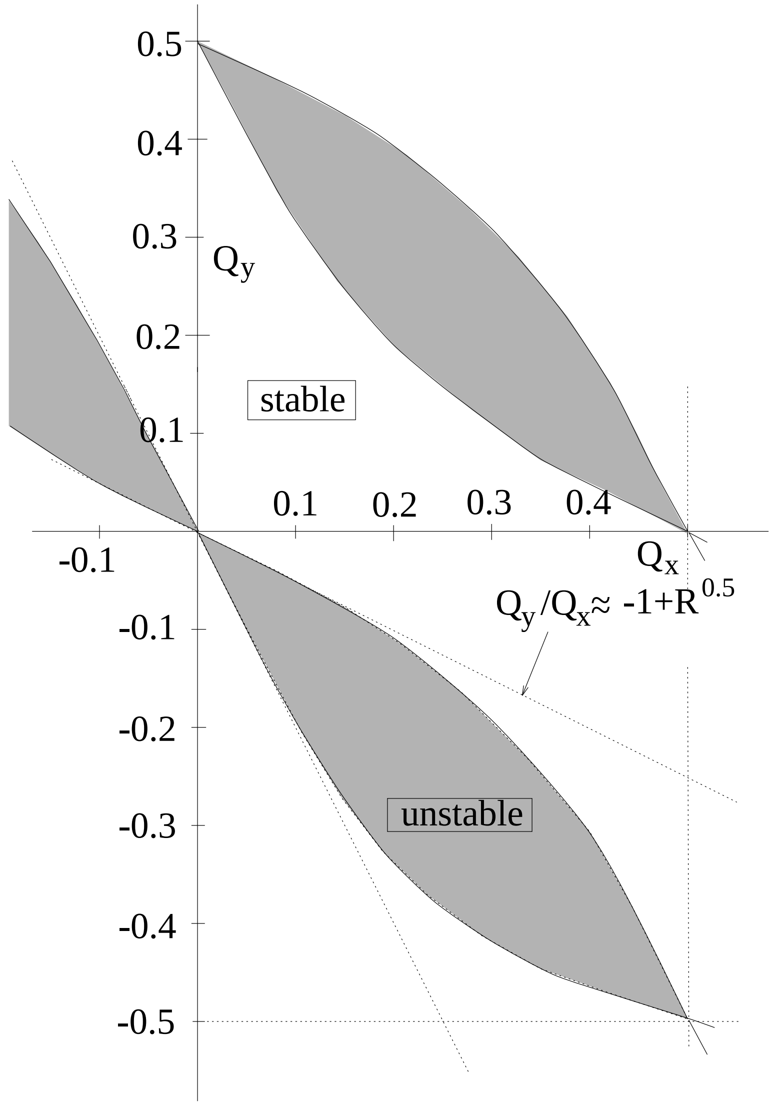

Figure 1: Stable and unstable regions of the Q_x, Q_y tune-plane for a grossly mismatched Möbius lattice. $\beta_y/\beta_x = 2$, $\alpha_x = \alpha_y = 0$.

difference equation can be derived:

$$X_{t+2} + (2 - \mathcal{E})\, X_t + X_{t-2} + \frac{1}{2}\begin{pmatrix} 0 \\ -\Delta x'_{t+2} + \Delta x'_{t-2} \end{pmatrix} =$$
$$= \frac{\mathbf{X}_1 \left(-\mathbf{YX} + \mathbf{X}^{-1}\mathbf{Y}^{-1}\right) \mathbf{X}_1^{-1}}{2} \begin{pmatrix} 0 \\ \Delta x'_t \end{pmatrix} - \mathbf{X}_1 \mathbf{Y}_2 \begin{pmatrix} 0 \\ \Delta y'_{t+1} \end{pmatrix} - \mathbf{X}_2^{-1} \mathbf{Y}_1^{-1} \begin{pmatrix} 0 \\ \Delta y'_{t-1} \end{pmatrix}$$
$$= \sin\left(\mu_x + \mu_y\right) \left[\begin{pmatrix} -\alpha_x^P & -\beta_x^P \\ \gamma_{Px} & \alpha_x^P \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x'_t \end{pmatrix} + \begin{pmatrix} -\beta_y^P & 0 \\ 0 & -\beta_y^P \end{pmatrix} \begin{pmatrix} \dfrac{\Delta y'}{y} \end{pmatrix}_0 X_t \right] + (\text{n.l.})_{t\pm 1}, \tag{5}$$

and Y_t satisfies a similar equation. In deriving the second equation, matching condition $R = 0$ has been assumed. The factor $(\Delta y'/y)_0$, independent of t and y, stands for the vertical focusing strength at P. Nonlinear deflections on turn t can be included in the factor $\Delta x'_t$, but nonlinear deflections on turns $t \pm 1$ are more complicated.

Knowing that the coefficient of X_t is equal to $-2\cos\mu^{(2)}$, where $\mu^{(2)}/(2\pi)$ is the tune of the twice-around machine, we have $\cos\mu^{(2)} = -1 + \mathcal{E}/2$, which agrees with Eq. (2), because $\cos(\mu^{(2)}) = \cos(4\pi Q_\pm)$. Ideally, there being no errors, all perturbation terms of Eq. (5) vanish. In that case, as is common with waves, all components satisfy the same equation, and most results follow from the equation for any one, say the one for x_t. With some loss of generality, taking $\alpha_x^P = \alpha_y^P = 0$, $\beta_x^P = \beta_y^P = 1$, a particular unperturbed solution can be written

$$X_t = a\cos\frac{\pi}{2}t \begin{pmatrix} \cos\frac{\mu_x+\mu_y}{2}t \\ -\sin\frac{\mu_x+\mu_y}{2}t \end{pmatrix}, \quad y_{t\mp 1} = \mp a\cos\frac{\pi}{2}t\ \cos\left(\frac{\mu_x+\mu_y}{2}t \mp \mu_{x(1,2)} \mp \mu_{y(2,1)}\right). \tag{6}$$

For steering errors and quadrupole errors, Eq. (5) can be solved in closed form. For nonlinear effects it can be solved perturbatively and iteratively, starting with Eq. (6) to approximate perturbation terms. Then, expanding by Fourier transformation in harmonics labeled by integer r, seeking a similarly expansion for x_t, and using the trigonometric identity

$$\cos\left(r\mu\left(t+2\right)\right) + \left(2-\mathcal{E}\right)\cos r\mu t + \cos\left(r\mu\left(t+2\right)\right) = 2\left(\cos\left(2r\mu\right) + 1 - \mathcal{E}/2\right)\cos r\mu t, \tag{7}$$

an improved solution is obtained. Since the coefficient of $\cos r\mu t$ in Eq. (7), is capable of becoming small or vanishing, and appears in the denominator of the improved solution, there is (as usual) the hazard of resonance at rational tunes.

Using this method of approximation to analyse various cases, some conclusions can be drawn. Spectra measured in the horizontal and vertical plane are the same, symmetric about 0.25 in linear approximation, and the spectra of nonlinear harmonics are also identical. Perturbations away from the ideal machine lattice influence every turn. For that reason the response to errors of the twice-around machine is not at all the same as would be the case if the machine truly had independent elements strung out along twice the circumference. Rather, since motion flips from plane to plane, a perturbing element causing horizontal force on one turn tends to cause vertical on the next and *vice versa*. For example, there is a strong tendency toward cancellation of tune shifts due to quadrupole perturbations; it can inferred from Eq. (5) and the cancellation is perfect if $\beta_x^P = \beta_y^P$. The head-on beam-beam interaction does not benefit from the cancellation of linear tune shifts just mentioned, because it is "focusing" in both planes. On the other hand, there is partial cancellation of "parasitic" beam-beam tune shifts due to beams that are separated, but share the same vacuum chamber, as for example at CESR, LEP, and the Fermilab Tevatron.

Turning to nonlinear effects, for colliding beams there is a benefit to round beams, no matter how they have been brought into existence, as mentioned previously. Other benefits result only because of the twist. If $Q_x + Q_y$ is close to an integer, making $\mathcal{E} << 1$, nonlinear perturbing elements with even symmetry (sextupoles being the leading example) are not resonant. This can be confirmed using Eq. (7) with $r = 2$, $\mu \approx \pi/2$. Furthermore, it is found that a single sextupole, located to satisfy tune condition $Q_{x1} + Q_{y2} \approx 0.5$ (which implies also $Q_{x2} + Q_{y1} \approx 0.5$) leads to cancellation and large dynamic aperture. In ordinary accelerators one is accustomed to being excluded from tunes close to integers because of "resonant" closed orbit deviation due to steering errors. In the Möbius accelerator $Q_x + Q_y$ close to an integer is not resonant for steering errors (since they have the same symmetry as sextupoles.)

On the other hand nonlinear perturbations with odd symmetry (head-on beam-beam interactions being the leading example) lead to vanishing denominators for $Q_x + Q_y$ close to an integer. However, if the beams are round, it is found that the numerators vanish for $Q_x + Q_y$ close to an integer, due to cancellation from successive turns. In fact, simulation shows that beam-beam forces, no matter how strong, (provided there is no other nonlinearity or physical aperture limitation) cause no particle loss in the Möbius lattice, no matter what the tunes or interaction locations. One can conjecture that stability near the resonance is enhanced by the cancellation just mentioned, and everywhere else because the perturbing force becomes small at large amplitudes. The cancellation that has been described is possible only because the beam-beam cubic force has been assumed to be "central", with x^3 and y^3 (and all other odd powers) having equal coefficients. This would be true only for round beams interacting at points with matched β-functions.

The influence of other potentially destructive nonlinear effects is presently unknown. These include beam-beam interactions with non-zero crossing angle (hard to analyse due to having neither even nor odd symmetry, nor vanishing on-axis current density), space charge effects (especially in low energy proton accelerators), and synchro-betatron effects.

There is another important advantage of the Möbius accelerator, arising from the need for "chromaticity compensation"; *i.e.* canceling tune dependencies on momentum, $\chi_x = d\mu_x/dp$ and $\chi_y = d\mu_y/dp$. (Uncompensated, they are naturally large and negative.) This compensation is necessary, for all high energy accelerators, to avoid the "head-tail" effect, and to avoid resonances by making the tune plane "footprint" small. Chromaticity compensation requires sextupoles in the lattice, and their presence necessarily limits the dynamic aperture. In the Möbius accelerator, because there is only one tune there is only one chromaticity. This means that all compensation can be performed in the "easy" direction—horizontal. Since both reasons for compensating chromaticity have to do with avoiding resonances that cause damage accumulating over many turns, compensation every two turns will be almost as effective as compensation every turn. Assuming this to be true, we can allow χ_x and χ_y for the untwisted lattice to deviate from zero provided their sum is held approximately constant. This is advantageous because the sextupole strength required for a given shift of χ_x is about half that required for the same shift of χ_y. Furthermore, when needed only in a single plane, all compensation can be performed with "non-interleaved" [2], π-separated, sextupole pairs.This issue of chromaticity compensation is by no means academic, either for synchrotron light sources or colliding beams. The factor of five reduction in emittance mentioned above was the result of applying these ideas (conceptually) to a CESR-like lattice. Furthermore, as stated before, colliding beam luminosities are controlled by aperture limitations of the lattice remote from the intersection region. This aperture is dominated (fundamentally anyway) by these very sextupoles. (Though the beam-beam force does not by itself lead to instability, it does send particles to amplitudes larger than would be true for single beam operation, and these particles can be "extracted" by the chromaticity compensation sextupoles.)

All that remains is to show how any ordinary circular accelerator can be converted inexpensively for Möbius operation. The simplest scheme conceptually uses a single solenoid of length L and longitudinal field B. Since the dominant, linearized, effect of such an element is to rotate transverse betatron amplitudes through angle KL, where $K = cB/(2pc/e)$. one requires $KL = \pi/2$. However such an element cannot simply be inserted into an available straight section, both because the proper beam matching will not in general be satisfied at that location, and because the linearized solenoid model includes half-drifts of length $L/2$ and a "lens", inverse focal length K^2/L, focusing in both planes. These could probably be accomodated, but there is a considerably simpler and more general purpose design using skew quadrupoles. It is illustrated in Fig. 2, and is based on a spectrometer design due to Kowalski and Enge.[3]

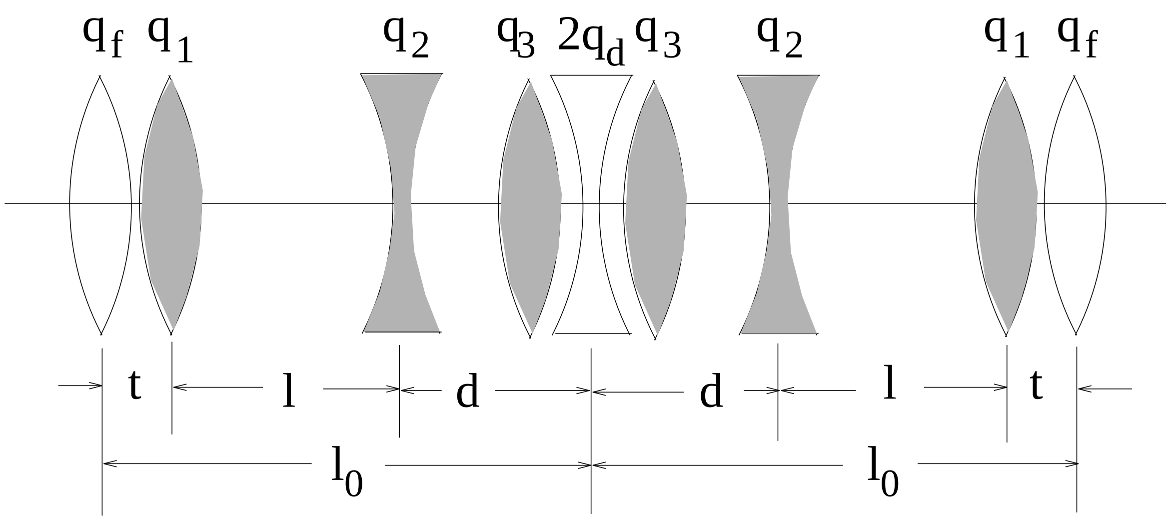

Figure 2: Lattice section needed to switch between ordinary and Möbius operation. For ordinary operation the unshaded elements are run as normal equal-tune FODO elements. For Möbius operation the central element q_d is turned off and the shaded, skew quadrupole elements are powered.

Most accelerators have simple FODO sections; one such section is indicated by the unshaded lenses (quadrupoles) indicated in the figure. The six, shaded, skew-quadrupoles interchange x and y. As well as performing this transformation, the line has an "effective" or "optical" length $\mathcal{L}$. Even while meeting the interchange requirement, $\mathcal{L}$ is to some extent adjustable. In particular it can be adjusted so that, $\mathcal{L} + 2t = l_0$, where t is an ideally negligible end length. With the optical length between quads q_f being l_0, what was originally a full FODO cell has become optically a half-cell which also interchanges x and y. The formulas that described the match of the original full cell, now demonstrate a matched interchange $\beta_x \leftrightarrow \beta_y$. Typical quadrupole to achieve this have strengths not exceeding three times the strengths of the original FODO quadrupoles. Practical conceptual designs exist for incorporating Möbius twists into both CESR and the Tevatron, and most of the features described in this report have been confirmed using realistic simulations.

References

1. A.N. Filippov *et. al.*, Proc. HEACC Conf. Hamburg, 1992, ed. J. Rossbach, p. 1145.
2. Oide, K. and Koiso, H., Phys. Rev. E 47, 2010 (1993).
3. Kowalski, S.K. and Enge, H., Proc. Int. Conf. Magnet. Technol. (Brookhaven), p.181 (1972).